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STOCHASTIC SHEAR THICKENING FLUIDS: STRONG CONVERGENCE OF THE GALERKIN APPROXIMATION AND THE ENERGY EQUALITY

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We consider a stochastic partial differential equation (SPDE) which describes the velocity field of a viscous, incompressible non-Newtonian fluid subject to a random force. Here, the extra stress tensor of the fluid is given by a polynomial of degree $p-1$ of the rate of strain tensor, while the colored noise is considered as a random force. We focus on the shear thickening case, more precisely, on the case $p \in [1+\frac{d}{2}, \frac{2d}{d-2})$, where d is the dimension of the space. We prove that the Galerkin scheme approximates the velocity field in a strong sense. As a consequence, we establish the energy equality for the velocity field.

1. Introduction. We consider a viscous, incompressible fluid whose motion is subject to a random force. The container of the fluid is supposed to be the torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0,1]^d$ as a part of idealization. For a differentiable vector field $v : \mathbb{T}^d \to \mathbb{R}^d$, which is interpreted as the velocity field of the fluid, we denote the *rate of strain tensor* by

$$e(v) = \left(\frac{\partial_i v_j + \partial_j v_i}{2}\right) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d. \tag{1.1}$$

We assume that the extra stress tensor

$$\tau(v) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$$

depends on $e(v)$ polynomially. More precisely, for $\nu > 0$ (the kinematic viscosity) and $p > 1$,

$$\tau(v) = 2\nu\big(1 + |e(v)|^2\big)^{(p-2)/2} e(v). \tag{1.2}$$

The linearly dependent case $p = 2$ is the *Newtonian fluid*, which is described by the Navier–Stokes equation, the special case of (1.13) and (1.14) below. On the other hand, both the *shear thinning* ($p < 2$) and the *shear thickening* ($p > 2$) cases are considered in many fields in science and engineering. For example, shear thinning fluids are used for automobile engine oil and pipeline for crude oil transportation,

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while applications of shear thickening fluids can be found in modeling of body armor and automobile four wheel driving systems.

We now explain the outline of the present paper before going through precise definitions; cf. Sections 1.1–1.4 below. The velocity field of the fluid $X_t : \mathbb{T}^d \to \mathbb{R}^d$ at time $t > 0$, given X_0 is described by the following SPDE:

$$\operatorname{div} X_t = 0, \tag{1.3}$$

$$\partial_t X_t + (X_t \cdot \nabla) X_t = -\nabla \Pi_t + \operatorname{div} \tau(X_t) + \partial_t W_t. \tag{1.4}$$

Here, and in what follows,

$$u \cdot \nabla = \sum_{j=1}^d u_j \partial_j \quad \text{and} \quad \operatorname{div} \tau(u) = \left(\sum_{j=1}^d \partial_j \tau_{ij}(u) \right)_{i=1}^d \tag{1.5}$$

for $u : \mathbb{T}^d \to \mathbb{R}^d$. Both the velocity field $X_t : \mathbb{T}^d \to \mathbb{R}^d$ and the pressure field $\Pi_t : \mathbb{T}^d \to \mathbb{R}$ are the unknown process in the SPDE. The Brownian motion W_t with values in $L_2(\mathbb{T}^d \to \mathbb{R}^d)$ (the set of vector fields on $\mathbb{T}^d$ with L_2 components) is added as the random force. Note also that the SPDE (1.3) and (1.4) for the case $p = 2$ is the stochastic Navier–Stokes equation [2, 3].

In [9], the following results are obtained for the SPDE (1.3) and (1.4) in consistency with the PDE case with nonrandom force [7].

- There exist weak solutions for $p \in I_d$, where I_d is defined as follows: by introducing $p_1(d) = \frac{3d}{d+2} \vee \frac{3d-4}{d}$, $p_2(d) = \frac{2d}{d-2}$ and $p_3(d) = \frac{3d-8+\sqrt{9d^2+64}}{2d}$, $I_d = (p_1(d), \infty)$ for $2 \le d \le 8$, $I_d = (p_1(d), p_2(d)) \cup (p_3(d), \infty)$ for $d = 9$ and $I_d = (p_3(d), \infty)$ for $d \ge 10$.
- The pathwise uniqueness of the solution holds for $p \ge 1 + \frac{d}{2}$.

We refer the readers to [9], Theorems 2.1.3 and 2.2.1, for more details of the above results.

In the case of stochastic Navier–Stokes equation, that is, the SPDE (1.3) and (1.4) with $p = 2$, the 2D (two-dimensional) case is much better understood than the higher-dimensional case. In particular, the weak solution is unique, which turns out to be a strong solution [6]. It is also known that the unique solution satisfies the energy *equality*, rather than merely an inequality as in the other dimensions [2, 6]. We note that these nice properties of the solution are obtained via the fact that, for the 2D stochastic Navier–Stokes equation, the Galerkin approximation (cf. Section 1.4 below) converges strongly enough.

Two progresses are made in this paper.

First is the generality. The above-mentioned nice properties possessed by the 2D stochastic Navier–Stokes equation are carried over to the SPDE (1.3) and (1.4) with $p \in [1 + \frac{d}{2}, \frac{2d}{d-2})$. We will do so by showing that the associated Galerkin approximation converges strongly enough.

The second progress made in this paper is that the method to prove the strong convergence of the Galerkin approximation is more direct than the ones previously used for 2D stochastic Navier–Stokes equation, for example, [6]. Our proof is based essentially only on the Gronwall's lemma. In particular, we do not need any compact embedding theorem for Sobolev-type spaces (e.g., [6], page 9, Lemma 2.5).

In the rest of this section, we introduce a series of definitions which we need to state our results precisely.

1.1. *Function spaces.* Let $\mathcal{V}$ be the set of $\mathbb{R}^d$-valued divergence free, mean-zero trigonometric polynomials, that is, the set of $v:\mathbb{T}^d \to \mathbb{R}^d$ of the following form:

$$v(x) = \sum_{z\in\mathbb{Z}^d\setminus\{0\}} \widehat{v}_z \psi_z(x), \qquad x \in \mathbb{T}^d, \tag{1.6}$$

where $\psi_z(x) = \exp(2\pi \mathbf{i} z \cdot x)$ and the coefficients $\widehat{v}_z \in \mathbb{C}^d$, $z \in \mathbb{Z}^d$ satisfy

$$\widehat{v}_z = 0 \qquad \text{except for finitely many } z, \tag{1.7}$$

$$\overline{\widehat{v}_z} = \widehat{v}_{-z} \qquad \text{for all } z, \tag{1.8}$$

$$z \cdot \widehat{v}_z = 0 \qquad \text{for all } z. \tag{1.9}$$

Note that (1.9) implies that

$$\operatorname{div} v = 0 \qquad \text{for all } v \in \mathcal{V}.$$

For $\alpha \in \mathbb{R}$ and $v \in \mathcal{V}$ we define

$$(1-\Delta)^{\alpha/2} v = \sum_{z\in\mathbb{Z}^d} (1 + 4\pi^2|z|^2)^{\alpha/2} \widehat{v}_z \psi_z.$$

We equip the torus $\mathbb{T}^d$ with the Lebesgue measure. For $p \in [1,\infty)$ and $\alpha \in \mathbb{R}$, we introduce

$$V_{p,\alpha} = \text{the completion of } \mathcal{V} \text{ with respect to the norm } \|\cdot\|_{p,\alpha}, \tag{1.10}$$

where

$$\|v\|_{p,\alpha}^p = \int_{\mathbb{T}^d} |(1-\Delta)^{\alpha/2} v|^p. \tag{1.11}$$

Then,

$$V_{p,\alpha+\beta} \subset V_{p,\alpha} \qquad \text{for } 1 \le p < \infty, \alpha \in \mathbb{R} \text{ and } \beta > 0, \tag{1.12}$$

and the inclusion $V_{p,\alpha+\beta} \to V_{p,\alpha}$ is compact if $1 < p < \infty$ ([8], page 23, (6.9)).

1.2. *The noise.* We need the following definition.

DEFINITION 1.1. Let $\Gamma : V_{2,0} \to V_{2,0}$ be a self-adjoint, nonnegative definite operator of trace class. A random variable $(W_t)_{t\geq 0}$ with values in $C([0,\infty) \to V_{2,0})$ is called a $V_{2,0}$-*valued Brownian motion* with the covariance operator Γ [abbreviated by BM$(V_{2,0}, \Gamma)$ below] if, for each $\varphi \in V_{2,0}$ and $0 \leq s < t$,

$$E[\exp(\mathbf{i}\langle\varphi, W_t - W_s\rangle)|(W_u)_{u\leq s}] = \exp\left(-\frac{t-s}{2}\langle\varphi, \Gamma\varphi\rangle\right) \qquad \text{a.s.}$$

1.3. *The SPDE.* Given an initial velocity $X_0 = \xi \in V_{2,0}$, the (random) time evolution of the velocity field $X = (X_t)_{t\geq 0}$ and the pressure field $\Pi = (\Pi_t)_{t\geq 0}$ is described by the following SPDE: for $t > 0$,

$$X_t \in V_{p,1} \cap V_{2,0}, \tag{1.13}$$

$$\partial_t X_t + (X_t \cdot \nabla)X_t = -\nabla\Pi_t + \operatorname{div}\tau(X_t) + \partial_t W_t. \tag{1.14}$$

The formal "time derivative" of W_t, a BM$(V_{2,0}, \Gamma)$, is added as the random force. Note that (1.13) implies that $\operatorname{div} X_t = 0$ in the distributional sense. As in the case of (stochastic) Navier–Stokes equation, we will reformulate the problem (1.13) and (1.14) into the one which does not contain the pressure. Let

$$b(v) = -(v \cdot \nabla)v + \operatorname{div}\tau(v), \qquad v \in \mathcal{V}. \tag{1.15}$$

Then, by integration by parts,

$$\langle\varphi, b(v)\rangle = \langle v, (v \cdot \nabla)\varphi\rangle - \langle e(\varphi), \tau(v)\rangle, \qquad \varphi \in \mathcal{V}. \tag{1.16}$$

We generalize the definition of $b(v)$ for $v \in V_{p,1} \cap V_{2,0}$ by regarding $b(v)$ as the linear functional on $\mathcal{V}$ defined by the right-hand side of (1.16). Let $\mathcal{P} : L_2(\mathbb{T}^d \to \mathbb{R}^d) \to V_{2,0}$ be the orthogonal projection. Then, formally,

$$\begin{aligned}
(1.14) \quad &\Longleftrightarrow \quad \partial_t X_t = -\nabla\Pi_t + b(X_t) + \partial_t W_t \\
&\Longrightarrow \quad \partial_t X_t = \mathcal{P}b(X_t) + \partial_t W_t \\
&\qquad\quad (\text{since } X_t, W_t \in V_{2,0}, \mathcal{P} \circ \nabla \equiv 0) \\
&\Longleftrightarrow \quad X_t = X_0 + \int_0^t \mathcal{P}b(X_s)\,ds + W_t.
\end{aligned} \tag{1.17}$$

We will refer to (1.13) and (1.17) as (SPLF)$_p$ (stochastic power law fluid). To give a more precise definition (Definition 1.2), we introduce a notation. For a Banach space S, we will denote by $L_{p,\text{loc}}(\mathbb{R}_+ \to S)$ the set of measurable functions $u : \mathbb{R}_+ \to S$ such that $\|u\|_S$ belongs to $L_p([0,T])$ for all $T \in (0,\infty)$, with the usual identification of any two elements which coincide a.e.

DEFINITION 1.2. Let (X, W) be a pair of processes such that W is a BM$(V_{2,0}, \Gamma)$. We say that (X, W) is a *weak solution* to (SPLF)$_p$ if the following two conditions are satisfied:

(a) Equation (1.13) holds in the sense that $t \mapsto X_t$ belongs to

$$L_{p,\mathrm{loc}}(\mathbb{R}_+ \to V_{p,1}) \cap L_{\infty,\mathrm{loc}}(\mathbb{R}_+ \to V_{2,0}) \cap C(\mathbb{R}_+ \to V_{p'\wedge 2,-\beta}) \tag{1.18}$$

for $\exists \beta > 0$, where $p' = \frac{p}{p-1}$.

(b) Equation (1.17) holds in the sense that

$$\langle \varphi, X_t \rangle = \langle \varphi, X_0 \rangle + \int_0^t \langle \varphi, b(X_s) \rangle \, ds + \langle \varphi, W_t \rangle \tag{1.19}$$

for all $\varphi \in \mathcal{V}$ and $t \geq 0$; cf. (1.16).

1.4. *The Galerkin approximation.* We now discuss a finite-dimensional approximation to $(\mathrm{SPLF})_p$.

For each $z \in \mathbb{Z}^d \setminus \{0\}$, let $\{e_{z,j}\}_{j=1}^{d-1}$ be an orthonormal basis of the hyperplane $\{x \in \mathbb{R}^d; z \cdot x = 0\}$ and let

$$\psi_{z,j}(x) = \begin{cases} \sqrt{2} e_{z,j} \cos(2\pi z \cdot x), & j = 1, \dots, d-1, \\ \sqrt{2} e_{z,j-d+1} \sin(2\pi z \cdot x), & j = d, \dots, 2d-2, \end{cases} \qquad x \in \mathbb{T}^d. \tag{1.20}$$

Then,

$$\{\psi_{z,j}; (z,j) \in (\mathbb{Z}^d \setminus \{0\}) \times \{1, \dots, 2d-2\}\}$$

is an orthonormal basis of $V_{2,0}$. We also introduce

$$\begin{aligned} \mathcal{V}_n &= \text{the linear span of } \{\psi_{z,j}; (z,j) \text{ with } z \in [-n,n]^d\}, \\ \mathcal{P}_n &= \text{the orthogonal projection: } V_{2,0} \to \mathcal{V}_n. \end{aligned} \tag{1.21}$$

Using the orthonormal basis (1.20), we identify $\mathcal{V}_n$ with $\mathbb{R}^N$, $N = \dim \mathcal{V}_n$. We suppose that:

► $\Gamma : V_{2,0} \to V_{2,0}$ is a self-adjoint, nonnegative definite operator of trace class such that $\Delta\Gamma = \Gamma\Delta$;

► $W = (W_t)_{t \geq 0}$ be a $\mathrm{BM}(V_{2,0}, \Gamma)$ defined on a probability space $(\Omega, \mathcal{F}, P)$; cf. Definition 1.2;

► ξ is a $V_{2,0}$-valued random variable defined on $(\Omega, \mathcal{F}, P)$ such that

$$m_0 = E[\|\xi\|_{2,0}^2] < \infty. \tag{1.22}$$

We note that the operator Γ has the following eigenfunction expansion [cf. (1.20)]:

$$\Gamma = \sum_{z,j} \gamma^{z,j} \langle \cdot, \psi_{z,j} \rangle \psi_{z,j} \qquad \text{with } \gamma^{z,j} = \langle \Gamma \psi_{z,j}, \psi_{z,j} \rangle. \tag{1.23}$$

We also note that $\mathcal{P}_n W_t$ is identified with an N-dimensional Brownian motion with covariance matrix $\Gamma\mathcal{P}_n$. We consider the following approximation of (1.17):

$$X_t^n = X_0^n + \int_0^t \mathcal{P}_n b(X_s^n)\,ds + \mathcal{P}_n W_t, \qquad t \ge 0, \tag{1.24}$$

where $X_0^n = \mathcal{P}_n \xi$. Let

$$X_t^{n,z,j} = \langle X_t^n, \psi_{z,j}\rangle \tag{1.25}$$

be the (z, j)-coordinate of X_t^n. Then, (1.24) reads

$$X_t^{n,z,j} = X_0^{n,z,j} + \int_0^t b^{z,j}(X_s^n)\,ds + W_t^{z,j}, \tag{1.26}$$

where

$$\begin{aligned} b^{z,j}(X_s^n) &= \langle X_s^n, (X_s^n \cdot \nabla)\psi_{z,j}\rangle - \langle \tau(X_s^n), e(\psi_{z,j})\rangle, \\ W_t^{z,j} &= \langle W_t, \psi_{z,j}\rangle. \end{aligned} \tag{1.27}$$

Note also that

$$X_t^{n,z,j} \equiv 0 \qquad \text{if } z \notin [-n, n]^d. \tag{1.28}$$

Let $W.$ and ξ be as above. We then define

$$\mathcal{G}_t^{\xi,W} = \sigma(\xi, W_s, s \le t), \qquad 0 \le t < \infty, \qquad \mathcal{G}_\infty^{\xi,W} = \sigma\Big(\bigcup_{t\ge 0} \mathcal{G}_t^{\xi,W}\Big),$$

$$\mathcal{N}^{\xi,W} = \{N \subset \Omega, \exists \widetilde{N} \in \mathcal{G}_\infty^{\xi,W}, N \subset \widetilde{N}, P(\widetilde{N}) = 0\}$$

and

$$\mathcal{F}_t^{\xi,W} = \sigma(\mathcal{G}_t^{\xi,W} \cup \mathcal{N}^{\xi,W}), \qquad 0 \le t < \infty. \tag{1.29}$$

The following existence and uniqueness result for the SDE (1.24) was obtained in [9].

THEOREM 1.3. *Let $W.$, ξ and $\mathcal{F}_t^{\xi,W}$ be as above. Then, for each $n \ge 1$, there exists a unique process $X.$ such that*:

(a) *X_t^n is $\mathcal{F}_t^{\xi,W}$-measurable for all $t \ge 0$;*
(b) (1.24) *is satisfied*;
(c) *for any $T > 0$,*

$$E\Big[\|X_T^n\|_2^2 + 2\int_0^T \langle e(X_t^n), \tau(X_t^n)\rangle\,dt\Big] = E[\|X_0^n\|_2^2] + \mathrm{tr}(\Gamma\mathcal{P}_n)T, \tag{1.30}$$

$$E\Big[\|X_T^n\|_2^2 + \frac{1}{C}\int_0^T \|X_t^n\|_{p,1}^p\,dt\Big] \le m_0 + \big(C + \mathrm{tr}(\Gamma)\big)T < \infty, \tag{1.31}$$

where $C = C(d, p) \in (0, \infty)$.

Suppose, in addition, that $p \geq \frac{2d}{d+2}$. *Then, for any* $T > 0$,

$$E\left[\sup_{t\leq T}\|X_t^n\|_2^2 + \int_0^T \|X_t^n\|_{p,1}^p\,dt\right] \leq (1+T)C' < \infty, \tag{1.32}$$

where $C' = C'(d, p, \Gamma, m_0) \in (0, \infty)$.

2. The strong convergence of the Galerkin approximation and the energy equality.

2.1. *Strong convergence of the Galerkin approximation.* We introduce

$$\lambda = \begin{cases} 0, & \text{if } d = 2, \\ \dfrac{2(3-p)^+}{dp - 3d + 4}, & \text{if } d \geq 3. \end{cases} \tag{2.1}$$

All the considerations in this article will be limited to the case $p > \frac{3d-4}{d}$ if $d \geq 3$ so that λ makes sense.

For $p \in [1 + \frac{d}{2}, \frac{2d}{d-2})$, the solution to $(\text{SPLF})_p$ is well behaved and is well approximated by the Galerkin approximation.

THEOREM 2.1. *Let* Γ, W *and* ξ *be as in Section* 1.4, *and let* X_t^n *be the unique solution to* (1.24); *cf. Theorem* 1.3. *Suppose additionally that*

$$d = 2, 3, 4 \quad \textit{and} \quad 1 + \frac{d}{2} \leq p < \frac{2d}{d-2}; \tag{2.2}$$

$$\textit{the operator } \Gamma\Delta \textit{ is of trace class}; \tag{2.3}$$

$$\begin{aligned} &\textit{the random variable } \xi \textit{ takes values in } V_{2,1} \quad \textit{and} \\ &m_1 \stackrel{\text{def}}{=} E[\|\xi\|_{2,1}^2] < \infty. \end{aligned} \tag{2.4}$$

Then, there exists a process $X = (X_t)_{t\geq 0}$ *on* $(\Omega, \mathcal{F}, P)$ *with the following properties for any* $T \in (0, \infty)$:

(a) *For any* $\alpha \in [0, 1)$, $X \in C([0, \infty) \to V_{2,\alpha})$ *and*

$$\sup_{0\leq t\leq T} \|X_t - X_t^n\|_{2,\alpha} \stackrel{n\to\infty}{\longrightarrow} 0 \qquad \textit{in probability}. \tag{2.5}$$

(b) *Let* $\alpha \in [0, 1)$ *if* $\lambda = 0$ [*cf.* (2.1)], *and let* $\alpha = 1 - \frac{2\lambda}{p} \in (0, 1)$ *if* $\lambda > 0$. *Then,* $X \in L_{2,\mathrm{loc}}([0, \infty) \to V_{2,1+\alpha})$ *and*

$$\int_0^T \|X_t - X_t^n\|_{2,1+\alpha}^2\,dt \stackrel{n\to\infty}{\longrightarrow} 0 \qquad \textit{in probability}. \tag{2.6}$$

(c) *For any* $\widetilde{p} \in [1, p)$, $X \in L_{\widetilde{p},\mathrm{loc}}([0, \infty) \to V_{\widetilde{p},1})$ *and*

$$\lim_{n\to\infty} E\left[\int_0^T \|X_t - X_t^n\|_{\widetilde{p},1}^{\widetilde{p}}\, dt\right] = 0. \tag{2.7}$$

We now explain the strategy for the proof of Theorem 2.1. Let $Z_t^{m,n} = X_t^m - X_t^n$. Then, the core of the proof is that

$$\sup_{0\le t\le T} \|Z_t^{m,n}\|_2 \stackrel{m,n\to\infty}{\longrightarrow} 0 \qquad \text{in probability.} \tag{2.8}$$

We will prove this by a series of elementary bounds (mainly, Gronwall's inequality) instead of functional analytic method based on compact embedding as in [6]. We have by Itô's formula (cf. Lemma 3.2 below for the detail), that

$$\begin{aligned} \|Z_t^{m,n}\|_2^2 &= \|(\mathcal{P}_m - \mathcal{P}_n)\xi\|_2^2 + \mathrm{tr}(\mathcal{P}_m\Gamma - \mathcal{P}_n\Gamma)t \\ &\quad + 2M_t^{m,n} + 2\int_0^t \langle Z_s^{m,n}, (\mathcal{P}_m - \mathcal{P}_n)b(X_s^n)\rangle\, ds \\ &\quad + 2\int_0^t \langle Z_s^{m,n}, b(X_s^m) - b(X_s^n)\rangle\, ds, \end{aligned} \tag{2.9}$$

where

$$M_t^{m,n} = \int_0^t \langle(\mathcal{P}_m - \mathcal{P}_n)Z_s^{m,n}, dW_s\rangle.$$

On the other hand, the following bound is known (cf. proofs of Theorem 4.29 of [7], pages 254 and 255, and Theorem 2.2.1 of [9]) for $p > \frac{d}{2}$ there exists $C \in (0, \infty)$ such that

$$\langle v - w, b(v) - b(w)\rangle \le C\|\nabla v\|_p^{2p/(2p-d)}\|v - w\|_2^2 \qquad \text{for all } v, w \in \mathcal{V}. \tag{2.10}$$

By (2.9) and (2.10), we observe that for $\forall t \in [0, T]$

$$\|Z_t^{m,n}\|_2^2 \le S_T^{m,n} + C\int_0^t \|\nabla X_s^m\|_p^{2p/(2p-d)}\|Z_s^{m,n}\|_2^2\, ds, \tag{2.11}$$

where

$$\begin{aligned} S_T^{m,n} = \|(\mathcal{P}_m - \mathcal{P}_n)\xi\|_2^2 + \mathrm{tr}(\mathcal{P}_m\Gamma - \mathcal{P}_n\Gamma)T + 2\sup_{0\le s\le T}|M_s^{m,n}| \\ + 2\int_0^T |\langle Z_s, (\mathcal{P}_m - \mathcal{P}_n)b(X_s^n)\rangle|\, ds. \end{aligned}$$

Thus, by Gronwall's lemma,

$$\begin{aligned} &\sup_{0\le t\le T} \|Z_t^{m,n}\|_2^2 \le S_T^{m,n}\exp(CR_T^m) \\ &\qquad\qquad \text{where } R_T^m = \int_0^t \|\nabla X_s^m\|_p^{2p/(2p-d)}\, ds. \end{aligned} \tag{2.12}$$

Since $\frac{2p}{2p-d} \le p (\Leftrightarrow p \ge 1 + \frac{d}{2})$, we see from (1.32) that $\{R_T^m\}_{m\ge1}$ are tight and so are $\{\exp(CR_T^m)\}_{m\ge1}$. Therefore, the convergence (2.8) follows if

$$S_T^{m,n} \overset{m,n\to\infty}{\longrightarrow} 0 \qquad \text{in probability.} \tag{2.13}$$

This is shown to be true for $1 + \frac{2d}{d+2} \le p < \frac{2d}{d-2}$; cf. Lemma 3.4 below. It is the most technical part of this article and requires a series of statements and bounds. The good news here is that each of them is elementary.

REMARKS. (1) In principle, the Galerkin approximation converges in stronger topology for larger p. It is thus natural that some lower bound of p [like $1 + \frac{d}{2} \le p$ in (2.2)] is required to show a result as above. To be precise, the bound $1 + \frac{d}{2} \le p$ is used to get (3.32) below. On the other hand, the upper bound on p in (2.2), $p < \frac{2d}{d-2}$ is assumed for a technical reason, which unfortunately does not seem easy to get rid of. This technical condition guarantees the continuous embedding of $V_{2,2}$ into $V_{p,\alpha}$ with $\alpha > 1$ and assumed rather commonly in the literature to control the $V_{p,\alpha}$-norm of the Galerkin approximation, for example, [7], page 222, (3.5) and [9], proof of Lemma 3.2.2. We will need $p < \frac{2d}{d-2}$ to be able to use (3.6) below, which is shown in [9].

(2) As mentioned in the Introduction, Theorem 2.1 and the following Corollary 2.2 can be thought of as an extension of the well-known case of 2D stochastic Navier–Stokes equation ($d = p = 2$) (see, e.g., [6], Theorem 2.6 and its proof). The results in the direction of Theorem 2.1 and the following Corollary 2.2 is also obtained in [1] for the 2D Navier–Stokes equation forced by the space–time white noise. In spite of the conceptual similarity of their result to ours, their technique, based on the Besov spaces, is much more involved. This is for the reason that, in contrast to the colored noise, the white noise is so rough that the solution is not expected to be accommodated in Sobolev spaces with positive differentiability indices.

The existence of the weak solution to the SPDE (1.13) and (1.14) in [9] includes the shear thinning case ($p < 2$). However, the weak solution discussed there is *not*, in general, a function of the initial data and the Brownian motion. On the other hand, with Theorem 2.1, it is almost straightforward to construct the weak solution to $(\mathrm{SPLF})_p$ as a function of the initial data and the Brownian motion.

COROLLARY 2.2. *Let Γ, W and ξ be as in Section 1.4 and suppose additionally that (2.2)–(2.4) hold true. Then, the process X in Theorem 2.1, coupled with W, is a weak solution to $(\mathrm{SPLF})_p$ such that*

$$X_0 = \xi; \tag{2.14}$$

$$X_t \text{ is } \mathcal{F}_t^{\xi,W}\text{-measurable} \qquad \text{for all } t \ge 0. \tag{2.15}$$

Moreover, for any $T > 0$,

$$E\left[\sup_{t \le T} \|X_t\|_2^2 + \int_0^T \|X_t\|_{p,1}^p \, dt\right] \le (1+T)C < \infty, \tag{2.16}$$

where $C = C(d, p, \Gamma, m_0) < \infty$.

We will derive Corollary 2.2 from (2.5) and (2.7); cf. Section 3.4.

2.2. *The energy equality.* The strong convergence of the Galerkin approximation proved in Theorem 2.1 has the following application.

THEOREM 2.3. *Let* Γ, W *and* ξ *be as in Section* 1.4 *and suppose additionally that* (2.2)–(2.4) *hold true. Then, the pathwise energy equality holds in the sense that there exists a martingale* M *with respect to the filtration* $\mathcal{F}_t^{\xi,W}$ *such that*

$$\begin{aligned} \frac{1}{2}\|X_t\|_2^2 = \frac{1}{2}\|X_0\|_2^2 - \int_0^t \langle e(X_s), \tau(X_s)\rangle \, ds \\ + \frac{1}{2}\operatorname{tr}(\Gamma)t + M_t, \qquad t \ge 0. \end{aligned} \tag{2.17}$$

In particular, the mean energy equality holds

$$\begin{aligned} \frac{1}{2}E[\|X_t\|_2^2] = \frac{1}{2}E[\|X_0\|_2^2] - E\left[\int_0^t \langle e(X_s), \tau(X_s)\rangle \, ds\right] \\ + \frac{1}{2}\operatorname{tr}(\Gamma)t, \qquad t \ge 0. \end{aligned} \tag{2.18}$$

We prove Theorem 2.3 by (2.5), (2.7) and (3.31) below; cf. Section 4.

REMARK. For the 2D stochastic Navier–Stokes equation ($d = p = 2$), (2.17) and (2.18) become, respectively,

$$\begin{aligned} \frac{1}{2}\|X_t\|_2^2 = \frac{1}{2}\|X_0\|_2^2 - \nu \int_0^t \|\nabla X_s\|_2^2 \, ds \\ + \frac{1}{2}\operatorname{tr}(\Gamma)t + M_t, \qquad t \ge 0, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \frac{1}{2}E[\|X_t\|_2^2] = \frac{1}{2}E[\|X_0\|_2^2] - \nu E\left[\int_0^t \|\nabla X_s\|_2^2 \, ds\right] \\ + \frac{1}{2}\operatorname{tr}(\Gamma)t, \qquad t \ge 0. \end{aligned} \tag{2.20}$$

2.3. *Remarks on the* 2D *stochastic Navier–Stokes equation*. In this subsection, we turn to the 2D stochastic Navier–Stokes equation, that is, the SPDE (1.13) and (1.14) for $d = p = 2$. We remark that some important results from the literature (e.g., [6], Sections 2.4 and 11.1) follow easily from the method of the present paper.

We suppose that:

- ▶ $d = p = 2$;
- ▶ Γ, W and ξ are as in Section 1.4;
- ▶ $X^n = (X^n_t)_{t\geq 0}$ is the unique solution to (1.24); cf. Theorem 1.3.

We also suppose that there is an $\alpha = 1, 2, \dots$ such that

(2.21) the operator $\Gamma(-\Delta)^\alpha$ is of trace class;

(2.22) the random variable ξ takes values in $V_{2,\alpha}$ and $E[\|\xi\|^2_{2,\alpha}] < \infty$.

Let $X = (X_t)_{t\geq 0}$ be the limit as $n \nearrow \infty$ of the process X^n as described in Theorem 2.1. Then, by Corollary 2.2, the pair (X, W) is identified with the unique weak solution to the SPDE (1.13) and (1.14). Moreover, by Theorem 2.3, the process X satisfies the energy equalities (2.19) and (2.20).

PROPOSITION 2.4. *Under the above assumptions, it holds for any* $T \in (0, \infty)$ *and* $\alpha_1 < \alpha$ *that*

$$\sup_{0\leq t\leq T} \|X^n_t\|^2_{2,\alpha} + \int_0^T \|X^n_t\|^2_{2,\alpha+1}\,dt, \qquad n = 1, 2, \dots \qquad \textit{are tight}; \tag{2.23}$$

$$\sup_{0\leq t\leq T} \|X^n_t - X_t\|^2_{2,\alpha_1} + \int_0^T \|X^n_t - X_t\|^2_{2,\alpha_1+1}\,dt \overset{n\nearrow\infty}{\longrightarrow} 0 \qquad \textit{in probability}. \tag{2.24}$$

Suppose, in particular, that (2.21) *and* (2.22) *are true for* $\alpha = 2$. *Then, the pathwise balance relation for the enstrophy holds in the sense that there exists a martingale* M *with respect to the filtration* $\mathcal{F}^{\xi,W}_t$ *such that*

$$\begin{aligned}\frac{1}{2}\|\nabla X_t\|^2_2 = \frac{1}{2}\|\nabla X_0\|^2_2 - \nu \int_0^t \|\Delta X_s\|^2_2\,ds \\ + \frac{1}{2}\operatorname{tr}(-\Gamma\Delta)t + M_t, \qquad t \geq 0.\end{aligned} \tag{2.25}$$

As a consequence,

$$\begin{aligned}\frac{1}{2}E[\|\nabla X_t\|^2_2] = \frac{1}{2}E[\|\nabla X_0\|^2_2] - \nu E\left[\int_0^t \|\Delta X_s\|^2_2\,ds\right] \\ + \frac{1}{2}\operatorname{tr}(-\Gamma\Delta)t, \qquad t \geq 0.\end{aligned} \tag{2.26}$$

We will prove Proposition 2.4 by (2.5) and (3.26) below; cf. Section 5.

REMARK. The mean balance relation for the enstrophy (2.26) can be used together with (2.20) to disprove Kolmogorov-type scaling law for 2D turbulent fluids ([4], page 11, Theorem 2.9).

3. Proof of Theorem 2.1. Let $n, m \in \mathbb{N}$, $n < m$ and

$$Z_t = Z_t^{m,n} \stackrel{\text{def}}{=} X_t^m - X_t^n. \tag{3.1}$$

To prove Theorem 2.1, it is enough to prove the following properties:

(a) For any $\alpha \in [0, 1)$,

$$\sup_{0 \le t \le T} \|Z_t^{m,n}\|_{2,\alpha} \stackrel{m,n \longrightarrow \infty}{\longrightarrow} 0 \qquad \text{in probability.} \tag{3.2}$$

(b) Let $\alpha \in [0, 1)$ if $\lambda = 0$ [cf. (2.1)] and let $\alpha = 1 - \frac{2\lambda}{p} \in (0, 1)$ if $\lambda > 0$. Then,

$$\int_0^T \|Z_t^{m,n}\|_{2,1+\alpha}^2 \, dt \stackrel{m,n \longrightarrow \infty}{\longrightarrow} 0 \qquad \text{in probability.} \tag{3.3}$$

(c) For any $\widetilde{p} \in [1, p)$,

$$E\left[\int_0^T \|Z_t^{m,n}\|_{\widetilde{p},1}^{\widetilde{p}} \, dt\right] \stackrel{m,n \longrightarrow \infty}{\longrightarrow} 0. \tag{3.4}$$

3.1. *Equation* (3.2) *implies equations* (3.3) *and* (3.4). We first prove (3.3) and (3.4) assuming (3.2). We will also need the following fact, which can be seen from [9], proof of Lemma 3.2.2.

LEMMA 3.1. (a) *Suppose that* $p \ge 2$ *if* $d = 2$ *and that* $p > p_3(d) \stackrel{\text{def}}{=} \frac{3d-8+\sqrt{9d^2+64}}{2d}$ *if* $d \ge 3$ [*note that* $p_3(d) \le 1 + \frac{2d}{d+2} \le 1 + \frac{d}{2}$]. *Then*, $\frac{2p}{p+2\lambda} > 1$ [*cf.* (2.1)] *and*

$$E\left[\int_0^T \|X_s^n\|_{2,2}^{2p/(p+2\lambda)} \, dt\right] \le C_T < \infty. \tag{3.5}$$

(b) *For* $2 \le p < \frac{2d}{d-2}$ *and* $\widetilde{p} \in (1, p)$, *there exists* $\alpha > 1$ *such that*

$$E\left[\int_0^T \|X_s^n\|_{p,\alpha}^{\widetilde{p}} \, dt\right] \le C_T < \infty. \tag{3.6}$$

PROOF OF (3.3). Let $\theta = \frac{1}{2-\alpha} \in (0, 1)$. Then, we have by interpolation that

$$\|Z_t^{m,n}\|_{2,1+\alpha}^2 \le \|Z_t^{m,n}\|_{2,\alpha}^{2-2\theta} \|Z_t^{m,n}\|_{2,2}^{2\theta}$$

and hence, that

$$\int_0^T \|Z_t^{m,n}\|_{2,1+\alpha}^2 \, dt \leq S_{m,n}^{2-2\theta} I_{m,n},$$

where

$$S_{m,n} = \sup_{t \leq T} \|Z_t^{m,n}\|_{2,\alpha} \quad \text{and} \quad I_{m,n} = \int_0^T \|Z_t^{m,n}\|_{2,2}^{2\theta} \, dt.$$

We note that $\frac{2\theta \leq 2p}{p+2\lambda}$. Since $S_{m,n} \stackrel{m,n \longrightarrow \infty}{\longrightarrow} 0$ in probability by (3.2) and $\{I_{m,n}\}_{m,n \geq 1}$ are tight by (3.5), we get (3.3). □

PROOF OF (3.4). By (3.3),

$$\int_0^T \|Z_t^{m,n}\|_{1,1} \, dt \stackrel{m,n \to \infty}{\longrightarrow} 0 \qquad \text{in probability } (P).$$

Moreover, the above random variables are uniformly integrable, since

$$E\left[\left(\int_0^T \|Z_t^{m,n}\|_{1,1} \, dt\right)^p\right] \stackrel{(1.32)}{\leq} C_T < \infty.$$

Therefore:

(1)

$$\lim_{m,n \to \infty} E\left[\int_0^T \|Z_t^{m,n}\|_{1,1} \, dt\right] = 0.$$

Let $m(\ell), n(\ell) \nearrow \infty$ be such that

(2)

$$\begin{aligned} \Phi_{\ell,t} &\stackrel{\text{def}}{=} |Z_t^{m(\ell),n(\ell)}| + |\nabla Z_t^{m(\ell),n(\ell)}| \\ &\stackrel{\ell \to \infty}{\longrightarrow} 0, \qquad dt|_{[0,T]} \times dx \times P\text{-a.e.}, \end{aligned}$$

where $dt|_{[0,T]} \times dx$ denotes the Lebesgue measure on $[0, T] \times \mathbb{T}^d$. Such sequences $m(\ell), n(\ell)$ exist by (1). The sequence $\{\Phi_{\ell,\cdot}\}_{\ell \geq 1}$ are uniformly integrable with respect to $dt|_{[0,T]} \times dx \times P$. In fact,

$$E\left[\int_0^T \int_{\mathbb{T}^d} \Phi_{\ell,t}^p \, dt\right] \stackrel{(1.32)}{\leq} C_T < \infty.$$

Therefore, (2) together with this uniform integrability implies (3.4) along the subsequence $m(\ell), n(\ell)$. Finally, we get rid of the subsequence, since the subsequence as $m(\ell), n(\ell)$ above can be chosen from any subsequence of m, n given in advance. □

3.2. *The bound by Gronwall's lemma.* We will prove (3.2) in Sections 3.2 and 3.3. We start with an easy Itô calculus. We write $|z|_\infty = \max_{1\le i\le d}|z_i|$ for $z=(z_1,\dots,z_d)\in\mathbb{R}^d$.

LEMMA 3.2.

$$\begin{aligned}\|Z_t\|_2^2 &= \|(\mathcal{P}_m-\mathcal{P}_n)\xi\|_2^2 + \operatorname{tr}(\mathcal{P}_m\Gamma-\mathcal{P}_n\Gamma)t \\ &\quad + 2M_t^{m,n} + 2\int_0^t \langle Z_s, (\mathcal{P}_m-\mathcal{P}_n)b(X_s^n)\rangle\, ds \\ &\quad + 2\int_0^t \langle Z_s, b(X_s^m)-b(X_s^n)\rangle\, ds,\end{aligned} \tag{3.7}$$

where

$$M_t^{m,n} = \sum_{\substack{z,j \\ n<|z|_\infty\le m}} \int_0^t Z_s^{z,j}\, dW_s^{z,j}. \tag{3.8}$$

PROOF. We write

$$Z_t = (\mathcal{P}_m-\mathcal{P}_n)\xi + \int_0^t \big(\mathcal{P}_m b(X_s^m) - \mathcal{P}_n b(X_s^n)\big)\, ds + (\mathcal{P}_m-\mathcal{P}_n)W_t.$$

Since

$$\|Z_t\|_2^2 = \sum_{z,j} |Z_t^{z,j}|^2,$$

we compute each summand. Recall that $n<m$. If $|z|_\infty \le n$, then

$$Z_t^{z,j} = \int_0^t \big(b^{z,j}(X_s^m) - b^{z,j}(X_s^n)\big)\, ds,$$

and thus,

$$|Z_t^{z,j}|^2 = 2\int_0^t Z_s^{z,j}\big(b^{z,j}(X_s^m) - b^{z,j}(X_s^n)\big)\, ds.$$

On the other hand, if $n<|z|_\infty\le m$, then

$$Z_t^{z,j} = \xi^{z,j} + \int_0^t b^{z,j}(X_s^m)\, ds + W_t^{z,j}.$$

With the martingale

$$M_t^{z,j} = \int_0^t Z_s^{z,j}\, dW_s^{z,j}$$

we have

$$|Z_t^{z,j}|^2 = |\xi_t^{z,j}|^2 + 2\int_0^t Z_s^{z,j} b^{z,j}(X_s^m)\,ds + 2M_t^{z,j} + \gamma^{z,j} t$$
$$= |\xi_t^{z,j}|^2 + 2\int_0^t Z_s^{z,j}\bigl(b^{z,j}(X_s^m) - b^{z,j}(X_s^n)\bigr)\,ds$$
$$+ 2\int_0^t Z_s^{z,j} b^{z,j}(X_s^n)\,ds + 2M_t^{z,j} + \gamma^{z,j} t,$$

where $\gamma^{z,j} = \langle \Gamma\psi_{z,j}, \psi_{z,j}\rangle$. Putting these together, we get

$$\|Z_t\|_2^2 = \|(\mathcal{P}_m - \mathcal{P}_n)\xi\|_2^2 + \mathrm{tr}(\mathcal{P}_m\Gamma - \mathcal{P}_n\Gamma)t + 2M_t^{m,n}$$
$$+ 2\sum_{\substack{z,j\\ n<|z|_\infty\le m}} \int_0^t Z_s^{z,j} b^{z,j}(X_s^n)\,ds$$
$$+ 2\sum_{\substack{z,j\\ |z|_\infty\le m}} \int_0^t Z_s^{z,j}\bigl(b^{z,j}(X_s^m) - b^{z,j}(X_s^n)\bigr)\,ds,$$

which is (3.7). □

LEMMA 3.3. *Referring to Lemma* 3.2, *let*

$$\begin{aligned} S_T^{m,n} &= \|(\mathcal{P}_m - \mathcal{P}_n)\xi\|_2^2 + \mathrm{tr}(\mathcal{P}_m\Gamma - \mathcal{P}_n\Gamma)T + 2\sup_{0\le s\le T}|M_s^{m,n}| \\ &\quad + 2\int_0^T |\langle Z_s, (\mathcal{P}_m - \mathcal{P}_n)b(X_s^n)\rangle|\,ds. \end{aligned} \tag{3.9}$$

Then, for $p > \frac{d}{2}$,

$$\sup_{0\le t\le T}\|Z_t\|_2^2 \le S_T^{m,n}\exp\biggl(C\int_0^T \|\nabla X_s^m\|_p^{2p/(2p-d)}\,ds\biggr). \tag{3.10}$$

PROOF. The lemma follows from Lemma 3.2, the known bound (2.10) and Gronwall's lemma, exactly as explained earlier; cf. (2.12). □

3.3. *Proof of* (3.2). The essential part of the proof of (3.2) is the following.

LEMMA 3.4. *For* $1 + \frac{2d}{d+2} \le p < \frac{2d}{d-2}$,

$$S_T^{m,n} \overset{m,n\to\infty}{\longrightarrow} 0 \qquad \textit{in probability},$$

where $S_T^{m,n}$ *is defined by* (3.9).

Most of this subsection is devoted to the proof of Lemma 3.4. Using Lemma 3.4, we will prove (3.2) at the end of this subsection.

Referring to (3.9), it is obvious that

$$\|(\mathcal{P}_m - \mathcal{P}_n)\xi\|_2^2 + \mathrm{tr}(\mathcal{P}_m\Gamma - \mathcal{P}_n\Gamma)T \longrightarrow 0, \qquad m, n \longrightarrow 0. \tag{3.11}$$

On the other hand, it is easy to prove that

$$E\Big[\sup_{0\le t\le T} |M_t^{m,n}|^2\Big] \longrightarrow 0, \qquad m, n \longrightarrow \infty. \tag{3.12}$$

To see this, we compute the quadratic variation of $M^{m,n}$,

$$\begin{aligned}\langle M^{m,n}\rangle_t &= \int_0^t \langle(\mathcal{P}_m\Gamma - \mathcal{P}_n\Gamma)X_s^m, X_s^m\rangle\, ds \\ &\le \|\mathcal{P}_m\Gamma - \mathcal{P}_n\Gamma\|_{2\to 2}\int_0^t \|X_s^m\|_2^2\, ds.\end{aligned}$$

Here, and in what follows, we denote the norm of the bounded operators on $V_{p,0}$ by

$$\|\cdot\|_{p\to p}. \tag{3.13}$$

We have that

$$\|\mathcal{P}_m\Gamma - \mathcal{P}_n\Gamma\|_{2\to 2}^2 \le \sum_{\substack{z,j \\ n<|z|_\infty\le m}} |\gamma^{z,j}|^2 \longrightarrow 0$$

and that

$$\sup_m E\Big[\int_0^t \|X_s^m\|_2^2\, ds\Big] \le C_t < \infty$$

by (1.32). Thus, by Doob's L^2-maximal inequality,

$$E\Big[\sup_{0\le t\le T} |M_t^{m,n}|^2\Big] \le 4E[\langle M^{m,n}\rangle_T] \longrightarrow 0.$$

Therefore, to prove Lemma 3.4, it is enough to show that

$$\int_0^T |\langle Z_s, (\mathcal{P}_m - \mathcal{P}_n)b(X_s^n)\rangle|\, ds \overset{m,n\longrightarrow\infty}{\longrightarrow} 0 \qquad \text{in probability}, \tag{3.14}$$

if $1 + \frac{2d}{d+2} \le p < \frac{2d}{d-2}$.

The rest of this section will be devoted to the proof of (3.14). We start by cutting the task into pieces. Since $(\mathcal{P}_m - \mathcal{P}_n)Z_s = (1 - \mathcal{P}_n)X_s^m$, we have

$$\begin{aligned}|\langle Z_s, (\mathcal{P}_m - \mathcal{P}_n)b(X_s^n)\rangle| &= |\langle(1 - \mathcal{P}_n)X_s^m, b(X_s^n)\rangle| \\ &\le \|(1 - \mathcal{P}_n)X_s^m\|_{p,1}\|b(X_s^n)\|_{p',-1}.\end{aligned} \tag{3.15}$$

With $\alpha > 1$ to be specified later on, we bound the first factor of (3.15) as follows:

$$\begin{aligned}\|(1-\mathcal{P}_n)X_s^m\|_{p,1} &= \|(1-\Delta)^{1/2}(1-\mathcal{P}_n)X_s^m\|_p \\ &= \big\|(1-\mathcal{P}_n)(1-\Delta)^{(1-\alpha)/2}(1-\Delta)^{\alpha/2}X_s^m\big\|_p \\ &\le \varepsilon_n\|X_s^m\|_{p,\alpha}\end{aligned} \tag{3.16}$$

$$\text{where } \varepsilon_n = \big\|(1-\mathcal{P}_n)(1-\Delta)^{(1-\alpha)/2}\big\|_{p\to p}.$$

As for the second factor of (3.15), we use [9], (1.31) and (1.32), to get

$$\begin{aligned}\|b(X_s^n)\|_{p',-1} &= \|-(X_s^n\cdot\nabla)X_s^n + \operatorname{div}\tau(X_s^n)\|_{p',-1} \\ &\le C\|X_s^n\|_{p,1}\|X_s^n\|_2 + C(1+\|\nabla X_s^n\|_p)^{p-1}.\end{aligned} \tag{3.17}$$

Putting (3.15)–(3.17) together, we have

$$\int_0^T |\langle Z_s, (\mathcal{P}_m-\mathcal{P}_n)b(X_s^n)\rangle|\,ds \le C\varepsilon_n(I_T^{m,n}+J_T^{m,n}),$$

where

$$\begin{aligned}I_T^{m,n} &= \int_0^T \|X_s^m\|_{p,\alpha}\|X_s^n\|_{p,1}\|X_s^n\|_2\,ds, \\ J_T^{m,n} &= \int_0^T \|X_s^m\|_{p,\alpha}(1+\|\nabla X_s^n\|_p)^{p-1}\,ds.\end{aligned} \tag{3.18}$$

We will prove (3.14) by showing that

$$\varepsilon_n \to 0 \qquad \text{for any } \alpha > 1; \tag{3.19}$$

$$\{I_T^{m,n}\}_{m,n}, \{J_T^{m,n}\}_{m,n} \text{ are tight} \qquad \text{for some } \alpha > 1. \tag{3.20}$$

Since $(1-\Delta)^{(1-\alpha)/2}: V_{p,0} \to V_{p,0}$ is compact for any $\alpha > 1$, (3.19) follows from Lemma 3.3.2.

LEMMA 3.5. *Let $G: V_{p,0} \to V_{p,0}$ be a compact operator. Then*

$$\lim_{n\to\infty} \|(1-\mathcal{P}_n)G\|_{p\to p} = 0.$$

PROOF. Since the projection $\mathcal{P}_n$ corresponds to the rectangular partial summation of the Fourier series, $\|\mathcal{P}_n\|_{p\to p}$ is bounded in n (see, e.g., [5], page 213, Theorem 3.5.7). Assuming this, the proof of the lemma is standard (compact uniform convergence of a series of equi-continuous functions, which converge on a dense set). □

We now turn to (3.20). We will use some facts from [9]. For $v \in \mathcal{V}$, we introduce

$$\mathcal{I}_p(v) = \int_{\mathbb{T}^d} (1+|e(v)|^2)^{(p-2)/2}|\nabla e(v)|^2, \tag{3.21}$$

$$\mathcal{K}(v) = \langle -\Delta v, (v\cdot\nabla v)v\rangle - \langle \tau(v), e(-\Delta v)\rangle + \tfrac{1}{2}\operatorname{tr}(-\Delta\Gamma\mathcal{P}_n). \tag{3.22}$$

Since $|\Delta v| \le |\nabla e(v)|$, we have

$$\|\Delta v\|_2^2 \le \mathcal{I}_p(v) \qquad \text{for } p \ge 2. \tag{3.23}$$

Then, we have from the proof of Lemma 3.2.3 in [9] that

$$\mathcal{K}(v) + c_1\mathcal{I}_p(v) \le C_1(1+\|\nabla v\|_2^2)^\lambda(1+\|\nabla v\|_p)^p, \tag{3.24}$$

$$E\left[\int_0^T \frac{\mathcal{I}_p(X_t^n)}{(1+\|X_s^n\|_2^2)^\lambda}\,dt\right] \le C_T < \infty. \tag{3.25}$$

Having prepared all the ingredients from [9], our starting point to prove (3.20) is the following tightness lemma (Lemma 3.6). In fact, this tightness, together with Lemma 3.1, is enough for the proof of (3.20) for $p=2$; cf. case 1 in the proof of Lemma 3.4 below.

LEMMA 3.6. *Let* $p \ge 1 + \frac{2d}{d+2} \ge 2$. *Then*

$$\sup_{0\le t\le T} \|X_t^n\|_{2,1}, \qquad n = 1, 2, \dots, \qquad \textit{are tight}. \tag{3.26}$$

PROOF. Note that $p \ge 1 + \frac{2d}{d+2} > \frac{3d-4}{d}$. For $x \ge 0$, let

$$f(x) = \begin{cases} \dfrac{1}{1-\lambda}(1+x)^{1-\lambda}, & \text{if } \lambda \ne 1, \\ \ln(1+x), & \text{if } \lambda = 1. \end{cases}$$

The condition $p \ge 1 + \frac{2d}{d+2}$ guarantees that $\lambda \in [0,1]$ and hence, that

$$0 \le f(x) \to \infty \qquad \text{as } x \to \infty.$$

Thus, taking (1.32) into account, it is enough to prove that

$$E\left[\sup_{0\le t\le T} f(\|\nabla X_t^n\|_2^2)\right] \le C_T < \infty. \tag{1}$$

We have by Itô's formula that

$$f(\|\nabla X_t^n\|_2^2) \le f(\|\nabla X_0^n\|_2^2) + N_t^n + 2\int_0^t \frac{\mathcal{K}(X_s^n)\,ds}{(1+\|\nabla X_s^n\|_2^2)^\lambda}, \tag{2}$$

where

$$N_t^n = \sum_{z,j}\int_0^t \frac{\Delta X_s^{n,z,j}}{(1+\|\nabla X_s^n\|_2^2)^\lambda}\,dW_s^{z,j};$$

cf. [9], proof of Lemma 3.2.3. We see from (3.24) that

$$\sup_{0\le t\le T}\int_0^t \frac{\mathcal{K}(X_s^n)\,ds}{(1+\|\nabla X_s^n\|_2^2)^\lambda} \le C_1\int_0^T (1+\|\nabla X_s^n\|_p)^p\,ds$$

and hence, that

(3)
$$E\left[\sup_{0\le t\le T}\int_0^t \frac{\mathcal{K}(X_s^n)\,ds}{(1+\|\nabla X_s^n\|_2^2)^{\lambda}}\right]\le C_T<\infty$$

by (1.32). On the other hand, we compute

$$\langle N^n\rangle_t=\sum_{z,j}\int_0^t \frac{(\Delta X_s^{n,z,j})^2}{(1+\|\nabla X_s^n\|_2^2)^{2\lambda}}\gamma^{z,j}\,ds\le\|\Gamma\|\int_0^t\frac{\|\Delta X_s^n\|_2^2}{(1+\|\nabla X_s^n\|_2^2)^{2\lambda}}\,ds.$$

Thus, by Doob's inequality, (3.23) and (3.25),

(4)
$$E\left[\sup_{0\le t\le T}|N_t^n|^2\right]\le 4E\langle N^n\rangle_T\le C_T<\infty.$$

We conclude (1) from (2)–(4). □

The following estimate plays a key role in the proof of (3.20) for $p>2$.

LEMMA 3.7. *Let*

$$\begin{aligned}&p>1, \qquad 2<p_1<\infty \qquad \text{if } d=2,\\ &p>\frac{3d-4}{d}, \qquad 2<p_1<p\frac{d}{d-2} \qquad \text{if } d\ge 3\end{aligned} \tag{3.27}$$

and let

$$p_2<p/\theta_1 \qquad \text{where } \theta_1=\frac{1/2-1/p_1}{1/2-(d-2)/(dp)}\in(0,1). \tag{3.28}$$

Then, for any $\delta>0$, *there are* $b, C\in(0,\infty)$ *such that for* $v\in\mathcal{V}$

$$\|\nabla v\|_{p_1}^{p_2}\le C\frac{\mathcal{I}_p(v)}{(1+\|\nabla v\|_2^2)^{\lambda}}+C(1+\|\nabla v\|_2^2)^b(1+\|\nabla v\|_p)^{\delta}, \tag{3.29}$$

where λ *is defined by* (2.1). *For* $d\ge 3$, *it is possible to take* $\delta=0$.

PROOF. Let $q=2$ for $d\ge 3$, $q\in(1,2)$ for $d=2$, $p_3=p\frac{d}{d-q}>p$. The choice of p_3 is made so that

(1)
$$\|\nabla v\|_{p_3}\le C\mathcal{I}_p(v)^{q/(2p)}(1+\|\nabla v\|_p)^{(2-q)/2}$$

cf. [7], page 227, (3.27).

Note also that the choice of θ_1 in (3.28) implies that

(2)
$$\frac{1}{p_1}=\frac{1-\theta_1}{2}+\frac{\theta_1}{p_3}.$$

With $\beta = \frac{1-\theta_1}{2} + \frac{\lambda q}{2p}\theta_1$ and an arbitrary $\theta_2 \in (0, 1)$, we have that

$$\begin{aligned}
\|\nabla v\|_{p_1} &\overset{(2)}{\leq} \|\nabla v\|_2^{1-\theta_1} \|\nabla v\|_{p_3}^{\theta_1} \\
&\overset{(1)}{\leq} C\|\nabla v\|_2^{1-\theta_1} \mathcal{I}_p(v)^{\theta_1 q/(2p)} (1 + \|\nabla v\|_p)^{((2-q)/2)\theta_1} \\
&\overset{\text{choice of } \beta}{=} C\biggl(\frac{\mathcal{I}_p(v)}{(1 + \|\nabla v\|_2^2)^\lambda}\biggr)^{\theta_1 q/(2p)} (1 + \|\nabla v\|_2^2)^\beta \\
&\qquad \times (1 + \|\nabla v\|_p)^{((2-q)/2)\theta_1} \\
&\overset{\theta_2 + (1-\theta_2) = 1}{\leq} C\biggl(\frac{\mathcal{I}_p(v)}{(1 + \|\nabla v\|_2^2)^\lambda}\biggr)^{(\theta_1/\theta_2)(q/(2p))} \\
&\qquad + C(1 + \|\nabla v\|_2^2)^{\beta/(1-\theta_2)} (1 + \|\nabla v\|_p)^{((2-q)/2)(\theta_1/(1-\theta_2))}
\end{aligned}$$

and hence, that

$$\|\nabla v\|_{p_1}^{p_2} \leq C\frac{\mathcal{I}_p(v)}{(1 + \|\nabla v\|_2^2)^\lambda} + C(1 + \|\nabla v\|_2^2)^b (1 + \|\nabla v\|_p)^\delta,$$

where

$$p_2 = \frac{\theta_2}{\theta_1}\frac{2p}{q}, \qquad b = \frac{\beta}{1-\theta_2} p_2, \qquad \delta = \frac{2-q}{2}\frac{\theta_1}{1-\theta_2} p_2.$$

In particular, for $d \geq 3$,

$$p_2 = \frac{\theta_2}{\theta_1} p, \qquad b = \frac{\beta}{1-\theta_2} p_2, \qquad \delta = 0.$$

Choosing θ_2 close to 1 (and then q close 2 if $d = 2$), we get the lemma. $\square$

Lemma 3.7 is used to obtain the following tightness lemma, which takes care of the case of $p > 2$.

LEMMA 3.8. *Suppose that*

$$\begin{aligned}
&p \geq 2, \qquad 2 < p_1 < \infty \qquad \text{if } d = 2, \\
&p \geq 1 + \frac{2d}{d+2}, \qquad 2 < p_1 < p\frac{d}{d-2} \qquad \text{if } d \geq 3
\end{aligned} \tag{3.30}$$

and that (3.28) *holds. Then*

$$\int_0^T \|\nabla X_t^n\|_{p_1}^{p_2}\, dt, \qquad n = 1, 2, \dots, \qquad \text{are tight.} \tag{3.31}$$

PROOF. By (3.29),

$$\int_0^T \|\nabla X_t^n\|_{p_1}^{p_2}\,dt \le C\int_0^T \frac{\mathcal{I}_p(X_t^n)}{(1+\|\nabla X_t^n\|_2^2)^\lambda}\,dt + C\sup_{0\le t\le T}(1+\|\nabla X_t^n\|_2^2)^b\int_0^T (1+\|\nabla X_t^n\|_p)^\delta\,dt.$$

The random variables on the right-hand side ($n = 1, 2, \dots$) are tight, because of (1.32), (3.25) and (3.26). □

PROOF OF LEMMA 3.4. As explained earlier [(3.11), Lemma 3.5], it is enough to show (3.20). We recall from (3.18) that

$$I_T^{m,n} = \int_0^T \|X_s^m\|_{p,\alpha}\|X_s^n\|_{p,1}\|X_s^n\|_2\,ds,$$

$$J_T^{m,n} = \int_0^T \|X_s^m\|_{p,\alpha}(1+\|\nabla X_s^n\|_p)^{p-1}\,ds.$$

Case 1 ($p = 2$).

$$I_T^{m,n} \le \sup_{0\le t\le T}\|X_t^n\|_{2,1}^2\int_0^T \|X_t^m\|_{2,\alpha}\,dt,$$

$$J_T^{m,n} \le \sup_{0\le t\le T}(1+\|\nabla X_t^n\|_2)\int_0^T \|X_t^m\|_{2,\alpha}\,dt.$$

By (3.6) and (3.26), the random variables on the right-hand side ($m, n = 1, 2, \dots$) are tight for some $\alpha > 1$.

Case 2 ($2 < p < \frac{2d}{d-2}$). As for $I_{m,n}$,

$$I_T^{m,n} \le \sup_{0\le t\le T}\|X_t^n\|_2\left(\int_0^T \|X_t^m\|_{p,\alpha}^{p'}\,dt\right)^{1/p'}\left(\int_0^T \|\nabla X_t^n\|_{p,1}^{p}\,dt\right)^{1/p}.$$

Note that $p' < 2 < p$, since $p > 2$. Thus, by (1.32) and (3.6), the random variables on the right-hand side ($m, n = 1, 2, \dots$) are tight for some $\alpha > 1$. As for $J_T^{m,n}$, we take $\widetilde{p} \in (1, p)$ so close to p that

$$p_2 \stackrel{\text{def}}{=} (p-1)\frac{\widetilde{p}}{\widetilde{p}-1} < p/\theta_1, \qquad \text{where } \theta_1 = \frac{1/2 - 1/p}{1/2 - (d-2)/(dp)} \in (0, 1).$$

Then

$$J_T^{m,n} \le \left(\int_0^T \|X_t^m\|_{p,\alpha}^{\widetilde{p}}\,dt\right)^{1/\widetilde{p}}\left(\int_0^T (1+\|\nabla X_t^n\|_p)^{p_2}\,dt\right)^{(\widetilde{p}-1)/\widetilde{p}}.$$

By (3.6) and (3.31), the random variables on the right-hand side ($m, n = 1, 2, \dots$) are tight for some $\alpha > 1$. □

PROOF OF (3.2). Since $p \geq 1 + \frac{d}{2}$, or equivalently, $\frac{2p}{2p-d} \leq p$,

$$R_T^m \stackrel{\text{def}}{=} \int_0^T \|\nabla X_s^m\|_p^{2p/(2p-d)}\, ds, \qquad m = 1, 2, \ldots, \qquad \text{are tight} \tag{3.32}$$

by (1.32), and so are $\exp(CR_T^m)$, $m = 1, 2, \ldots$. Thus, by Lemma 3.4,

$$\sup_{0 \leq t \leq T} \|Z_t^{m,n}\|_2^2 \leq S_T^{m,n} \exp(CR_T^m) \stackrel{m,n\longrightarrow\infty}{\longrightarrow} 0 \qquad \text{in probability.}$$

Therefore, we get (3.2) for $\alpha = 0$. We get (3.2) for $\alpha \in (0, 1)$ by interpolation and (3.26). □

3.4. *Proof of Corollary* 2.2. We only have to prove (1.19) and (2.16). By (1.24) and integration by parts, we have for all $\varphi \in \mathcal{V}$ and $t \geq 0$

$$\begin{aligned} \langle \varphi, X_t^n \rangle &= \langle \mathcal{P}_n \varphi, \xi \rangle + \int_0^t \big(\langle X_s^n, (X_s^n \cdot \nabla)\varphi \rangle - \langle e(\varphi), \tau(X_s^n) \rangle \big)\, ds \\ &\quad + \langle \mathcal{P}_n \varphi, W_t \rangle. \end{aligned} \tag{3.33}$$

Now, we have by (2.5) that

$$\sup_{0 \leq t \leq T} |\langle \varphi, X_t^n - X_t \rangle| \stackrel{n\nearrow\infty}{\longrightarrow} 0 \qquad \text{in probability.}$$

On the other hand, we have by (2.7) and the argument of [9], Lemma 4.1.1, that

$$\int_0^T |\langle X_t^n, (X_t^n \cdot \nabla)\varphi \rangle - \langle X_t, (X_t \cdot \nabla)\varphi \rangle|\, dt \stackrel{n\nearrow\infty}{\longrightarrow} 0 \qquad \text{in probability,}$$

$$\int_0^T |\langle e(\varphi), \tau(X_t^n) - \tau(X_t) \rangle|\, dt \stackrel{n\nearrow\infty}{\longrightarrow} 0 \qquad \text{in } L_1(P).$$

Therefore, we get (1.19) via (3.33). The bound (2.16) follows from (1.32) by Fatou's lemma.

4. Proof of Theorem 2.3.

4.1. *The strategy.* Note that

$$\|X_t^n\|_2^2 = \sum_{z,j} |X^{n,z,j}|^2.$$

Applying Itô's formula to $|X^{n,z,j}|^2$ and using (1.26), we see that

$$\begin{aligned} |X_t^{n,z,j}|^2 &= |X_0^{n,z,j}|^2 + 2\int_0^t X_s^{n,z,j}\, dW_s^{z,j} + 2\int_0^t X_s^{n,z,j} b_s^{z,j}(X_s^n)\, ds \\ &\quad + \langle \psi_{z,j}, \Gamma \psi_{z,j} \rangle t. \end{aligned} \tag{4.1}$$

Thus,

$$\|X_t^n\|_2^2 - \|X_0^n\|_2^2 = 2M_t^n + 2\int_0^t \langle X_s^n, b(X_s^n)\rangle\, ds + \mathrm{tr}(\Gamma\mathcal{P}_n)t,$$

where

$$M_t^n = \sum_{z,j} \int_0^t X_s^{n,z,j}\, dW_s^{z,j}. \tag{4.2}$$

We now recall that

$$\langle w, (v\cdot\nabla)w\rangle = 0, \tag{4.3}$$

$v \in \mathcal{V}$ and $w \in C^1(\mathbb{T}^d \to \mathbb{R}^d)$. Since

$$\langle v, b(v)\rangle \overset{(1.16), (4.3)}{=} -\langle \tau(v), e(v)\rangle,$$

we have

$$\|X_t^n\|_2^2 - \|X_0^n\|_2^2 = 2M_t^n - 2\int_0^t \langle \tau(X_s^n), e(X_s^n)\rangle\, ds + \mathrm{tr}(\Gamma\mathcal{P}_n)t. \tag{4.4}$$

Thus, Theorem 2.3 follows from the following two lemmas.

LEMMA 4.1. *Referring to* (4.2), *there exists a martingale M such that*

$$\lim_{n\to\infty} E\Big[\sup_{0\le t\le T} |M_t^n - M_t|\Big] = 0 \qquad \textit{for any } T \in (0,\infty). \tag{4.5}$$

LEMMA 4.2. *For any $T \in (0,\infty)$,*

$$\int_0^T |\langle e(X_s^n), \tau(X_s^n)\rangle - \langle e(X_s), \tau(X_s)\rangle|\, ds \overset{n\nearrow\infty}{\longrightarrow} 0 \qquad \textit{in probability } (P). \tag{4.6}$$

4.2. *Proof of Lemma* 4.1. It is enough to show that

(1)
$$\lim_{m,n\to\infty} E\Big[\sup_{0\le t\le T} |M_t^n - M_t^m|\Big] = 0.$$

By the Burkholder–Davis–Gundy inequality,

$$E\Big[\sup_{0\le t\le T} |M_t^n - M_t^m|\Big] \le C E[\langle M^n - M^m\rangle_T^{1/2}].$$

We may assume $m > n$. Then, for any $t > 0$,

$$M_t^m - M_t^n = \sum_{\substack{z,j \\ n<|z|_\infty\le m}} \int_0^t X_s^{m,z,j}\, dW_s^{z,j} + \sum_{\substack{z,j \\ |z|_\infty\le n}} \int_0^t (X_s^{m,z,j} - X_s^{n,z,j})\, dW_s^{z,j}$$

and thus,

$$\begin{aligned}\langle M^m - M^n\rangle_t &= \sum_{\substack{z,j\\ n<|z|_\infty\le m}} \int_0^t (X_s^{m,z,j})^2\gamma^{z,j}\,ds\\ &\quad + \sum_{\substack{z,j\\ |z|_\infty\le n}} \int_0^t (X_s^{m,z,j} - X_s^{n,z,j})^2\gamma^{z,j}\,ds\\ &\le Q_t + R_t,\end{aligned}$$

where

$$Q_t = \int_0^t \|(1-\mathcal{P}_n)\sqrt{\Gamma}X_s^m\|_2^2\,ds, \qquad R_t = \int_0^t \|\sqrt{\Gamma}(X_s^m - X_s^n)\|_2^2\,ds.$$

By (1.32), we have

$$E[Q_T] \le \|(1-\mathcal{P}_n)\sqrt{\Gamma}\|_{2\to 2}^2 \int_0^T E[\|X_s^m\|_2^2]\,ds \le \|(1-\mathcal{P}_n)\sqrt{\Gamma}\|_{2\to 2}^2 C_T \stackrel{n\nearrow\infty}{\longrightarrow} 0.$$

On the other hand, we see from (2.5) that

$$E[R_T^{1/2}] \le \|\sqrt{\Gamma}\|_{2\to 2} E\left[\left(\int_0^T \|X_s^m - X_s^n\|_2^2\,ds\right)^{1/2}\right] \stackrel{m,n\nearrow\infty}{\longrightarrow} 0.$$

Putting things together, we get (1).

4.3. *Proof of Lemma* 4.2. We write

$$\begin{aligned}&\langle\tau(X_s), e(X_s)\rangle - \langle\tau(X_s^n), e(X_s^n)\rangle\\ &\qquad = \langle\tau(X_s) - \tau(X_s^n), e(X_s)\rangle + \langle\tau(X_s^n), e(X_s) - e(X_s^n)\rangle.\end{aligned}$$

In view of this, we will prove (4.6) by showing that

(1)
$$\int_0^T |\langle\tau(X_s) - \tau(X_s^n), e(X_s)\rangle|\,ds \stackrel{n\nearrow\infty}{\longrightarrow} 0 \qquad \text{in probability } (P),$$

(2)
$$\int_0^T |\langle\tau(X_s^n), e(X_s) - e(X_s^n)\rangle|\,ds \stackrel{n\nearrow\infty}{\longrightarrow} 0 \qquad \text{in probability } (P).$$

To show (1), we note that

$$\big|(1+|x|^2)^{(p-2)/2}x - (1+|y|^2)^{(p-2)/2}y\big| \le C|x-y|(1+|x|+|y|)^{p-2}, \qquad x, y \in \mathbb{R}^d \otimes \mathbb{R}^d.$$

Therefore, with $p_1 \in (1, p)$ and $p_1' = \frac{p_1}{p_1 - 1}$,

$$\begin{aligned}\int_0^T |\langle \tau(X_s) - \tau(X_s^n), e(X_s)\rangle| \, ds \\ &\leq C \int_0^T ds \int_{\mathbb{T}^d} |e(X_s) - e(X_s^n)| \big(1 + |e(X_s^n)| + |e(X_s)|\big)^{p-1} \\ &\leq C I_n^{1/p_1} (I_n')^{1/p_1'},\end{aligned}$$

where

$$\begin{aligned} I_n &= \int_0^T ds \int_{\mathbb{T}^d} |e(X_s) - e(X_s^n)|^{p_1}, \\ I_n' &= \int_0^T ds \int_{\mathbb{T}^d} \big(1 + |e(X_s^n)| + |e(X_s)|\big)^{(p-1)p_1'}. \end{aligned}$$

Note that $(p-1)p_1' \searrow p$ as $p_1 \nearrow p$. Thus, for p_1 sufficiently close to p, $\{I_n'\}_{n \geq 1}$ are tight by (3.31). On the other hand, $I_n \to 0$ in probability (P) for any $p_1 < p$ by (2.7). Thus, we get (1).

As for (2), with $p_1 \in (1, p)$ and $p_1' = \frac{p_1}{p_1 - 1}$ again,

$$\int_0^T ds \int_{\mathbb{T}^d} |\langle \tau(X_s^n), e(X_s) - e(X_s^n)\rangle| \, ds \leq J_n^{1/p_1} (J_n')^{1/p_1'},$$

where

$$\begin{aligned} J_n &= \int_0^T ds \int_{\mathbb{T}^d} |e(X_s) - e(X_s^n)|^{p_1}, \\ J_n' &= \int_0^T ds \int_{\mathbb{T}^d} |\tau(X_s^n)|^{p_1'} \leq \nu \int_0^T ds \int_{\mathbb{T}^d} \big(1 + |e(X_s^n)|\big)^{(p-1)p_1'}. \end{aligned}$$

As in the proof of (1), for p_1 sufficiently close to p, $\{J_n'\}_{n \geq 1}$ are tight by (3.31), and $J_n \to 0$ in probability (P) for any $p_1 < p$ by (2.7). Thus, we get (2).

5. Proof of Proposition 2.4.

5.1. *Proofs of* (2.23) *and* (2.24). Note that for $\alpha = 0, 1, 2, \ldots$

$$\|\nabla^\alpha v\|_2^2 = \langle v, (-\Delta)^\alpha v\rangle = \sum_{z,j} (-4\pi^2 |z|^2)^\alpha \langle v, \psi_{z,j}\rangle^2, \qquad v \in \mathcal{V}.$$

By plugging $v = X_t^n$ into the above identity, and using (4.1), we obtain that

$$\begin{aligned} \|\nabla^\alpha X_t^n\|_2^2 = \|\nabla^\alpha X_0^n\|_2^2 + 2M_t^n + 2\int_0^t \langle (-\Delta)^\alpha X_s^n, b(X_s^n)\rangle \, ds \\ + \operatorname{tr}(\Gamma(-\Delta)^\alpha \mathcal{P}_n) t, \end{aligned} \tag{5.1}$$

where

$$M_t^n = \sum_{z,j} \int_0^t (-\Delta)^\alpha X_s^{n,z,j} \, dW_s^{z,j}. \tag{5.2}$$

Since we assume (2.21), we may repeat the proof of Lemma 4.1, with Γ replaced by $\Gamma(-\Delta)^\alpha$ to obtain the following lemma.

LEMMA 5.1. *Referring to* (5.2), *there exists a martingale M such that*

$$\lim_{n\to\infty} E\Big[\sup_{0\le t\le T} |M_t^n - M_t|\Big] = 0 \qquad \textit{for any } T \in (0,\infty). \tag{5.3}$$

We now continue on (5.1). For $p = 2$, we have for $v \in \mathcal{V}$ that

$$\begin{aligned} \langle (-\Delta)^\alpha v, b(v)\rangle &= \langle (-\Delta)^\alpha v, (v\cdot\nabla)v\rangle + \nu\langle (-\Delta)^\alpha v, \Delta v\rangle \\ &= \langle (-\Delta)^\alpha v, (v\cdot\nabla)v\rangle - \nu\|\nabla^{\alpha+1} v\|_2^2. \end{aligned} \tag{5.4}$$

Moreover, we have for $d = 2$ that

$$|\langle (-\Delta)^\alpha v, (v\cdot\nabla)v\rangle| \le C_1 \|\nabla^{\alpha+1} v\|_2^{(2\alpha-1)/\alpha} \|\nabla v\|_2^{(\alpha+1)/\alpha}. \tag{5.5}$$

This follows from the argument in the proof of (2.27), [6], page 17. By (5.5) and Young inequality, we obtain that

$$|\langle (-\Delta)^\alpha v, (v\cdot\nabla)v\rangle| \overset{(2\alpha-1)/(2\alpha)+1/(2\alpha)=1}{\le} \frac{\nu}{2}\|\nabla^{\alpha+1} v\|_2^2 + C_2\|\nabla v\|_2^{2\alpha+2}. \tag{5.6}$$

By (5.1) and (5.4),

$$\begin{aligned} &\|\nabla^\alpha X_t^n\|_2^2 + 2\nu \int_0^t \|\nabla^{\alpha+1} X_s^n\|_2^2 \, ds \\ &\qquad = \|\nabla^\alpha X_0^n\|_2^2 + 2M_t^n + 2\int_0^t \langle (-\Delta)^\alpha X_s^n, (X_s^n\cdot\nabla) X_s^n\rangle \, ds \\ &\qquad\quad + \operatorname{tr}(\Gamma(-\Delta)^\alpha \mathcal{P}_n)t. \end{aligned} \tag{5.7}$$

Therefore, by (5.6),

$$\begin{aligned} &\|\nabla^\alpha X_t^n\|_2^2 + \nu \int_0^t \|\nabla^{\alpha+1} X_s^n\|_2^2 \, ds \\ &\qquad \le \|\nabla^\alpha X_0^n\|_2^2 + 2M_t^n + C_2\int_0^t \|\nabla X_s^n\|_2^{2\alpha+2} \\ &\qquad\quad + \operatorname{tr}(\Gamma(-\Delta)^\alpha \mathcal{P}_n)t. \end{aligned} \tag{5.8}$$

We conclude the tightness (2.23) from (5.8), using (2.21), (2.22), Lemmas 3.6 and 5.1. The convergence (2.24) follows from (2.5) and (2.23) via interpolation.

5.2. *The pathwise balance relation for the enstrophy.* Here, we prove that the process defined by (2.25) is a martingale. Since

$$\langle \Delta v, (v \cdot \nabla) v \rangle = 0 \qquad \text{for } v \in \mathcal{V} \qquad \text{cf. [7], page 225, (3.20),}$$

we set $\alpha = 1$ in (5.7) to get

$$\|\nabla X_t^n\|_2^2 + 2\nu \int_0^t \|\Delta X_s^n\|_2^2 \, ds = \|\nabla X_0^n\|_2^2 + 2M_t^n + \operatorname{tr}(\Gamma(-\Delta)\mathcal{P}_n)t, \tag{5.9}$$

where

$$M_t^n = \sum_{z,j} \int_0^t (-\Delta) X_s^{n,z,j} \, dW_s^{z,j},$$

for which Lemma 5.1 (with $\alpha = 1$) is valid. Since we assume (2.21) and (2.22) with $\alpha = 2$, we have by (2.24) that

$$\sup_{0 \le t \le T} \|X_t^n - X_t\|_{2,1}^2 + \int_0^T \|X_t^n - X_t\|_{2,2}^2 \, dt \stackrel{n \nearrow \infty}{\longrightarrow} 0 \qquad \text{in probability.} \tag{5.10}$$

Therefore, we let $n \nearrow \infty$ in (5.9) to see that

$$\|\nabla X_t\|_2^2 + 2\nu \int_0^t \|\Delta X_s\|_2^2 \, ds = \|\nabla X_0\|_2^2 + 2M_t + \operatorname{tr}(\Gamma(-\Delta))t, \qquad t \ge 0.$$

This means that the process $M.$ defined by (2.25) is exactly the martingale obtained in Lemma 5.1 (with $\alpha = 1$).

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