

OUTPERFORMING THE MARKET PORTFOLIO WITH A GIVEN PROBABILITY

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Our goal is to resolve a problem proposed by Fernholz and Karatzas [On optimal arbitrage (2008) Columbia Univ.]: to characterize the minimum amount of initial capital with which an investor can beat the market portfolio with a certain probability, as a function of the market configuration and time to maturity. We show that this value function is the smallest nonnegative viscosity supersolution of a nonlinear PDE. As in Fernholz and Karatzas [On optimal arbitrage (2008) Columbia Univ.], we do not assume the existence of an equivalent local martingale measure, but merely the existence of a local martingale deflator.

1. Introduction. In this paper we consider the quantile hedging problem when the underlying market does not have an equivalent martingale measure. Instead, we assume that there exists a *local martingale deflator* (a strict local martingale which, when multiplied by the asset prices, yields a positive local martingale). We characterize the value function as the smallest nonnegative viscosity supersolution of a fully nonlinear partial differential equation. This resolves the open problem proposed in the final section of [13]; also see pages 61 and 62 of [38].

Our framework falls under the umbrella of the stochastic portfolio theory of Fernholz and Karatzas (see, e.g., [17–19]) and the benchmark approach of Platen [35]. In this framework, the linear partial differential equation that the superhedging price satisfies does not have a unique solution; see, for example, [14, 15, 18] and [39]. Similar phenomena occur when the asset prices have *bubbles*: an equivalent local martingale measure exists, but the asset prices under this measure are strict local martingales; see, for example, [6, 8, 10, 24, 26] and [27]. A related series of papers [1, 4, 11, 25, 32, 33] and [40], addressed the issue of bubbles in the context of stochastic volatility models. In particular, Bayraktar, Kardaras and Xing

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[4] gave necessary and sufficient conditions for linear partial differential equations appearing in the context of stochastic volatility models to have a unique solution.

In contrast, we show that the quantile hedging problem, which is equivalent to an optimal control problem, is the smallest nonnegative viscosity supersolution to a fully nonlinear PDE. As in the linear case, these PDEs may not have a unique solution, and, therefore, an alternative characterization for the value function needs to be provided. Recently, the authors of [5, 16] and [29] also considered stochastic control problems in this framework. The first reference solves the classical utility maximization problem; the second one solves the optimal stopping problem; whereas the third one determines the optimal arbitrage under model uncertainty, which is equivalent to solving a zero-sum stochastic game.

The structure of the paper is simple: in Section 2, we formulate the problem. In this section we also discuss the implications of assuming the existence of a local martingale deflator. In Section 3, we generalize the results of [20] on quantile hedging, in particular the Neyman–Pearson lemma. We also prove other properties of the value function such as convexity. Section 4 is where we give the PDE characterization of the value function.

2. The model. We consider a financial market with a bond which is always equal to 1, and d stocks $X = (X_1, \dots, X_d)$ which satisfy

$$(2.1) \quad \begin{aligned} dX_i(t) &= X_i(t) \left(b_i(X(t)) dt + \sum_{k=1}^d s_{ik}(X(t)) dW_k(t) \right), \\ i &= 1, \dots, d, X(0) = x = (x_1, \dots, x_d), \end{aligned}$$

where $W(\cdot) := (W_1(\cdot), \dots, W_d(\cdot))$ is a d -dimensional Brownian motion.

Following the set up in [14], Section 8, we make the following assumption.

ASSUMPTION 2.1. Let $b_i : (0, \infty)^d \rightarrow \mathbb{R}$ and $s_{ik} : (0, \infty)^d \rightarrow \mathbb{R}$ be continuous functions. Set $b(\cdot) = (b_1(\cdot), \dots, b_d(\cdot))'$ and $s(\cdot) = (s_{ij}(\cdot))_{1 \leq i, j \leq d}$, which we assume to be invertible for all $x \in (0, \infty)^d$. We also assume that (2.1) has a weak solution that is unique in distribution for every initial value. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space specified by a weak solution. Another assumption we will impose is that

$$(2.2) \quad \sum_{i=1}^d \int_0^T (|b_i(X(t))| + a_{ii}(X(t)) + \theta_i^2(X(t))) dt < \infty, \quad \mathbb{P}\text{-a.s.},$$

where $\theta(\cdot) := s^{-1}(\cdot)b(\cdot)$, $a_{ij}(\cdot) := \sum_{k=1}^d s_{ik}(\cdot)s_{jk}(\cdot)$.

We will denote by $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ the right-continuous version of the natural filtration generated by $X(\cdot)$, and by \mathbb{G} the \mathbb{P} -augmentation of the filtration \mathbb{F} . Thanks to Assumption 2.1, the Brownian motion $W(\cdot)$ of (2.1) is adapted to \mathbb{G} (see, e.g., [14],

Section 2), every local martingale of \mathbb{F} has the martingale representation property; that is, it can be represented as a stochastic integral, with respect to $W(\cdot)$, of some \mathbb{G} -progressively measurable integrand (see, e.g., the discussion on page 1185 in [14]), the solution of (2.1) takes values in the positive orthant and the exponential local martingale

$$(2.3) \quad Z(t) := \exp \left\{ - \int_0^t \theta(X(s))' dW(s) - \frac{1}{2} \int_0^t |\theta(X(s))|^2 ds \right\}, \quad 0 \leq t < \infty,$$

the so-called *deflator* is well defined. We do not exclude the possibility that $Z(\cdot)$ is a strict local martingale.

Let \mathcal{H} be the set of \mathbb{G} -progressively measurable processes $\pi : [0, T) \times \Omega \rightarrow \mathbb{R}^d$, which satisfies

$$\int_0^T (|\pi(t)' \mu(X(t))| + \pi(t)' \alpha(X(t)) \pi(t)) dt < \infty, \quad \mathbb{P}\text{-a.s.},$$

in which $\mu = (\mu_1, \dots, \mu_d)$ and $\sigma = (\sigma_{ij})_{1 \leq i, j \leq d}$ with $\mu_i(x) = b_i(x)x_i$, $\sigma_{ik}(x) = s_{ik}(x)x_i$ and $\alpha(x) = \sigma(x)\sigma(x)'$.

At time t , an investor invests $\pi_i(t)$ proportion of his wealth in the i th stock. The proportion $1 - \sum_{i=1}^d \pi_i(t)$ gets invested in the bond. For each $\pi \in \mathcal{H}$ and initial wealth $y \geq 0$ the associated wealth process will be denoted by $Y^{y, \pi}(\cdot)$. This process solves

$$dY^{y, \pi}(t) = Y^{y, \pi}(t) \sum_{i=1}^d \pi_i(t) \frac{dX_i(t)}{X_i(t)}, \quad Y^{y, \pi}(0) = y.$$

It can be easily seen that $Z(\cdot)Y^{y, \pi}(\cdot)$ is a positive local martingale for any $\pi \in \mathcal{H}$. Let $g : (0, \infty)^d \rightarrow (0, \infty)$ be a measurable function satisfying

$$(2.4) \quad \mathbb{E}[Z(T)g(X(T))] < \infty$$

and define

$$V(T, x, 1) := \inf\{y > 0 : \exists \pi(\cdot) \in \mathcal{H} \text{ s.t. } Y^{y, \pi}(T) \geq g(X(T))\}.$$

Thanks to Assumption 2.1, we have that $V(T, x, 1) = \mathbb{E}[Z(T)g(X(T))]$; see, for example, [18], Section 10. Note that if g has linear growth, then (2.4) is satisfied since the process ZX is a positive supermartingale.

2.1. *A digression: What does the existence of a local martingale deflator entail?* Although we do not assume the existence of equivalent local martingale measures, we assume the existence of a local martingale deflator. This is equivalent to the *No Unbounded Profit with Bounded Risk* (NUPBR) condition; see [29], Theorem 4.12. NUPBR is defined as follows: a sequence (π^n) of admissible portfolios is said to generate a UPBR if $\lim_{m \rightarrow \infty} \sup_n \mathbb{P}[Y^{1, \pi^n}(T) > m] > 0$. If no

such sequence exists, then we say that NUPBR holds; see [29], Proposition 4.2. In fact, the so-called *No Free Lunch with Vanishing Risk* (NFLVR) is equivalent to NUPBR plus the classical *no-arbitrage* assumption. Thus, in our setting (since we assumed the existence of local martingale deflators) although arbitrages exist they remain on the level of “cheap thrills,” which was coined by [34]. (Note that the results of Karatzas and Kardaras [29] also imply that one does not need NFLVR for the portfolio optimization problem of an individual to be well defined. One merely needs the NUPBR condition to hold.) The failure of no-arbitrage means that the money market is not an optimal investment and is dominated by other investments. It follows that a short position in the money market and long position in the dominating assets leads one to arbitrage. However, one cannot scale the arbitrage and make an arbitrary profit because of the admissibility constraint, which requires the wealth to be positive. This is what is contained in NUPBR, which holds in our setting. Also, see [31], where these issues are further discussed.

3. On quantile hedging. In this section, we develop new probabilistic tools to extend results of Föllmer and Leukert [20] on quantile hedging to settings where equivalent martingale measures need not exist. This is not only mathematically intriguing, but also economically important because it admits arbitrage in the market, which opens the door to the notion of optimal arbitrage, recently introduced in Fernholz and Karatzas [14]. The tools in this section facilitate the discussion of quantile hedging under the context of optimal arbitrage, leading us to generalize the results of [14] on this sort of probability-one outperformance.

We will try to determine

$$(3.1) \quad V(T, x, p) = \inf\{y > 0 \mid \exists \pi \in \mathcal{H} \text{ s.t. } \mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} \geq p\}$$

for $p \in [0, 1]$. Note that the set on which we take infimum in (3.1) is nonempty. Indeed, under condition (2.4), there exists $\pi \in \mathcal{H}$ such that $Y^{y,\pi}(T) = g(X(T))$ a.s., where $y := \mathbb{E}[Z(T)g(X(T))]$; see, for example, [18], Section 10. It follows that for any $p \in [0, 1]$,

$$\mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} = 1 \geq p.$$

Also observe that

$$\begin{aligned} \tilde{V}(T, x, p) &:= \frac{V(T, x, p)}{g(x)} \\ &= \inf\{r > 0 \mid \exists \pi \in \mathcal{H} \text{ s.t. } \mathbb{P}\{Y^{rg(x),\pi}(T) \geq g(X(T))\} \geq p\}. \end{aligned}$$

When $g(x) = \sum_{i=1}^d x_i$, observe that $\tilde{V}(T, x, 1)$ is equal to equation (6.1) of [14], the smallest relative amount to beat the market capitalization $\sum_{i=1}^d X_i(T)$.

REMARK 3.1. Clearly,

$$(3.2) \quad 0 = V(T, x, 0) \leq V(T, x, p) \nearrow V(T, x, 1) \leq g(x) \quad \text{as } p \rightarrow 1.$$

By analogy with [20], we shall present a probabilistic characterization of $V(T, x, p)$. First, we will generalize the Neyman–Pearson lemma (see, e.g., [21], Theorem A.28) in the next result.

LEMMA 3.1. *Suppose that Assumption 2.1 holds, and g satisfies (2.4). Let $A \in \mathcal{F}_T$ satisfy*

$$(3.3) \quad \mathbb{P}(A) \geq p.$$

Then

$$(3.4) \quad V(T, x, p) \leq \mathbb{E}[Z(T)g(X(T))1_A].$$

Furthermore, if $A \in \mathcal{F}_T$ satisfies (3.3) with equality and

$$(3.5) \quad \operatorname{ess\,sup}_A\{Z(T)g(X(T))\} \leq \operatorname{ess\,inf}_{A^c}\{Z(T)g(X(T))\},$$

then A satisfies (3.4) with equality.

PROOF. Under Assumption 2.1, since $g(X(T))1_A \in \mathcal{F}_T$ satisfies condition (2.4), it is replicable with initial capital $y := \mathbb{E}[Z(T)g(X(T))1_A]$; see, for example, Section 10.1 of [18]. That is, there exists $\pi \in \mathcal{H}$ such that $Y^{y,\pi}(T) = g(X(T))1_A$ a.s. Now if $\mathbb{P}(A) \geq p$, we have $\mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} = \mathbb{P}\{1_A \geq 1\} \geq p$. Then it follows from (3.1) that $V(T, x, p) \leq y = \mathbb{E}[Z(T)g(X(T))1_A]$.

Now, take an arbitrary pair (y_0, π_0) of initial capital and admissible portfolio that replicates $g(X(T))$ with probability greater than or equal to p ; that is,

$$\mathbb{P}\{B\} \geq p \quad \text{where } B \triangleq \{Y^{y_0,\pi_0}(T) \geq g(X(T))\}.$$

Let $A \in \mathcal{F}_T$ satisfy $p = \mathbb{P}(A) \leq \mathbb{P}(B)$ and (3.5). To prove equality in (3.4), it is enough to show that

$$y_0 \geq \mathbb{E}[Z(T)g(X(T))1_A],$$

which can be shown as follows:

$$\begin{aligned} y_0 &\geq \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)] = \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)1_B] + \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)1_{B^c}] \\ &\geq \mathbb{E}[Z(T)g(X(T))1_B] = \mathbb{E}[Z(T)g(X(T))1_{A \cap B}] + \mathbb{E}[Z(T)g(X(T))1_{A^c \cap B}] \\ &\geq \mathbb{E}[Z(T)g(X(T))1_{A \cap B}] + \mathbb{P}(A^c \cap B) \operatorname{ess\,inf}_{A^c \cap B}\{Z(T)g(X(T))\} \\ &\geq \mathbb{E}[Z(T)g(X(T))1_{A \cap B}] + \mathbb{P}(A \cap B^c) \operatorname{ess\,sup}_{A \cap B^c}\{Z(T)g(X(T))\} \\ &\geq \mathbb{E}[Z(T)g(X(T))1_{A \cap B}] + \mathbb{E}[Z(T)g(X(T))1_{A \cap B^c}] \\ &= \mathbb{E}[Z(T)g(X(T))1_A], \end{aligned}$$

where in the fourth inequality we use the following two observations: First, $\mathbb{P}(A^c \cap B) = \mathbb{P}(A \cup B) - \mathbb{P}(A) \geq \mathbb{P}(A \cup B) - \mathbb{P}(B) = \mathbb{P}(B^c \cap A)$. Second,

$$\begin{aligned} \operatorname{ess\,inf}_{A^c \cap B} \{Z(T)g(X(T))\} &\geq \operatorname{ess\,inf}_{A^c} \{Z(T)g(X(T))\} \\ &\geq \operatorname{ess\,sup}_A \{Z(T)g(X(T))\} \\ &\geq \operatorname{ess\,sup}_{A \cap B^c} \{Z(T)g(X(T))\}, \end{aligned}$$

in which the second inequality follows from (3.5). \square

Let $F(\cdot)$ be the cumulative distribution function of $Z(T)g(X(T))$, and, for any $a \in \mathbb{R}_+$, define

$$A_a := \{\omega : Z(T)g(X(T)) < a\}, \quad \partial A_a := \{\omega : Z(T)g(X(T)) = a\},$$

and let \bar{A}_a denote $A_a \cup \partial A_a$; that is,

$$(3.6) \quad \bar{A}_a = \{\omega : Z(T)g(X(T)) \leq a\}.$$

Taking $A = \bar{A}_a$ in Lemma 3.1, we see that (3.5) is satisfied. It follows that

$$(3.7) \quad V(T, x, F(a)) = \mathbb{E}[Z(T)g(X(T))1_{\bar{A}_a}].$$

On the other hand, taking $A = A_a$, we see that (3.5) is again satisfied. We therefore obtain

$$(3.8) \quad V(T, x, F(a-)) = \mathbb{E}[Z(T)g(X(T))1_{A_a}].$$

The last two equalities imply the following relationship:

$$(3.9) \quad \begin{aligned} V(T, x, F(a)) &= V(T, x, F(a-)) + a\mathbb{P}\{\partial A_a\} \\ &= V(T, x, F(a-)) + a(F(a) - F(a-)). \end{aligned}$$

Next, we will determine $V(T, x, p)$ for $p \in (F(a-), F(a))$ when $F(a-) < F(a)$.

PROPOSITION 3.1. *Suppose Assumption 2.1 holds. Fix an $(x, p) \in (0, \infty)^d \times [0, 1]$.*

(i) *There exists $A \in \mathcal{F}_T$ satisfying (3.3) with equality and (3.5). As a result, (3.4) holds with equality.*

(ii) *If $F^{-1}(p) := \{s \in \mathbb{R}_+ : F(s) = p\} = \emptyset$, then letting $a := \inf\{s \in \mathbb{R}_+ : F(s) > p\}$ we have*

$$(3.10) \quad \begin{aligned} V(T, x, p) &= V(T, x, F(a-)) + a(p - F(a-)) \\ &= V(T, x, F(a)) - a(F(a) - p). \end{aligned}$$

PROOF. (i) If there exists $a \in \mathbb{R}$ such that either $F(a) = p$ or $F(a-) = p$, then we can take $A = A_a$ or $A = \bar{A}_a$, thanks to (3.7) and (3.8). In the rest of the proof we will assume that $F^{-1}(p) = \emptyset$.

Let \tilde{W} be a Brownian motion with respect to \mathbb{F} , and define $B_b = \{\omega : \frac{\tilde{W}(T)}{\sqrt{T}} < b\}$. Let us define $f(\cdot)$ by $f(b) = \mathbb{P}\{\partial A_a \cap B_b\}$. The function f satisfies $\lim_{b \rightarrow -\infty} f(b) = 0$ and $\lim_{b \rightarrow \infty} f(b) = \mathbb{P}(\partial A_a)$. Moreover, the function $f(\cdot)$ is continuous and nondecreasing. Right continuity can be shown as follows: for $\varepsilon > 0$,

$$0 \leq f(b + \varepsilon) - f(b) = \mathbb{P}(\partial A_a \cap B_{b+\varepsilon}) - \mathbb{P}(\partial A_a \cap B_b) \leq \mathbb{P}(B_{b+\varepsilon} \cap B_b^c).$$

The right continuity follows from observing that the last expression goes to zero as $\varepsilon \rightarrow 0$. One can show left continuity of $f(\cdot)$ in a similar fashion.

Since $0 < p - \mathbb{P}(A_a) < \mathbb{P}(\partial A_a)$, thanks to the above properties of f , there exists $b^* \in \mathbb{R}$ satisfying $f(b^*) = p - \mathbb{P}(A_a)$.

Define $A := A_a \cup (\partial A_a \cap B_{b^*})$. Observe that $\mathbb{P}(A) = \mathbb{P}(A_a) + \mathbb{P}(\partial A_a \cap B_{b^*}) = p$ and that A satisfies (3.5).

(ii) This follows immediately from (1):

$$\begin{aligned} V(T, x, p) &= \mathbb{E}[Z(T)g(X(T))1_A] \\ &= \mathbb{E}[Z(T)g(X(T))1_{A_a}] + \mathbb{E}[Z(T)g(X(T))1_{\partial A_a \cap B_{b^*}}] \\ &= V(T, x, F(a-)) + a\mathbb{P}(\partial A_a \cap B_{b^*}) \\ &= V(t, x, F(a-)) + a(p - F(a-)). \end{aligned} \quad \square$$

REMARK 3.2. Note that when Z is a martingale, using the Neyman–Pearson lemma, it was shown in [20] that

$$(3.11) \quad V(T, x, p) = \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi] = \mathbb{E}[Z(T)g(X(T))\varphi^*],$$

where

$$(3.12) \quad \mathcal{M} = \{\varphi : \Omega \rightarrow [0, 1] | \mathcal{F}_T \text{ measurable, } \mathbb{E}[\varphi] \geq p\}.$$

The randomized test function φ^* is not necessarily an indicator function. Using Lemma 3.1 and the fine structure of the filtration \mathcal{F}_T , we provide in Proposition 3.1 another optimizer of (3.11) which is an indicator function.

PROPOSITION 3.2. *Suppose Assumption 2.1 holds. Then, the map $p \mapsto V(T, x, p)$ is convex and continuous on the closed interval $[0, 1]$. Hence, $V(T, x, p) \leq pV(T, x, 1) \leq pg(x)$ for all $p \in [0, 1]$.*

PROOF. By Proposition 3.1, for any $p \in [0, 1]$ there exists $A \in \mathcal{F}_T$ such that

$$V(T, x, p) = \mathbb{E}[Z(T)g(X(T))1_A] \leq \mathbb{E}[Z(T)g(X(T))] < \infty.$$

Then thanks to a theorem by Ostroski (see [9], page 12), to show the convexity it suffices to demonstrate the midpoint convexity

$$(3.13) \quad \frac{V(T, x, p_1) + V(T, x, p_2)}{2} \geq V\left(T, x, \frac{p_1 + p_2}{2}\right) \quad \text{for all } 0 \leq p_1 < p_2 \leq 1.$$

Denote $\tilde{p} \triangleq \frac{p_1 + p_2}{2}$. It follows from Proposition 3.1 that there exist $A_1 \subset \tilde{A} \subset A_2$ with $\mathbb{P}(A_1) = p_1 < \mathbb{P}(\tilde{A}) = \tilde{p} < \mathbb{P}(A_2) = p_2$ satisfying (3.5),

$$V(T, x, p_i) = \mathbb{E}[Z(T)g(X(T))1_{A_i}], \quad i = 1, 2,$$

and

$$V(T, x, \tilde{p}) = \mathbb{E}[Z(T)g(X(T))1_{\tilde{A}}].$$

By (3.5),

$$\begin{aligned} \text{ess inf}\{Z(T)g(X(T))1_{A_2 \cap \tilde{A}^c}\} &\geq \text{ess inf}\{Z(T)g(X(T))1_{\tilde{A}^c}\} \\ &\geq \text{ess sup}\{Z(T)g(X(T))1_{\tilde{A}}\} \\ &\geq \text{ess sup}\{Z(T)g(X(T))1_{\tilde{A} \cap A_1^c}\}, \end{aligned}$$

which implies that

$$\mathbb{E}[Z(T)g(X(T))1_{A_2 \cap \tilde{A}^c}] \geq \mathbb{E}[Z(T)g(X(T))1_{\tilde{A} \cap A_1^c}].$$

As a result,

$$\begin{aligned} \mathbb{E}[Z(T)g(X(T))1_{A_2}] - \mathbb{E}[Z(T)g(X(T))1_{\tilde{A}}] \\ \geq \mathbb{E}[Z(T)g(X(T))1_{\tilde{A}}] - \mathbb{E}[Z(T)g(X(T))1_{A_1}], \end{aligned}$$

which is equivalent to (3.13).

Now thanks to convexity, we immediately have that $p \mapsto V(T, x, p)$ is continuous on $[0, 1)$. It remains to show that it is continuous from the left at $p = 1$; but this is indeed true because

$$\begin{aligned} \lim_{a \rightarrow \infty} V(T, x, F(a)) &= \lim_{a \rightarrow \infty} \mathbb{E}[Z(T)g(X(T))1_{\{Z(T)g(X(T)) \leq a\}}] \\ &= \mathbb{E}[Z(T)g(X(T))] = V(T, x, 1), \end{aligned}$$

where the second equality is due to the dominated convergence theorem. \square

EXAMPLE 3.1. Consider a market with a single stock, whose dynamics follow a three-dimensional Bessel process, that is,

$$dX(t) = \frac{1}{X(t)} dt + dW(t), \quad X_0 = x > 0,$$

and let $g(x) = x$. In this case $Z(t) = x/X(t)$, which is the classical example for a strict local martingale; see [28]. On the other hand, $Z(t)X(t) = x$ is a martingale. Thanks to Proposition 3.1 there exists a set $A \in \mathcal{F}_T$ with $\mathbb{P}(A) = p$ such that

$$V(T, x, p) = \mathbb{E}[Z(T)X(T)1_A] = px.$$

In [20], the following result was proved when Z is a martingale. Here, we generalize this result to the case where Z is only a local martingale.

PROPOSITION 3.3. *Under Assumption 2.1*

$$(3.14) \quad V(T, x, p) = \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi],$$

where \mathcal{M} is defined in (3.12).

PROOF. Thanks to Proposition 3.1 there exists a set $A \in \mathcal{F}_T$ satisfying $\mathbb{P}(A) = p$ and (3.5) such that $V(T, x, p) = \mathbb{E}[Z(T)g(X(T))1_A]$. Since $1_A \in \mathcal{M}$, clearly

$$V(T, x, p) \geq \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi].$$

For the other direction, it is enough to show that for any $\varphi \in \mathcal{M}$, we have

$$\mathbb{E}[Z(T)g(X(T))1_A] \leq \mathbb{E}[Z(T)g(X(T))\varphi].$$

Indeed, since the left-hand side is actually $V(T, x, p)$, we can get the desired result by taking infimum on both sides over $\varphi \in \mathcal{M}$.

Letting $M = \text{ess sup}_A\{Z(T)g(X(T))\}$, we observe that

$$\begin{aligned} & \mathbb{E}[Z(T)g(X(T))\varphi] - \mathbb{E}[Z(T)g(X(T))1_A] \\ &= \mathbb{E}[Z(T)g(X(T))\varphi 1_A] + \mathbb{E}[Z(T)g(X(T))\varphi 1_{A^c}] - \mathbb{E}[Z(T)g(X(T))1_A] \\ &= \mathbb{E}[Z(T)g(X(T))\varphi 1_{A^c}] - \mathbb{E}[Z(T)g(X(T))1_A(1 - \varphi)] \\ &\geq \text{ess inf}_{A^c}\{Z(T)g(X(T))\}\mathbb{E}[\varphi 1_{A^c}] - M\mathbb{E}[1_A(1 - \varphi)] \\ &\geq M\mathbb{E}[\varphi 1_{A^c}] - M\mathbb{E}[1_A(1 - \varphi)] \quad [\text{by (3.5)}] \\ &= M\mathbb{E}[\varphi] - M\mathbb{E}[1_A] \geq 0. \end{aligned} \quad \square$$

3.1. *A digression: Representation of V as a stochastic control problem.* For $p \in [0, 1]$, we introduce an additional controlled state variable

$$(3.15) \quad P_\alpha^p(s) = p + \int_0^s \alpha(r)' dW(r), \quad s \in [0, T],$$

where $\alpha(\cdot)$ is a \mathbb{G} -progressively measurable \mathbb{R}^d -valued process satisfying the integrability condition $\int_0^T |\alpha(s)|^2 ds < \infty$ a.s. such that P_α^p takes values in $[0, 1]$. We will denote the class of such processes by \mathcal{A} . Note that \mathcal{A} is nonempty, as the constant control $\alpha(\cdot) \equiv (0, \dots, 0) \in \mathbb{R}^d$ obviously lies in \mathcal{A} . The next result obtains an alternative representation for V in terms of P_α^p .

PROPOSITION 3.4. *Under Assumption 2.1,*

$$(3.16) \quad V(T, x, p) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}[Z(T)g(X(T))P_\alpha^p(T)] < \infty.$$

PROOF. The finiteness follows from (2.4). Define

$$\tilde{\mathcal{M}} := \{\varphi : \Omega \rightarrow [0, 1] | \mathcal{F}_T \text{ measurable, } \mathbb{E}[\varphi] = p\}.$$

Thanks to Proposition 3.1, there exists a set $A \in \mathcal{F}_T$ satisfying $\mathbb{P}(A) = p$ and (3.5) such that

$$V(T, x, p) = \mathbb{E}[Z(T)g(X(T))1_A] \geq \inf_{\varphi \in \tilde{\mathcal{M}}} \mathbb{E}[Z(T)g(X(T))\varphi].$$

Since the opposite inequality follows immediately from Proposition 3.3, we conclude that

$$V(T, x, p) = \inf_{\varphi \in \tilde{\mathcal{M}}} \mathbb{E}[Z(T)g(X(T))\varphi].$$

Therefore, it is enough to show that $\tilde{\mathcal{M}}$ satisfies $\tilde{\mathcal{M}} = \{P_\alpha^p(T) | \alpha \in \mathcal{A}\}$. The inclusion $\tilde{\mathcal{M}} \supset \{P_\alpha^p(T) | \alpha \in \mathcal{A}\}$ is clear. To show the other inclusion we will use the Martingale representation theorem: for any $\varphi \in \tilde{\mathcal{M}}$ there exists a \mathbb{G} -progressively measurable \mathbb{R}^d -valued process $\psi(\cdot)$ satisfying $\int_0^T |\psi(s)|^2 ds < \infty$ a.s. such that

$$\mathbb{E}[\varphi | \mathcal{F}_t] = p + \int_0^t \psi(s)' dW(s), \quad t \in [0, T].$$

Note that since φ takes values in $[0, 1]$, so does $\mathbb{E}[\varphi | \mathcal{F}_t]$ for all $t \in [0, T]$. Then we see that $\mathbb{E}[\varphi | \mathcal{F}_t]$ satisfies (3.15) with $\alpha(\cdot) = \psi(\cdot) \in \mathcal{A}$. \square

4. The PDE characterization.

4.1. *Notation.* We denote by $X^{t,x}(\cdot)$ the solution of (2.1) starting from x at time t and by $Z^{t,x,z}(\cdot)$ the solution of

$$(4.1) \quad dZ(s) = -Z(s)\theta(X^{t,x}(s))' dW(s), \quad Z(t) = z.$$

Define the process $Q^{t,x,q}(\cdot)$ by

$$(4.2) \quad Q^{t,x,q}(\cdot) := \frac{1}{Z^{t,x,(1/q)}(\cdot)}, \quad q \in (0, \infty).$$

Then we see from (4.1) that $Q(\cdot)$ satisfies

$$(4.3) \quad \frac{dQ(s)}{Q(s)} = |\theta(X^{t,x}(s))|^2 ds + \theta(X^{t,x}(s))' dW(s), \quad Q^{t,x,q}(t) = q.$$

We then introduce the value function

$$U(t, x, p) := \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))\varphi],$$

where \mathcal{M} is defined in (3.12). Note that the original value function V can be written in terms of U as $V(T, x, p) = U(0, x, p)$.

We also consider the Legendre transform of U with respect to the p variable. To make the discussion clear, however, let us first extend the domain of the map $p \mapsto U(t, x, p)$ from $[0, 1]$ to the entire real line \mathbb{R} by setting

$$(4.4) \quad U(t, x, p) = 0 \quad \text{for } p < 0,$$

$$(4.5) \quad U(t, x, p) = \infty \quad \text{for } p > 1.$$

Then the Legendre transform of U with respect to p is well defined:

$$(4.6) \quad w(t, x, q) := \sup_{p \in \mathbb{R}} \{pq - U(t, x, p)\} = \begin{cases} \infty, & \text{if } q < 0; \\ \sup_{p \in [0, 1]} \{pq - U(t, x, p)\}, & \text{if } q \geq 0. \end{cases}$$

From Proposition 3.2, we already know that $p \mapsto U(t, x, p)$ is convex and continuous on $[0, 1]$. Since $U(t, x, 0) = 0$, we see from (4.4) and (4.5) that $p \mapsto U(t, x, p)$ is continuous on $(-\infty, 1]$ and lower semicontinuous on \mathbb{R} . Moreover, considering that $p \mapsto U(t, x, p)$ is increasing on $[0, 1]$, we conclude that $p \mapsto U(t, x, p)$ is also convex on \mathbb{R} . Now thanks to [42], Section 6.18, the convexity and the lower semicontinuity of $p \mapsto U(t, x, p)$ on \mathbb{R} imply that the double transform of U is indeed equal to U itself. That is, for any $(t, x, p) \in [0, T] \times (0, \infty)^d \times \mathbb{R}$,

$$U(t, x, p) = \sup_{q \in \mathbb{R}} \{pq - w(t, x, q)\} = \sup_{q \geq 0} \{pq - w(t, x, q)\},$$

where the second equality is a consequence of (4.6).

In this section, we also consider the function

$$(4.7) \quad \begin{aligned} \tilde{w}(t, x, q) &:= \mathbb{E}[Z^{t,x,1}(T)(Q^{t,x,q}(T) - g(X^{t,x}(T)))^+] \\ &= \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))^+] \end{aligned}$$

for any $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$. We will show that $w = \tilde{w}$ and derive various properties of \tilde{w} .

REMARK 4.1. From the definition of \tilde{w} in (4.7), \tilde{w} is the upper hedging price for the contingent claim $(Q^{t,x,q}(T) - g(X^{t,x}(T)))^+$, and potentially solves the linear PDE

$$(4.8) \quad \partial_t \tilde{w} + \frac{1}{2} \text{Tr}(\sigma \sigma' D_x^2 \tilde{w}) + \frac{1}{2} |\theta|^2 q^2 D_q^2 \tilde{w} + q \text{Tr}(\sigma \theta D_{xq} \tilde{w}) = 0.$$

This is not, however, a traditional Black–Scholes-type equation because it is degenerate on the entire space $(x, q) \in (0, \infty)^d \times (0, \infty)$. Consider the following function v which takes values in the space of $(d + 1) \times d$ matrices:

$$v(\cdot) := \begin{bmatrix} s(\cdot)_{d \times d} \\ \theta(\cdot)'_{1 \times d} \end{bmatrix}.$$

Degeneracy can be seen by observing that $v(x)v(x)'$ is only positive semi-definite for all $x \in (0, \infty)^d$. Or, one may observe degeneracy by noting that there are $d + 1$ risky assets, X_1, \dots, X_d and Q , with only d independent sources of uncertainty, W_1, \dots, W_d . As a result, the existence of classical solutions to (4.8) cannot be guaranteed by standard results for parabolic equations. Indeed, under the setting of Example 3.1, we have

$$\tilde{w}(t, x, q) = \mathbb{E}[(q - Z^{t,x,1}(T)X^{t,x}(T))^+] = (q - x)^+,$$

which is not smooth.

4.2. *Elliptic regularization.* In this subsection, we will approximate \tilde{w} by a sequence of smooth functions \tilde{w}_ε , constructed by elliptic regularization. We will then derive some properties of \tilde{w}_ε and investigate the relation between \tilde{w} and \tilde{w}_ε . Finally, we will show that $\tilde{w} = w$, which validates the construction of \tilde{w}_ε .

To perform elliptic regularization under our setting, we need to first introduce a product probability space. Recall that we have been working on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, given by a weak solution to the SDE (2.1). Now consider the sample space $\Omega^B := C([0, T]; \mathbb{R})$ and the canonical process $B(\cdot)$. Let \mathbb{F}^B be the filtration generated by B and \mathbb{P}^B be the Wiener measure on (Ω^B, \mathbb{F}^B) . We then introduce the product probability space $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$, with $\bar{\Omega} := \Omega \times \Omega^B$, $\bar{\mathbb{F}} := \mathbb{F} \times \mathbb{F}^B$ and $\bar{\mathbb{P}} := \mathbb{P} \times \mathbb{P}^B$. For any $\bar{\omega} \in \bar{\Omega}$, we write $\bar{\omega} = (\omega, \omega^B)$, where $\omega \in \Omega$ and $\omega^B \in \Omega^B$. Also, we denote by $\bar{\mathbb{E}}$ the expectation taken under $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$.

For any $\varepsilon > 0$, introduce the process $Q_\varepsilon^{t,x,q}(\cdot)$ which satisfies the following dynamics:

$$(4.9) \quad \begin{aligned} \frac{dQ_\varepsilon(s)}{Q_\varepsilon(s)} &= |\theta(X^{t,x}(s))|^2 ds + \theta(X^{t,x}(s))' dW(s) + \varepsilon dB(s), \\ Q_\varepsilon^{t,x,q} &= q \in (0, \infty). \end{aligned}$$

Then under the probability space $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$, we have $d + 1$ risky assets, the d stocks X_1, \dots, X_d and Q_ε . Define

$$\bar{s} := \left[\begin{array}{ccc|c} s_{11} & \cdots & s_{1d} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ s_{d1} & \cdots & s_{dd} & 0 \\ \hline \theta_1 & \cdots & \theta_d & \varepsilon \end{array} \right], \quad \bar{b} := \left[\begin{array}{c} b_1 \\ \vdots \\ b_d \\ \hline |\theta|^2 \end{array} \right]$$

and

$$\bar{a} := \bar{s}\bar{s}' = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1d} & | \\ \vdots & \ddots & \vdots & s\theta \\ a_{d1} & \cdots & a_{dd} & | \\ \hline - & \theta's' & - & |\theta|^2 + \varepsilon^2 \end{array} \right].$$

Since we assume that the matrix s has full rank (Assumption 2.1), \bar{s} has full rank by definition. It follows that \bar{a} is positive definite. Now we can define the corresponding market price of risk under $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ as $\bar{\theta} := \bar{s}^{-1}\bar{b}$, and the corresponding deflator $\bar{Z}(\cdot)$ under $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ as the solution of

$$(4.10) \quad d\bar{Z}(s) = -\bar{Z}(s)\bar{\theta}(X^{t,x}(s))' d\bar{W}(s), \quad \bar{Z}^{t,x,z}(t) = z,$$

where $\bar{W} := (W_1, \dots, W_d, B)$ is a $(d + 1)$ -dimensional Brownian motion. Observe that

$$\bar{\theta} = \left[\begin{array}{c|c} s^{-1} & O_{d \times 1} \\ \hline -\frac{1}{\varepsilon}\theta' s^{-1} & \frac{1}{\varepsilon} \end{array} \right] \left[\begin{array}{c} b \\ |\theta|^2 \end{array} \right] = \left[\begin{array}{c} \theta \\ 0 \end{array} \right].$$

This implies that (4.10) coincides with (4.1). Thus, we conclude that $\bar{Z}(\cdot) = Z(\cdot)$. Finally, let us introduce the function

$$\tilde{w}_\varepsilon(t, x, q) := \bar{\mathbb{E}}[\bar{Z}^{t,x,1}(T)(Q_\varepsilon^{t,x,q}(T) - g(X^{t,x}(T)))^+]$$

for any $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$. By (4.9) and (4.3), we see that the processes $Q_\varepsilon(\cdot)$ and $Q(\cdot)$ have the following relation:

$$(4.11) \quad Q_\varepsilon^{t,x,q}(s) = Q^{t,x,q}(s) \exp\{-\frac{1}{2}\varepsilon^2(s - t) + \varepsilon(B(s) - B(t))\}, \quad s \in [t, T].$$

It then follows from (4.11), the fact that $\bar{Z}(\cdot) = Z(\cdot)$ and the definition of \tilde{w}_ε that

$$(4.12) \quad \tilde{w}_\varepsilon(t, x, q) = \bar{\mathbb{E}}[(q \exp\{-\frac{1}{2}\varepsilon^2(T - t) + \varepsilon(B(T) - B(t))\} - Z^{t,x,1}(T)g(X^{t,x}(T)))^+].$$

ASSUMPTION 4.1. The functions θ_i and s_{ij} are locally Lipschitz, for all $i, j \in \{1, \dots, d\}$.

LEMMA 4.1. Under Assumption 4.1, we have that $\tilde{w}_\varepsilon \in C^{1,2,2}((0, T) \times (0, \infty)^d \times (0, \infty))$ and satisfies the PDE

$$(4.13) \quad \partial_t \tilde{w}_\varepsilon + \frac{1}{2} \text{Tr}(\sigma \sigma' D_x^2 \tilde{w}_\varepsilon) + \frac{1}{2}(|\theta|^2 + \varepsilon^2)q^2 D_q^2 \tilde{w}_\varepsilon + q \text{Tr}(\sigma \theta D_{xq} \tilde{w}_\varepsilon) = 0,$$

$(t, x, q) \in (0, T) \times (0, \infty)^d \times (0, \infty)$, with the boundary condition

$$(4.14) \quad \tilde{w}_\varepsilon(T, x, q) = (q - g(x))^+.$$

PROOF. Since \bar{a} is positive definite and continuous, it must satisfy the following ellipticity condition: for every compact set $K \subset (0, \infty)^d$, there exists a positive constant C_K such that

$$(4.15) \quad \sum_{i=1}^{d+1} \sum_{j=1}^{d+1} \bar{a}_{ij}(x) \xi_i \xi_j \geq C_K |\xi|^2$$

for all $\xi \in \mathbb{R}^{d+1}$ and $x \in K$; see, for example, [23], Lemma 3. Under Assumption 4.1 and (4.15), the smoothness of \tilde{w}_ε and the PDE (4.13) follow immediately from [39], Theorem 4.2. Finally, note that \tilde{w}_ε satisfies the boundary condition by definition. \square

PROPOSITION 4.1. *For any $(t, x) \in [0, T] \times (0, \infty)^d$, the map $q \mapsto \tilde{w}_\varepsilon(t, x, q)$ is strictly convex on $(0, \infty)$. More precisely, the map $q \mapsto D_q \tilde{w}_\varepsilon(t, x, q)$ is strictly increasing on $(0, \infty)$ with*

$$\lim_{q \downarrow 0} D_q \tilde{w}_\varepsilon(t, x, q) = 0 \quad \text{and} \quad \lim_{q \rightarrow \infty} D_q \tilde{w}_\varepsilon(t, x, q) = 1.$$

PROOF. We will first compute $D_q \tilde{w}_\varepsilon(t, x, q)$, and then show that it is strictly increasing in q from 0 to 1. Let $L_\varepsilon(t, T) := \exp(-\frac{1}{2}\varepsilon^2(T-t) + \varepsilon(B(T) - B(t)))$ and $\tilde{A}_a := \{\bar{\omega} : Z^{t,x,1}(T)g(X^{t,x}(T)) \leq aL_\varepsilon(t, T)\}$ for $a \geq 0$. Fix an arbitrary $q > 0$. For any $\delta > 0$, define

$$E^\delta := \{\bar{\omega} : qL_\varepsilon(t, T) < Z^{t,x,1}(T)g(X^{t,x}(T)) \leq (q + \delta)L_\varepsilon(t, T)\}.$$

Note that by construction, \tilde{A}_q and E^δ are disjoint, and $\tilde{A}_{q+\delta} = \tilde{A}_q \cup E^\delta$. It follows that

$$\begin{aligned} & \frac{1}{\delta} [\tilde{w}_\varepsilon(t, x, q + \delta) - \tilde{w}_\varepsilon(t, x, q)] \\ &= \frac{1}{\delta} \{ \bar{\mathbb{E}}[((q + \delta)L_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))) 1_{\tilde{A}_{q+\delta}}] \\ & \quad - \bar{\mathbb{E}}[(qL_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))) 1_{\tilde{A}_q}] \} \\ &= \frac{1}{\delta} \{ \bar{\mathbb{E}}[((q + \delta)L_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))) 1_{\tilde{A}_q}] \\ & \quad + \bar{\mathbb{E}}[((q + \delta)L_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))) 1_{E^\delta}] \\ & \quad - \bar{\mathbb{E}}[(qL_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))) 1_{\tilde{A}_q}] \} \\ &= \bar{\mathbb{E}}[L_\varepsilon(t, T) 1_{\tilde{A}_q}] + \frac{1}{\delta} \bar{\mathbb{E}}[((q + \delta)L_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))) 1_{E^\delta}]. \end{aligned}$$

By the definition of E^δ ,

$$\begin{aligned} 0 &\leq \frac{1}{\delta} \bar{\mathbb{E}}[((q + \delta)L_\varepsilon(t, T) - Z^{t,x,1}(T)g(X^{t,x}(T))) 1_{E^\delta}] \\ &\leq \frac{1}{\delta} \bar{\mathbb{E}}[\delta L_\varepsilon(t, T) 1_{E^\delta}] \\ &= \bar{\mathbb{E}}[L_\varepsilon(t, T) 1_{E^\delta}] \rightarrow 0 \quad \text{as } \delta \downarrow 0, \end{aligned}$$

where we use the dominated convergence theorem. We therefore conclude that

$$D_q \tilde{w}_\varepsilon(t, x, q) = \lim_{\delta \downarrow 0} \frac{1}{\delta} [\tilde{w}_\varepsilon(t, x, q + \delta) - \tilde{w}_\varepsilon(t, x, q)] = \bar{\mathbb{E}}[L_\varepsilon(t, T) 1_{\tilde{A}_q}].$$

Thanks to the dominated convergence theorem again, we have

$$\lim_{q \downarrow 0} \bar{\mathbb{E}}[L_\varepsilon(t, T) 1_{\tilde{A}_q}] = 0 \quad \text{and} \quad \lim_{q \rightarrow \infty} \bar{\mathbb{E}}[L_\varepsilon(t, T) 1_{\tilde{A}_q}] = \bar{\mathbb{E}}[L_\varepsilon(t, T)] = 1.$$

It remains to prove that $D_q \tilde{w}_\varepsilon(t, x, q) = \bar{\mathbb{E}}[L_\varepsilon(t, T) 1_{\tilde{A}_q}]$ is strictly increasing in q . Note that it is enough to show that the event E^δ has positive probability for all $\delta > 0$. Under the integrability condition (2.2), the deflator $Z(\cdot)$ is strictly positive with probability 1; see, for example, [2], Section 6. It follows from our assumptions on g [see (2.4) and the line before it] that

$$0 < Z^{t,x,1}(T)g(X^{t,x}(T)) < \infty, \quad \mathbb{P}\text{-a.s.}$$

This implies that

$$(4.16) \quad -\infty < \log Z^{t,x,1}(T)g(X^{t,x}(T)) < \infty, \quad \bar{\mathbb{P}}\text{-a.s.}$$

Now, from (4.16) and the definitions E^δ and L_ε , we see that $\bar{\mathbb{P}}(E^\delta)$ equals to the probability of the event

$$\left\{ \bar{\omega} : \frac{\varepsilon}{2}(T - t) + \frac{1}{\varepsilon} \log \frac{Z^{t,x,1}(T)g(X^{t,x}(T))}{q + \delta} \leq B(T) - B(t) < \frac{\varepsilon}{2}(T - t) + \frac{1}{\varepsilon} \log \frac{Z^{t,x,1}(T)g(X^{t,x}(T))}{q} \right\}.$$

Thanks to Fubini’s theorem, this probability is strictly positive. \square

We investigate the relation between \tilde{w} and \tilde{w}_ε in the following result.

LEMMA 4.2. *The functions \tilde{w} and \tilde{w}_ε satisfy the following relations:*

(i) *For any $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$,*

$$\tilde{w}(t, x, q) = \lim_{\varepsilon \downarrow 0} \tilde{w}_\varepsilon(t, x, q).$$

(ii) *For any compact subset $E \subset (0, \infty)$, \tilde{w}_ε converges to \tilde{w} uniformly on $[0, T] \times (0, \infty)^d \times E$. Moreover, for any $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$,*

$$(4.17) \quad \tilde{w}(t, x, q) = \lim_{(\varepsilon, t', x', q') \rightarrow (0, t, x, q)} \tilde{w}_\varepsilon(t', x', q').$$

PROOF. (i) By (4.11), we observe that

$$\begin{aligned}
 & \bar{\mathbb{E}} \left[\sup_{\varepsilon \in (0,1]} Z^{t,x,1}(T) Q_\varepsilon^{t,x,q}(T) \right] \\
 &= \bar{\mathbb{E}} \left[\sup_{\varepsilon \in (0,1]} q \exp \left\{ -\frac{1}{2} \varepsilon^2 (T-t) + \varepsilon (B(T) - B(t)) \right\} \right] \\
 &\leq q \bar{\mathbb{E}} \left[\sup_{\varepsilon \in (0,1]} \exp \{ \varepsilon (B(T) - B(t)) \} \right] \\
 (4.18) \quad &\leq q \bar{\mathbb{E}} \left[\sup_{\varepsilon \in (0,1]} \exp \{ \varepsilon (B(T) - B(t)) \} 1_{\{B(T)-B(t) \geq 0\}} \right] \\
 &\quad + q \bar{\mathbb{E}} \left[\sup_{\varepsilon \in (0,1]} \exp \{ \varepsilon (B(T) - B(t)) \} 1_{\{B(T)-B(t) < 0\}} \right] \\
 &\leq q \bar{\mathbb{E}} [\exp \{ B(T) - B(t) \}] + q \\
 &= q \left(\exp \left\{ \frac{1}{2} (T-t) \right\} + 1 \right) < \infty.
 \end{aligned}$$

Then it follows from the dominated convergence theorem that

$$\begin{aligned}
 \lim_{\varepsilon \downarrow 0} \tilde{w}_\varepsilon(t, x, q) &= \lim_{\varepsilon \downarrow 0} \bar{\mathbb{E}} \left[\left(q \exp \left\{ -\frac{1}{2} \varepsilon^2 (T-t) + \varepsilon (B(T) - B(t)) \right\} \right. \right. \\
 &\quad \left. \left. - Z^{t,x,1}(T) g(X^{t,x}(T)) \right)^+ \right] \\
 &= \bar{\mathbb{E}} \left[(q - Z^{t,x,1}(T) g(X^{t,x}(T)))^+ \right] \\
 &= \mathbb{E} \left[(q - Z^{t,x,1}(T) g(X^{t,x}(T)))^+ \right] \\
 &= \tilde{w}(t, x, q),
 \end{aligned}$$

where the third equality is due to the fact that $Z^{t,x,1}(T)g(X^{t,x}(T))$ depends only on $w \in \Omega$.

(ii) From (4.7), (4.12) and the observation that $|(a - b)^+ - (c - b)^+| \leq |a - c|$ for any $a, b, c \in \mathbb{R}$,

$$\begin{aligned}
 & |\tilde{w}_\varepsilon(t, x, q) - \tilde{w}(t, x, q)| \\
 &\leq q \bar{\mathbb{E}} \left| \exp \left\{ -\frac{1}{2} \varepsilon^2 (T-t) + \varepsilon (B(T) - B(t)) \right\} - 1 \right| \\
 (4.19) \quad &\leq q \bar{\mathbb{E}} \left[\exp \left\{ \frac{\varepsilon^2}{2} (T-t) + \varepsilon |B(T) - B(t)| \right\} - 1 \right] \\
 &= q \left[(1 + \Phi(\varepsilon \sqrt{T-t}) - \Phi(-\varepsilon \sqrt{T-t})) e^{\varepsilon^2 (T-t)} - 1 \right] \\
 &\leq q \left[(1 + \Phi(\varepsilon \sqrt{T}) - \Phi(-\varepsilon \sqrt{T})) e^{\varepsilon^2 T} - 1 \right],
 \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Note that the second line of (4.19) follows from the inequality $|e^v - 1| \leq e^{|v|} - 1$ for $v \in \mathbb{R}$; this inequality holds because if $v < 0$, $|e^v - 1| = 1 - e^v = (e^{-v} - 1)e^v \leq e^{-v} - 1 = e^{|v|} - 1$ and if $v \geq 0$, $|e^v - 1| = e^v - 1 = e^{|v|} - 1$. We can then conclude from (4.19) that \tilde{w}_ε converges to \tilde{w} uniformly on $[0, T] \times (0, \infty)^d \times E$, for any compact subset E of $(0, \infty)$. Now, by Lemma 4.1 \tilde{w}_ε is continuous on $(0, T) \times (0, \infty)^d \times (0, \infty)$. Then as a result of uniform convergence, \tilde{w} must be continuous on the same domain. Noting that

$$|\tilde{w}_\varepsilon(t', x', q') - \tilde{w}(t, x, q)| \leq |\tilde{w}_\varepsilon(t', x', q') - \tilde{w}(t', x', q')| + |\tilde{w}(t', x', q') - \tilde{w}(t, x, q)|,$$

we see that (4.17) follows from the continuity of \tilde{w} and the uniform convergence of \tilde{w}_ε to \tilde{w} on $[0, T] \times (0, \infty)^d \times E$ for any compact subset E of $(0, \infty)$. \square

Thanks to the stability of viscosity solutions, we have the following result immediately.

PROPOSITION 4.2. *Under Assumption 4.1, we have that \tilde{w} is a continuous viscosity solution to*

$$(4.20) \quad \partial_t \tilde{w} + \frac{1}{2} \text{Tr}(\sigma \sigma' D_x^2 \tilde{w}) + \frac{1}{2} |\theta|^2 q^2 D_q^2 \tilde{w} + q \text{Tr}(\sigma \theta D_{xq} \tilde{w}) = 0$$

for $(t, x, q) \in (0, T) \times (0, \infty)^d \times (0, \infty)$, with the boundary condition

$$(4.21) \quad \tilde{w}(T, x, q) = (q - g(x))^+.$$

PROOF. By Lemmas 4.1 and 4.2(ii), the viscosity solution property follows as a direct application of [41], Proposition 2.3, and the boundary condition holds trivially from the definition of \tilde{w} . \square

Now we want to relate to \tilde{w} to w . Given $(t, x) \in [0, T] \times (0, \infty)^d$, recall the notation in Section 3: for any $a \geq 0$, $\bar{A}_a := \{\omega : Z^{t,x,1}(T)g(X^{t,x}(T)) \leq a\}$; also, $F(\cdot)$ again denotes the cumulative distribution function of $Z^{t,x,1}(T)g(X^{t,x}(T))$. We first present another representation for \tilde{w} as follows.

LEMMA 4.3. *For any $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$, we have*

$$\max_{a \geq 0} \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}] = \tilde{w}(t, x, q).$$

PROOF. Let us first take $a < q$. Since $\bar{A}_a \subset \bar{A}_q$ and $q - Z^{t,x,1}(T)g(X^{t,x}(T)) \geq 0$ on \bar{A}_q ,

$$\begin{aligned} \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}] &\leq \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}] \\ &= \tilde{w}(t, x, q). \end{aligned}$$

Now consider $a > q$. Set $F := \{\omega : q < Z^{t,x,1}(T)g(X^{t,x}(T)) \leq a\}$. Observing that \bar{A}_q and F are disjoint, and $\bar{A}_a = \bar{A}_q \cup F$, we have

$$\begin{aligned} & \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}] \\ &= \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}] + \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_F] \\ &\leq \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}] = \tilde{w}(t, x, q), \end{aligned}$$

where the inequality is due to the fact that $q - Z^{t,x,1}(T)g(X^{t,x}(T)) < 0$ on F . \square

Next, we will argue that w and \tilde{w} are equal.

PROPOSITION 4.3. $w(t, x, q) = \tilde{w}(t, x, q)$, for all $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$.

PROOF. Given $p \in [0, 1]$, there exists $a \geq 0$ such that $F(a-) \leq p \leq F(a)$. We can take two nonnegative numbers λ_1 and λ_2 with $\lambda_1 + \lambda_2 = 1$ such that

$$(4.22) \quad p = \lambda_1 F(a) + \lambda_2 F(a-).$$

Observe that $p - F(a-) = \lambda_1(F(a) - F(a-))$. Plugging this into the first line of (3.10), we get

$$(4.23) \quad U(t, x, p) = U(t, x, F(a-)) + \lambda_1 a(F(a) - F(a-)).$$

Also note from (3.10) that

$$a(F(a) - F(a-)) = U(t, x, F(a)) - U(t, x, F(a-)).$$

Plugging this back into (4.23), we obtain

$$(4.24) \quad U(t, x, p) = \lambda_1 U(t, x, F(a)) + \lambda_2 U(t, x, F(a-)).$$

It then follows from (4.22) and (4.24) that

$$\begin{aligned} & pq - U(t, x, p) \\ (4.25) \quad &= \lambda_1[F(a)q - U(t, x, F(a))] + \lambda_2[F(a-)q - U(t, x, F(a-))] \\ &\leq \max\{F(a)q - U(t, x, F(a)), F(a-)q - U(t, x, F(a-))\}. \end{aligned}$$

Choose a sequence $a_n \in [a/2, a)$ such that $a_n \rightarrow a$ from the left as $n \rightarrow \infty$. Thanks to Proposition 3.2, $p \mapsto U(t, x, p)$ is continuous on $[0, 1]$. We can therefore select a subsequence of a_n (without relabelling) such that, for any $n \in \mathbb{N}$,

$$F(a-) - F(a_n) < \frac{1}{n} \quad \text{and} \quad U(t, x, F(a_n)) - U(t, x, F(a-)) < \frac{1}{n}.$$

It follows that, for any $n \in \mathbb{N}$,

$$F(a-)q - U(t, x, F(a-)) < F(a_n)q - U(t, x, F(a_n)) + \frac{1+q}{n},$$

which yields

$$\begin{aligned}
 & F(a-)q - U(t, x, F(a-)) \\
 (4.26) \quad & \leq \limsup_{n \rightarrow \infty} \left\{ F(a_n)q - U(t, x, F(a_n)) + \frac{1+q}{n} \right\} \\
 & \leq \sup_{n \in \mathbb{N}} F(a_n)q - U(t, x, F(a_n)).
 \end{aligned}$$

Combining (4.25) and (4.26), we obtain

$$pq - U(t, x, p) \leq \sup_{\delta \in [a/2, a]} F(\delta)q - U(t, x, F(\delta)) \leq \sup_{\delta \geq 0} F(\delta)q - U(t, x, F(\delta)).$$

This implies

$$w(t, x, q) = \sup_{p \in [0, 1]} \{pq - U(t, x, p)\} \leq \sup_{a \geq 0} \{F(a)q - U(t, x, F(a))\}.$$

Since $F(a) \in [0, 1]$ for all $a \geq 0$, the opposite inequality is trivial. We therefore conclude

$$(4.27) \quad w(t, x, q) = \sup_{p \in [0, 1]} \{pq - U(t, x, p)\} = \sup_{a \geq 0} \{F(a)q - U(t, x, F(a))\}.$$

Now, thanks to (3.7), we have

$$\begin{aligned}
 (4.28) \quad & F(a)q - U(t, x, F(a)) = F(a)q - \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))1_{\bar{A}_a}] \\
 & = \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}].
 \end{aligned}$$

It follows from (4.27), (4.28) and Lemma 4.3 that

$$w(t, x, q) = \max_{a \geq 0} \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}] = \tilde{w}(t, x, q). \quad \square$$

REMARK 4.2. Since $w = \tilde{w}$, we immediately have the following result from Proposition 4.2: w is a continuous viscosity solution to (4.20) on $(0, T) \times (0, \infty)^d \times (0, \infty)$ with the boundary condition (4.21).

4.3. *Viscosity supersolution property of U .* Let us extend the domain of the map $q \mapsto \tilde{w}_\varepsilon(t, x, q)$ from $(0, \infty)$ to the entire real line \mathbb{R} by setting $\tilde{w}_\varepsilon(t, x, 0) = 0$ and $\tilde{w}_\varepsilon(t, x, q) = \infty$ for $q < 0$. In this subsection, we consider the Legendre transform of \tilde{w}_ε with respect to the q variable

$$U_\varepsilon(t, x, p) := \sup_{q \in \mathbb{R}} \{pq - \tilde{w}_\varepsilon(t, x, q)\} = \sup_{q \geq 0} \{pq - \tilde{w}_\varepsilon(t, x, q)\}.$$

We will first show that U_ε is a classical solution to a nonlinear PDE. Then we will relate U_ε to U and derive the viscosity supersolution property of U .

PROPOSITION 4.4. *Under Assumption 4.1, we have that $U_\varepsilon \in C^{1,2,2}((0, T) \times (0, \infty)^d \times (0, 1))$ and satisfy the equation*

$$\begin{aligned}
 0 &= \partial_t U_\varepsilon + \frac{1}{2} \text{Tr}[\sigma \sigma' D_{xx} U_\varepsilon] \\
 (4.29) \quad &+ \inf_{a \in \mathbb{R}^d} \left((D_{xp} U_\varepsilon)' \sigma a + \frac{1}{2} |a|^2 D_{pp} U_\varepsilon - \theta' a D_p U_\varepsilon \right) \\
 &+ \inf_{b \in \mathbb{R}^d} \left(\frac{1}{2} |b|^2 D_{pp} U_\varepsilon - \varepsilon D_p U_\varepsilon \mathbf{1}' b \right),
 \end{aligned}$$

where $\mathbf{1} := (1, \dots, 1)' \in \mathbb{R}^d$, with the boundary condition

$$(4.30) \quad U_\varepsilon(T, x, p) = pg(x).$$

Moreover, $U_\varepsilon(t, x, p)$ is strictly convex in the p variable for $p \in (0, 1)$, with

$$(4.31) \quad \lim_{p \downarrow 0} D_p U_\varepsilon(t, x, p) = 0 \quad \text{and} \quad \lim_{p \uparrow 1} D_p U_\varepsilon(t, x, p) = \infty.$$

PROOF. Since from Proposition 4.1 the function $q \mapsto D_q \tilde{w}_\varepsilon(t, x, q)$ is strictly increasing on $(0, \infty)$ with

$$\lim_{q \downarrow 0} D_q \tilde{w}_\varepsilon(t, x, q) = 0 \quad \text{and} \quad \lim_{q \rightarrow \infty} D_q \tilde{w}_\varepsilon(t, x, q) = 1,$$

its inverse function $p \mapsto H(t, x, p)$ is well defined on $(0, 1)$. Moreover, considering that $\tilde{w}_\varepsilon(t, x, q)$ is smooth on $(0, T) \times (0, \infty)^d \times (0, \infty)$, $U_\varepsilon(t, x, p)$ is smooth on $(0, T) \times (0, \infty)^d \times (0, 1)$ and can be expressed as

$$\begin{aligned}
 (4.32) \quad U_\varepsilon(t, x, p) &= \sup_{q \geq 0} \{pq - \tilde{w}_\varepsilon(t, x, q)\} \\
 &= pH(t, x, p) - \tilde{w}_\varepsilon(t, x, H(t, x, p));
 \end{aligned}$$

see, for example, [37]. By direct calculations, we have

$$\begin{aligned}
 D_p U_\varepsilon(t, x, p) &= H(t, x, p), \\
 D_{pp} U_\varepsilon(t, x, p) &= D_p H(t, x, p) = \frac{1}{D_{qq} \tilde{w}_\varepsilon(t, x, H(t, x, p))}, \\
 D_x U_\varepsilon(t, x, p) &= -D_x \tilde{w}_\varepsilon(t, x, H(t, x, p)), \\
 (4.33) \quad D_{xx} U_\varepsilon(t, x, p) &= -D_{xx} \tilde{w}_\varepsilon(t, x, H(t, x, p)) \\
 &\quad + \frac{1}{D_{pp} U_\varepsilon(t, x, p)} (D_{px} U_\varepsilon)(D_{px} U_\varepsilon)', \\
 D_{px} U_\varepsilon(t, x, p) &= -D_{qx} \tilde{w}_\varepsilon(t, x, H(t, x, p)) D_{pp} U_\varepsilon(t, x, p), \\
 \partial_t U_\varepsilon(t, x, p) &= -\partial_t \tilde{w}_\varepsilon(t, x, H(t, x, p)).
 \end{aligned}$$

In particular, we see that $U_\varepsilon(t, x, p)$ is strictly convex in p for $p \in (0, 1)$ and satisfies (4.31). Now by setting $q := H(t, x, p)$, we deduce from (4.13) that

$$\begin{aligned}
 0 &= -\partial_t \tilde{w}_\varepsilon - \frac{1}{2} \text{Tr}[\sigma \sigma' D_{xx} \tilde{w}_\varepsilon] \\
 &\quad - \frac{1}{2} (|\theta|^2 + \varepsilon^2) q^2 D_{qq} \tilde{w}_\varepsilon - q \text{Tr}[\sigma \theta D_{xq} \tilde{w}_\varepsilon] \\
 &= \partial_t U_\varepsilon + \frac{1}{2} \text{Tr}[\sigma \sigma' D_{xx} U_\varepsilon] - \frac{1}{2 D_{pp} U_\varepsilon} \text{Tr}[\sigma \sigma' (D_{px} U_\varepsilon) (D_{px} U_\varepsilon)'] \\
 &\quad - \frac{1}{2} (|\theta|^2 + \varepsilon^2) \frac{(D_p U_\varepsilon)^2}{D_{pp} U_\varepsilon} + \frac{D_p U_\varepsilon}{D_{pp} U_\varepsilon} \text{Tr}[\sigma \theta D_{px} U_\varepsilon] \\
 (4.34) \quad &= \partial_t U_\varepsilon + \frac{1}{2} \text{Tr}[\sigma \sigma' D_{xx} U_\varepsilon] \\
 &\quad + \left((D_{xp} U_\varepsilon)' \sigma a^* + \frac{1}{2} |a^*|^2 D_{pp} U_\varepsilon - \theta' a^* D_p U_\varepsilon \right) \\
 &\quad + \left(\frac{1}{2} |b^*|^2 D_{pp} U_\varepsilon - \varepsilon D_p U_\varepsilon \mathbf{1}' b^* \right) \\
 &= \partial_t U_\varepsilon + \frac{1}{2} \text{Tr}[\sigma \sigma' D_{xx} U_\varepsilon] \\
 &\quad + \inf_{a \in \mathbb{R}^d} \left((D_{xp} U_\varepsilon)' \sigma a + \frac{1}{2} |a|^2 D_{pp} U_\varepsilon - \theta' a D_p U_\varepsilon \right) \\
 &\quad + \inf_{b \in \mathbb{R}^d} \left(\frac{1}{2} |b|^2 D_{pp} U_\varepsilon - \varepsilon D_p U_\varepsilon \mathbf{1}' b \right),
 \end{aligned}$$

where the minimizers a^* and b^* are defined by

$$\begin{aligned}
 a^*(t, x, p) &:= \frac{D_p U_\varepsilon(t, x, p)}{D_{pp} U_\varepsilon(t, x, p)} \theta(x) - \frac{1}{D_{pp} U(t, x, p)} \sigma'(x) D_{px} U_\varepsilon(t, x, p), \\
 b^*(t, x, p) &:= \varepsilon \frac{D_p U_\varepsilon(t, x, p)}{D_{pp} U_\varepsilon(t, x, p)} \mathbf{1}.
 \end{aligned}$$

Finally, observe that for any $p \in (0, 1)$, the maximum of $pq - (q - g(x))^+$ is attained at $q = g(x)$. Therefore, by (4.14)

$$\begin{aligned}
 U_\varepsilon(T, x, p) &= \sup_{q \geq 0} \{pq - \tilde{w}_\varepsilon(T, x, p)\} = \sup_{q \geq 0} \{pq - (q - g(x))^+\} \\
 &= pg(x). \quad \square
 \end{aligned}$$

Now we intend to use the stability of viscosity solutions to derive the supersolution property of U . We first have the following observation.

LEMMA 4.4. For any $(t, x, p) \in [0, T] \times (0, \infty)^d \times \mathbb{R}$, we have

$$\liminf_{(\varepsilon, \tilde{t}, \tilde{x}, \tilde{p}) \rightarrow (0, t, x, p)} U_\varepsilon(\tilde{t}, \tilde{x}, \tilde{p}) = U(t, x, p).$$

PROOF. As a consequence of Lemma 4.2(ii), $\tilde{w}_\varepsilon(t, x, q)$ is continuous at $(\varepsilon, t, x, q) \in [0, \infty) \times [0, T] \times (0, \infty)^d \times (0, \infty)$. This implies that $U_\varepsilon(t, x, p) = \sup_{q \geq 0} \{pq - \tilde{w}_\varepsilon(t, x, q)\}$ is lower semicontinuous at $(\varepsilon, t, x, p) \in [0, \infty) \times [0, T] \times (0, \infty)^d \times \mathbb{R}$. It follows that

$$\begin{aligned} \liminf_{(\varepsilon, \tilde{t}, \tilde{x}, \tilde{p}) \rightarrow (0, t, x, p)} U_\varepsilon(\tilde{t}, \tilde{x}, \tilde{p}) &= \sup_{q \geq 0} \{pq - \tilde{w}(t, x, q)\} = \sup_{q \geq 0} \{pq - w(t, x, q)\} \\ &= U(t, x, p), \end{aligned}$$

where the second equality follows from Proposition 4.3. \square

Before we state the supersolution property for U , let us first introduce some notation. For any $(x, \beta, \gamma, \lambda) \in (0, \infty)^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, define

$$G(x, \beta, \gamma, \lambda) := \inf_{a \in \mathbb{R}^d} \left(\lambda' \sigma(x) a + \frac{1}{2} |a|^2 \gamma - \beta \theta(x)' a \right).$$

We also consider the lower semicontinuous envelope of G

$$G_*(x, \beta, \gamma, \lambda) := \liminf_{(\tilde{x}, \tilde{\beta}, \tilde{\gamma}, \tilde{\lambda}) \rightarrow (x, \beta, \gamma, \lambda)} G(\tilde{x}, \tilde{\beta}, \tilde{\gamma}, \tilde{\lambda}).$$

Observe that, by definition,

$$(4.35) \quad G_*(x, \beta, \gamma, \lambda) = \begin{cases} G(x, \beta, \gamma, \lambda), & \text{if } \gamma > 0; \\ -\infty, & \text{if } \gamma \leq 0. \end{cases}$$

PROPOSITION 4.5. Under Assumption 4.1, U is a lower semicontinuous viscosity supersolution to the equation

$$(4.36) \quad 0 \geq \partial_t U + \frac{1}{2} \text{Tr}[\sigma \sigma' D_{xx} U] + G_*(x, D_p U, D_{pp} U, D_{xp} U)$$

for $(t, x, p) \in (0, T) \times (0, \infty)^d \times (0, 1)$, with the boundary condition

$$(4.37) \quad U(T, x, p) = pg(x).$$

PROOF. Note that the lower semicontinuity of U is a consequence of Lemma 4.4, and the boundary condition (4.37) comes from the fact that $w = \tilde{w}$ and the definition of \tilde{w} as the following calculation demonstrates:

$$\begin{aligned} U(T, x, p) &= \sup_{q \geq 0} \{pq - w(T, x, p)\} = \sup_{q \geq 0} \{pq - \tilde{w}(T, x, p)\} \\ &= \sup_{q \geq 0} \{pq - (q - g(x))^+\} = pg(x). \end{aligned}$$

Let us now turn to the PDE characterization inside the domain of U . Set $\bar{x} := (t, x, p)$. Let φ be a smooth function such that $U - \varphi$ attains a local minimum at $\bar{x}_0 = (t_0, x_0, p_0) \in (0, T) \times (0, \infty)^d \times (0, 1)$ and $U(\bar{x}_0) = \varphi(\bar{x}_0)$. Note from (4.35) that as $D_{pp}\varphi(\bar{x}_0) \leq 0$, we must have $G_*(x_0, D_p\varphi, D_{pp}\varphi, D_{xp}\varphi) = -\infty$. Thus, the viscosity supersolution property (4.36) is trivially satisfied. We therefore assume in the following that $D_{pp}\varphi(\bar{x}_0) > 0$.

Let $F_\varepsilon(\bar{x}, \partial_t U_\varepsilon(\bar{x}), D_p U_\varepsilon(\bar{x}), D_{pp} U_\varepsilon(\bar{x}), D_{xp} U_\varepsilon(\bar{x}), D_{xx} U_\varepsilon(\bar{x}))$ denote the right-hand side of (4.29). Observe from the calculation in (4.34) that as $\gamma > 0$,

$$F_\varepsilon(\bar{x}, \alpha, \beta, \gamma, \lambda, A) = \alpha + \frac{1}{2} \text{Tr}[\sigma(x)\sigma(x)'A] - \frac{1}{2\gamma} \text{Tr}[\sigma(x)\sigma(x)'\lambda\lambda'] - \frac{\beta^2}{2\gamma} (|\theta(x)|^2 + \varepsilon^2) + \frac{\beta}{\gamma} \text{Tr}[\sigma(x)\theta(x)\lambda].$$

This shows that F_ε is continuous at every $(\varepsilon, \bar{x}, \alpha, \beta, \gamma, \lambda, A)$ as long as $\gamma > 0$. It follows that for any $z = (\bar{x}, \alpha, \beta, \gamma, \lambda, A)$ with $\gamma > 0$, we have

$$\begin{aligned} F_*(z) &:= \liminf_{(\varepsilon, z') \rightarrow (0, z)} F_\varepsilon(z') = F_0(z) \\ (4.38) \quad &= \alpha + \frac{1}{2} \text{Tr}[\sigma(x)\sigma(x)'A] \\ &\quad + \inf_{a \in \mathbb{R}^d} \left(\lambda' \sigma(x) a + \frac{1}{2} |a|^2 \gamma - \theta(x)' a \beta \right). \end{aligned}$$

Since we have $U(\bar{x}) = \liminf_{(\varepsilon, \bar{x}') \rightarrow (0, \bar{x})} U_\varepsilon(\bar{x}')$ from Lemma 4.4, we may use the same argument in [41], Proposition 2.3, and obtain that

$$F_*(\bar{x}_0, \partial_t \varphi(\bar{x}_0), D_p \varphi(\bar{x}_0), D_{pp} \varphi(\bar{x}_0), D_{xp} \varphi(\bar{x}_0), D_{xx} \varphi(\bar{x}_0)) \leq 0.$$

Considering that $D_{pp}\varphi(\bar{x}_0) > 0$, we see from (4.38) and (4.35) that this is the desired supersolution property. \square

A few remarks are in order:

REMARK 4.3. Results similar to Proposition 4.5 were proved by [7], with stronger assumptions [such as the existence of an equivalent martingale measure and the existence of a unique strong solution to (2.1)], using the stochastic target formulation. Here, we first observe that the Legendre transform of U is equal to \tilde{w} and that \tilde{w} can be approximated by \tilde{w}_ε , which is a classical solution to a linear PDE and is strictly convex in q ; then, we apply the Legendre duality argument, as carried out in [30], to show that U_ε , the Legendre transform of \tilde{w}_ε , is a classical solution to a nonlinear PDE. Finally, the stability of viscosity solutions leads to the viscosity supersolution property of U .

REMARK 4.4. Instead of relying on the Legendre duality we could directly apply the dynamic programming principle of [22] for weak solutions to the formulation in Section 3.1. The problem with this approach is that it requires some growth conditions on the coefficients of (2.1), which would rule out the possibility of arbitrage, the thing we are interested in and want to keep in the scope of our discussion.

REMARK 4.5. Under our assumptions, the solution of (4.36) may not be unique as pointed out below:

(i) Let us consider the PDE satisfied by the superhedging price $U(t, x, 1)$,

$$(4.39) \quad 0 = v_t + \frac{1}{2} \text{Tr}(\sigma \sigma' D_x^2 v) \quad \text{on } (0, T) \times (0, \infty)^d,$$

$$(4.40) \quad v(T-, x) = g(x) \quad \text{on } (0, \infty)^d.$$

Unless additional boundary conditions are specified, this PDE may have multiple solutions. The role of additional boundary conditions in identifying $(t, x) \rightarrow U(t, x, 1)$ as the unique solution of the above Cauchy problem is discussed in Section 4 of [4]. Also see [36] for a similar discussion on boundary conditions for degenerate parabolic problems on bounded domains.

Even when additional boundary conditions are specified, the growth of σ might lead to the loss of uniqueness; see, for example, [6] and Theorem 4.8 of [4] which give necessary and sufficient conditions on the uniqueness of Cauchy problems in one and two-dimensional settings in terms of the growth rate of its coefficients. We also note that [14] develops necessary and sufficient conditions for uniqueness, in terms of the attainability of the boundary of the positive orthant by an auxiliary diffusion (or, more generally, an auxiliary Itô) process.

(ii) Let $\Delta U(t, x, 1)$ be the difference of two solutions of (4.39)–(4.40). Then both $U(t, x, p)$ and $U(t, x, p) + \Delta U(t, x, 1)$ are solutions of (4.36) (along with its boundary conditions). As a result, whenever (4.39) and (4.40) have multiple solutions, so does the PDE (4.36) for the value function U .

4.4. *Characterizing the value function U .* We intend to characterize U_ε as the smallest solution among a particular class of functions, as specified below in Proposition 4.6. Then, considering that $\liminf_{(\varepsilon, \tilde{t}, \tilde{x}, \tilde{p}) \rightarrow (0, t, x, p)} U_\varepsilon(\tilde{t}, \tilde{x}, \tilde{p}) = U(t, x, p)$ from Lemma 4.4, this gives a characterization for U . In determining U numerically, one could use U_ε as a proxy for U for small enough ε . Additionally, we will characterize U as the smallest nonnegative supersolution of (4.36) in Proposition 4.7.

PROPOSITION 4.6. *Suppose that Assumption 4.1 holds. Let $u: [0, T] \times (0, \infty)^d \times [0, 1] \mapsto [0, \infty)$ be of class $\mathcal{C}^{1,2,2}((0, T) \times (0, \infty)^d \times (0, 1))$ such that $u(t, x, 0) = 0$, and $u(t, x, p)$ is strictly convex in p for $p \in (0, 1)$ with*

$$(4.41) \quad \lim_{p \downarrow 0} D_p u(t, x, p) = 0 \quad \text{and} \quad \lim_{p \uparrow 1} D_p u(t, x, p) = \infty.$$

If u satisfies the partial differential inequality

$$(4.42) \quad \begin{aligned} 0 \geq & \partial_t u + \frac{1}{2} \text{Tr}[\sigma \sigma' D_{xx} u] + \inf_{a \in \mathbb{R}^d} \left((D_x p u)' \sigma a + \frac{1}{2} |a|^2 D_{pp} u - \theta' a D_p u \right) \\ & + \inf_{b \in \mathbb{R}^d} \left(\frac{1}{2} |b|^2 D_{pp} u - \varepsilon D_p u \mathbf{1}' b \right), \end{aligned}$$

where $\mathbf{1} := (1, \dots, 1)' \in \mathbb{R}^d$, with the boundary condition

$$(4.43) \quad u(T, x, p) = pg(x),$$

then $u \geq U_\varepsilon$.

PROOF. Let us extend the domain of the map $p \mapsto u(t, x, p)$ from $[0, 1]$ to the entire real line \mathbb{R} by setting $u(t, x, p) = 0$ for $p < 0$ and $u(t, x, p) = \infty$ for $p > 1$. Then we can define the Legendre transform of u with respect to the p variable

$$(4.44) \quad \begin{aligned} w^u(t, x, q) &:= \sup_{p \in \mathbb{R}} \{pq - u(t, x, p)\} \\ &= \sup_{p \in [0, 1]} \{pq - u(t, x, p)\} \geq 0 \quad \text{for } q \geq 0, \end{aligned}$$

where the positivity comes from the condition $u(t, x, 0) = 0$. First, observe that since u is nonnegative, we must have

$$(4.45) \quad w^u(t, x, q) \leq \sup_{p \in [0, 1]} pq = q \quad \text{for any } q \geq 0.$$

Next, we derive the boundary condition of w^u from (4.43) as

$$(4.46) \quad \begin{aligned} w^u(T, x, q) &= \sup_{p \in [0, 1]} \{pq - u(T, x, p)\} = \sup_{p \in [0, 1]} \{pq - pg(x)\} \\ &= (q - g(x))^+. \end{aligned}$$

Now, since $u(t, x, p)$ is strictly convex in p for $p \in (0, 1)$ and satisfies (4.41), we can express w^u as

$$w^u(t, x, q) = J(t, x, q)q - u(t, x, J(t, x, q)) \quad \text{for } q \in (0, \infty),$$

where $q \mapsto J(\cdot, q)$ is the inverse function of $p \mapsto D_p u(\cdot, p)$. We can therefore compute the derivatives of $w^u(t, x, q)$ in terms of those of $u(t, x, J(t, x, q))$, as carried out in (4.33). We can then perform the same calculation in (4.34) (but going backward), and deduce from (4.42) that for any $(t, x, q) \in (0, T) \times (0, \infty)^d \times (0, \infty)$,

$$(4.47) \quad \begin{aligned} 0 \leq & \partial_t w^u + \frac{1}{2} \text{Tr}[\sigma \sigma' D_{xx} w^u] + \frac{1}{2} (|\theta|^2 + \varepsilon^2) q^2 D_{qq} w^u \\ & + q \text{Tr}[\sigma \theta D_{xq} w^u]. \end{aligned}$$

PROOF. Let us denote by w^u the Legendre transform of u with respect to p . By the same argument in the proof of Proposition 4.6, we can show that (4.44), (4.45) and (4.46) are true. Moreover, as demonstrated in [7], Section 4, by using the supersolution property of u we may show that w^u is an upper semicontinuous viscosity subsolution on $(0, T) \times (0, \infty)^d \times (0, \infty)$ to the equation

$$(4.50) \quad \partial_t w^u + \frac{1}{2} \text{Tr}(\sigma \sigma' D_x^2 w^u) + \frac{1}{2} |\theta|^2 q^2 D_q^2 w^u + q \text{Tr}(\sigma \theta D_{xq} w^u) = 0.$$

Let $\rho(t, x, q)$ be a nonnegative C^∞ function supported in $\{(t, x, q) : t \in [0, 1], |(x, q)| \leq 1\}$ with unit mass. Without loss of generality, set $w^u(t, x, q) = 0$ for $(t, x, q) \in \mathbb{R}^{d+2} \cap ([0, T] \times (0, \infty)^d \times (0, \infty))^c$. Then for any $(t, x, q) \in \mathbb{R}^{d+2}$, define

$$w_\delta^u(t, x, q) := \rho^\delta * w^u \quad \text{where } \rho^\delta(t, x, q) := \frac{1}{\delta^{d+2}} \rho\left(\frac{t}{\delta^2}, \frac{x}{\delta}, \frac{q}{\delta}\right).$$

By definition, w_δ^u is C^∞ . Moreover, it can be shown that w_δ^u is a subsolution to (4.50) on $(0, T) \times (0, \infty)^d \times (0, \infty)$; see, for example, (3.23) and (3.24) in [12], Section 3.3.2, and [3], Lemma 2.7. Set $\bar{x} = (t, x, q)$. By (4.45), we see from the definition of w_δ^u that

$$(4.51) \quad w_\delta^u(\bar{x}) = \int_{\mathbb{R}^{d+2}} \rho^\delta(y) w^u(\bar{x} - y) dy \leq (q + \delta) \int_{\mathbb{R}^{d+2}} \rho^\delta(y) dy = q + \delta.$$

Also, the continuity of w^u implies that $w_\delta^u \rightarrow w^u$ for every $(t, x, q) \in [0, T] \times (0, \infty)^d \times (0, \infty)$. Considering that w_δ^u is a classical subsolution to (4.50), we have

$$(4.52) \quad \begin{aligned} &w_\delta^u(t, x, q) \\ &\leq \mathbb{E}[Z^{t,x,1}(T \wedge \tau_n) w_\delta^u(T \wedge \tau_n, X^{t,x}(T \wedge \tau_n), Q^{t,x,q}(T \wedge \tau_n))] \end{aligned} \quad \text{for } n \in \mathbb{N},$$

where $\tau_n := \inf\{s \geq t : |X^{t,x}(s)| > n \text{ or } |Q^{t,x,q}(s)| > n\}$. For each fixed $n \in \mathbb{N}$, thanks to (4.51), we may apply the dominated convergence theorem as we take the limit $\delta \rightarrow 0$ in (4.52). We thus get

$$(4.53) \quad \begin{aligned} &w^u(t, x, q) \\ &\leq \mathbb{E}[Z^{t,x,1}(T \wedge \tau_n) w^u(T \wedge \tau_n, X^{t,x}(T \wedge \tau_n), Q^{t,x,q}(T \wedge \tau_n))]. \end{aligned}$$

Now by applying the reverse of Fatou's lemma (see, e.g., [43], page 53) to (4.53), we have

$$\begin{aligned} w^u(t, x, q) &\leq \mathbb{E}\left[Z^{t,x,1}(T) \limsup_{n \rightarrow \infty} w^u(T \wedge \tau_n, X^{t,x}(T \wedge \tau_n), Q^{t,x,q}(T \wedge \tau_n))\right] \\ &\leq \mathbb{E}[Z^{t,x,1}(T) w^u(T, X^{t,x}(T), Q^{t,x,q}(T))] \\ &\leq \mathbb{E}[Z^{t,x,1}(T) (Q^{t,x,q}(T) - g(X^{t,x}(T)))^+] \\ &= w(t, x, q), \end{aligned}$$

where the second inequality follows from the upper semicontinuity of w^u , and the third inequality is due to (4.46). Finally, we conclude that

$$u(t, x, p) = \sup_{q \geq 0} \{pq - w^u(t, x, q)\} \geq \sup_{q \geq 0} \{pq - w(t, x, q)\} = U(t, x, p),$$

where the first equality is guaranteed by the convexity and the lower semicontinuity of u . \square

One should note that U_ε and U satisfy the assumptions stated in Propositions 4.6 and 4.7, respectively. Therefore, one can indeed see these results as PDE characterizations of the functions U_ε and U .

In this paper, under the context where equivalent martingale measures need not exist, we discuss the quantile hedging problem and focus on the PDE characterization for the minimum amount of initial capital required for quantile hedging. An interesting problem following this is the construction of the corresponding quantile hedging portfolio. We leave this problem open for future research.

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