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SMALL-TIME ASYMPTOTICS FOR FAST MEAN-REVERTING STOCHASTIC VOLATILITY MODELS[1]

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In this paper, we study stochastic volatility models in regimes where the maturity is small, but large compared to the mean-reversion time of the stochastic volatility factor. The problem falls in the class of averaging/homogenization problems for nonlinear HJB-type equations where the "fast variable" lives in a noncompact space. We develop a general argument based on viscosity solutions which we apply to the two regimes studied in the paper. We derive a large deviation principle, and we deduce asymptotic prices for out-of-the-money call and put options, and their corresponding implied volatilities. The results of this paper generalize the ones obtained in Feng, Forde and Fouque [*SIAM J. Financial Math.* **1** (2010) 126–141] by a moment generating function computation in the particular case of the Heston model.

1. Introduction. On one hand, the theory of large deviations has been recently applied to local and stochastic volatility models [1, 2, 4, 5, 20] and has given very interesting results on the behavior of implied volatilities near maturity. (An implied volatility is the volatility parameter needed in the Black–Scholes formula in order to match a call option price; it is common practice to quote prices in volatility through this transformation.) In the context of stochastic volatility models, the rate function involved in the large deviation estimates is given in terms of a distance function, which in general cannot be calculated in closed form. For particular models, such as the SABR model [19, 21], approximations obtained by expansion techniques have been proposed; see also [18, 22, 28]. Semi closed form expressions for short time implied volatilities have been obtained in [15].

On the other hand, multi-factor stochastic volatility models have been studied during the last ten years by many authors (see, e.g., [8, 16, 18, 27, 29]). They are quite efficient in capturing the main features of implied volatilities known as smiles and skews, but they are usually not simple to calibrate. In the presence of separated time scales, an asymptotic theory has been proposed in [16, 17]. It has

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the advantage of capturing the main effects of stochastic volatility through a small number of group parameters arising in the asymptotic. The fast time scale expansion is related to the ergodic property of the corresponding fast mean-reverting stochastic volatility factor.

It is natural to try to combine these two modeling aspects and limiting results, by considering short maturity options computed with fast mean-reverting stochastic volatility models, in such a way that maturity is of order $\varepsilon \ll 1$, and the mean-reversion time, δ, of volatility is even smaller of order $\delta = \varepsilon^2$ (fast mean-reversion) or $\delta = \varepsilon^4$ (ultra-fast mean-reversion).

In [12], the authors studied the particular case of the Heston model in the regime $\delta = \varepsilon^2$ by an explicit computation of the moment generating function of the stock price and its asymptotic analysis.

In this paper, we establish a large deviation principle for general stochastic volatility models in the two regimes of fast and ulta-fast mean-reversion, and we derive asymptotic smiles/skews. For such general dynamics, a moment generating function approach is no longer available. Our problem falls in the class of homogenization/averaging problems for nonlinear HJB-type equations where the "fast variable" lives in a noncompact space. We develop a general argument based on viscosity solutions which we apply to the two regimes studied in the paper. Viscosity solution techniques have been used in averaging of nonlinear HJB equations over noncompact space in [3]. However, the techniques in [3] were proved for a certain class of nonlinear HJB equations which does not include our case. In this paper, we develop a method more general than [3]. In particular, it can be used to treat the problems in [5], but not vice versa.

We start by considering the following stochastic differential equations modeling the evolution of the stock price (S_t) under a risk-neutral pricing probability measure, and with a stochastic volatility determined by a process (Y_t):

$$dS_t = rS_t\,dt + \sigma(Y_t)S_t\,dW_t^{(1)}, \tag{1.1a}$$

$$dY_t = \frac{1}{\delta}(m - Y_t)\,dt + \frac{\nu}{\sqrt{\delta}}Y_t^{\beta}\,dW_t^{(2)}, \tag{1.1b}$$

where $m \in \mathbb{R}, r, \nu > 0$, $W^{(1)}$ and $W^{(2)}$ are standard Brownian motions with $\langle W^{(1)}, W^{(2)}\rangle_t = \rho t$, with $|\rho| < 1$ constant. The process (Y_t) is a fast mean-reverting process with rate of mean reversion $1/\delta$ $(\delta > 0)$. The parameters β and $\sigma(y)$ are chosen to satisfy the following.

ASSUMPTION 1.1. We assume that:

(1) $\beta \in \{0\} \cup [\frac{1}{2}, 1)$;

(2) in the case of $\beta = 1/2$, we require $m > \nu^2/2$ and $Y_0 > 0$ a.s., in the case of $1/2 < \beta < 1$, we require $m > 0$ and $Y_0 > 0$ a.s.;

(3) $\sigma(y) \in C(\mathbb{R}; \mathbb{R}_+)$ satisfies

$$\sigma(y) \leq C(1 + |y|^\sigma)$$

for some constants $C > 0$ and σ with $0 \leq \sigma < 1 - \beta$.

These assumptions ensure existence and uniqueness of a strong solution of (1.1). This can be seen as a combination of existence of martingale problem solution (e.g., Theorem 5.3.10 in Ethier and Kurtz [9]) and the Yamada–Watanabe theory for 1-D diffusions (e.g., Chapter 5, Karatzas and Shreve [23]). In particular, Assumption 1.1(2) ensures that, in the case $\beta \in [\frac{1}{2}, 1)$, $Y_t > 0$ a.s. for all $t \geq 0$ (see Appendix A). In the case $\beta = 0$, Y is an Ornstein–Uhlenbeck (OU) process with a natural state space $(-\infty, \infty)$. In order to present both model cases using one simple set of notation, we denote the state space for Y as E_0 with $E_0 := \mathbb{R}$ if $\beta = 0$ and $E_0 := (0, \infty)$ when $\beta \in [\frac{1}{2}, 1)$.

Note that the Heston model, for which $\beta = 1/2$ and $\sigma(y) = \sqrt{y}$, does not satisfy these assumptions, but it has been treated separately in [12] by explicit computation of the moment generating function.

The infinitesimal generator of the Y process, when $\delta = 1$, can be identified with the following differential operator on the class of smooth test functions vanishing off compact sets:

$$B := (m - y)\partial_y + \tfrac{1}{2}\nu^2|y|^{2\beta}\partial_{yy}^2. \tag{1.2}$$

Following the general theory of 1-D diffusion (e.g., Karlin and Taylor [24], page 221), we introduce the so called scale and speed measure of the (Y_t) process,

$$s(y) := \exp\left\{-\int_1^y \frac{2(m-z)}{\nu^2|z|^{2\beta}}\,dz\right\}, \qquad m(y) := \frac{1}{\nu^2|y|^{2\beta}s(y)}.$$

Denoting $dS(y) := s(y)\,dy$ and $dM(y) := m(y)\,dy$, we then have

$$Bf(y) = \frac{1}{2}\frac{d}{dM}\left[\frac{df(y)}{dS}\right]. \tag{1.3}$$

Under Assumption 1.1 there exists a unique probability measure

$$\pi(dy) := Z^{-1}m(y)\,dy, \qquad Z := \int_{E_0} m(y)\,dy < \infty \tag{1.4}$$

such that $\int Bf\,d\pi = 0$ for all $f \in C_c^2(E_0)$. See Appendix C.

By a change of variable $X_t = \log S_t$, we have

$$dX_t = \bigl(r - \tfrac{1}{2}\sigma^2(Y_t)\bigr)\,dt + \sigma(Y_t)\,dW_t^{(1)}.$$

In order to study small time behavior of the system, we rescale time $t \mapsto \varepsilon t$ for $0 < \varepsilon \ll 1$; denoting the rescaled processes by $X_{\varepsilon,\delta,t}$ and $Y_{\varepsilon,\delta,t}$, we have, in distribution,

$$dX_{\varepsilon,\delta,t} = \varepsilon\left(r - \frac{1}{2}\sigma^2(Y_{\varepsilon,\delta,t})\right)dt + \sqrt{\varepsilon}\sigma(Y_{\varepsilon,\delta,t})\,dW_t^{(1)}, \tag{1.5a}$$

$$dY_{\varepsilon,\delta,t} = \frac{\varepsilon}{\delta}(m - Y_{\varepsilon,\delta,t})\,dt + \nu\sqrt{\frac{\varepsilon}{\delta}}Y_{\varepsilon,\delta,t}^{\beta}\,dW_t^{(2)}. \tag{1.5b}$$

We are interested in understanding the two-scale $\varepsilon, \delta \to 0$ limit behavior of option prices and its implication to implied volatility. In this paper, we restrict our attention to the following two regimes:

$$\delta = \varepsilon^4 \quad \text{and} \quad \delta = \varepsilon^2.$$

In view of [12], to obtain a large deviation estimate of option prices, it is sufficient to obtain a large deviation principle (LDP) for $\{X_{\varepsilon,\delta,t} : \varepsilon > 0\}$. By Bryc's inverse Varadhan lemma [7] (Theorem 4.4.2), we know that the key step is proving convergence of the following functionals:

$$u_{\varepsilon,\delta}(t, x, y) := \varepsilon \log E\big[e^{\varepsilon^{-1}h(X_{\varepsilon,\delta,t})} | X_{\varepsilon,\delta,0} = x, Y_{\varepsilon,\delta,0} = y\big], \qquad h \in C_b(\mathbb{R}), \tag{1.6}$$

to some quantity independent of y. The rate function in the LDP is then given in terms of a variational formula involving the limit of the functionals $u_{\varepsilon,\delta}$.

For each $h \in C_b(\mathbb{R})$, the function $u_{\varepsilon,\delta}$ satisfies a nonlinear partial differential equation given in (3.4). In Section 3.2, we use heuristic arguments to obtain PDEs that characterize the limit of these $u_{\varepsilon,\delta}$. Proving this convergence rigorously, however, is nontrivial. Intuitively we know that, as Y has a mean reversion rate $1/\delta$ and $\delta \ll \varepsilon$, the effect of the Y process should get averaged out. To be exact, the form of nonlinear operator (3.5) indicates that convergence of $u_{\varepsilon,\delta}$ is an averaging problem (over the fast y variable) for Hamilton–Jacobi equations. Such problems, in the context of compact state space for the averaging variable, can be handled by extending standard linear equation techniques using viscosity solution language. The Y process in this article lies in E_0, which is $\mathbb{R}$ in the case of $\beta = 0$ and $(0, \infty)$ in other cases. E_0 is a noncompact space, and therein lies an additional difficulty.

We adapt methods developed in Feng and Kurtz [13]. Indeed, an abstract method for large deviation for sequence of Markov processes, based on convergence of HJB equation, is developed fully in [13]. The two schemes treated in this article are of the nature of Examples 1.8 and 1.9, introduced in Chapter 1, and proved in detail in Chapter 11 of [13]. In this article, we not only present a direct proof, but also introduce some argument to further simplify [13] in the setting of multi-scale. This is possible in a large part due to the locally compact state space and mean-reverting nature of the process Y.

In particular, modulo technical subtleties in verification of conditions, the setup of Section 11.6 in [13] corresponds to the large deviation result in our case of

$\delta = \varepsilon^2$. Since E_0 is locally compact, and we only deal with PDEs instead of abstract operator equations, great simplification of [13] can be achieved through the use of a special class of test functions. See Conditions 4.1 and 4.2. The techniques we introduce (Lemmas 4.1 and 4.2) are not limited to averaging problems, but are also applicable to problems of homogenization, which we will not delve into in this article. The rigorous justification of convergence of $u_{\varepsilon,\delta}$ is shown in Section 5.

The main results of the paper are stated in Section 2. Theorem 2.1 is a rare event large deviation-type estimate corresponding to short time, out-of-the-money option pricing. Corollary 2.1 and Theorem 2.2 give asymptotics of option price and implied volatility, respectively, for such situations. The proofs are given in the sections that follow, starting with heuristic proofs in Section 3.2 and finishing with rigorous justifications in Sections 4 and 5. The technical results in Lemmas 4.1 and 4.2 may be of independent interest.

2. Main results. Observe that in the SDE (1.5), while the scaled log stock price process runs on a time scale of order ε, the scaled Y process runs on a time scale of order ε/δ. This is due to the extremely short mean-reversion time, $\delta = \varepsilon^r$ ($r = 2, 4$), of the $Y_{\varepsilon,\delta,\cdot}$ process. Thus, as ε approaches zero, long-time behavior of the unscaled Y process comes into play. This long-time behavior of the Y process manifests itself in the large deviation principle (LDP) of the scaled log stock price via the quantities $\overline{\sigma}^2$ and $\overline{H}_0$ defined below. Define

$$\overline{\sigma}^2 := \int \sigma^2(y)\pi(dy); \tag{2.1}$$

the average of the volatility function $\sigma^2(\cdot)$ with respect to the invariant distribution of Y. Recall B, the generator of the Y process, defined in (1.2). Define the perturbed generator

$$B^p g(y) = Bg(y) + \rho\sigma\nu y^\beta p\, \partial_y g(y), \qquad g \in C_c^2(E_0). \tag{2.2}$$

Let Y^p be the process corresponding to generator B^p, and define

$$\overline{H}_0(p) := \limsup_{T\to+\infty} \sup_{y\in E_0} T^{-1} \log E\bigl[e^{(1/2)|p|^2 \int_0^T \sigma^2(Y_s^p)\,ds} | Y_0^p = y\bigr]. \tag{2.3}$$

Y^p has strong enough ergodic properties that the limit above does not depend upon y even if we omitted the $\sup_{y\in E_0}$; and, in fact, the $\limsup_{T\to\infty}$ can be replaced with $\lim_{T\to\infty}$ in the above definition. We will justify this fact in the rigorous derivations. By Girsanov's transformation

$$\overline{H}_0(p) = \limsup_{T\to+\infty} T^{-1} \log E[e^{\int_0^T \rho p\sigma(Y_s)\,dW^{(2)}(s) + ((1-\rho^2)/2)|p|^2 \int_0^T \sigma^2(Y_s)\,ds}], \tag{2.4}$$

where Y is the process with generator B. From this expression, we see that $\overline{H}_0$ is convex and superlinear in p. $\overline{H}_0(p)$ is the scaled limit of the log moment generating function of a function of occupation measures of the process Y^p. As such, it has an equivalent representation in terms of the rate function for the LDP of occupation measures of Y^p. This equivalent representation of $\overline{H}_0$ is given in (5.12) in Section 5.2.

Having defined these crucial terms, we proceed to the statement of our results.

THEOREM 2.1 (Large deviation). *Assume* $X_{\varepsilon,\varepsilon^r,0} = x_0$ *and* $Y_{\varepsilon,\varepsilon^r,0} = y_0$ *where* $r = 2, 4$ *and suppose that Assumption* 1.1 *holds. For* $x \in \mathbb{R}$, *let*

$$I_4(x; x_0, t) := \frac{|x_0 - x|^2}{2\overline{\sigma}^2 t}, \tag{2.5}$$

where $\overline{\sigma}$ *is defined in* (2.1) *and*

$$I_2(x; x_0, t) := t\overline{L}_0\left(\frac{x_0 - x}{t}\right), \tag{2.6}$$

where $\overline{L}_0$ *is the Legendre transform of* $\overline{H}_0$ *defined in* (2.3).

Then, for each regime $r \in \{2, 4\}$, *for every fixed* $t > 0$ *and* $x_0 \in \mathbb{R}$, $y_0 \in E_0$, *a large deviation principle* (*LDP*) *holds for* $\{X_{\varepsilon,\varepsilon^r,t} : \varepsilon > 0\}$ *with speed* $1/\varepsilon$ *and good rate function* $I_r(x; x_0, t)$. *In particular,*

$$\lim_{\varepsilon \to 0} \varepsilon \log P(X_{\varepsilon,\varepsilon^r,t} > x) = -I(x; x_0, t) \qquad \text{when } x > x_0. \tag{2.7}$$

Similarly, when $x < x_0$, *we have*

$$\lim_{\varepsilon \to 0} \varepsilon \log P(X_{\varepsilon,\varepsilon^r,t} < x) = -I(x; x_0, t). \tag{2.8}$$

REMARK 2.1. The rate functions $I_r(x; x_0, t)$, in both regimes, are convex, continuous functions of x and $I_r(x_0; x_0, t) = 0$.

REMARK 2.2. In the case $\delta = \varepsilon^4$, observe that the rate function I_4, in (2.5), is the same as the rate function for the Black–Scholes model with constant volatility $\overline{\sigma}$. In other words, in the ultra fast regime, to the leading order, it is the same as averaging first and then taking the short maturity limit.

REMARK 2.3. In the case $\delta = \varepsilon^2$, no explicit formula for the rate function is obtained. However, an explicit formula of the rate function is obtained for the Heston model in [12] which corroborates the formula in (2.6). The Heston model per se does not fall in the category of stochastic volatility models covered in this paper, but direct computation of $\overline{H}_0$, given by (2.3) and $\overline{L}_0$, its Legendre transform, is possible for this model.

Let $S_0 > 0$ be the initial value of stock price, and let $X_{\varepsilon,\varepsilon^r,0} = x_0 = \log S_0$. The asymptotic behavior of the price of out-of-the-money European call option with strike price K and short maturity time $T = \varepsilon t$ is given in the following corollary. We only consider out-of-the-money call options by taking

$$S_0 < K \quad \text{or} \quad x_0 < \log K. \tag{2.9}$$

The other case, $S_0 > K$, is easily deduced by considering out-of-the-money European put options and using put-call parity.

COROLLARY 2.1 (Option price). *For fixed $t > 0$,*

$$\lim_{\varepsilon \to 0^+} \varepsilon \log E[e^{-r\varepsilon t}(S_{\varepsilon,\varepsilon^r,t} - K)^+] = -I_r(\log K; x_0, t)$$

for $r = 2, 4$.

Denote the Black–Scholes implied volatility for out-of the-money European call option, with strike price K, by $\sigma_{r,\varepsilon}(t, \log K, x_0)$, where $r = 2, 4$ correspond to the two regimes. By the same argument used in [12], we get an asymptotic formula for implied volatility:

THEOREM 2.2 (Implied volatilities).

$$\lim_{\varepsilon \to 0^+} \sigma^2_{r,\varepsilon}(t, \log K, x_0) = \frac{(\log K - x_0)^2}{2I_r(\log K; x_0, t)t}.$$

REMARK 2.4. In the case $\delta = \varepsilon^4$, the implied volatility is $\overline{\sigma}$, which is obtained by averaging the volatility term $\sigma^2(y)$ with respect to the equilibrium measure for Y. It is likely that more features of the Y process, beyond its equilibrium, will be manifested in higher order terms of implied volatility. Studying the next order term of implied volatility is a topic for future research.

REMARK 2.5. The limit of at-the-money implied volatility, that is, $\lim_{\varepsilon \to 0} \sigma^2_{r,\varepsilon}(t, x_0, x_0)$, is obtained as in [12], Lemma 2.6. However, the continuity of the limiting implied volatility at $\log K = x_0$ is not obvious in the $r = 2$ case. We discuss this at the end of Section 6.3.

3. Preliminaries. The process $(X_{\varepsilon,\delta}, Y_{\varepsilon,\delta})$ is Markovian, and can be identified through a martingale problem given by generator

$$\begin{aligned} A_{\varepsilon,\delta} f(x,y) = \varepsilon\Big(\Big(r - \frac{1}{2}\sigma^2(y)\Big)\partial_x f(x,y) + \frac{1}{2}\sigma^2(y)\,\partial^2_{xx} f(x,y)\Big) \\ + \frac{\varepsilon}{\delta} Bf(x,y) + \frac{\varepsilon}{\sqrt{\delta}}\rho\sigma(y)\nu y^\beta\,\partial^2_{xy} f(x,y), \end{aligned} \tag{3.1}$$

where $f \in C^2_c(\mathbb{R} \times E_0)$. Recall that B is given by (1.2). Let $g \in C_b(\mathbb{R})$ and define

$$v_{\varepsilon,\delta}(t,x,y) := E[g(X_{\varepsilon,\delta,t}) | X_{\varepsilon,\delta,0} = x, Y_{\varepsilon,\delta,0} = y]. \tag{3.2}$$

In general, $v_{\varepsilon,\delta} \in C_b([0,T] \times \mathbb{R} \times E_0)$. If, moreover, $v_{\varepsilon,\delta} \in C^{1,2}([0,T] \times \mathbb{R} \times \mathbb{R})$, then it solves the following Cauchy problem in classical sense:

$$\partial_t v = A_{\varepsilon,\delta} v \qquad \text{in } (0,T] \times \mathbb{R} \times E_0; \tag{3.3a}$$

$$v(0,x,y) = g(x), \qquad (x,y) \in \mathbb{R} \times E_0. \tag{3.3b}$$

3.1. *Logarithmic transformation method.* Recall the definition of $u_{\varepsilon,\delta}$ in (1.6). That is, $u_{\varepsilon,\delta} := \varepsilon \log v_{\varepsilon,\delta}$ when $g(x) = e^{\varepsilon^{-1} h(x)}$, $h \in C_b(\mathbb{R})$, in (3.2). By (3.3) and some calculus, at least informally, (3.4) below is satisfied. This is the logarithmic transform method by Fleming and Sheu. See Chapters VI and VII in [14]. In general, in the absence of knowledge on smoothness of $v_{\varepsilon,\delta}$, we can only conclude that $u_{\varepsilon,\delta}$ solves the Cauchy problem (3.4) in the sense of viscosity solution (Definition 4.1). In addition to Fleming and Soner [14], such arguments can also be found in Section 5 of Feng [11].

LEMMA 3.1. *For $h \in C_b(\mathbb{R})$, $u_{\varepsilon,\delta}$ defined as in* (1.6), *is a bounded continuous function satisfying the following nonlinear Cauchy problem in viscosity solution sense*:

$$\partial_t u = H_{\varepsilon,\delta} u \qquad \textit{in } (0,T] \times \mathbb{R} \times E_0; \tag{3.4a}$$

$$u(0,x,y) = h(x), \qquad (x,y) \in \mathbb{R} \times E_0. \tag{3.4b}$$

In the above,

$$\begin{aligned} H_{\varepsilon,\delta} u(t,x,y) &= \varepsilon e^{-\varepsilon^{-1} u} A_{\varepsilon,\delta} e^{\varepsilon^{-1} u}(t,x,y) \\ &= \varepsilon \biggl(\biggl(r - \frac{1}{2}\sigma^2(y) \biggr) \partial_x u + \frac{1}{2}\sigma^2(y)\, \partial_{xx}^2 u \biggr) \\ &\quad + \frac{1}{2} |\sigma(y)\, \partial_x u|^2 + \frac{\varepsilon^2}{\delta} e^{-\varepsilon^{-1} u} B e^{\varepsilon^{-1} u} \\ &\quad + \rho \sigma(y) \nu y^\beta \biggl(\frac{\varepsilon}{\sqrt{\delta}}\, \partial_{xy}^2 u + \frac{1}{\sqrt{\delta}}\, \partial_x u\, \partial_y u \biggr), \end{aligned} \tag{3.5}$$

where

$$\frac{\varepsilon^2}{\delta} e^{-\varepsilon^{-1} u} B e^{\varepsilon^{-1} u} = \frac{\varepsilon}{\delta} B u + \delta^{-1} \frac{1}{2} |\nu y^\beta\, \partial_y u|^2.$$

Note that $H_{\varepsilon,\delta}$ only operates on the spatial variables x and y.

3.2. *Heuristic expansion.* By Bryc's inverse Varadhan lemma (e.g., Theorem 4.4.2 of [7]), we know that convergence of $u_{\varepsilon,\delta}$ is a necessary condition to obtain the LDP for $\{X_{\varepsilon,\delta,t} : \varepsilon > 0\}$. In this section, we describe heuristically PDEs characterizing $u_{\varepsilon,\delta}$ in the limit and the nature of convergence itself.

Henceforth, for notational simplicity, we will drop the subscript δ and write u_ε and H_ε for $u_{\varepsilon,\delta}$ and $H_{\varepsilon,\delta}$, respectively. We begin by the following heuristic expansion of u_ε in integer powers of ε:

$$u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4 + \cdots \tag{3.6}$$

in both regimes. The $u_i, i = 0, 1, \dots,$ are functions of t, x, y. In this heuristic section, we make reasonable choices of u_i which a posteriori, following a rigorous proof of the convergence of u_ε in Section 5, are shown to be the right choice.

3.2.1. *The case of* $\delta = \varepsilon^4$. Computation of $H_\varepsilon u_\varepsilon$ [see (3.5)] reveals that, in this scale, the fast process Y oscillates so fast that averaging occurs up to terms of order ε^2. Namely, $u_0 = u_0(t, x)$, $u_1 = u_1(t, x)$ and $u_2 = u_2(t, x)$ will not depend on y. To see this, we equate coefficients of powers of ε in $\partial_t u_\varepsilon = H_\varepsilon u_\varepsilon$.

Terms of $O(\frac{1}{\varepsilon^4})$ satisfy

$$0 = \tfrac{1}{2}\nu^2 y^{2\beta}(\partial_y u_0)^2,$$

so we choose u_0 independent of y. With this choice of u_0 the equation for the coefficients of the next order terms, which is of $O(\frac{1}{\varepsilon^2})$, reduces to

$$0 = Bu_1 + \tfrac{1}{2}\nu^2 y^{2\beta}(\partial_y u_1)^2.$$

This equation is satisfied by choosing u_1 independent of y. With this choice of u_1, the equation for coefficients of the next order terms, of $O(\frac{1}{\varepsilon})$, becomes

$$0 = Bu_2.$$

By choosing u_2 independent of y the last equation is satisfied.

Thus, by these choices of u_0, u_1 and u_2 independent of y, it follows that

$$\begin{aligned} H_\varepsilon u_\varepsilon(x, y) = {} & \tfrac{1}{2}|\sigma(y)\,\partial_x u_0|^2 + Bu_3 \\ & + \varepsilon\big(\sigma^2(y)\,\partial_x u_0\,\partial_x u_1 + \tfrac{1}{2}\sigma^2(y)\,\partial_{xx} u_0 \\ & \qquad + \big(r - \tfrac{1}{2}\sigma^2(y)\big)\,\partial_x u_0 + \nu\rho\sigma(y)y^\beta\,\partial_x u_0\,\partial_y u_3 + Bu_4\big) \\ & + o(\varepsilon). \end{aligned}$$

The ε^0 order terms then satisfy

$$\partial_t u_0(t, x) = \tfrac{1}{2}|\partial_x u_0(t, x)|^2\sigma^2(y) + Bu_3(t, x, y),$$

that is,

$$Bu_3(t, x, y) = \partial_t u_0(t, x) - \tfrac{1}{2}|\partial_x u_0(t, x)|^2\sigma^2(y).$$

The above is a Poisson equation for u_3 with respect to the operator B in the y variable. We impose the condition that the right-hand side is centered with respect to the invariant distribution π [given in (1.4)]. This ensures a solution to the Poisson equation, which is unique up to a constant in y. See Appendix B for growth estimates of the solution. Therefore we get

$$\partial_t u_0(t,x) = \frac{1}{2}|\overline{\sigma}\,\partial_x u_0(t,x)|^2;$$

where

$$\overline{\sigma}^2 = \int \sigma^2(y)\pi(dy).$$

Thus the leading order term in the heuristic expansion satisfies

$$\partial_t u_0 = \overline{H}_0 u_0(x), \qquad t > 0; \tag{3.7a}$$

$$u_0(0,x) = h(x), \tag{3.7b}$$

where

$$\overline{H}_0 u_0(x) := \frac{1}{2}|\overline{\sigma}\,\partial_x u_0(x)|^2.$$

3.2.2. *The case of* $\delta = \varepsilon^2$. When δ goes to zero at a slower rate ε^2, limits become very different and more features in the Y process (rather than just its equilibrium) is retained. We observe that while u_0 is independent of y as in the faster scaling regime, u_1 may now depend on y. Equating coefficients of $O(\varepsilon^{-2})$ in $\partial_t u_\varepsilon = H_\varepsilon u_\varepsilon$ we get

$$0 = \frac{1}{2}\nu^2 y^{2\beta}(\partial_y u_0)^2,$$

and so we choose $u_0 = u_0(t,x)$ independent of y. Then $H_\varepsilon u_\varepsilon$ reduces to

$$\begin{aligned} H_\varepsilon u_\varepsilon(t,x,y) = {}& \frac{1}{2}|\sigma(y)\,\partial_x u_0|^2 + \rho\sigma(y)\nu y^\beta\,\partial_x u_0\,\partial_y u_1 + e^{-u_1}Be^{u_1} \\ & + \varepsilon\big(\sigma^2(y)\,\partial_x u_0\,\partial_x u_1 + \tfrac{1}{2}\sigma^2(y)\,\partial_{xx}u_0 + \big(r - \tfrac{1}{2}\sigma^2(y)\big)\,\partial_x u_0 \\ & \quad + Bu_2 + \nu y^{2\beta}\,\partial_y u_1\,\partial_y u_2 + \rho\sigma(y)\nu y^\beta\,\partial_{xy}u_1 \\ & \quad\quad + \rho\sigma(y)\nu y^\beta\,\partial_x u_1\,\partial_y u_1 + \rho\sigma(y)\nu y^\beta\,\partial_x u_0\,\partial_y u_2\big) \\ & + o(\varepsilon). \end{aligned}$$

The leading order terms should satisfy

$$\begin{aligned} \partial_t u_0(t,x) = {}& \frac{1}{2}|\partial_x u_0(t,x)|^2\sigma^2(y) + \rho\nu\sigma(y)y^\beta\,\partial_x u_0(t,x)\,\partial_y u_1(t,x,y) \\ & + e^{-u_1}Be^{u_1}(t,x,y). \end{aligned} \tag{3.8}$$

We will rewrite the above equation as an eigenvalue problem. Recall B, the generator of the Y process defined in (1.2) and the perturbed generator B^p defined in (2.2). Then

$$e^{-u_1}Be^{u_1} + \rho\sigma(y)\nu y^\beta\,\partial_x u_0\,\partial_y u_1 = e^{-u_1}B^{\partial_x u_0(t,x)}e^{u_1}. \tag{3.9}$$

Fix t and x, and rewrite (3.8) in terms of the perturbed generator (3.9).

$$e^{-u_1} B^{\partial_x u_0(t,x)} e^{u_1}(t,x,y) + \frac{1}{2}|\partial_x u_0(t,x)|^2 \sigma^2(y) = \partial_t u_0(t,x).$$

Multiplying the above equation by e^{u_1}, we get the eigenvalue problem

$$(B^{\partial_x u_0} + V) g(y) = \lambda g(y), \tag{3.10}$$

where $V(\cdot) = \frac{1}{2}|\partial_x u_0(t,x)|^2 \sigma^2(\cdot)$ is a multiplicative potential operator, $g(\cdot) = e^{u_1(t,x,\cdot)}$ and $\lambda(t,x) = \partial_t u_0(t,x)$. Choose u_1 such that (λ, g) is the solution to the principal (positive) eigenvalue problem (3.10). Note that the dependence of the eigenvalue, λ, on t and x is only through $\partial_x u_0$. If (3.10) can be solved with a nice g, then we have

$$\lambda(t,x) = \overline{H}_0(\partial_x u_0), \tag{3.11}$$

where $\overline{H}_0$ is defined as (2.3). The leading order terms then satisfy

$$\partial_t u_0(t,x) = \overline{H}_0(\partial_x u_0(t,x)). \tag{3.12}$$

Constructing a classical solution for (3.10) is a considerably hard problem, even in the 1-D situation. If (3.10) can be solved with a nice g, then (2.3) always holds with the $\overline{H}_0$ given by (3.11). The converse is not always true. Especially, (2.3) says nothing about the eigenfunction g. However, we only need the definition in (2.3) in rigorous treatment of the problem. We will show (in Section 5.2) that (3.12) is the limit equation where $\overline{H}_0$ is given by (2.3) irrespective of whether a solution to the eigenvalue problem (3.10) exists or does not.

To summarize,

$$\partial_t u_0(t,x) = \overline{H}_0(\partial_x u_0(t,x)), \qquad t > 0; \tag{3.13a}$$

$$u_0(0,x) = h(x), \tag{3.13b}$$

where $\overline{H}_0$ is given by (2.3) or (2.4).

4. Convergence of HJB equations. The results of this section can be independently read from the rest of the article.

We reformulate and simplify some techniques, regarding multi-scale convergence of HJB equations, introduced in [13]. Compared with [13], the simplification makes ideas more transparent and readily applicable. These are made possible because we are dealing with Euclidean state spaces which are locally compact. All these results are generalizations of Barles–Perthame's half-relaxed limit argument first introduced in single scale, compact state space setting.

Let $E \subset \mathbb{R}^m$, $E_0 \subset \mathbb{R}^n$ and $E' := E \times E_0 \subset \mathbb{R}^d$ where $d = m + n$. A typical element in E is denoted as x, and a typical element in E' is denoted as $z = (x, y)$ with $x \in E$ and $y \in E_0$. We denote a class of compact sets in E'

$$\mathcal{Q} := \{K \times \tilde{K}: \text{compact } K \subset\subset E, \text{compact } \tilde{K} \subset\subset E_0\}.$$

We specify a family of differential operators next. Let Λ be an index set and

$$H_i(x, p, P; \alpha): E \times \mathbb{R}^m \times M_{m\times m} \times \Lambda \mapsto \mathbb{R}, \qquad i = 0, 1;$$

$$H_\varepsilon(z, p, P): E' \times \mathbb{R}^d \times M_{d\times d} \mapsto \mathbb{R}$$

be continuous. For each $f \in C^2(\mathbb{R}^d)$, let $\nabla f(x) \in \mathbb{R}^d$ and $D^2 f(x) \in M_{d\times d}$, respectively, denote gradient and Hessian matrix evaluated at x. We consider a sequence of differential operators

$$H_\varepsilon f(z) := H_\varepsilon(z, \nabla f(z), D^2 f(z))$$

for f belongs to the following two domains:

$$D_{\varepsilon,+} := \{f : f \in C^2(E'),\ f \text{ has compact finite level sets}\};$$

$$D_{\varepsilon,-} := -D_{\varepsilon,+} := \{-f : f \in C^2(E'),\ f \text{ has compact finite level sets}\}.$$

We will separately consider these two domains depending on the situation of sub- or super-solution. We also define domains D_+, D_- similarly replacing E' by E.

We will give conditions where $u_\varepsilon(t, z) = u_\varepsilon(t, x, y)$ solving

$$\partial_t u_\varepsilon(t, z) = H_\varepsilon(z, \nabla u_\varepsilon(t, z), D^2 u_\varepsilon(t, z)) \tag{4.1}$$

converging to $u(t, x)$ which is a sub-solution to

$$\partial_t u(t, x) \le \inf_{\alpha\in\Lambda} H_0(x, \nabla u(t, x), D^2 u(t, x); \alpha) \tag{4.2}$$

and a super-solution to

$$\partial_t u(t, x) \ge \sup_{\alpha\in\Lambda} H_1(x, \nabla u(t, x), D^2 u(t, x); \alpha). \tag{4.3}$$

The meaning of sub- super-solutions is defined as follows (as, e.g., in Fleming and Soner [14]).

DEFINITION 4.1 (Viscosity sub- super-solutions). We call a bounded measurable function u a viscosity sub-solution to (4.2) [resp., super-solution to (4.3)], if u is upper semicontinuous (resp., lower semicontinuous), and for each

$$u_0(t, x) = \phi(t) + f_0(x), \qquad \phi \in C^1(\mathbb{R}_+),\ f_0 \in D_+,$$

and each $x_0 \in E$ satisfying $u - u_0$ has a local maximum [resp., each

$$u_1(t, x) = \phi(t) + f_1(x), \qquad \phi \in C^1(\mathbb{R}_+),\ f_1 \in D_-,$$

and each $x_0 \in E$ satisfying $u - u_1$ has a local minimum] at x_0, we have

$$\partial_t u_0(t_0, x_0) - \inf_{\alpha\in\Lambda} H_0(x_0, \nabla u_0(t_0, x_0), D^2 u_0(t_0, x_0); \alpha) \le 0,$$

respectively,

$$\partial_t u_1(t_0, x_0) - \sup_{\alpha\in\Lambda} H_1(x_0, \nabla u_1(t_0, x_0), D^2 u_1(t_0, x_0); \alpha) \ge 0.$$

If a function is both a sub- as well as a super-solution, then it is a solution.

We will assume the following two conditions.

CONDITION 4.1 (limsup convergence of operators). For each $f_0 \in D_+$ and each $\alpha \in \Lambda$, there exists $f_{0,\varepsilon} \in D_{\varepsilon,+}$ (may depend on α) such that:

(1) for each $c > 0$, there exists $K \times \tilde{K} \in \mathcal{Q}$ satisfying

$$\{(x, y) : H_\varepsilon f_{0,\varepsilon}(x, y) \geq -c\} \cap \{(x, y) : f_{0,\varepsilon}(x, y) \leq c\} \subset K \times \tilde{K};$$

(2) for each $K \times \tilde{K} \in \mathcal{Q}$,

$$\lim_{\varepsilon \to 0} \sup_{(x,y) \in K \times \tilde{K}} |f_{0,\varepsilon}(x, y) - f_0(x)| = 0; \tag{4.4}$$

(3) whenever $(x_\varepsilon, y_\varepsilon) \in K \times \tilde{K} \in \mathcal{Q}$ satisfies $x_\varepsilon \to x$,

$$\limsup_{\varepsilon \to 0} H_\varepsilon f_{0,\varepsilon}(x_\varepsilon, y_\varepsilon) \leq H_0(x, \nabla f_0(x), D^2 f_0(x); \alpha). \tag{4.5}$$

CONDITION 4.2 (liminf convergence of operators). For each $f_1 \in D_-$ and each $\alpha \in \Lambda$, there exists $f_{1,\varepsilon} \in D_{\varepsilon,-}$ (may depend on α) such that:

(1) for each $c > 0$, there exists $K \times \tilde{K} \in \mathcal{Q}$ satisfying

$$\{(x, y) : H_\varepsilon f_{1,\varepsilon}(x, y) \leq c\} \cap \{(x, y) : f_{1,\varepsilon}(x, y) \geq -c\} \subset K \times \tilde{K};$$

(2) for each $K \times \tilde{K} \in \mathcal{Q}$,

$$\lim_{\varepsilon \to 0} \sup_{(x,y) \in K \times \tilde{K}} |f_1(x) - f_{1,\varepsilon}(x, y)| = 0;$$

(3) whenever $(x_\varepsilon, y_\varepsilon) \in K \times \tilde{K} \in \mathcal{Q}$, and $x_\varepsilon \to x$,

$$\liminf_{\varepsilon \to 0} H_\varepsilon f_{1,\varepsilon}(x_\varepsilon, y_\varepsilon) \geq H_1(x, \nabla f_1(x), D^2 f_1(x); \alpha).$$

Let u_ε be the viscosity solutions to (4.1); we define

$$u_3(t, x) := \sup\Big\{\limsup_{\varepsilon \to 0+} u_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) : \exists (t_\varepsilon, x_\varepsilon, y_\varepsilon) \in [0, T] \times K \times \tilde{K},$$
$$(t_\varepsilon, x_\varepsilon) \to (t, x), K \times \tilde{K} \in \mathcal{Q}\Big\},$$
$$u_4(t, x) := \inf\Big\{\liminf_{\varepsilon \to 0+} u_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon) : \exists (t_\varepsilon, x_\varepsilon, y_\varepsilon) \in [0, T] \times K \times \tilde{K},$$
$$(t_\varepsilon, x_\varepsilon) \to (t, x), K \times \tilde{K} \in \mathcal{Q}\Big\},$$

and $\overline{u} = u_3^*$ the upper semicontinuous regularization of u_3 and $\underline{u} = (u_4)_*$ the lower semicontinuous regularization of u_4.

LEMMA 4.1. *Suppose that* $\sup_{\varepsilon > 0} \|u_\varepsilon\|_\infty < \infty$. *Then*:

(1) *under Condition* 4.1, $\overline{u}$ *is a sub-solution to* (4.2);
(2) *under Condition* 4.2, $\underline{u}$ *is a super-solution to* (4.3).

PROOF. Let $u_0(t,x) = \phi(t) + f_0(x)$ for a fixed $\phi \in C^1(\mathbb{R}_+)$ and $f_0 \in D_+$. Let (t_0, x_0) be a local maximum of $\overline{u} - u_0$, $t_0 > 0$. We can modify f_0 and ϕ if necessary so that (t_0, x_0) is a strict global maximum, for instance, by taking $\tilde{f}_0(x) = f_0(x) + k|x - x_0|^4$ and $\tilde{\phi}(t) = \phi(t) + k|t - t_0|^2$ for $k > 0$ large enough. Note that such modification has the property that

$$\lim_{\varepsilon \to 0+} \sup_{|x - x_0| < \varepsilon} |\nabla \tilde{f}_0(x) - \nabla f_0(x_0)| + |D^2 \tilde{f}_0(x) - D^2 f_0(x_0)| = 0.$$

Let $\tilde{u}_0 = \tilde{\phi} + \tilde{f}_0$.

Let $\alpha \in \Lambda$ be given. We now take $u_{0,\varepsilon}(t,z) = \tilde{\phi}(t) + f_{0,\varepsilon}(z)$ where $f_{0,\varepsilon}$ is the approximate of $\tilde{f}_0$ in Condition 4.1. Since u_ε is bounded, and $u_{0,\varepsilon}$ has compact level sets, there exists $(t_\varepsilon, z_\varepsilon) \in [0,T] \times E'$ such that

$$(u_\varepsilon - u_{0,\varepsilon})(t_\varepsilon, z_\varepsilon) \geq (u_\varepsilon - u_{0,\varepsilon})(t,z) \qquad \text{for } (t,z) \in [0,T] \times E' \tag{4.6}$$

and

$$\partial_t \tilde{\phi}(t_\varepsilon) - H_\varepsilon f_{0,\varepsilon}(z_\varepsilon) \leq 0. \tag{4.7}$$

The above implies $\inf_\varepsilon H_\varepsilon f_{0,\varepsilon}(z_\varepsilon) > -\infty$. We verify next that $f_{0,\varepsilon}(z_\varepsilon) < c < \infty$. Then by Condition 4.1(1), there exists $K \times \tilde{K} \in \mathcal{Q}$ such that $z_\varepsilon = (x_\varepsilon, y_\varepsilon) \in K \times \tilde{K}$.

Take a $(\hat{t}, \hat{x})$ such that $\tilde{u}_0(\hat{t}, \hat{x}) < \infty$. Take $\hat{z} = (\hat{x}, \hat{y})$ for some $\hat{y} \in E_0$. Then

$$u_{0,\varepsilon}(\hat{t}, \hat{z}) = \tilde{\phi}(\hat{t}) + f_{0,\varepsilon}(\hat{z}) \to \tilde{\phi}(\hat{t}) + f_0(\hat{x}) = \tilde{u}_0(\hat{t}, \hat{x}) < \infty.$$

Combined with (4.6),

$$u_{0,\varepsilon}(t_\varepsilon, z_\varepsilon) \leq 2 \sup_{\varepsilon > 0} \|u_\varepsilon\|_\infty + \sup_{\varepsilon > 0} u_{0,\varepsilon}(\hat{t}, \hat{z}) < \infty,$$

and $\sup_{\varepsilon > 0} f_{0,\varepsilon}(z_\varepsilon) < \infty$ follows.

Since $K \times \tilde{K}$ is compact in E', there exists a subsequence of $\{(t_\varepsilon, z_\varepsilon)\}$ (to simplify, we still use the ε to index it) and a $(\tilde{t}_0, \tilde{x}_0) \in [0,T] \times E$ such that $t_\varepsilon \to \tilde{t}_0$ and $x_\varepsilon \to \tilde{x}_0$. Such $(\tilde{t}_0, \tilde{x}_0)$ has to be the unique global maximizer (t_0, x_0) for $\overline{u} - \tilde{u}_0$ that appeared earlier. This is because, by using $x_\varepsilon \to \tilde{x}_0$ and $z_\varepsilon = (x_\varepsilon, y_\varepsilon)$, the definition of $\overline{u}$ and (4.4), from (4.6) we have

$$(\overline{u} - u_0)(\tilde{t}_0, \tilde{x}_0) \geq (\overline{u} - u_0)(t,x) \qquad \forall (t,x). \tag{4.8}$$

Now, from (4.7) and (4.5), we also have

$$\partial_t u_0(t_0, x_0) \leq H_0(x_0, \nabla f_0(x_0), D^2 f_0(x_0); \alpha).$$

Note that t_0, x_0 and u_0 are all chosen prior to, and independent of, α. We can take $\inf_{\alpha \in \Lambda}$ on both sides to get

$$\partial_t u_0(t_0, x_0) - \inf_{\alpha \in \Lambda} H_0(x_0, \nabla u_0(t_0, x_0), D^2 u_0(t_0, x_0); \alpha) \leq 0.$$

The proof that $\underline{u}$ is a super-solution of (4.3) under Condition 4.2 follows similarly. □

LEMMA 4.2. *Suppose that the conditions in Lemma 4.1 hold and that there exists $h \in C_b(E)$ such that*

$$\lim_{\varepsilon\to 0} \sup_{(x,y)\in K\times\tilde{K}} |h(x) - u_\varepsilon(0,x,y)| = 0 \qquad \forall K \times \tilde{K} \in \mathcal{Q}.$$

Further suppose that for any sub-solution $u_0(t,x)$ of (4.2) with $u_0(0,x) = h(x)$ and super-solution u_1 of (4.3) with $u_1(0,x) = h(x)$, we have

$$u_0(t,x) \le u_1(t,x), \qquad (t,x) \in [0,T] \times E.$$

That is, a comparison principle holds for sub-solutions of (4.2) and super-solutions of (4.3) with initial data h.

Then $u = \overline{u} = \underline{u}$ and

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \sup_{(x,y)\in K\times\tilde{K}} |u(t,x) - u_\varepsilon(t,x,y)| = 0 \qquad \forall K \times \tilde{K} \in \mathcal{Q}.$$

5. Rigorous justification of expansions. To rigorously prove the convergence of operators H_ε given by (3.5) to operators $\overline{H}_0$ obtained by heuristic arguments in Section 3.2, we rely on and extend results developed in [13]. An exposition of the relevant results from [13] was laid out in Section 4. In this section we verify Conditions 4.1 and 4.2 and prove the comparison principle in Lemma 4.2. We will adhere to the notation used in Section 4.

Conditions 4.1 and 4.2 require us to carefully choose a class of perturbed test functions with an index set Λ and a family of operators $\{H_0(\cdot;\alpha), H_1(\cdot;\alpha); \alpha \in \Lambda\}$ to obtain viscosity sub- and super-solution estimates of u_0, the limit of u_ε. This technique was first introduced in [13] and illustrated through examples in Chapter 11 of that book. Our presentation simplifies the technique in the context of application here. We will make the sub-solution estimate given by $H_0(\cdot,\alpha)$ tight, by inf-ing over α, hence introducing yet another operator H_0. Similarly, we sup over α to tighten up the super-solution type estimate provided by $H_1(\cdot,\alpha)$ which introduces operator H_1.

Let

$$\zeta(y) := |y - m|^\zeta, \tag{5.1}$$

where $\zeta > 0$ is any number satisfying $2\sigma < \zeta < 2(1-\beta)$ with σ and β given as in Assumption 1.1. Throughout the two regimes ($\delta = \varepsilon^4, \varepsilon^2$), we take the index set

$$\Lambda := \{\alpha = (\xi,\theta) : \xi \in C_c^2(E_0), 0 < \theta < 1\};$$

and define two domains

$$D_+ := \{f : f(x) = \varphi(x) + \gamma \log(1 + |x|^2); \varphi \in C_c^2(\mathbb{R}), \gamma > 0\}$$

and

$$D_- := \{f : f(x) = \varphi(x) - \gamma \log(1 + |x|^2); \varphi \in C_c^2(\mathbb{R}), \gamma > 0\}.$$

A collection of compact sets in $\mathbb{R} \times E_0$ is defined by

$$\mathcal{Q} := \{K \times \tilde{K}\text{: compact } K \subset\subset \mathbb{R}, \tilde{K} \subset\subset E_0\}.$$

5.1. *Case* $\delta = \varepsilon^4$. For each $f = f(x) \in D_+$, and each $\alpha = (\xi, \theta) \in \Lambda$, we let

$$g(y) := \xi(y) + \theta\zeta(y)$$

and define perturbed test function

$$f_\varepsilon(x, y) := f(x) + \varepsilon^3 g(y) = f(x) + \varepsilon^3\xi(y) + \varepsilon^3\theta\zeta(y).$$

Note that $\|\partial_x f\|_\infty + \|\partial_{xx}^2 f\|_\infty < \infty$. Then

$$\begin{aligned} H_\varepsilon f_\varepsilon(x, y) = {} & \varepsilon\big[\big(r - \tfrac{1}{2}\sigma^2(y)\big)\,\partial_x f + \tfrac{1}{2}\sigma^2(y)\,\partial_{xx}^2 f\big] + \tfrac{1}{2}\sigma^2(y)|\partial_x f|^2 \\ & + B\xi(y) + \theta B\zeta(y) + \tfrac{1}{2}\varepsilon^2\nu^2 y^{2\beta}|\partial_y\xi(y) + \theta\,\partial_y\zeta(y)|^2 \\ & + \varepsilon\rho\sigma(y)\nu y^\beta\,\partial_x f\big(\partial_y\xi(y) + \partial_y\zeta(y)\big). \end{aligned}$$

The choice of the number ζ in definition of the function $\zeta(y)$ in (5.1) guarantees that $B\zeta(y) \le -C\zeta(y)$. Moreover, with the earlier assumption that $0 \le \sigma < 1 - \beta$, the growth of $\zeta(y)$ as $|y| \to \infty$ dominates the growth in y of all other terms in $H_\varepsilon f_\varepsilon$. Therefore, there exist constants $c_0, c_1 > 0$ with

$$H_\varepsilon f_\varepsilon(x, y) \le \tfrac{1}{2}|\sigma(y)\,\partial_x f(x)|^2 + B\xi(y) - \theta c_0\zeta(y) + \varepsilon c_1.$$

In addition,

$$f_\varepsilon(x, y) = f(x) + \varepsilon^3 g(y) \ge f(x) - \varepsilon^3\|\xi\|_\infty.$$

Furthermore, for each $c > 0$, we can find $K \times \tilde{K} \in \mathcal{Q}$, such that

$$\{(x, y) : H_\varepsilon f_\varepsilon(x, y) \ge -c\} \cap \{(x, y) : f_\varepsilon(x, y) \le c\} \subset K \times \tilde{K} \tag{5.2}$$

verifying Condition 4.1(1). The rest of Condition 4.1 can be verified by taking

$$H_0(x, p; \xi, \theta) = \sup_{y \in E_0}\left(\frac{1}{2}|\sigma(y)p|^2 + B\xi(y) - \theta c_0\zeta(y)\right).$$

We define

$$\begin{aligned} H_0 f(x) &:= \inf_{\alpha \in \Lambda} H_0(x, \partial_x f(x); \alpha) \\ &= \inf_{0<\theta<1}\ \inf_{\xi \in C_c^2(E_0)}\ \sup_{y \in E_0}\left(\frac{1}{2}|\sigma(y)\,\partial_x f(x)|^2 + B\xi(y) - \theta c_0\zeta(y)\right). \end{aligned}$$

Similarly, for $f \in D_-$, $\alpha = (\xi, \theta) \in \Lambda$, we can choose

$$f_\varepsilon(x, y) = f(x) + \varepsilon^3\xi(y) - \varepsilon^3\theta\zeta(y).$$

Then Condition 4.2 holds for the choice of

$$H_1(x,p;\xi,\theta) = \inf_{y\in\mathbb{R}}\left(\frac{1}{2}|\sigma(y)p|^2 + B\xi(y) + \theta c_0\zeta(y)\right).$$

We define

$$\begin{aligned}H_1 f(x) &:= \sup_{\alpha\in\Lambda} H_1(x, \partial_x f(x);\alpha)\\ &= \sup_{0<\theta<1}\sup_{\xi\in C_c^2(E_0)}\inf_{y\in E_0}\left(\frac{1}{2}|\sigma(y)\,\partial_x f(x)|^2 + B\xi(y) + \theta c_0\zeta(y)\right).\end{aligned}$$

Next, to verify Lemma 4.2, we estimate $H_0 f$ from above and $H_1 f$ from below using some simple quantity.

LEMMA 5.1.

$$H_0 f(x) \le \tfrac{1}{2}|\overline{\sigma}\,\partial_x f(x)|^2, \qquad f\in D_+;$$
$$H_1 f(x) \ge \tfrac{1}{2}|\overline{\sigma}\,\partial_x f(x)|^2, \qquad f\in D_-.$$

We note that H_0, H_1 have different domains D_+ and D_-, respectively, $D_+\cap D_- = \varnothing$.

PROOF. The key to obtaining the estimates in the statement of the lemma is the Poisson equation,

$$B\chi(y) = \tfrac{1}{2}|p|^2\big(\overline{\sigma}^2 - \sigma^2(y)\big), \tag{5.3}$$

where B is the differential operator (generator of Y) defined in (1.2). We will need growth estimates for χ. In the case of $\beta = 0$ (i.e., Y is an O–U process), Section 5.2.2 of Fouque, Papanicolaou and Sircar [16] contains such estimates. Specifically, if $\sigma(y)$ is bounded, $|\chi(y)| \le C(1+\log(1+|y|))$; if $\sigma(y)$ has polynomial growth, χ has polynomial growth estimates of the same order. The following growth estimates for the situation $\frac{1}{2}\le\beta<1$ are derived in Appendix B:

$$|\chi'(y)| \le C_1 y^{2\sigma-1} \qquad \text{as } y\to\infty, \text{for some positive constant } C_1. \tag{5.4}$$

Therefore $|\chi(y)| \le C(1+\log(1+|y|))$ if $\sigma(y)$ is bounded and $|\chi(y)| \le \tilde{C}(1+y^{2\sigma})$ when $0<\sigma<1-\beta$.

We will make use of χ as a test function in the expressions for $H_0 f$ and $H_1 f$. However, χ does not have compact support. We choose a cut-off function φ to approximate it using localization arguments. Let nonnegative $\varphi(y)\in C^\infty(E_0)$ be such that $\varphi(y) = 1$ when $|y|\le 1$ and 0 when $|y|>2$. We take a sequence of $\xi_n(y) = \varphi(\frac{y}{n})\chi(y)$, which are truncated versions of χ. Then

$$\begin{aligned}B\xi_n(y) &= \varphi\left(\frac{y}{n}\right)B\chi(y) + (m-y)\chi(y)n^{-1}\varphi'\left(\frac{y}{n}\right)\\ &\quad + \frac{1}{2}\nu^2 y^{2\beta}\chi(y)n^{-2}\varphi''\left(\frac{y}{n}\right) + \nu^2 y^{2\beta}\chi'(y)n^{-1}\varphi'\left(\frac{y}{n}\right).\end{aligned}$$

Suppose $\sigma > 0$. Noting that $|\varphi(y)|$, $|\varphi'(y)|$ and $|\varphi''(y)|$ are uniformly bounded and are 0 when $|y| > 2$, and using the growth estimates (5.4) for χ and χ', we get

$$\begin{aligned}|B\xi_n(y)| &\le cy^{2\sigma}\left(1 + \frac{(m-y)}{n} + \left(\frac{y}{n}\right)^{2\beta} n^{2\beta-2} + y^{\beta-1}\left(\frac{y}{n}\right)^{\beta} n^{\beta-1}\right)1_{\{y/n\le 2\}} \\ &\le cy^{2\sigma} \qquad \text{for all } n.\end{aligned}$$

In the above, we used the fact that $\frac{y}{n} \le 2$ and $\beta - 1 < 0$. Similarly, if $\sigma(y)$ is bounded, that is, $\sigma = 0$, we get $|B\xi_n(y)|$ is uniformly bounded for all n. Therefore, for large y, $\zeta(y)$ dominates $B\xi_n(y)$ uniformly in n in the following sense: there exists a sub-linear function $\psi : \mathbb{R} \mapsto \mathbb{R}_+$ such that

$$\sup_{n=1,2,\dots} |B\xi_n(y)| \le \psi(\zeta(y)).$$

With the above estimate, we have

$$\begin{aligned}H_0 f(x) &\le \limsup_{n\to\infty} \inf_{0<\theta<1} \sup_{y\in E_0}\left(\frac{1}{2}|\sigma(y)\,\partial_x f(x)|^2 + B\xi_n(y) - \theta c_0\zeta(y)\right) \\ &\le \frac{1}{2}|\overline{\sigma}\,\partial_x f(x)|^2.\end{aligned}$$

Similarly, one can prove the case for $H_1 f$. □

By standard viscosity solution theory (e.g., [6]), the comparison principle holds for sub-solutions and super-solutions of

$$\begin{aligned}\partial_t u_0 &= \tfrac{1}{2}|\overline{\sigma}\,\partial_x u_0|^2, \qquad t > 0; \\ u_0(0,x) &= h(x),\end{aligned}$$

and the solution is uniquely given by the Lax formula (see [10]),

$$u_0(t,x) = \sup_{x'\in\mathbb{R}}\left\{h(x') - \frac{|x-x'|^2}{2\overline{\sigma}^2 t}\right\}. \tag{5.5}$$

Putting together the above result and Lemmas 4.1 and 4.2, we get:

LEMMA 5.2.

$$\lim_{\varepsilon\to 0+} \sup_{|t|+|x|+|y|<c} |u_\varepsilon(t,x,y) - u_0(t,x)| = 0 \qquad \forall c > 0,$$

where u_0 is the solution of (3.7) *and is given by* (5.5).

5.2. *Case* $\delta = \varepsilon^2$. For each $f = f(x) \in D_+$ and $\alpha = (\xi, \theta) \in \Lambda$, we choose our perturbed test function as

$$f_\varepsilon(x, y) := f(x) + \varepsilon g(y),$$

where $g(y) = (1-\theta)\xi(y) + \theta\zeta(y)$; $\zeta(y)$ is defined as before in (5.1). Then

$$\begin{aligned} H_\varepsilon f_\varepsilon(x, y) &= \varepsilon\big[\big(r - \tfrac{1}{2}\sigma^2(y)\big)\,\partial_x f + \tfrac{1}{2}\sigma^2(y)\,\partial^2_{xx} f\big] + \tfrac{1}{2}\sigma^2(y)|\partial_x f|^2 \\ &\quad + e^{-g(y)} B^{\partial_x f(x)} e^g(y) \\ &\le \varepsilon\big[\big(r - \tfrac{1}{2}\sigma^2(y)\big)\,\partial_x f + \tfrac{1}{2}\sigma^2(y)\,\partial^2_{xx} f\big] + \tfrac{1}{2}\sigma^2(y)|\partial_x f|^2 \\ &\quad + (1-\theta)e^{-\xi} B^{\partial_x f} e^{\xi}(y) + \theta e^{-\zeta} B^{\partial_x f} e^{\zeta}(y), \end{aligned}$$

where $B^{\partial_x f(x)}$ is the perturbed generator defined in (2.2). Recall that $\|\partial_x f\|_\infty + \|\partial^2_{xx} f\|_\infty < \infty$ by the choice of domain D_+. We can thus find a constant $c_0 > 0$ such that

$$H_\varepsilon f_\varepsilon(x, y) \le \tfrac{1}{2}|\sigma(y)\partial_x f(x)|^2 + (1-\theta)e^{-\xi} B^{\partial_x f} e^{\xi}(y) + \theta e^{-\zeta} B^{\partial_x f} e^{\zeta}(y) + \varepsilon c_0.$$

Note that

$$e^{-\zeta} B^{\partial_x f(x)} e^{\zeta}(y) = B\zeta(y) + \rho\sigma(y)\nu y^\beta\,\partial_x f(x)\,\partial_y\zeta(y) + \tfrac{1}{2}\nu^2 y^{2\beta}|\partial_y\zeta(y)|^2,$$

where

$$B\zeta(y) = -\zeta\cdot|y-m|^\zeta + \tfrac{1}{2}\nu^2 y^{2\beta}\zeta(\zeta-1)|y-m|^{\zeta-2}. \tag{5.6}$$

The term $-\zeta(y)$ in $B\zeta(y)$ dominates growth in y from all other terms in $H_\varepsilon f_\varepsilon$ as $|y| \to \infty$. Since $\zeta(y) \to \infty$ as $|y| \to \infty$, $H_\varepsilon f_\varepsilon(x, y) \to -\infty$ as $|y| \to \infty$. We also have $f_\varepsilon(x, y) = f(x) + \varepsilon g(y) \ge f(x) - \varepsilon\|\xi\|_\infty$. Therefore, for each $c > 0$, we can find $K \times \tilde{K} \in \mathcal{Q}$, such that

$$\{(x, y) : H_\varepsilon f_\varepsilon(x, y) \ge -c\} \cap \{(x, y) : f_\varepsilon(x, y) \le c\} \subset K \times \tilde{K} \tag{5.7}$$

verifying Condition 4.1(1).

The super-solution case follows similarly, where we define the perturbed test function as $f_\varepsilon(x, y) = f(x) + \varepsilon(1+\theta)\xi(y) - \varepsilon\theta\zeta(y)$, for each $f \in D_-$ and $(\xi, \theta) \in \Lambda$.

Take

$$H_0(x, p; \xi, \theta) := \sup_{y \in E_0}\left(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)e^{-\xi} B^p e^{\xi}(y) + \theta e^{-\zeta} B^p e^{\zeta}(y)\right),$$

$$H_1(x, p; \xi, \theta) := \inf_{y \in E_0}\left(\frac{1}{2}|\sigma(y)p|^2 + (1+\theta)e^{-\xi} B^p e^{\xi}(y) - \theta e^{-\zeta} B^p e^{\zeta}(y)\right)$$

and

$$H_0 f(x) := \inf_{0<\theta<1}\inf_{\xi \in C_c^\infty(E_0)} H_0(x, \partial_x f; \xi, \theta),$$

$$H_1 f(x) := \sup_{0<\theta<1}\sup_{\xi \in C_c^\infty(E_0)} H_1(x, \partial_x f; \xi, \theta).$$

Conditions 4.1 and 4.2 are satisfied by these choices of H_0 and H_1. Note that, although $\frac{1}{2}|\sigma(y)p|^2$ is not bounded in y, its growth is at most $|y|^{2\sigma}$ and is dominated by the growth of $\zeta(y)$ for $|y|$ large enough.

To verify Lemma 4.2, we develop useful sharp estimates for H_0 and H_1 next. Denote

$$T(t)g(y) := E[g(Y_t)|Y(0) = y], \qquad g \in C_b(E_0),$$

and let $\mathbb{B}$ be the weak infinitesimal generator for semigroup $\{T(t) : t \geq 0\}$ in $C_b(E_0)$ (see page 244 of [13] for a definition of a weak infinitesimal generator). Let $D^{++}(\mathbb{B})$ denote the domain of $\mathbb{B}$ with functions strictly bounded from below by a positive constant. Similarly define notations for $\mathbb{B}^p$, the weak infinitesimal generator corresponding to the process Y^p introduced in Section 3.2.2. For each $g \in D^{++}(\mathbb{B}^p) \subset C_b(E_0)$, since $\zeta > 2\sigma$, there exists compact $K \subset\subset E_0$ with

$$\begin{aligned}\sup_{y\in E_0}&\left(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)\frac{\mathbb{B}^p g}{g}(y) + \theta e^{-\zeta}B^p e^{\zeta}(y)\right)\\ &= \sup_{y\in K}\left(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)\frac{\mathbb{B}^p g}{g}(y) + \theta e^{-\zeta}B^p e^{\zeta}(y)\right).\end{aligned}$$

For each $\varepsilon > 0$, by truncating and mollifying g, we can find a $\xi := \xi_\varepsilon \in C_c^\infty(E_0)$ such that

$$H_0(x, p; \xi, \theta) \leq \varepsilon + \sup_{y\in K}\left(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)\frac{\mathbb{B}^p g}{g}(y) + \theta e^{-\zeta}B^p e^{\zeta}(y)\right).$$

Denote $p = \partial_x f(x)$. Then

$$\begin{aligned}H_0 f(x) \leq \inf_{0<\theta<1}\inf_{g\in D^{++}(\mathbb{B}^p)}\sup_{y\in E_0}&\left(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)\frac{\mathbb{B}^p g}{g}(y)\right.\\ &\left.+\theta e^{-\zeta}B^p e^{\zeta}(y)\right).\end{aligned} \tag{5.8}$$

Similarly, we have

$$\begin{aligned}H_1 f(x) \geq \sup_{0<\theta<1}\sup_{g\in D^{++}(\mathbb{B}^p)}\inf_{y\in E_0}&\left(\frac{1}{2}|\sigma(y)p|^2 + (1+\theta)\frac{\mathbb{B}^p g}{g}(y)\right.\\ &\left.-\theta e^{-\zeta}B^p e^{\zeta}(y)\right).\end{aligned} \tag{5.9}$$

We define $I_B(\cdot; p) : \mathcal{P}(E_0) \mapsto \mathbb{R} \cup \{+\infty\}$ by

$$I_B(\mu; p) := -\inf_{g\in D^{++}(\mathbb{B}^p)}\int_{E_0}\frac{\mathbb{B}^p g}{g}\,d\mu \wedge \int_{E_0} e^{-\zeta(y)}B^p e^{\zeta(y)}\,d\mu(y).$$

However, we can find a sequence $\{g_n\} \subset D^{++}(\mathbb{B}^p)$ [take, e.g., $g_n := e^{\zeta_n}$ where $\zeta_n \in C_c^2(E_0)$ are some smooth truncations of ζ], such that

$$\int_{E_0} e^{-\zeta(y)} B^p e^{\zeta(y)}\, d\mu(y) \geq \limsup_{n\to\infty} \int_{E_0} \frac{\mathbb{B}^p g_n}{g_n}\, d\mu.$$

Therefore we have

$$I_B(\mu; p) = -\inf_{g\in D^{++}(\mathbb{B}^p)} \int_{E_0} \frac{\mathbb{B}^p g}{g}\, d\mu. \tag{5.10}$$

Recall that Y^p denotes the process corresponding to generator B^p (or, equivalently, $\mathbb{B}^p$). It can be directly verified that Y^p has a unique stationary distribution π^p and that Y^p is reversible with respect to it (see Appendix C of this article). Let

$$\mathcal{E}^p(f, g) := -\int f \mathbb{B}^p g\, d\pi^p$$

be the Dirichlet form for Y^p. By the material in Section 7 of Stroock [30] (particularly Theorem 7.44; note that the diffusion generated by B^p has transition density with respect to Lebesgue measure, e.g., Theorem 4.3.5 of Knight [25]), we get

$$I_B(\mu; p) = \mathcal{E}^p\left(\sqrt{\frac{d\mu}{d\pi^p}}, \sqrt{\frac{d\mu}{d\pi^p}}\right) = \frac{\nu^2}{2}\int_0^\infty y^{2\beta}\left|\partial_y\sqrt{\frac{d\mu}{d\pi^p}}(y)\right|^2 \pi^p(dy); \tag{5.11}$$

see Appendix C.3 for the last equality above. If μ in $I_B(\mu; p)$ is not absolutely continuous with respect to π^p, then the right-hand quantity in (5.11) is viewed as $+\infty$. Again through Theorem 7.44 of [30], we also get that $\overline{H}_0$, defined in (2.3), can be expressed as

$$\begin{aligned}\overline{H}_0(p) &= \sup_{\mu\in\mathcal{P}(\mathbb{R}_+)}\left(\frac{|p|^2}{2}\int_{\mathbb{R}_+}\sigma^2\, d\mu - I_B(\mu; p)\right)\\ &= \sup_{h\in L^2(\pi^p), \|h\|_{L^2(\pi^p)}=1}\left(\frac{|p|^2}{2}\int_{\mathbb{R}_+}\sigma^2(y)h^2(y)\pi^p(dy)\right.\\ &\qquad\left. - \frac{\nu^2}{2}\int_0^\infty y^{2\beta}|\partial h(y)|^2\pi^p(dy)\right).\end{aligned} \tag{5.12}$$

As in Lemma 11.35 of [13],

$$\inf_{0<\theta<1}\inf_{g\in D^{++}(\mathbb{B}^p)}\sup_{y\in E_0}\left(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)\frac{\mathbb{B}^p g}{g}(y) + \theta e^{-\zeta} B^p e^{\zeta}(y)\right) = \overline{H}_0(p).$$

Using (5.8), this immediately gives

$$H_0 f(x) \leq \overline{H}_0(\partial f(x)), \qquad f \in D_+.$$

We will prove a similar inequality estimate for H_1, hence giving the following:

LEMMA 5.3.

$$H_1 f(x) \geq \overline{H}_0(\partial f(x)), \qquad f \in D_-,$$
$$H_0 f(x) \leq \overline{H}_0(\partial f(x)), \qquad f \in D_+.$$

It remains to prove the estimate for H_1. By the proof of Lemma B.10 of [13],

$$\begin{aligned} &\sup_{0<\theta<1} \sup_{g\in D^{++}(\mathbb{B}^p)} \inf_{y\in\mathbb{R}_+} \left(\frac{1}{2}|\sigma(y)p|^2 + (1+\theta)\frac{\mathbb{B}^p g}{g}(y) - \theta e^{-\zeta} B^p e^{\zeta}(y)\right) \\ &\quad \geq \inf_{\nu\in\mathcal{P}(\mathbb{R}_+),\langle\zeta,\nu\rangle<+\infty} \liminf_{t\to\infty} t^{-1}\log E^{\nu}\bigl[e^{(1/2)|p|^2\int_0^t \sigma^2(Y_s^p)\,ds}\bigr]. \end{aligned} \tag{5.13}$$

We show that:

LEMMA 5.4.

$$\liminf_{t\to+\infty} t^{-1}\log E\bigl[e^{(1/2)|p|^2\int_0^t\sigma^2(Y_s^p)\,ds}|Y_0^p = y\bigr] \geq \overline{H}_0(p). \tag{5.14}$$

PROOF. The proof of (5.14) follows essentially the same argument used in Example B.14 in the Appendix of [13], which we will outline. Two ingredients need to be emphasized. First, for each μ with $I_B(\mu; p) < \infty$, by a mollification and truncation argument, we can find a sequence $\mu_n(dy) = \frac{e^{h_n(y)}}{\int e^{h_n}\,d\pi^p}\,d\pi^p(y)$ with $h_n + c_n \in C_c^\infty(E_0)$ for some constant c_n, such that $\lim_{n\to\infty} I_B(\mu_n; p) = I_B(\mu; p)$. Second, for every $y \in E_0$ and every $h \in C_c^\infty(E_0)$, the following ergodic theorem holds:

$$\lim_{t\to\infty} \frac{1}{t} E\biggl[\int_0^t \sigma^2(\tilde{Y}_s^h)\,ds|\tilde{Y}_0^h = y\biggr] = \int_{-\infty}^{\infty} \sigma^2(z)\,d\tilde{\pi}^h(z), \tag{5.15}$$

where

$$d\tilde{Y}_s^h = \bigl((m - \tilde{Y}_s^h) + \rho p\sigma(\tilde{Y}_s^h)\nu(\tilde{Y}_s^h)^\beta + \nu^2(\tilde{Y}_s^h)^{2\beta}\,\partial h(\tilde{Y}_s^h)\bigr)\,ds + \nu(\tilde{Y}_s^h)^\beta\,dW_s^2,$$

and where $\tilde{\pi}^h$ is the unique stationary distribution of $\tilde{Y}^h$. We will prove (5.15) in Lemma 5.5.

The process $\tilde{Y}^h$ is Y^p under the Girsanov transformation of measures

$$\frac{dP^h}{dP}\bigg|_{\mathcal{F}_t} = \exp\biggl\{h(Y_t) - h(Y_0) - \int_0^t e^{-h} B^p e^h(Y_s)\,ds\biggr\},$$

where P and P^h refer to the probability measures of the processes Y^p and $\tilde{Y}^h$, respectively. The invariant distribution of $\tilde{Y}^h$ is then

$$d\tilde{\pi}^h = \frac{e^{2h}\,d\pi^p}{\int e^{2h}\,d\pi^p}.$$

We can write

$$
\begin{aligned}
\liminf_{t\to+\infty} & \frac{1}{t}\log E^P\left[\exp\left\{\frac{1}{2}|p|^2\int_0^t \sigma^2(Y_s^p)\,ds\right\}\Big|Y_0^p=y\right]\\
&= \lim_{t\to\infty}\frac{1}{t}\log E^{P^h}\left[\exp\left\{\frac{1}{2}|p|^2\int_0^t \sigma^2(\tilde{Y}_s^h)\,ds\right.\right.\\
&\qquad\qquad - \left(h(\tilde{Y}_t^h) - h(\tilde{Y}_0^h)\right.\\
&\qquad\qquad\qquad \left.\left.\left. - \int_0^t e^{-h}B^p e^h(\tilde{Y}_s^h)\,ds\right)\right\}\Big|\tilde{Y}_0^h=y\right]\\
&\geq \lim_{t\to\infty}\frac{1}{t}E^{P^h}\left[\frac{1}{2}|p|^2\int_0^t \sigma^2(\tilde{Y}_s^h)\,ds\right.\\
&\qquad - \left(h(\tilde{Y}_t^h) - h(\tilde{Y}_0^h)\right.\\
&\qquad\qquad \left.\left. - \int_0^t e^{-h}B^p e^h(\tilde{Y}_s^h)\,ds\right)\Big|\tilde{Y}_0^h=y\right]\\
&\qquad \text{(by Jensen's inequality)}\\
&= \frac{1}{2}|p|^2\int_{-\infty}^{\infty}\sigma^2(z)\,d\tilde{\pi}^h(z)\\
&\qquad + \int_{-\infty}^{\infty} e^{-h}B^p e^h(z)\,d\tilde{\pi}^h(z) \qquad \text{(by ergodicity of } \tilde{Y}^h)\\
&= \frac{1}{2}|p|^2\int_{-\infty}^{\infty}\sigma^2(z)\,d\tilde{\pi}^h(z) - \mathcal{E}^p\left(\sqrt{\frac{d\tilde{\pi}^h}{d\pi^p}},\sqrt{\frac{d\tilde{\pi}^h}{d\pi^p}}\right)\\
&= \frac{1}{2}|p|^2\int_{-\infty}^{\infty}\sigma^2(z)\,d\tilde{\pi}^h(z) - I(\tilde{\pi}^h;p).
\end{aligned}
$$

By arbitrariness of h, (5.14) follows. To complete the proof, we finally check that:

LEMMA 5.5. *Equation* (5.15) *holds*.

PROOF. By Itô's formula,

$$E[\zeta(\tilde{Y}_t^h)] = E[\zeta(\tilde{Y}_0^h)] + E\left[\int_0^t \tilde{B}^h\zeta(\tilde{Y}_s^h)\right]ds,$$

where $\tilde{B}^h\zeta(y) = (m - y + \rho p\sigma(y)\nu y^\beta + \nu^2 y^{2\beta}\,\partial_y h(y))\zeta'(y) + \frac{1}{2}\nu^2 y^{2\beta}\zeta''(y)$. As in (5.6), $-\zeta(y)$ is the dominating growth term in $\tilde{B}^h\zeta(y)$. Therefore, defining a family of mean occupation measure,

$$\tilde{\pi}^h(t,y,A) := E\left[t^{-1}\int_0^t \mathbf{1}_{\{\tilde{Y}_s^h\in A\}}\,ds\Big|\tilde{Y}_0^h=y\right],$$

we have that

$$\sup_{t>0} \int_z \zeta(z)\tilde{\pi}^h(t,y,dz) = \sup_{t>0} t^{-1} E\left[\int_0^t \zeta(\tilde{Y}_s^h)\,ds \Big| \tilde{Y}_0^h = y\right] \le C(y; h(\cdot)) < \infty.$$

Hence $\{\tilde{\pi}^h(t,y,\cdot) : t > 0\}$ is tight and along convergent subsequences and corresponding limiting point $\tilde{\pi}^h$, we have

$$E\left[t^{-1}\int_0^t \varphi(\tilde{Y}_s^h)\,ds \Big| \tilde{Y}_0^h = y\right] \to \int_z \varphi\,d\tilde{\pi}^h, \qquad \varphi \in C_b(E_0). \tag{5.16}$$

Such $\tilde{\pi}^h$ is necessarily a stationary distribution satisfying $\int \tilde{B}^h \psi\,d\tilde{\pi}^h = 0$ for all $\psi \in C_c^2(E_0)$. Uniqueness of such probability measure can be proved by an argument similar to the one in Appendix C. We thus conclude that there is only one such $\tilde{\pi}^h$ and that convergence (5.16) occurs along the whole sequence, not just subsequences. Furthermore, the growth of σ^2 is dominated by ζ, and so by uniform integrability argument, (5.15) holds. □

Now (5.9), (5.13) and (5.14) together give us the estimate for H_1 in Lemma 5.3.

From (2.4), we see that $\overline{H}_0(p)$ is convex in $p \in \mathbb{R}$. Let us denote its Legendre transform as $\overline{L}_0$, then we have the following.

LEMMA 5.6. *The unique viscosity solution to* (3.13) *is*

$$u_0(t,x) := \sup_{x' \in \mathbb{R}} \left\{ h(x') - t\overline{L}_0\left(\frac{x - x'}{t}\right)\right\}. \tag{5.17}$$

Moreover, u_ε *converges uniformly over compact sets in* $[0,T] \times \mathbb{R} \times E_0$ *to* u_0.

PROOF. We know that u_0, defined by (5.17), solves (3.13) by the Lax formula. That u_0 is the unique solution follows from standard viscosity comparison principle with convex Hamiltonians. The convergence result follows from multi-scale viscosity convergence results developed in Section 4, Lemmas 4.1 and 4.2. □

6. Large deviation, asymptotic for option prices and implied volatilities. We finish the proof of Theorem 2.1, Corollary 2.1 and Theorem 2.2.

6.1. *A large deviation theorem.*

PROOF OF THEOREM 2.1. From the previous section we have $u_\varepsilon(t,x,y) \to u_0(t,x)$ as $\varepsilon \to 0$ for each fixed $(t,x,y) \in [0,T] \times \mathbb{R} \times E_0$. All we need is exponential tightness of $\{X_{\varepsilon,\delta,t}\}$ to apply Bryc's lemma and to conclude our proof. This is obtained as follows.

Let $f(x) = \log(1 + x^2)$ and $\zeta(y)$ be defined as in (5.1). Take

$$f_\varepsilon(x,y) = \begin{cases} f(x) + \varepsilon^3\zeta(y), & \text{for the case } \delta = \varepsilon^4, \\ f(x) + \varepsilon\zeta(y), & \text{for the case } \delta = \varepsilon^2. \end{cases}$$

Note that $f(x)$ is an increasing function of $|x|$ and $\zeta(\cdot) \geq 0$; therefore, for any $c > 0$ there exists a compact set $K_c \subset \mathbb{R}$ such that $f_\varepsilon(x, y) > c$ when $x \notin K_c$. We next compute $H_\varepsilon f_\varepsilon(x, y)$ [see (3.5)]. Observe that since $\|\partial_x f\|_\infty + \|\partial^2_{xx} f\|_\infty < \infty$, by our choice of $\zeta(\cdot)$, $H_\varepsilon f_\varepsilon(x, y) \to -\infty$ as $|y| \to \infty$. Therefore $\sup_{x\in R, y\in R} H_\varepsilon f_\varepsilon(x, y) = C < \infty$. For simplicity, we denote $X_{\varepsilon,\delta,t}$ by $X_{\varepsilon,t}$. The P and E below denote probability and expectation conditioned on (X, Y) starting at (x, y).

$$
\begin{aligned}
P(X_{\varepsilon,t} \notin K_c)&e^{(c-f_\varepsilon(x,y)-tC)/\varepsilon} \\
&\leq E\biggl[\exp\biggl\{\frac{f_\varepsilon(X_{\varepsilon,t}, Y_{\varepsilon,t})}{\varepsilon} - \frac{f_\varepsilon(x, y)}{\varepsilon} \\
&\qquad\qquad - \int_0^t e^{-f_\varepsilon(X_{\varepsilon,s},Y_{\varepsilon,s})/\varepsilon} A_\varepsilon e^{f_\varepsilon(X_{\varepsilon,s},Y_{\varepsilon,s})/\varepsilon}\, ds\biggr\}\biggr] \\
&\leq 1.
\end{aligned}
$$

In the above inequalities, the term within expectation in the second line is a non-negative local martingale (and hence a supermartingale); see [9], Lemma 4.3.2. We apply the optional sampling theorem to get the last inequality above. Therefore

$$
\varepsilon \log P(X_{\varepsilon,t} \notin K_c) \leq tC + f_\varepsilon(x, y) - c \leq \mathrm{const} - c
$$

giving us exponential tightness of $X_{\varepsilon,t}$.

Let $u_0^{h,r}$ denote the limit of $u_{\varepsilon,\delta}$ when $u_{\varepsilon,\delta}(0, x, y) = h(x)$ and $\delta = \varepsilon^r$, $r = 2, 4$. Applying Bryc's lemma we get, $\{X_{\varepsilon,\varepsilon^r,t}\}$ for $r = 2, 4$ satisfies a LDP with speed $1/\varepsilon$ and rate function

$$
I_r(x; x_0, t) := \sup_{h\in C_b(R)} \{h(x) - u_0^{h,r}(t, x_0)\}. \tag{6.1}
$$

In Appendix D we check that $I_2(x; x_0, t) = t\overline{L}_0(\frac{x_0-x}{t})$ where $\overline{L}$ is the Legendre transform of $\overline{H}_0$ defined in (2.3), and $I_4 = \frac{|x_0-x|^2}{2\overline{\sigma}^2 t}$. □

6.2. *Option prices.*

PROOF OF COROLLARY 2.1. We follow the proof of Corollary 1.3 in [12] and show that $\lim_{\varepsilon\to 0^+} \varepsilon \log E[(S_{\varepsilon,t} - K)^+]$ is bounded above and below by $-I_r(\log K; x_0, t)$.

Recall that we are considering out-of-the-money call options and hence $x_0 < \log K$ [see (2.9)]. Since our rate functions $I_r(x; x_0, t)$, for both $r = 2, 4$, are non-negative, convex functions with $I_r(x_0; x_0, t) = 0$, they are consequently monotonically increasing functions of x when $x \geq x_0$. Using this fact and the continuity of the rate functions, the proof of the lower bound follows verbatim from the proof in [12]. We refer the reader to [12] for details.

The upper bound follows from [12] once we justify the following limit: for any $p > 1$,

$$\lim_{\varepsilon \to 0^+} \varepsilon \log E[S^p_{\varepsilon,\delta,t}] = 0 \qquad \text{for both } \delta = \varepsilon^4 \text{ and } \delta = \varepsilon^2. \tag{6.2}$$

Recall the operator $A_{\varepsilon,\delta}$ defined at the beginning of Section 3. By a slight abuse of notation, we can use $A_{\varepsilon,\delta}$ to denote the operator acting on the unbounded function e^{px} given below:

$$A_{\varepsilon,\delta} e^{px} = \varepsilon\big((r - \tfrac{1}{2}\sigma^2(y))pe^{px} + \tfrac{1}{2}\sigma^2(y)p^2 e^{px}\big).$$

Let

$$M_t := \exp\biggl\{pX_{\varepsilon,\delta,t} - pX_{\varepsilon,\delta,0} - \int_0^t e^{-pX_{\varepsilon,\delta,s}} A_{\varepsilon,\delta} e^{pX_{\varepsilon,\delta,s}}\, ds\biggr\}.$$

Then M_t is a nonnegative local martingale (supermartingale); this follows from the proof of [9], Lemma 4.3.2. By the optional sampling theorem,

$$EM_t \le 1.$$

Recall that $X_{\varepsilon,\delta,t} = \log S_{\varepsilon,\delta,t}$, then

$$\begin{aligned} E[S^{p/2}_{\varepsilon,\delta,t}] &= E[e^{p/2 X_{\varepsilon,\delta,t}}] \\ &\le (EM_t)^{1/2}\biggl(E\biggl[\exp\biggl\{pX_{\varepsilon,\delta,0} + \int_0^t e^{-pX_{\varepsilon,\delta,s}} A_{\varepsilon,\delta} e^{pX_{\varepsilon,\delta,s}}\, ds\biggr\}\biggr]\biggr)^{1/2} \\ &\qquad \text{(by Hölder's inequality)} \\ &\le 1 \cdot e^{px_0/2}\biggl(E\biggl[\exp\biggl\{\int_0^t e^{-pX_{\varepsilon,\delta,s}} A_{\varepsilon,\delta} e^{pX_{\varepsilon,\delta,s}}\, ds\biggr\}\biggr]\biggr)^{1/2}. \end{aligned} \tag{6.3}$$

We simplify and bound the right-hand side of the above inequality:

$$\begin{aligned} &E\biggl[\exp\biggl\{\int_0^t e^{-pX_{\varepsilon,\delta,s}} A_{\varepsilon,\delta} e^{pX_{\varepsilon,\delta,s}}\, ds\biggr\}\biggr] \\ &\quad = E\biggl[\exp\biggl\{\int_0^t \varepsilon\biggl(\biggl(r - \frac{1}{2}\sigma^2(Y_{\varepsilon,\delta,s})\biggr)p + \frac{1}{2}\sigma^2(Y_{\varepsilon,\delta,s})p^2\biggr)\, ds\biggr\}\biggr] \\ &\quad = e^{\varepsilon r p t} E\biggl[\exp\biggl\{\delta(p^2 - p)\int_0^{\varepsilon t/\delta} \sigma^2(Y_{\varepsilon,\delta,(\delta/\varepsilon)u})\, du\biggr\}\biggr] \\ &\qquad \biggl(\text{by change of variable } u = \frac{\varepsilon}{\delta}s; \text{ recall that } \delta = \varepsilon^2 \text{ or } \varepsilon^4\biggr) \\ &\quad = e^{\varepsilon r p t} E\biggl[\exp\biggl\{\delta(p^2 - p)\int_0^{\varepsilon t/\delta} \sigma^2(Y_u)\, du\biggr\}\biggr], \end{aligned}$$

where Y_u is the process with generator B given in (1.2). By convexity of exponential functions we get

$$E\left[\exp\left\{\int_0^t e^{-pX_{\varepsilon,\delta,s}} A_{\varepsilon,\delta} e^{pX_{\varepsilon,\delta,s}}\, ds\right\}\right] \leq e^{\varepsilon rpt} E\left[\frac{\delta}{t\varepsilon}\int_0^{\varepsilon t/\delta} \exp\{t\varepsilon(p^2-p)\sigma^2(Y_u)\}\, du\right]. \tag{6.4}$$

Since $\delta = \varepsilon^2$ or ε^4, $\varepsilon/\delta \to \infty$ as $\varepsilon \to 0$. Therefore, by the ergodicity of Y and $\exp\{t(p^2-p)\sigma^2(y)\} \in L^1(d\pi)$ [this follows from an argument similar to proof of Lemma 5.5; note that $\sigma < 1-\beta$ by Assumption 1.1(3)], the right-hand side of the above inequality (6.4) is uniformly bounded for all $\varepsilon > 0$. Putting this together with (6.3), we get (6.2). □

6.3. *Implied volatilities*.

PROOF OF THEOREM 2.2. Recall that $X_{\varepsilon,t} = \log S_{\varepsilon,t}$ and $x_0 = \log S_0$. Note that we have dropped the subscript δ in the notation and the dependence on $\delta = \varepsilon^4$ or ε^2 should be understood by context. Our first step is to show that

$$\lim_{\varepsilon\to 0^+} \sigma_\varepsilon(t, \log K, x_0)\sqrt{\varepsilon t} = 0. \tag{6.5}$$

Once we have shown this, the rest of the proof is identical to that of Corollary 1.4 in [12].

By the definition of implied volatility,

$$E[(S_{\varepsilon,t}-K)^+] = e^{r\varepsilon t} S_0 \Phi\left(\frac{x_0 - \log K + r\varepsilon t + \sigma_\varepsilon^2 \varepsilon t/2}{\sigma_\varepsilon\sqrt{\varepsilon t}}\right) - K\Phi\left(\frac{x_0 - \log K + r\varepsilon t - \sigma_\varepsilon^2 \varepsilon t/2}{\sigma_\varepsilon\sqrt{\varepsilon t}}\right), \tag{6.6}$$

where Φ is the Gaussian cumulative distribution function. Let $l \geq 0$ be the limit of $\sigma_\varepsilon\sqrt{\varepsilon t}$ along a converging subsequence. If $\lim_{\varepsilon\to 0^+}$ of the left-hand side of (6.6) is 0, then l satisfies

$$S_0\Phi\left(\frac{x_0-\log K}{l} + \frac{l}{2}\right) - K\Phi\left(\frac{x_0 - \log K}{l} - \frac{l}{2}\right) = 0.$$

The only solution of the above equation is $l = 0$, and thus we get (6.5).

We therefore need to prove

$$\lim_{\varepsilon\to 0^+} E[(S_{\varepsilon,t}-K)^+] = 0. \tag{6.7}$$

By (1.5a) we have

$$S_{\varepsilon,t} - K = S_0 - K + \varepsilon\int_0^t rS_{\varepsilon,t}\, dt + \sqrt{\varepsilon}\int_0^t S_{\varepsilon,t}\sigma(Y_{\varepsilon,t})\, dW_t^{(1)}.$$

It can be verified that $E[(S_{\varepsilon,t} - K) - (S_0 - K)]^2 \to 0$, as $\varepsilon \to 0$, for both cases $\delta = \varepsilon^4$ and $\delta = \varepsilon^2$. Therefore

$$\lim_{\varepsilon \to 0^+} E[(S_{\varepsilon,t} - K)^+] = E[(S_0 - K)^+] = 0$$

as $S_0 < K$ (this is an out-of-the-money call option).

The same formula is obtained when $S_0 > K$ by considering out-of-the-money put options. We finally turn our attention to at-the-money implied volatility. The asymptotic limit of at-the-money (ATM) volatility can be shown to be $\overline{\sigma}^2$, that is,

$$\lim_{\varepsilon \to 0} \sigma^2_{r,\varepsilon}(t, \log K, x_0) = \overline{\sigma}^2 \qquad \text{when } x_0 = \log K; r = 2, 4,$$

by a similar argument as in [12], Lemma 2.6. The continuity, at-the-money, of the limiting implied volatility, that is,

$$\lim_{|\log K - x_0| \to 0} \frac{(\log K - x_0)^2}{2 I_r(\log K, x_0, t) t} = \overline{\sigma}^2$$

is obvious in the $r = 4$ regime, but is more involved in the $r = 2$ regime. We conjecture that it is true, that is,

$$\lim_{z \to 0} \frac{z^2}{2t^2 \overline{L}_0(z/t)} = \overline{\sigma}^2, \tag{6.8}$$

and we briefly indicate an outline of the proof. Let

$$\Lambda_T(p) := T^{-1} \log E\left[e^{\int_0^T \rho p \sigma(Y_s)\, dW_s^{(2)} + ((1-\rho^2)/2)|p|^2 \int_0^T \sigma^2(Y_s)\, ds}\right],$$

so that $\overline{H}_0(p) = \lim_{T\to\infty} \Lambda(p)$. The result (6.8) follows if $\overline{H}_0(p)$ is twice differentiable in a neighborhood of $p = 0$ and $H_0''(0) = \frac{\overline{\sigma}^2}{2}$. It can easily be checked that $\lim_{T\to\infty} \Lambda_T''(0) = \frac{\overline{\sigma}^2}{2}$. The main difficulty is to get a uniform bound on $\Lambda_T'''(p)$ for all T and in a neighborhood of $p = 0$. Obtaining such a uniform bound on $\Lambda_T'''(p)$ involves tedious calculations but should follow from the multiplicative ergodic properties of the Y process (see [26]). $\square$

In the following Appendix, we collect some material regarding 1-D diffusions Y and technical but elementary estimates.

APPENDIX A: POSITIVITY OF THE Y PROCESS

In this section we prove positivity of the Y process when $\frac{1}{2} < \beta < 1$ in (1.1b). Assume $m > 0$ and $Y_0 > 0$. Recall the scale function $s(y)$ defined in the Introduction, and let $S(y) = \int_1^y s(y)\, dy$. By Lemma 6.1(ii) in Karlin and Taylor [24], to prove that Y_t remains positive a.s. for all $t \geq 0$, it is sufficient to show that

$$\lim_{\varepsilon \to 0^+} S(\varepsilon) = -\infty.$$

For $0 < \varepsilon \ll 1$,

$$-S(\varepsilon) = \int_\varepsilon^1 s(y)\,dy = \int_\varepsilon^1 \exp\left\{-\int_1^y \frac{2(m-z)}{\nu^2|z|^{2\beta}}\,dz\right\}dy$$

$$= C\int_\varepsilon^1 \exp\left\{\frac{2m}{\nu^2(2\beta-1)y^{2\beta-1}} + \frac{y^{2-2\beta}}{\nu^2(1-\beta)}\right\}dy$$

(where C is a positive constant and $2\beta-1, 1-\beta > 0$)

$$= \int_{2\varepsilon}^1 (\text{positive integrand})\,dy$$

$$+ C\int_\varepsilon^{2\varepsilon} \exp\left\{\frac{2m}{\nu^2(2\beta-1)y^{2\beta-1}} + \frac{y^{2-2\beta}}{\nu^2(1-\beta)}\right\}dy$$

$$\geq C\varepsilon \exp\left\{\frac{2m}{\nu^2(2\beta-1)(2\varepsilon)^{2\beta-1}}\right\} \to +\infty$$

as $\varepsilon \to 0^+$, provided $m > 0$. Therefore $\lim_{\varepsilon\to 0^+} S(\varepsilon) = -\infty$.

APPENDIX B: GROWTH ESTIMATES FOR SOLUTIONS TO POISSON EQUATIONS

Assume χ satisfies the Poisson equation

$$B\chi(y) = \tfrac{1}{2}|p|^2\big(\overline{\sigma}^2 - \sigma^2(y)\big),$$

where $\overline{\sigma}^2$, defined in (2.1), is the average of $\sigma^2(y)$ with respect to the invariant distribution $\pi(dy)$, given in (1.4), of the Y process. In this section we find growth estimates for χ.

The right-hand side of the above Poisson equation is centered with respect to the invariant distribution $\pi(dy) = \frac{m(y)}{Z}\,dy$ [given in (1.4)], and so

$$\int_0^\infty m(z)\big(\overline{\sigma}^2 - \sigma^2(z)\big)\,dz = 0, \tag{B.1}$$

where

$$m(y) = \frac{1}{\nu^2 y^{2\beta}} \exp\left\{\int_1^y \frac{2(m-z)}{\nu^2 z^{2\beta}}\,dz\right\}.$$

By (1.3),

$$\chi(y) := \int dS(y) \int_0^y |p|^2\big(\overline{\sigma}^2 - \sigma^2(z)\big)\,dM(z)$$

$$= \int \frac{1}{y^{2\beta} m(y)}\left[\int^y \frac{|p|^2 m(z)(\overline{\sigma}^2 - \sigma^2(z))}{\nu^2}\,dz\right]dy$$

is a solution up to a constant, and so

$$\chi'(y) = \frac{|p|^2}{\nu^2 y^{2\beta} m(y)} \left[\int_0^y m(z)\big(\overline{\sigma}^2 - \sigma^2(z)\big)\, dz \right]$$
$$= -\frac{|p|^2}{\nu^2 y^{2\beta} m(y)} \left[\int_y^\infty m(z)\big(\overline{\sigma}^2 - \sigma^2(z)\big)\, dz \right].$$

The last equality is by the centering condition (B.1). Given the bounds on $\sigma(y)$ in Assumption 1.1(3), we can compute the following bounds where the constants, denoted by c, are positive and vary from line to line:

$$|\chi'(y)| \le \frac{c|p|^2}{\nu^2 y^{2\beta} m(y)} \int_y^\infty z^{2\sigma} m(z)\, dz$$
$$= \frac{c|p|^2 e^{\alpha y^{1-2\beta}}}{\nu^2 e^{-(y^{2-2\beta})/(\nu^2(1-\beta))}} \int_y^\infty z^{2\sigma - 2\beta} e^{-\alpha z^{1-2\beta}} e^{-(z^{2-2\beta})/(\nu^2(1-\beta))}\, dz,$$

where $\alpha = \frac{2m}{\nu^2(2\beta-1)} > 0$. Bounding $e^{-\alpha z^{1-2\beta}}$ above by 1 we get

$$|\chi'(y)| \le \frac{c|p|^2 e^{\alpha y^{1-2\beta}}}{\nu^2 e^{-(y^{2-2\beta})/(\nu^2(1-\beta))}} \int_y^\infty z^{2\sigma-2\beta} e^{-(z^{2-2\beta})/(\nu^2(1-\beta))}\, dz$$
$$= \frac{c|p|^2 e^{\alpha y^{1-2\beta}}}{\nu^2 e^{-(y^{2-2\beta})/(\nu^2(1-\beta))}} \int_{y^{2-2\beta}}^\infty u^{(2\sigma-1)/(2-2\beta)} \exp\left\{-\frac{u}{\nu^2(1-\beta)}\right\} du$$

(by change of variable $u = z^{2-2\beta}$)

$$\le \frac{c|p|^2 e^{\alpha y^{1-2\beta}}}{\nu^2 e^{-(y^{2-2\beta})/(\nu^2(1-\beta))}} \left[y^{2\sigma-1} \exp\left\{-\frac{y^{2-2\beta}}{\nu^2(1-\beta)}\right\}\right].$$

In the last inequality we used $\int_a^\infty [u^{(2\sigma-1)/(2-2\beta)} e^{-u/(\nu^2(1-\beta))}]\, du \le \nu^2(1-\beta) a^{(2\sigma-1)/(2-2\beta)} e^{-a/(\nu^2(1-\beta))}$ (since $\frac{2\sigma-1}{2-2\beta} < 0$). Therefore

$$|\chi'(y)| \le \frac{c|p|^2 e^{\alpha y^{1-2\beta}}}{\nu^2} y^{2\sigma-1} \sim c|p|^2 y^{2\sigma-1} \qquad \text{as } y \to \infty,$$

since $e^{\alpha y^{1-2\beta}} \sim O(1)$ as $y \to \infty$.

APPENDIX C: Y^p PROCESS

Fix $p \in \mathbb{R}$. Denote $\mu_p(y) := (m - y) + \rho p \sigma(y) \nu y^\beta$, and let Y^p be the process with generator

$$B^p g = \mu_p(y)\, \partial_y g + \tfrac{1}{2} \nu^2 y^{2\beta}\, \partial_{yy}^2 g, \qquad g \in C_c^2(E_0).$$

In this section we calculate the unique stationary distribution and Dirichlet form of the process Y^p, and we show that it is a reversible process. To this end, we first compute the scale function and speed measure.

The scale function and speed measure for the Y^p process are given by

$$s_p(y) = \exp\left\{-\int_1^y \frac{2\mu_p(z)}{\nu^2 z^{2\beta}}\right\} \quad \text{and} \quad m_p(y) = \frac{2}{\nu^2 y^{2\beta} s_p(y)}.$$

Evaluating the integral in $s_p(y)$ we get (the C below denotes a positive constant that varies from line to line)

$$s_p(y) = \begin{cases} C\exp\left\{-\dfrac{2m\log y}{\nu^2} + \dfrac{y^{2-2\beta}}{\nu^2(1-\beta)} - \dfrac{2\rho p}{\nu}J\right\}, & \text{if } \beta = \dfrac{1}{2}, \\ C\exp\left\{\dfrac{2m}{\nu^2(2\beta-1)y^{2\beta-1}} + \dfrac{y^{2-2\beta}}{\nu^2(1-\beta)} - \dfrac{2\rho p}{\nu}J\right\}, & \\ & \text{if } \beta \in 0 \cup \left(\dfrac{1}{2}, 1\right), \end{cases}$$

where

$$J(y) = \int^y \frac{\sigma(z)}{z^\beta}\,dz.$$

Due to bounds on σ given in Assumption 1.1(3), there exist $C_1, C_2 > 0$ such that

$$C_1 y^{1-\beta} \le J(y) \le C_2 y^{1-\beta+\sigma},$$

where

$$\begin{cases} 0 < 1-\beta \le 1-\beta+\sigma \le 1, & \text{if } \frac{1}{2} \le \beta < 1, \\ 1 = 1-\beta \le 1-\beta+\sigma < 2, & \text{if } \beta = 0. \end{cases}$$

Therefore

$$\text{(C.1)} \qquad \begin{cases} \dfrac{1}{s_p(y)} \to 0 \text{ when } y \to 0 \text{ or } y \to \infty, & \text{if } \dfrac{1}{2} \le \beta < 1, \\ \dfrac{1}{s_p(y)} \to 0 \text{ when } |y| \to \infty, & \text{if } \beta = 0. \end{cases}$$

Define for $y \in E_0$,

$$S_p(y) := \int_1^y s_p(z)\,dz.$$

Observe that $S_p(y) \to -\infty$ as y approaches the left endpoint of E_0 and $S_p(y) \to +\infty$ as $y \to \infty$.

C.1. Stationary distribution. Let π^p be an invariant distribution of the process Y^p. Suppose it has density function $\Psi(y)$, that is, $d\pi^p(y) = \Psi(y)\,dy$, then Ψ is uniquely determined as the solution of

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}(\nu^2 y^{2\beta}\Psi(y)) - \frac{\partial}{\partial y}(\mu_p(y)\Psi(y)) = 0$$

satisfying $\Psi(y) \geq 0$ for all y and $\int_{E_0} \Psi(y)\,dy = 1$. Solving the above differential equation, we get $\Psi(y) = m_p(y)[C_1 S_p(y) + C_2]$. Since Ψ is nonnegative, and $S_p(y) \to -\infty$ as y approaches the left boundary of E_0, we take $C_1 = 0$. The other constant C_2 is uniquely determined by the condition $\int_{E_0} \Psi(y)\,dy = 1$. Therefore π^p is the unique invariant distribution of Y^p and is given by

$$d\pi^p(y) = \Psi(y)\,dy = \frac{m_p(y)}{Z_1}\,dy = \frac{2}{Z_1 \nu^2 y^{2\beta} s_p(y)}\,dy \qquad \text{for } y \in E_0, \tag{C.2}$$

where $Z_1 = \int_{E_0} m_p(y)\,dy$.

C.2. Reversibility. Let $\varphi, \psi \in C_c^2(E_0)$, then

$$\begin{aligned}\int_{E_0} \psi B^p \varphi\, d\pi^p &= \frac{1}{Z_1}\int_{E_0} \psi \left[\frac{1}{2}\nu^2 y^{2\beta}\varphi'' + \mu_p(y)\varphi'\right]\frac{2}{\nu^2 y^{2\beta} s_p(y)}\,dy \\ &= \frac{1}{Z_1}\int_{E_0} \psi \left[\varphi'' e^{\int^y 2\mu_p(y)/(\nu^2 y^{2\beta})} + \frac{2\mu_p}{\nu^2 y^{2\beta}}\varphi' e^{\int^y 2\mu_p(y)/(\nu^2 y^{2\beta})}\right] dy \\ &= \frac{1}{Z_1}\int_{E_0} \psi\, \frac{d}{dy}\left(\frac{\varphi'}{s_p(y)}\right) dy.\end{aligned}$$

Integrating by parts twice and using the boundary conditions (C.1), we get

$$\int_{E_0} \psi B^p \varphi\, d\pi^p = \frac{1}{Z_1}\int_{E_0} \varphi\, \frac{d}{dy}\left(\frac{\psi'}{s_p(y)}\right) dy = \int_{E_0} \varphi B^p \psi\, d\pi^p.$$

C.3. Dirichlet form. By similar calculations as before, when proving reversibility, we get, for $f, g \in L^2(\pi^p)$,

$$\begin{aligned}\mathcal{E}^p(f,g) &:= -\int_{E_0} f B^p g\, d\pi^p \\ &= -\frac{1}{Z_1}\int_{E_0} f(y)\, \frac{d}{dy}\left(\frac{g'(y)}{s_p(y)}\right) dy \\ &= \frac{1}{Z_1}\int_{E_0} f'(y) g'(y)\, \frac{1}{s_p(y)}\,dy \\ &= \frac{\nu^2}{2}\int_{E_0} y^{2\beta} f'(y) g'(y)\, d\pi^p(y),\end{aligned}$$

where we integrated by parts once and used (C.1) in the second last line.

APPENDIX D: RATE FUNCTION FORMULAS

Recall the following characterization of the rate functions given in (6.1):

$$I_r(x; x_0, t) = \sup_{h \in C_b(\mathbb{R})} \{h(x) - u_0^{h,r}(t, x_0)\},$$

where $r = 2, 4$ correspond to the two regimes $\delta = \varepsilon^2$ and $\delta = \varepsilon^4$, respectively. The $u_0^{h,r}$ are given in (5.17) and (5.5), respectively, as

$$u_0^{h,2}(t, x_0) = \sup_{x' \in \mathbb{R}} \left\{ h(x') - t\overline{L}\left(\frac{x_0 - x'}{t}\right) \right\},$$

$$u_0^{h,4}(t, x_0) = \sup_{x' \in \mathbb{R}} \left\{ h(x') - \left(\frac{|x_0 - x'|^2}{2\overline{\sigma}^2 t}\right) \right\}.$$

For notational convenience, we will drop the subscript r in I_r, and, in the case $r = 4$, we will denote the term $(\frac{|x_0 - x'|^2}{2\overline{\sigma}^2 t})$ by $t\overline{L}(\frac{x_0 - x'}{t})$. The rate functions can then be rewritten as

$$I(x; x_0, t) = \sup_{h \in C_b(\mathbb{R})} \inf_{x' \in \mathbb{R}} \left\{ h(x) - h(x') + t\overline{L}\left(\frac{x_0 - x'}{t}\right) \right\}$$

for both regimes $r = 2$ and $r = 4$.

LEMMA D.1.

$$I(x; x_0, t) = t\overline{L}\left(\frac{x_0 - x}{t}\right).$$

PROOF. Note that for both cases $r = 2, 4$, $\overline{L}_0$ is convex, $\overline{L}_0(0) = 0$ and $\overline{L}_0$ is a nonnegative function. This is obvious for the case $r = 4$. We can deduce this in the $r = 2$ case since $\overline{H}_0(p)$ [defined in (2.3)] is convex and $\overline{H}_0(0) = 0$.

Re-write

$$\begin{aligned} I(x; x_0, t) &= t\overline{L}_0\left(\frac{x_0 - x}{t}\right) \\ &\quad + \sup_{h \in C_b(\mathbb{R})} \inf_{x' \in \mathbb{R}} \left\{ h(x) - h(x') + t\overline{L}_0\left(\frac{x_0 - x'}{t}\right) - t\overline{L}_0\left(\frac{x_0 - x}{t}\right) \right\} \\ &= t\overline{L}_0\left(\frac{x_0 - x}{t}\right) + J, \end{aligned}$$

where $J = \sup_{h \in C_b(\mathbb{R})} J_h$ and $J_h = \inf_{x' \in \mathbb{R}} \{h(x) - h(x') + t\overline{L}_0(\frac{x_0 - x'}{t}) - t \times \overline{L}_0(\frac{x_0 - x}{t})\}$. Taking $x' = x$ in the inf we get $J_h \leq 0$ and therefore

$$J \leq 0. \tag{D.1}$$

Note that x_0 and x are fixed. Define a function $h^* \in C_b(\mathbb{R})$ as follows:

$$h^*(x') = t\overline{L}_0\left(\frac{x_0 - x'}{t}\right) \wedge t\overline{L}_0\left(\frac{x_0 - x}{t}\right).$$

Then

$$J_{h^*} = 0$$

and consequently

$$J \geq 0. \tag{D.2}$$

By (D.1) and (D.2), $J = 0$ and we get

$$I(x; x_0, t) = t\overline{L}_0\left(\frac{x_0 - x}{t}\right). \qquad \square$$

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