The Annals of Applied Probability
2004, Vol. 14, No. 3, 1577–1602
DOI 10.1214/105051604000000387

BEHAVIOR DOMINATED BY SLOW PARTICLES IN A DISORDERED ASYMMETRIC EXCLUSION PROCESS

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We study the large space and time scale behavior of a totally asymmetric, nearest-neighbor exclusion process in one dimension with random jump rates attached to the particles. When slow particles are sufficiently rare, the system has a phase transition. At low densities there are no equilibrium distributions, and on the hydrodynamic scale the initial profile is transported rigidly. We elaborate this situation further by finding the correct order of the correction from the hydrodynamic limit, together with distributional bounds averaged over the disorder. We consider two settings, a macroscopically constant low density profile and the outflow from a large jam.

1. Introduction. We study a simple model of single-lane traffic, a system known in the interacting particle systems literature as the totally asymmetric nearest-neighbor exclusion process. In the traffic interpretation the particles in the process represent vehicles that occupy the points (sites) of the one-dimensional integer lattice $\mathbf{Z}$. The particles move to the right by executing nearest-neighbor jumps after exponentially distributed random waiting times. The continuous waiting time distribution has the convenient effect that simultaneous jump attempts never happen. The exclusion rule means that jumps to already occupied sites are prohibited. If a particle attempts to jump but the site to its right is already taken, the particle simply stays put and waits for its next jump attempt. So the particles, or vehicles, never pass each other.

The special feature we add to the process is random rates. This means that the mean waiting time between successive jump attempts varies from particle to particle. These mean waiting times will be chosen randomly at time zero, and then kept fixed as the dynamics is run. Since the exclusion rule prevents faster particles from overtaking slower particles, the system has the potential to produce large platoons of particles trapped behind unusually slow particles. This paper studies some aspects of this clustering phenomenon.

Received March 2003; revised August 2003.

[1]Supported in part by the Summer Internship Program in Probability and Stochastic Processes, UW–Madison, NSF Grant DMS-00-98605.

[2]Supported in part by NSF Grant DMS-01-26775.

AMS 2000 subject classification. 60K35.

Key words and phrases. Asymmetric exclusion process, hydrodynamic limit, random rates.

Next we introduce notation and give a technically more precise description of the process. We label the particles by integers in an increasing fashion. The position of particle i at time t is denoted by an integer-valued random variable $\sigma_i(t)$. The exclusion rule stipulates that $\sigma_i(t) < \sigma_{i+1}(t)$ for all $i \in \mathbf{Z}$ and all $t \geq 0$.

At the outset each particle σ_i receives its jump rate p_i, which then remains fixed throughout the dynamics. The rates $\mathbf{p} = \{p_i\}$ are i.i.d. random variables with common distribution F. F is supported on $(c, 1]$ for some $c > 0$, and we take c to be the left endpoint of the support of F. In other words, $F(p) = 0$ for $p < c$, $F(p) > 0$ for $p > c$, and $F(1) = 1$. We also assume $F(c) = 0$ so no particle has c as its intrinsic jump rate.

Once the rates have been fixed and an initial configuration $\sigma = (\sigma_i : i \in \mathbf{Z})$ specified, the process $\sigma(t) = (\sigma_i(t) : i \in \mathbf{Z})$ evolves in the usual way: each particle σ_i carries its own Poisson clock of rate p_i, and whenever the clock rings, σ_i advances one step to the right provided the next site to the right is vacant.

It is also useful to consider the gaps $\eta_i(t) = \sigma_{i+1}(t) - \sigma_i(t) - 1$. The process $\eta(t) = (\eta_i(t) : i \in \mathbf{Z})$ is a zero-range process with random rates attached to the spatial positions. The jump rule is that whenever a particle is present at position i $[\eta_i \geq 1]$, one particle is moved from i to $i - 1$ at rate p_i. We can also view this system as a series of tandem queues where queue i is served at rate p_i, and customers departing queue i immediately join queue $i - 1$. The gap variable $\eta_i(t)$ is the queue length and the particle increment $\sigma_i(t) - \sigma_i(0)$ is the departure process from queue i.

Fix the rates $\mathbf{p} = \{p_i\}$. Given any $a \in [0, c]$, the product distribution $P^{\mathbf{p}}$ with geometric marginals

$$P^{\mathbf{p}}[\eta_i = k] = \left(1 - \frac{a}{p_i}\right)\left(\frac{a}{p_i}\right)^k, \qquad k = 0, 1, 2, \ldots, \tag{1.1}$$

is an invariant distribution for the gap process $\eta(t)$. In this equilibrium, each particle motion is marginally a Poisson process with rate a. More precisely, for each i, the increment $\sigma_i(t) - \sigma_i(s)$ is Poisson with mean $a(t - s)$. This is a consequence of Burke's theorem from queueing theory, according to which the departure process of an $M/M/1$ queue in equilibrium is a Poisson process.

When F is suitably chosen, this model manifests a phase transition. Here is a way to approach it. Given $a \in [0, c]$, the (annealed) mean gap in equilibrium is

$$u = \int E^{\mathbf{p}}[\eta_i]\, F^{\otimes \mathbf{Z}}(d\mathbf{p}) = \int_{(c,1]} \frac{a}{p - a}\, dF(p).$$

The common velocity a of the particles cannot exceed c because there are particles whose intrinsic rates come arbitrarily close to c. Thus the maximal mean gap u^* is defined by letting $a \nearrow c$; in other words,

$$u^* = \int_{(c,1]} \frac{c}{p - c}\, dF(p). \tag{1.2}$$

If this integral is finite, there is a critical gap size $u^* < \infty$ such that the geometric product equilibrium distributions do not exist for mean gaps $u > u^*$. Equivalently, there is a positive critical density $\rho^* = (1 + u^*)^{-1}$ for the exclusion particles such that the product equilibria for the gaps do not exist at low densities $\rho < \rho^*$. One interesting question is the behavior of the system at low densities.

This system attracted interest in both the theoretical physics and mathematics literature, starting from the mid-1990s. It appears that the invariant distributions (1.1) have been discovered several times independently. Among the early ones was Evans [3, 4], who derived the invariant distributions for the disordered exclusion model in both continuous and discrete time. Independently, Krug and Ferrari [7] studied the phase transition of the continuous-time model and interpreted the results in various physical contexts such as traffic flow and directed polymers. In general, on the physics side, there is wide interest in particle systems as simple models of traffic flow and other "single file" systems. We refer the reader to [8] for a review of particle systems in traffic modeling. The state of the art in traffic modeling with exclusion type systems is the Gray–Griffeath model [6], which is an exclusion process whose jump rates depend on nearby sites.

Returning to the disordered exclusion, on the mathematical side, Benjamini, Ferrari and Landim [2] first proved hydrodynamic limits for several asymmetric exclusion and zero-range processes with random rates. However, their assumptions specifically ruled out the phase transition.

A complete hydrodynamic limit theorem for the model studied here was proved by Seppäläinen and Krug [11]. For the case $\rho^* > 0$, the result was the following. If the initial distributions have a macroscopic profile below ρ^*, then on the hydrodynamic scale the initial macroscopic profile is rigidly translated at speed c. In particular, if the system has initially a spatially homogeneous particle distribution with density $\rho < \rho^*$ (such as ergodic gaps with mean $u > u^*$), a tagged particle satisfies

$$t^{-1}\sigma_i(t) \to c \qquad \text{as } t \to \infty.$$

Subsequently Andjel, Ferrari, Guiol and Landim [1] proved a weak convergence result for the low density regime. Start the system so that the gaps are ergodic with mean $u > u^*$. Then the gap process converges weakly to the maximal invariant distribution, in other words, to the product distribution with marginals as in (1.1) with $a = c$.

The hydrodynamic limit and the weak limit suggest the following picture. Let us follow particle σ_0 that initially starts at the origin. The other particles are distributed so that the gaps are, for example, i.i.d. with mean $u > u^*$, and then initially particle density is $\rho < \rho^*$. As t grows, particle $\sigma_0(t)$ experiences an increasing density around itself, and correspondingly its advance is slowed down. The reason is that σ_0 is part of an ever-growing "platoon" of particles, headed by an especially slow particle. As this platoon catches up with slower platoons ahead of it, it grows and slows down even more. As $t \to \infty$, the particle density around $\sigma_0(t)$ approaches the critical density ρ^*, and simultaneously its motion slows down to rate c. However, all this must happen at a scale below the hydrodynamic, because the hydrodynamic limit reveals only the trivial final behavior.

The purpose of this paper is to quantify the slowdown experienced by $\sigma_0(t)$ when the system starts at low density. Technically speaking, we are seeking the next-order term in the hydrodynamic limit. We find that by time t, $\sigma_0(t)$ has traveled a distance $ct + w(t)t^{(\nu+1)/(\nu+2)}$, where $\nu > 0$ is an exponent characterizing the tail of $F(p)$ as $p \searrow c$, and $w(t)$ is a random quantity, which becomes strictly positive and is tight as $t \to \infty$. We do not have a precise limiting distribution for $w(t)$. Our bounds suggest that, for large t, the tail of $w(t)$ behaves like $\exp\{-C(u-u^*)^{-1}w^{2+\nu}\}$ for some constant C. These results are for annealed distributions, in other words, for probabilities where the random rates have been averaged out.

Following the nonrigorous picture sketched above, proofs of the estimates proceed by bounding the rate of the slowest particle in a suitable range ahead of $\sigma_0(t)$. The technical side of the proofs involves couplings of various kinds between several processes with different rates and/or initial distributions.

We also address another question which is related, and partly uses the same tools for the proof, as the main result. Suppose the exclusion process starts with all sites in $(-\infty, 0]$ occupied and all sites in $[1, \infty)$ vacant. The traffic version of this setup is outflow from a large jam: initially vehicles are packed at maximal density 1 to the left of the origin, and we follow the evolution of the density profile of the vehicles on a macroscopic scale. As time t increases to ∞, the t^{-1}-scaled density profile of vehicles approaches a particular deterministic function supported on the interval $(-\infty, c]$. It follows from this that the number X_t of particles that are in (ct, ∞) at time t must satisfy $X_t = o(t)$. We find bounds on the true size of X_t. This question is not restricted to the situation where $u^* < \infty$. It makes sense whenever $F(c) = 0$ because then every particle is attempting to jump at a rate strictly higher than c. Then presumably X_t is unbounded as t increases.

2. Results.

The basic assumption is on the tail of $F(p)$ as $p \searrow c$.

There exist constants $-1 < \nu < \infty$ and $0 < \kappa < \infty$ such that

$$\lim_{p \searrow c} \frac{F(p)}{(p-c)^{\nu+1}} = \kappa. \tag{2.1}$$

If the reader prefers a concrete example, let F have density $f(p) = \kappa(\nu+1) \times (p-c)^\nu$ on some interval $(c, c+\varepsilon)$. At $\nu = -1$, the distribution F has a jump of size κ at c, so there is a positive density κ of particles with minimal rate c. The behaviors we look at become simple. Values $\nu < -1$ are of course not possible. Recall the definition (1.2) of the critical gap u^*. An integration by parts checks that, under assumption (2.1), $\nu > 0$ is equivalent to $u^* < \infty$.

First we look at the slowdown phenomenon in low density. We specify that particle σ_0 starts at the origin $[\sigma_0 = 0]$. Initial locations $(\sigma_i : i \neq 0)$ of the other particles are determined by taking the initial gaps $\{\eta_i\}$ to be i.i.d. random variables with common mean $u = E\eta_i > u^*$ and finite variance. Then set

$$\sigma_i = i + \sum_{j=0}^{i-1} \eta_j \qquad \text{for } i > 0 \quad \text{and} \quad \sigma_i = i + \sum_{j=i}^{-1} \eta_j \qquad \text{for } i < 0.$$

Our results are bounds on the "annealed" distributions of the quantities of interest. This means that while the process is run with fixed rates $\mathbf{p} = \{p_i\}$, we look at the average of all the processes for different choices of $\mathbf{p}$, but with the fixed initial distribution for (σ_i). The symbol P will denote this probability measure which represents the random choice of rates, the random initial configuration (σ_i) and the random exclusion evolution.

Notationally it is convenient to use

$$\alpha = \frac{1}{\nu + 2},$$

so that in particular the power of the correction is

$$1 - \alpha = \frac{\nu + 1}{\nu + 2}.$$

Set also

$$A(\nu) = \frac{(\nu + 2)^{\nu+2}}{(\nu + 1)^{\nu+1}}.$$

THEOREM 1. *Assume* (2.1) *with* $\nu > 0$. *Let the initial gaps* $\{\eta_i\}$ *be i.i.d. random variables with common mean* $u = E\eta_i > u^*$ *and finite variance. The following bounds are valid for any* $0 < z < \infty$:

$$\limsup_{t\to\infty} P\left(\frac{\sigma_0(t) - ct}{t^{1-\alpha}} > z\right) \le \exp\left\{-A(\nu)^{-1} \frac{\kappa}{u - u^*} z^{\nu+2}\right\} \tag{2.2}$$

and

$$\liminf_{t\to\infty} P\left(\frac{\sigma_0(t) - ct}{t^{1-\alpha}} > z\right) \ge \exp\left\{-\frac{\kappa}{u - u^*} z^{\nu+2}\right\}. \tag{2.3}$$

Next we consider the situation where initially all sites in $(-\infty, 0]$ are occupied by particles, and all sites in $[1, \infty)$ are vacant. This could be thought of as an outflow from a large jam. Now there is always a rightmost particle, so we label the particles with nonpositive integers in increasing order. We drop the generic σ notation, and for this special situation denote the locations of the particles at time t by

$$\cdots < \xi_{-2}(t) < \xi_{-1}(t) < \xi_0(t).$$

The initial locations are $\xi_i(0) = i$ for $i \le 0$. Particle ξ_i jumps at rate p_i independently drawn from distribution F.

This system has a hydrodynamic limit which can be expressed in terms of the empirical measure as follows: for a compactly supported continuous test function ϕ,

$$\lim_{t\to\infty} t^{-1} \sum_{i\le 0} \phi\bigl(t^{-1}\xi_i(t)\bigr) = \int_{\mathbf{R}} \phi(x) r(x)\, dx$$

almost surely. The limiting density $r(x)$ is supported on $(-\infty, c]$. (The reader can find more information about the limit and $r(x)$ in [11].) For the homogeneous exclusion with constant rates 1, this is Rost's classical result [9], with a piecewise linear profile

$$r_1(x) = \begin{cases} 1, & x \le -1, \\ \frac{1}{2}(1-x), & -1 < x \le 1, \\ 0, & x > 1. \end{cases}$$

The random rates produce the following qualitative difference with the homogeneous case. In the homogeneous case with rates 1, the lead particle $\xi_0(t)$ is a Poisson process of rate 1, and so its location is $t + O(t^{1/2})$. In other words, its location coincides with the right edge of the hydrodynamic front. However, in the disordered system the lead particle is a Poisson process of rate p_0, which under assumption (2.1) is strictly greater than c. Thus $\xi_0(t)$ and in fact a large number of particles are ahead of the hydrodynamic front whose right edge at time t is at ct. The second question we address is to bound the number of these particles.

Let X_t be the number of particles that are beyond point ct at time t; in other words,

$$X_t = 0 \vee \sup\{k \ge 1 : \xi_{-k+1}(t) > ct\}.$$

THEOREM 2. *Assume* (2.1) *with* $\nu > 0$. *Then for all* $b > 0$,

$$\limsup_{t\to\infty} P\{X_t > bt^{1-\alpha}\} \le \exp\{-A(\nu)^{-1}\kappa b^{\nu+2}\}$$

and

$$\liminf_{t\to\infty} P\{X_t > bt^{1-\alpha}\} \ge \exp\{-A(\nu)(1+u^*)^{\nu+1}\kappa b^{\nu+2}\}.$$

When $\nu \le 0$, we no longer have a finite critical gap size u^*. Theorem 1 fails, not just because $u > u^*$ is no longer possible, but because in equilibrium $\sigma_0(t)$ is a Poisson process and has fluctuations on the scale $t^{1/2}$.

The phenomenon described by Theorem 2 is not restricted to $\nu > 0$. With $-1 < \nu \le 0$, it is still the case that many particles advance ahead of the hydrodynamic front, as no particle has the lower bound c as its actual rate.

For $\nu = 0$, our result is the same as for $\nu > 0$ but with a logarithmic weakening in the lower bound. This seems an artifact of our proof, so it is not clear whether this is the true state of affairs. Note that at $\nu = 0$ we have $\alpha = 1 - \alpha = 1/2$, matching with diffusive fluctuations.

THEOREM 3. *Assume* (2.1) *with* $\nu = 0$. *Let* $\varepsilon > 0$ *be arbitrarily small and* $0 < a < \infty$ *arbitrarily large. If* $0 < b < \infty$ *is large enough, then for all large enough* t,

$$P\{at^{1/2}(\log t)^{-1} \leq X_t \leq bt^{1/2}\} \geq 1 - \varepsilon.$$

X_t changes behavior for $\nu < 0$, and is of smaller order than $O(t^{1-\alpha})$. Unfortunately, we do not have matching upper and lower bounds. As $\nu \searrow -1$ ($\alpha \nearrow 1$), the ratio of the upper and lower bound exponents becomes 1.

THEOREM 4. *Assume* (2.1) *with* $-1 < \nu < 0$. *Let* $\varepsilon > 0$. *If* $0 < b < \infty$ *is large enough, then for all large enough* t,

$$P\{X_t \leq bt^{(1+\nu)/2}\} \geq 1 - \varepsilon.$$

If $0 < a < \infty$ *is small enough, then for all large enough* t,

$$P\{X_t \geq at^{(1+\nu)/(3+\nu)}\} \geq 1 - \varepsilon.$$

The upper bound is on the boundary of conflicting with Gaussian fluctuations of the Poisson clocks. For large b, with high probability the slowest particle among $bt^{(1+\nu)/2}$ particles has rate at most $c + qt^{-1/2}$ for a small $q > 0$. Consequently, the number of jump attempts experienced by this slow particle by time t is Poisson with mean $ct + qt^{1/2}$. This can be brought below ct by a fluctuation of order $t^{1/2}$ in the clock. Thus there is some chance that this particle does not reach ct by time t. To improve the probability to $1 - \varepsilon$, we choose b and q so that there is a large enough number of slow particles. The lower bound meets this "Gaussian border" only in the limit $\nu \searrow -1$.

3. Variational representations. In this section we run through notions which have been elaborated elsewhere [11]. The purpose is to establish the conventions followed in this paper, which in some cases deviate slightly from those used before. Let an arbitrary initial configuration $\sigma = \{\sigma_i\}$ be given, random or deterministic. Fix the rates $\{p_i\}$. The process $\sigma(t) = \{\sigma_i(t)\}$ is constructed with the usual graphical representation, by attaching a rate p_i homogeneous Poisson process $N_i = (N_i(t) : t \geq 0)$ to each particle σ_i.

Construct an auxiliary family $\{\zeta^i(t)\}$ of exclusion processes by stipulating that, at time $t = 0$, their initial locations are

$$\zeta_j^i(0) = \sigma_i + j \qquad \text{for } j \leq 0.$$

Only particle indices $j \le 0$ are used for the auxiliary processes. The jumps of the particles ζ^i_j are defined by

$$\zeta^i_j \text{ attempts to jump whenever Poisson clock } N_{i+j} \text{ rings.}$$

This translation of the index of the clock has the effect that, for any fixed k, particles $\{\sigma_k, \zeta^i_{k-i} : i \ge k\}$ make jump attempts at the same times, namely when clock N_k rings.

Process $\zeta^i(t)$ has initially all sites in $(-\infty, \sigma_i]$ occupied and all sites in $[\sigma_i + 1, \infty)$ vacant. From this observation one can see that the variational equation

$$\sigma_k(t) = \inf_{i\,:\,i \ge k} \zeta^i_{k-i}(t) \tag{3.1}$$

is valid at $t = 0$. Then one proves it by induction on jumps for all times t.

In Theorems 2–4 we consider the system $\xi(t)$ that starts exactly as $\zeta^i(t)$ but centered at the origin. Let

$$\xi^i_j(t) = \zeta^i_j(t) - \sigma_i.$$

Then the processes $\xi^i(t)$ are copies of $\xi(t)$, except that the rates $\{p_i\}$ have been shifted in space. Of course this does not affect the distribution of $\xi^i(t)$ when the rates are averaged out. We will find it convenient to use the variational equality (3.1) also in the form

$$\sigma_k(t) = \inf_{i\,:\,i \ge k} \{\sigma_i + \xi^i_{k-i}(t)\}. \tag{3.2}$$

Exclusion processes can be represented by interface processes. Suppose an interface process is given in terms of a height function $i \mapsto h_i(t)$ from $\mathbf{Z}$ into $\mathbf{Z}$. This means that at time t the interface is the graph of the function $h(t)$, so that $h_i(t)$ is the vertical coordinate of the location of the interface over site i. We impose the condition $h_i \le h_{i+1}$ on admissible height functions. Dynamics are defined by stipulating that, if $N_i(t) = N_i(t-) + 1$, then

$$h_i(t) = h_i(t-) + 1 \qquad \text{provided } h_i(t-) \le h_{i+1}(t-) - 1.$$

In other words, height h_i jumps up at rate p_i, provided it does not go above its right neighbor. Obviously, we can map between $\sigma(t)$ and $h(t)$ by

$$\sigma_i(t) = h_i(t) + i.$$

Precisely speaking, if the processes $\sigma(t)$ and $h(t)$ are coupled so that this equality is true at $t = 0$, then it remains true for all $t \ge 0$.

The gap process $\eta(t) = \{\eta_i(t)\}$ is defined in terms of these processes by

$$\eta_i(t) = \sigma_{i+1}(t) - \sigma_i(t) - 1 = h_{i+1}(t) - h_i(t).$$

The variational equation for the height process takes this form. Let $Z^i(t)$ be an interface process with these properties: initially

$$Z^i_j = 0 \qquad \text{for } j \le i \quad \text{and} \quad Z^i_j = \infty \qquad \text{for } j > i.$$

Dynamically,

Z^i_j takes its jump commands from Poisson process N_j for all i and j.

Then

$$h_k(t) = \inf_{i\,:\,i \ge k} \{h_i + Z^i_k(t)\}. \tag{3.3}$$

There is no translation in (3.3) because each column of the height processes $h(t)$ and $Z^i(t)$ reads the same clock. Since $\sigma_0(t) = h_0(t)$, we can use the variational formula

$$\sigma_0(t) = \inf_{i\,:\,i \ge 0} \{h_i + Z^i_0(t)\} \tag{3.4}$$

in the proof of Theorem 1 where we follow the evolution of $\sigma_0(t)$.

4. Proof of Theorem 1.

We begin with the key lemma that points the way to controlling the behavior of the system by looking at the slowest rate in a suitable range of indices. For fixed positive q_1 and q_2, and a positive real parameter N, let

$$J(N) = \inf\{i \ge 0 : p_i \le c + q_2 N^{-\alpha}\} \tag{4.1}$$

and define the event

$$\begin{aligned} D(N) &= \{p_i > c + q_2 N^{-\alpha} \text{ for } 0 \le i \le [q_1 N^{1-\alpha}]\} \\ &= \{J(N) > q_1 N^{1-\alpha}\}. \end{aligned} \tag{4.2}$$

LEMMA 1. *Assume* (2.1) *and recall that the rates* $\{p_i\}$ *are i.i.d. with common distribution* F. *For fixed* $q_1, q_2 > 0$,

$$\lim_{N \to \infty} P(D(N)) = \exp\{-\kappa q_1 q_2^{\nu+1}\}. \tag{4.3}$$

PROOF. Let $\delta > 0$. For p sufficiently close to c,

$$(\kappa - \delta)(p - c)^{\nu+1} \le F(p) \le (\kappa + \delta)(p - c)^{\nu+1}. \tag{4.4}$$

Due to the independence of the rates p_i, we have

$$P(D(N)) = \bigl(1 - F(c + q_2 N^{-\alpha})\bigr)^{[q_1 N^{1-\alpha}]}. \tag{4.5}$$

This yields the upper and lower bounds

$$\bigl(1 - (\kappa \pm \delta) q_2^{\nu+1} N^{-\alpha(\nu+1)}\bigr)^{[q_1 N^{1-\alpha}]}$$

for $P(D(N))$. Let $N \to \infty$ and then $\delta \to 0$ to obtain the limit (4.3). $\square$

4.1. *Proof of the upper bound in Theorem* 1. The upper bound (2.2) follows from this proposition.

PROPOSITION 1. *Suppose the initial gaps* $\{\eta_i\}$ *are an i.i.d. sequence with common mean* $u > u^*$ *and finite variance*. *Let* $q_1, q_2 > 0$. *Then, for any* $\delta > 0$,

$$\limsup_{t\to\infty} P\bigl[\sigma_0(t) \geq ct + \bigl(q_1(u-u^*) + q_2\bigr)t^{1-\alpha} + \delta t^{1-\alpha}\bigr] \leq \lim_{t\to\infty} P(D(t)).$$

Before proving Proposition 1, let us observe how it implies the upper bound (2.2). Together with Lemma 1, the proposition gives

$$\limsup_{t\to\infty} P[\sigma_0(t) \geq ct + zt^{1-\alpha}] \leq \exp\{-\kappa q_1 q_2^{\nu+1}\}$$

for any q_1, q_2 such that $z = q_1(u-u^*) + q_2 + \delta$. Minimize the right-hand side of the inequality subject to this constraint on q_1, q_2. Then let $\delta \to 0$.

The remainder of this section proves Proposition 1.

LEMMA 2. *Consider an arbitrary process* $\sigma(t)$. *Let* $K > 1$. *Then*

$$\lim_{t\to\infty} P\biggl[\sigma_0(t) = \min_{0\leq j\leq Kt}\{h_j + Z_0^j(t)\}\biggr] = 1.$$

PROOF. From the definition of the process $Z_i^{[Kt]}(\cdot)$, initially at time zero $Z_i^{[Kt]}(0) = 0$ for $i \leq [Kt]$. Variable $Z_{[Kt]}^{[Kt]}$ is the first to jump, after which $Z_{[Kt]-1}^{[Kt]}$ may jump, then $Z_{[Kt]-2}^{[Kt]}$, and so on. Consequently, the time T when variable $Z_0^{[Kt]}$ takes its first jump up is a sum of independent exponential waiting times with rates $p_{[Kt]}, p_{[Kt]-1}, p_{[Kt]-2}, \dots, p_0$. Let $\varepsilon > 0$. Since each rate p_i is bounded above by 1, $T \leq (K-\varepsilon)t$ with probability that vanishes exponentially fast as $t \to \infty$. If we take $0 < \varepsilon < K-1$, we conclude that

$$P\bigl\{Z_0^{[Kt]}(t) > 0\bigr\} \to 0$$

exponentially fast.

To prove the lemma, it suffices to show that $Z_0^{[Kt]}(t) = 0$ implies

$$\sigma_0(t) = \min_{0\leq j\leq Kt}[h_j + Z_0^j(t)].$$

Let $i > [Kt]$. Then, since $Z_0^i \geq 0$ always and the height h_i is nondecreasing in i,

$$h_i + Z_0^i(t) \geq h_i \geq h_{[Kt]} = h_{[Kt]} + Z_0^{[Kt]}(t).$$

This shows that indices $i > [Kt]$ cannot contribute to the infimum in the variational formula. $\square$

LEMMA 3. *Consider two processes* σ *and* $\tilde{\sigma}$ *whose initial gaps are i.i.d. with common mean* $E\eta_i = E\tilde{\eta}_i = u$ *and finite variances. Couple the initial configurations so that they are independent, but give the processes the same rates* $\{p_i\}$ *and the same Poisson clocks. Then for* $\delta > 0$,

$$\lim_{t\to\infty} P[\sigma_0(t) \ge \tilde{\sigma}_0(t) + \delta t^{1-\alpha}] = 0.$$

PROOF. Let $K > 1$ and define the event

$$A(K,t) = \left\{\sigma_0(t) = \min_{0\le j\le Kt}[h_j + Z_0^j(t)] \text{ and } \tilde{\sigma}_0(t) = \min_{0\le j\le Kt}[\tilde{h}_j + Z_0^j(t)]\right\}.$$

By Lemma 2, $P(A(K,t)) \to 1$ as $t \to \infty$. On $A(K,t)$,

$$\begin{aligned}\sigma_0(t) &= \min_{0\le j\le Kt}\{h_j + Z_0^j(t)\} = \min_{0\le j\le Kt}\{h_j - \tilde{h}_j + \tilde{h}_j + Z_0^j(t)\}\\ &\le \tilde{\sigma}_0(t) + \max_{0\le j\le Kt}\{h_j - \tilde{h}_j\}.\end{aligned}$$

By Kolmogorov's inequality,

$$P\left[\max_{0\le j\le Kt}\{h_j - \tilde{h}_j\} \ge \delta t^{1-\alpha}\right] \le \frac{Kt\,\mathrm{Var}[\eta_1] + Kt\,\mathrm{Var}[\tilde{\eta}_1]}{\delta^2 t^{2(1-\alpha)}}.$$

As $2(1-\alpha) = 2(\nu+1)/(\nu+2) > 1$, this last expression vanishes as $t \to \infty$. Consequently,

$$P[\sigma_0(t) \ge \tilde{\sigma}_0(t) + \delta t^{1-\alpha}] \le P\big(A(K,t)^c\big) + P\left[\max_{0\le j\le Kt}\{h_j - \tilde{h}_j\} \ge \delta t^{1-\alpha}\right]$$

gives the conclusion by letting $t \to \infty$. $\square$

Now define a particular mean u initial system as follows. Fix a number $\bar{u} < u^*$, and let $\bar{a}$ be the equilibrium velocity corresponding to average gap $\bar{u}$, defined by

$$\bar{u} = \int_{(c,1]} \frac{\bar{a}}{p - \bar{a}}\, dF(p).$$

For each realization $\mathbf{p}$ of the rates, let $\{\bar{\eta}_i\}$ have the nonstationary geometric product equilibrium distribution

$$P^{\mathbf{p}}[\bar{\eta}_i = k] = \left(1 - \frac{\bar{a}}{p_i}\right)\left(\frac{\bar{a}}{p_i}\right)^k.$$

Then $E\bar{\eta}_i = \bar{u}$, and the $\{\bar{\eta}_i\}$ are i.i.d. when the random rates are averaged out. We chose $\bar{u}$ strictly less than u^* because then

$$E[\bar{\eta}_i^2] = \int_{(c,1]} \left\{2\left(\frac{\bar{a}}{p-\bar{a}}\right)^2 + \frac{\bar{a}}{p-\bar{a}}\right\} dF(p) < \infty.$$

This finite variance is necessary so we can apply the previous Lemma 3. We cannot use equilibrium gaps at mean u^* because they have infinite variance if $0 < \nu \le 1$.

Let $\{\gamma_i\}$ be an i.i.d. sequence of nonnegative integer-valued random variables, independent of $\{\bar\eta_i\}$, and with common mean $E\gamma_i = u - \bar u$. Assume the $\{\gamma_i\}$ have finite variance. Define

$$\tilde\eta_i = \bar\eta_i + \gamma_i.$$

Then $\{\tilde\eta_i\}$ are i.i.d. with common mean u. Let $\tilde\sigma(t)$ denote the process with $\tilde\sigma_0(0) = 0$ and initial gaps $\{\tilde\eta_i\}$.

LEMMA 4. *For any* $\delta > 0$,

$$\limsup_{t\to\infty} P\big[\tilde\sigma_0(t) \ge ct + \big(q_1(u-\bar u) + q_2\big)t^{1-\alpha} + \delta t^{1-\alpha}\big] \le \limsup_{t\to\infty} P(D(t)).$$

PROOF. Thinking of the zero-range process of the gap evolution, couple the processes $\tilde\eta(t) = \{\tilde\eta_i(t)\}$ and $\bar\eta(t) = \{\bar\eta_i(t)\}$ via the basic coupling, so that $\bar\eta_i(t) \le \tilde\eta_i(t)$ for all i and t. This entails having $\tilde\sigma_i$ and $\bar\sigma_i$ read the same Poisson clocks for each i.

Let

$$J(t) = \inf\{i \ge 0 : p_i \le c + q_2 t^{-\alpha}\}.$$

The variable $J(t)$ depends only on the rates. Since $\tilde\sigma_{J(t)}(t) - \tilde\sigma_{J(t)}(0)$ is stochastically dominated by a mean $ct + q_2 t^{1-\alpha}$ Poisson random variable, the event

$$B_1(t) = \big\{\tilde\sigma_{J(t)}(t) \le \tilde\sigma_{J(t)}(0) + ct + q_2 t^{1-\alpha} + \delta t^{1-\alpha}/4\big\}$$

satisfies $P(B_1(t)^c) \to 0$. Let

$$B_2(t) = \big\{\tilde\sigma_{J(t)}(0) \le J(t)(u+1) + J(t)\delta/(4q_1)\big\}.$$

By the weak law of large numbers, $P(B_2(t)^c) \to 0$ because $J(t) \to \infty$ almost surely. By the connection between particles $\tilde\sigma_i(t)$ and gaps $\tilde\eta_i(t)$, and by the coupling with $\bar\eta_i(t)$,

$$\begin{aligned}
\tilde\sigma_0(t) &= \tilde\sigma_{J(t)}(t) - \sum_{i=0}^{J(t)-1} \tilde\eta_i(t) - J(t)\\
&\le \tilde\sigma_{J(t)}(t) - \sum_{i=0}^{J(t)-1} \bar\eta_i(t) - J(t).
\end{aligned}$$

By stationarity, $\bar\eta(t) = \{\bar\eta_i(t)\}$ has the same distribution for all $t \ge 0$, under any fixed $\mathbf{p}$.

Now combine the inequalities. On the event

$$A(t) = \big\{\tilde\sigma_0(t) \ge ct + \big(q_1(u-\bar u) + q_2\big)t^{1-\alpha} + \delta t^{1-\alpha}\big\},$$

we have

$$\tilde{\sigma}_{J(t)}(t) \geq \sum_{i=0}^{J(t)-1} \bar{\eta}_i(t) + J(t) + ct + \big(q_1(u-\bar{u}) + q_2\big)t^{1-\alpha} + \delta t^{1-\alpha}.$$

Consequently, on $A(t) \cap B_1(t)$ we have

$$\tilde{\sigma}_{J(t)}(0) \geq \sum_{i=0}^{J(t)-1} \bar{\eta}_i(t) + J(t) + q_1(u-\bar{u})t^{1-\alpha} + \tfrac{3}{4}\delta t^{1-\alpha}.$$

Next, on $A(t) \cap B_1(t) \cap B_2(t)$ we have

$$\sum_{i=0}^{J(t)-1} \bar{\eta}_i(t) \leq J(t)u - q_1(u-\bar{u})t^{1-\alpha} - \frac{3}{4}\delta t^{1-\alpha} + \frac{J(t)\delta}{4q_1}.$$

And finally, on the event $D(t)^c$, $J(t) \leq q_1 t^{1-\alpha}$, and so as our last inequality, on $A(t) \cap B_1(t) \cap B_2(t) \cap D(t)^c$ we have

$$\sum_{i=0}^{J(t)-1} \bar{\eta}_i(t) \leq J(t)\bar{u} - \tfrac{1}{2}\delta t^{1-\alpha}.$$

To summarize,

$$\begin{aligned} P\big[\tilde{\sigma}_0(t) &\geq ct + \big(q_1(u-\bar{u}) + q_2\big)t^{1-\alpha} + \delta t^{1-\alpha}\big] \\ &\leq P(D(t)) + P\big(B_1(t)^c\big) + P\big(B_2(t)^c\big) \\ &\quad + P\left(D(t)^c \cap \left\{\sum_{i=0}^{J(t)-1} \bar{\eta}_i(t) \leq J(t)\bar{u} - \delta t^{1-\alpha}/2\right\}\right). \end{aligned}$$

The conclusion follows because on $D(t)^c$, $J(t) \leq q_1 t^{1-\alpha}$ while still $J(t) \to \infty$, so the last probability vanishes as $t \to \infty$. $\square$

Now we prove Proposition 1. Fix $\bar{u} < u^*$ so that

$$q_1(u^* - \bar{u}) < \delta/4.$$

Define the processes $\tilde{\sigma}(t)$ and $\bar{\sigma}(t)$ as was done for Lemma 4. Couple all three processes $(\sigma(t), \tilde{\sigma}(t), \bar{\sigma}(t))$ so that the initial gaps of $\sigma(t)$ are independent of the initial gaps of the other two, and all read the same Poisson clocks. By the choice of $\bar{u}$,

$$\begin{aligned} P\big[\sigma_0(t) &\geq ct + \big(q_1(u-u^*) + q_2\big)t^{1-\alpha} + \delta t^{1-\alpha}\big] \\ &\leq P\big[\tilde{\sigma}_0(t) \geq ct + \big(q_1(u-\bar{u}) + q_2\big)t^{1-\alpha} + \delta t^{1-\alpha}/2\big] \\ &\quad + P[\sigma_0(t) \geq \tilde{\sigma}_0(t) + \delta t^{1-\alpha}/2]. \end{aligned}$$

Let $t \to \infty$ and apply the lemmas.

4.2. *Proof of the lower bound in Theorem* 1. The lower bound will follow from proving this proposition.

PROPOSITION 2. *Suppose the initial gaps are i.i.d. with common mean $u > u^*$ and finite variance. Given positive q_1, q_2, let*

$$r = \min\{q_2, (u - u^*)q_1\}.$$

Then for any $\delta > 0$,

$$\liminf_{t\to\infty} P\{\sigma_0(t) \geq ct + rt^{1-\alpha} - \delta t^{1-\alpha}\} \geq \exp\{-\kappa q_1 q_2^{\nu+1}\}. \tag{4.6}$$

The lower bound (2.3) will follow from this proposition the same way the upper bound (2.2) followed from Proposition 1. Namely, for a given z, maximize the right-hand side of (4.6) subject to $r - \delta = z$, and then let $\delta \to 0$.

To prove Proposition 2, we start with the variational equation and split it into two separate ranges:

$$\sigma_0(t) = \inf_{j\geq 0}\{h_j(0) + Z_0^j(t)\} = \min\{S_1(t), S_2(t)\},$$

where

$$S_1(t) = \inf_{0\leq j\leq q_1 t^{1-\alpha}}\{h_j(0) + Z_0^j(t)\} \quad \text{and} \quad S_2(t) = \inf_{j > q_1 t^{1-\alpha}}\{h_j(0) + Z_0^j(t)\}.$$

We shall show that

$$\liminf_{t\to\infty} P\{S_1(t) \geq ct + q_2 t^{1-\alpha} - \delta t^{1-\alpha}\} \geq \lim_{t\to\infty} P(D(t)) \tag{4.7}$$

and

$$\lim_{t\to\infty} P\{S_2(t) \geq ct + q_1(u - u^*)t^{1-\alpha} - \delta t^{1-\alpha}\} = 1. \tag{4.8}$$

Together with Lemma 1, these imply (4.6).

Proof of lower bound, part 1. In this section we prove (4.7) for $S_1(t)$.

PROPOSITION 3. *Let $q_1, q_2, \delta > 0$. There exists an event $B(t)$ such that $P(B(t)^c) \to 0$ and*

$$\{S_1(t) \geq ct + q_2 t^{1-\alpha} - \delta t^{1-\alpha}\} \supseteq D(t) \cap B(t).$$

Lower bound (4.7) follows from this proposition. The rest of this section proves the proposition. Pick a further constant q_3 such that

$$0 < q_3 < q_2 < q_3 + \delta/4.$$

We shall couple $\sigma(t)$ with a faster process $\hat{\sigma}(t)$ whose jump rates $\hat{p}_i$ are given by

$$\hat{p}_i = p_i \vee (c + q_2 N^{-\alpha}).$$

Process $\hat{\sigma}(t)$ will be in equilibrium so that each particle $\hat{\sigma}_i(t)$ jumps as a Poisson process with rate

$$\hat{a} = c + q_3 N^{-\alpha}.$$

To achieve this, the gap process $\hat{\eta}(t) = \{\hat{\eta}_i(t)\}$ has to have the appropriate geometric product equilibrium distribution. Given $\mathbf{p}$, $\{\hat{\eta}_i\}$ are independent with geometric marginals

$$P^{\mathbf{p}}[\hat{\eta}_i = k] = \left(1 - \frac{\hat{a}}{\hat{p}_i}\right)\left(\frac{\hat{a}}{\hat{p}_i}\right)^k, \qquad k = 0, 1, 2, \dots.$$

Note that this is sensible because $\hat{a} < \hat{p}_i$ for each i by the assumption $q_3 < q_2$. The processes $\hat{\eta}(t)$ and $\hat{\sigma}(t)$ depend on N, but we suppress this dependence from the notation.

The mean gap for the $\hat{\sigma}(t)$ process is

$$\hat{u} = E[\hat{\eta}_i] = \int_{(c,1]} \frac{\hat{a}}{\hat{p} - \hat{a}} \, dF(p).$$

LEMMA 5. *The mean gap $\hat{u}$ converges to u^* as $N \to \infty$.*

PROOF. The integral comes in two parts:

$$\begin{aligned}
\hat{u} &= \int_{(c, c+q_2N^{-\alpha}]} \frac{\hat{a}}{\hat{p} - \hat{a}} \, dF(p) + \int_{(c+q_2N^{-\alpha}, 1]} \frac{\hat{a}}{\hat{p} - \hat{a}} \, dF(p) \\
&= \frac{c + q_3 N^{-\alpha}}{(q_2 - q_3)N^{-\alpha}} \cdot F(c + q_2 N^{-\alpha}) \\
&\quad + \int_{(c,1]} \frac{c + q_3 N^{-\alpha}}{p - c - q_3 N^{-\alpha}} \mathbf{1}_{(c+q_2N^{-\alpha},1]}(p) \, dF(p).
\end{aligned}$$

The first term on the last line vanishes as $N \to \infty$ by hypothesis (2.1). To the second term we apply dominated convergence. The integrand converges to $c/(p - c)$ for each fixed $p \in (c, 1]$, and satisfies the bound

$$\frac{c + q_3 N^{-\alpha}}{p - c - q_3 N^{-\alpha}} \mathbf{1}_{(c+q_2N^{-\alpha},1]}(p) \le \frac{q_2}{q_2 - q_3} \cdot \frac{c + q_3}{p - c}$$

if $N \ge 1$. The last upper bound is integrable under $dF(p)$, again by assumption (2.1). $\square$

For higher moments of $\hat{\eta}_i$ we develop a bound.

LEMMA 6. *For $k \ge 1$ and $N \ge 4$,*

$$E[\hat{\eta}_i^k] \le C N^{\alpha(k-1-\nu)^+} \log N.$$

C is a constant that depends on k and all the other constants in the problem, but not on N.

PROOF. For a fixed $\mathbf{p}$, properties of a geometric distribution give

$$E^{\mathbf{p}}[\hat{\eta}_i^k] \le C_0 + C_1(E^{\mathbf{p}}[\hat{\eta}_i])^k \le C_0 + C_1(\hat{p}_i - \hat{a})^{-k}$$

for constants C_0, C_1 that depend on k. It remains to show

$$\int_{(c,1]} (\hat{p} - \hat{a})^{-k}\, dF(p) \le C N^{\alpha(k-1-\nu)^+} \log N.$$

This integral is decomposed as

$$(q_2 - q_3)^{-k} N^{k\alpha} \int_{(c,c+q_2N^{-\alpha}]} dF(p) + \int_{(c+q_2N^{-\alpha},1]} (p - \hat{a})^{-k}\, dF(p). \tag{4.9}$$

Apply assumption (2.1) to the first integral. In the second integral, observe that

$$(p - \hat{a})^{-k} \le q_4(p - c)^{-k} \qquad \text{for } q_4 = \left(\frac{q_2}{q_2 - q_3}\right)^k.$$

Subsume the constants q_i into constants C_i. Thus the next upper bound is of the form

$$C_0 N^{\alpha(k-1-\nu)} + C_1 \int_{(c+q_2N^{-\alpha},1]} (p - c)^{-k}\, dF(p). \tag{4.10}$$

Pick C_2 and $\delta > 0$ so that $F(p) \le C_2(p - c)^{\nu+1}$ for $c < p \le c + \delta$. In the second term, the integral over $(c + \delta, 1]$ is bounded by a constant. Over $(c + q_2N^{-\alpha}, c + \delta]$ integrate by parts:

$$\begin{aligned}\int_{(c+q_2N^{-\alpha},c+\delta]} \frac{dF(p)}{(p - c)^k} &\le F(c + \delta)\delta^{-k} - \int_{(c+q_2N^{-\alpha},c+\delta]} F(p)\, d\{(p - c)^{-k}\} \\ &\le C_3 + C_2 k \int_{c+q_2N^{-\alpha}}^{c+\delta} (p - c)^{\nu-k}\, dp.\end{aligned}$$

Consider different cases for the last integral. If $\nu > k - 1$, it is bounded by a constant. If $\nu = k - 1$, it is bounded by $C_3 + C_4 \log N$. Finally, in the case $\nu < k - 1$, it is bounded by $C_5 N^{(k-1-\nu)\alpha}$. In all cases the bound given in the statement of the lemma works. $\square$

Couple $\{\eta_i\}$ and $\{\hat{\eta}_i\}$ so that they are mutually independent.

LEMMA 7. *For any $q > 0$, $\delta > 0$,*

$$\lim_{N\to\infty} P\left\{ \inf_{0 \le j \le qN^{1-\alpha}} [h_j - \hat{h}_j] < -\delta N^{1-\alpha} \right\} = 0.$$

Note that the height function $\hat{h}$ changes with N in the statement above.

PROOF. Take N large enough so that $u - \hat{u} > 0$, which can be achieved by Lemma 5 and the assumption $u > u^*$. Then the probability in the statement of the lemma is bounded above by

$$P\left\{\inf_{0\le j\le qN^{1-\alpha}}[h_j - \hat{h}_j - j(u-\hat{u})] < -\delta N^{1-\alpha}\right\}$$
$$\le \delta^{-2}N^{-2(1-\alpha)}\cdot qN^{1-\alpha}(\mathrm{Var}[\eta_0]+\mathrm{Var}[\hat{\eta}_0]),$$

where we used Kolmogorov's inequality. By the previous lemma, $\mathrm{Var}[\hat{\eta}_0] \le CN^{\alpha(1-\nu)^+}\log N$, while $\mathrm{Var}[\eta_0]$ is a constant. As $\alpha(1-\nu)^+ < 1-\alpha$ for all $\nu > 0$, the probability vanishes as $N \to \infty$. $\square$

Now we turn to $S_1(t)$. Consider first a fixed t. Set $N = t$, and as above construct the equilibrium process $\hat{\sigma}(\cdot)$ with rates $\hat{p}_i$. Also, let $\widehat{Z}^j$ denote the corner processes run with the $\hat{p}_i$ rates. On the event $D(t)$, we have

$$\begin{aligned} S_1(t) &= \min_{0\le j\le q_1t^{1-\alpha}}\{h_j + \widehat{Z}_0^j(t)\} \\ &= \min_{0\le j\le q_1t^{1-\alpha}}\{h_j - \hat{h}_j + \hat{h}_j + \widehat{Z}_0^j(t)\} \\ &\ge \min_{0\le j\le q_1t^{1-\alpha}}\{h_j - \hat{h}_j\} + \hat{\sigma}_0(t), \end{aligned}$$

because

$$\hat{h}_j + \widehat{Z}_0^j(t) \ge \hat{\sigma}_0(t)$$

for each $j \ge 0$. Consequently,

$$\begin{aligned} &\{S_1(t) \ge ct + q_2t^{1-\alpha} - \delta t^{1-\alpha}\} \\ &\quad\supseteq D(t)\cap\left\{\min_{0\le j\le q_1t^{1-\alpha}}\{h_j-\hat{h}_j\} \ge -\tfrac{1}{2}\delta t^{1-\alpha}, \hat{\sigma}_0(t) \ge ct + q_2t^{1-\alpha} - \tfrac{1}{2}\delta t^{1-\alpha}\right\} \\ &\quad\equiv D(t)\cap B(t), \end{aligned}$$

where the last identity means that the event $B(t)$ is defined by the previous expression in braces. For the complement,

$$P\big(B(t)^c\big) \le P\left\{\min_{0\le j\le q_1t^{1-\alpha}}\{h_j-\hat{h}_j\} < -\tfrac{1}{2}\delta t^{1-\alpha}\right\}$$
$$+ P\big\{\hat{\sigma}_0(t) < ct + q_2t^{1-\alpha} - \tfrac{1}{2}\delta t^{1-\alpha}\big\}.$$

The probabilities above vanish as $t \to \infty$, the first by Lemma 7. For the second probability, note that $\hat{\sigma}_0(t)$ is Poisson distributed with mean

$$\hat{a}t = ct + q_3t^{1-\alpha} > ct + q_2t^{1-\alpha} - \tfrac{1}{4}\delta t^{1-\alpha}.$$

Since $1-\alpha > 1/2$, the deviation $\frac{1}{4}\delta t^{1-\alpha}$ has zero probability in the $t \to \infty$ limit. This completes the proof of Proposition 3.

Proof of lower bound, part 2. In this section we complete the proof of Proposition 2 by proving (4.8).

PROPOSITION 4. *Given* $\varepsilon, \delta > 0$,

$$P\{S_2(t) \geq ct + (u - u^* - \delta)q_1 t^{1-\alpha}\} \geq 1 - \varepsilon$$

for all large enough t.

PROOF. Let $\{\eta_i^*\}$ be the independent mean u^* equilibrium gaps, so given $\mathbf{p}$,

$$P^{\mathbf{p}}[\eta_i^* = k] = \left(1 - \frac{c}{p_i}\right)\left(\frac{c}{p_i}\right)^k, \qquad k \geq 0.$$

Let $\sigma^*(t)$ be the equilibrium process where particle $\sigma_0^*(t)$ is a rate c Poisson process. Couple the processes $\sigma(t)$ and $\sigma^*(t)$ so that they read the same Poisson clocks but their initial states are independent.

By the Strong Law of Large Numbers,

$$M^{-1}(h_M - h_M^*) \underset{M\to\infty}{\longrightarrow} u - u^* \qquad \text{almost surely.}$$

Note that here we do not need finite variance, which the h^* height function would not possess if $0 < \nu \leq 1$. Shrink δ if necessary so that $0 < \delta < u - u^*$. Pick $M_0 = M_0(\delta, \varepsilon)$ such that

$$P\{h_j - h_j^* \geq j(u - u^* - \delta/2) \text{ for all } j \geq M_0\} \geq 1 - \varepsilon/2.$$

Since $1 - \alpha > 1/2$, there exists a t_0 such that

$$P\{\sigma_0^*(t) \geq ct - q_1 t^{1-\alpha}\delta/2\} \geq 1 - \varepsilon/2$$

for all $t \geq t_0$. Now with probability at least $1 - \varepsilon$, for $t \geq t_0$ such that $q_1 t^{1-\alpha} > M_0$,

$$\begin{aligned}
S_2(t) &= \inf_{j \geq q_1 t^{1-\alpha}} \{h_j + Z_0^j(t)\} \\
&= \inf_{j \geq q_1 t^{1-\alpha}} \{h_j - h_j^* + h_j^* + Z_0^j(t)\} \\
&\geq \inf_{j \geq q_1 t^{1-\alpha}} \{h_j - h_j^*\} + \sigma_0^*(t) \geq q_1 t^{1-\alpha}(u - u^* - \delta/2) + ct - q_1 t^{1-\alpha}\delta/2 \\
&= ct + q_1 t^{1-\alpha}(u - u^* - \delta). \qquad \square
\end{aligned}$$

5. Proof of Theorem 2. We begin with the upper bound. Let $b > 0$, $0 < \theta < 1$, $0 < \varepsilon < \theta b$ and $q_2 = \theta b - \varepsilon$. Let $\bar{p}$ be the minimal rate among $p_{-[bt^{1-\alpha}]}, \ldots, p_{-[\theta b t^{1-\alpha}]}$, and I an index such that $p_I = \bar{p}$. Let $Y(t)$ be a Poisson variable with mean $ct + q_2 t^{1-\alpha}$. If $\bar{p} \leq c + q_2 t^{-\alpha}$, $Y(t)$ dominates the number of jump attempts particle ξ_I experiences during time interval $[0, t]$. By the particle ordering, $\xi_{-[bt^{1-\alpha}]}(t) \ge ct$ implies $\xi_I(t) \ge ct$, and thereby ξ_I must have at least $ct + \theta bt^{1-\alpha}$ jump attempts. We get the bound

$$\begin{aligned}P\{X_t > bt^{1-\alpha}\} &\le P\bigl\{\xi_{-[bt^{1-\alpha}]}(t) \ge ct\bigr\}\\ &\le P\{\bar{p} > c + q_2 t^{-\alpha}\} + P\{Y(t) \ge ct + \theta bt^{1-\alpha}\}.\end{aligned}$$

Since $1-\alpha > 1/2$, the last probability vanishes as $t \to \infty$. By Lemma 1 we get

$$\limsup_{t\to\infty} P\{X_t > bt^{1-\alpha}\} \le \exp\bigl(-\kappa(1-\theta)bq_2^{\nu+1}\bigr).$$

Let $\varepsilon \searrow 0$ so that $q_2 \nearrow \theta b$, and then choose $\theta = (\nu+1)/(\nu+2)$.

The lower bound of Theorem 2 comes from the lower bound of Theorem 1. Pick a density $u > u^*$, and let the initial gaps $\{\eta_i\}$ be bounded i.i.d. random variables with mean u. By the variational formula (3.2),

$$\sigma_0(t) = \inf_{j\ge 0}\{\sigma_j + \xi_{-j}^j(t)\},$$

where $\xi^j(t)$ is a version of the $\xi(t)$ process with translated rates. Let $b > 0$, and then pick $\theta > b(u+1)$. Let $j = [bt^{1-\alpha}]$. Then

$$\xi_{-j}^j(t) \ge \sigma_0(t) - \sigma_j \ge ct + \bigl(\sigma_0(t) - ct\bigr) - \sigma_j.$$

The annealed distribution of the process $\xi^j(t)$ is the same as that of $\xi(t)$. Consequently,

$$\begin{aligned}P\{X_t > bt^{1-\alpha}\} &\ge P\bigl\{\xi_{-[bt^{1-\alpha}]}(t) > ct\bigr\}\\ &\ge P\left\{\frac{\sigma_0(t) - ct}{t^{1-\alpha}} > \theta, \sigma_{[bt^{1-\alpha}]} < \theta t^{1-\alpha}\right\}.\end{aligned}$$

By the law of large numbers, $t^{-1+\alpha}\sigma_{[bt^{1-\alpha}]} \to b(u+1)$, and so by Theorem 1,

$$\begin{aligned}\liminf_{t\to\infty} P\{X_t > bt^{1-\alpha}\} &\ge \liminf_{t\to\infty} P\left\{\frac{\sigma_0(t) - ct}{t^{1-\alpha}} > \theta\right\}\\ &\ge \exp\left\{-\frac{\kappa}{u-u^*}\theta^{\nu+2}\right\}.\end{aligned}$$

Maximize the last lower bound over θ and u subject to $u > u^*$ and $\theta > b(u+1)$.

6. Proof of Theorem 3. The argument for the upper bound is similar to the previous one. Now $\nu = 0$ and $1-\alpha = 1/2$. Let $\varepsilon > 0$ be small. By the central limit theorem, we can fix a large $1 < M < \infty$ so that, if $Y(t)$ is a Poisson random variable with mean $ct + t^{1-\alpha}$, then

$$P[Y(t) \ge ct + Mt^{1-\alpha}] \le \varepsilon/4$$

for all large enough t. Given ε and M, choose $0 < q_2 < 1 < M < q < b$ so that

$$\exp(-\kappa q q_2^{\nu+1}) \geq 1 - \varepsilon/16$$

and

$$\exp(-\kappa b q_2^{\nu+1}) \leq \varepsilon/16.$$

Let

$$J(t) = \inf\{i \geq 0 : p_{-i} \leq c + q_2 t^{-\alpha}\}.$$

By Lemma 1 we have $t_0 < \infty$, so that

$$P\{qt^{1-\alpha} < J(t) < bt^{1-\alpha}\} \geq 1 - \varepsilon/4$$

for all $t \geq t_0$. Suppose this event happens. Then if $\xi_{-[bt^{1-\alpha}]}(t) \geq ct$, also $\xi_{-J(t)}(t) \geq ct$, and particle $\xi_{-J(t)}$ has had to cover distance $ct + J(t) \geq ct + qt^{1-\alpha}$. The increment $\xi_{-J(t)}(t) - \xi_{-J(t)}(0)$ is stochastically bounded by the variable $Y(t)$ defined above. So for large enough t,

$$\begin{aligned} P\{X_t > bt^{1-\alpha}\} &\leq P\{\xi_{-[bt^{1-\alpha}]}(t) \geq ct\} \\ &\leq P\{\xi_{-J(t)}(t) \geq ct\} + \varepsilon/4 \\ &\leq P\{Y(t) \geq ct + qt^{1-\alpha}\} + \varepsilon/4 \\ &\leq \varepsilon/2. \end{aligned}$$

We prove the lower bound by comparison with a faster system in equilibrium. Let $0 < a < \infty$ be fixed. Given $\varepsilon > 0$, pick $1 < w < \infty$ large enough so that

$$P[Y(N) > EY(N) - wN^{1/2}] \geq 1 - \varepsilon/4 \tag{6.1}$$

for large enough N, for a Poisson variable $Y(N)$ with mean $cN + 2wN^{1/2}$. Later we have to increase w further.

Let $q_2 = 4w$, and define faster rates by $\hat{p}_i = p_i \vee (c + q_2 N^{-1/2})$. Consider N large enough to have $\hat{p}_i < 1$. Let $\hat{\sigma}(t)$ be a process run with rates $\hat{p}_i$ and in equilibrium, so that $\hat{\sigma}_0(t)$ is a Poisson process with rate

$$a = c + 2wN^{-1/2}.$$

The gap process $\hat{\eta}(t)$ then has a product distribution with independent geometric marginals

$$P^{\mathbf{p}}[\hat{\eta}_i = k] = \left(1 - \frac{a}{\hat{p}_i}\right)\left(\frac{a}{\hat{p}_i}\right)^k, \qquad k = 0, 1, 2, \ldots.$$

The annealed mean gap is

$$u = E[\hat{\eta}_i] = \int_{(c,1]} \frac{a}{\hat{p} - a}\, dF(p)$$

and the annealed variance is bounded as in

$$\mathrm{Var}[\hat{\eta}_i] \le E[\hat{\eta}_i^2] = 2\int_{(c,1]} \left(\frac{a}{\hat{p}-a}\right)^2 dF(p) + u.$$

LEMMA 8. *There is a constant C that depends only on the distribution F such that, for large enough N,*

$$u \le C \log N \quad \textit{and} \quad \mathrm{Var}[\hat{\eta}_i] \le C N^{1/2}.$$

PROOF. First for the mean. Integrate by parts, and use assumption (2.1) to pick $0 < \delta < 1$ such that

$$F(p) \le (\kappa+1)(p-c) \qquad \text{for } c < p < c+\delta.$$

Then note that

$$p - a \ge \frac{q_2 - 2w}{q_2}(p-c) = \frac{1}{2}(p-c) \qquad \text{for } p \ge c + q_2 N^{-1/2}.$$

Carrying out these steps yields

$$\begin{aligned}
u &= \frac{aF(c+q_2N^{-1/2})}{c+q_2N^{-1/2}-a} + \int_{(c+q_2N^{-1/2},1]} \frac{a}{p-a}\, dF(p) \\
&= \frac{aF(c+q_2N^{-1/2})}{c+q_2N^{-1/2}-a} \\
&\quad + \left\{ \frac{aF(1)}{1-a} - \frac{aF(c+q_2N^{-1/2})}{c+q_2N^{-1/2}-a} - \int_{(c+q_2N^{-1/2},1]} F(p)\, d\left(\frac{a}{p-a}\right) \right\} \\
&= \frac{a}{1-a} + a\int_{c+q_2N^{-1/2}}^{1} \frac{F(p)}{(p-a)^2}\, dp \\
&\le \frac{a}{1-a} + 4(\kappa+1)a \int_{c+q_2N^{-1/2}}^{c+\delta} \frac{dp}{p-c} + 4a\int_{c+\delta}^{1} \frac{F(p)}{(p-c)^2}\, dp \\
&\le \frac{a}{1-a} + 4(\kappa+1)a(\log\delta - \log q_2 N^{-1/2}) + 4a\delta^{-1} \\
&\le \frac{1+c}{1-c} + 2(\kappa+1)\log N + 4\delta^{-1} \\
&\le C\log N.
\end{aligned}$$

In the second to last step, we took N large enough so that

$$a = c + q_2 N^{-1/2} \le \frac{1+c}{2} \le 1.$$

If $N \ge 3$, in the last step we can take

$$C = \frac{1+c}{1-c} + 2(\kappa+1) + 4\delta^{-1},$$

which depends only on the distribution F.

Following the same pattern for $E[\hat{\eta}_i^2]$ shows that, after integration by parts, the main part is the integral

$$a^2 \int_{c+q_2 N^{-1/2}}^{1} \frac{F(p)}{(p-a)^3}\, dp$$
$$\leq 8a^2(\kappa+1) \int_{c+q_2 N^{-1/2}}^{c+\delta} \frac{dp}{(p-c)^2} + 8a^2 \int_{c+\delta}^{1} \frac{F(p)}{(p-c)^3}\, dp.$$

The desired bound follows as above. □

Let $\hat{\hat{\xi}}(t)$ denote a ξ-type process run with rates $\hat{p}_i$. Let

$$j(N) = [aN^{1/2}(\log N)^{-1}].$$

From the variational coupling (3.2), we have

$$\hat{\hat{\xi}}^{j(N)}_{-j(N)}(t) \geq \hat{\sigma}_0(t) - \hat{\sigma}_{j(N)}$$
$$= ct + 2wN^{-1/2}t + \big(\hat{\sigma}_0(t) - (ct + 2wN^{-1/2}t)\big) - \hat{\sigma}_{j(N)}.$$

The processes $\hat{\hat{\xi}}(t)$ and $\hat{\sigma}(t)$ depend on N but we suppress this from the notation. Set time $t = N$. Note that when the random rates are averaged out, processes $\hat{\hat{\xi}}^{j(N)}(t)$ and $\hat{\hat{\xi}}(t)$ have the same distribution. We get this bound:

$$P\{\hat{\hat{\xi}}_{-j(N)}(N) > cN\}$$
$$\geq P\{\hat{\sigma}_0(N) - (cN + 2wN^{1/2}) > -wN^{1/2}, \hat{\sigma}_{j(N)} < wN^{1/2}\}$$
$$\geq P\{\hat{\sigma}_0(N) - (cN + 2wN^{1/2}) > -wN^{1/2}\} - P\{\hat{\sigma}_{j(N)} \geq wN^{1/2}\}.$$

The next to last probability is at least $1 - \varepsilon/4$ for large N by (6.1). It remains to show that the last probability vanishes as $N \to \infty$. From the annealed perspective, $\hat{\sigma}_{j(N)}$ is a sum of i.i.d.'s, so its mean and variance are bounded as follows:

$$E\hat{\sigma}_{j(N)} = j(N)(u+1) \leq CaN^{1/2}$$

and

$$\mathrm{Var}[\hat{\sigma}_{j(N)}] = j(N)\,\mathrm{Var}[\hat{\eta}_0] \leq CaN(\log N)^{-1}.$$

At this point we need to increase our original choice of w to guarantee that $w > 2Ca$, where a is given in the beginning of the proof and C is the constant that appears in Lemma 8. Then Chebychev's inequality gives

$$P\{\hat{\sigma}_{j(N)} \geq wN^{1/2}\} \leq P\{\hat{\sigma}_{j(N)} \geq E\hat{\sigma}_{j(N)} + CaN^{1/2}\}$$
$$\leq \frac{CaN(\log N)^{-1}}{C^2a^2N},$$

which vanishes as $N \to \infty$. We can conclude that, for large N,

$$P\{\hat{\xi}_{-j(N)}(N) > cN\} > 1 - \varepsilon/3.$$

Finally we make contact with $\xi(N)$. Given q_2 chosen above, pick $q_1 > 0$ small enough so that $\exp(-\kappa q_1 q_2) > 1 - \varepsilon/7$. Let $D(N)$ be the event

$$D(N) = \{p_i = \hat{p}_i \text{ for } -[q_1 N^{1/2}] \le i \le 0\}.$$

By Lemma 1, $P(D(N)) > 1 - \varepsilon/6$ for large enough N. On the event $D(N)$, $\xi_i(t) = \hat{\xi}_i(t)$ for $-[q_1 N^{1/2}] \le i \le 0$ and all $t \ge 0$, so in particular for $i = j(N)$ if N is large enough. Consequently,

$$\begin{aligned}
&P\{X_N \ge aN^{1/2}(\log N)^{-1}\} \\
&\quad \ge P\{\xi_{-J(N)}(N) > cN\} \\
&\quad \ge P(\{\xi_{-J(N)}(N) > cN\} \cap D(N)) \\
&\quad = P(\{\hat{\xi}_{-J(N)}(N) > cN\} \cap D(N)) \\
&\quad \ge P\{\hat{\xi}_{-j(N)}(N) > cN\} - P(D(N)^c) \\
&\quad > 1 - \varepsilon/3 - \varepsilon/6 \\
&\quad = 1 - \varepsilon/2.
\end{aligned}$$

This completes the proof of Theorem 3.

7. Proof of Theorem 4.

7.1. *Proof of the upper bound of Theorem* 4. The upper bound is proved by comparison with independent particles. Let

$$\beta = \frac{1-\alpha}{2\alpha} = \frac{1+\nu}{2}.$$

For $b > 0$ and $q_2 > 0$, define

$$K_t = \sum_{i=-[bt^\beta]+1}^{0} \mathbf{1}\{p_i \le c + q_2 t^{-1/2}\}.$$

LEMMA 9. *Let $\{Y_j(t)\}$ be independent copies of a Poisson random variable with mean $ct + q_2 t^{1/2}$, independent of the rates $\{p_i\}$ and thereby independent of K_t. Then, given $\varepsilon > 0$, if q_2 is small enough while $bq_2^{\nu+1}$ is large enough,*

$$P\{Y_j(t) \ge ct \text{ for } 1 \le j \le K_t\} < \varepsilon$$

for all large enough t.

PROOF. Fix a small $0 < \delta < 1/2$. Fix a positive integer m large enough so that $(\frac{1}{2} + \delta)^m < \varepsilon/2$. Pick $\varepsilon_0 > 0$ small enough so that

$$P(\chi \geq -\varepsilon_0) < (1 + \delta)/2$$

for a standard normal χ. Let $q_2 < \varepsilon_0 \sqrt{c}$.

By assumption (2.1), for large t, K_t is stochastically dominated by a binomial random variable with $[bt^\beta]$ trials and success probability $(\kappa + 1)q_2^{\nu+1} t^{-(1+\nu)/2}$. Such a variable converges weakly to a Poisson with mean $b(\kappa + 1)q_2^{\nu+1}$ as $t \to \infty$. Thus we may fix b large enough so that

$$P(K_t \leq m) < \varepsilon/2$$

for large enough t.

By the choice of q_2 and the definition of $Y(t) = Y_1(t)$,

$$P\{Y(t) \geq ct\} \leq P\left\{\frac{Y(t) - EY(t)}{\sqrt{\operatorname{Var} Y(t)}} \geq -\varepsilon_0\right\}.$$

Then by the central limit theorem, for large enough t,

$$P\{Y(t) \geq ct\} \leq P(\chi \geq -\varepsilon_0) + \delta/2 < 1/2 + \delta.$$

Finally, as the $Y_j(t)$ are i.i.d. and independent of K_t,

$$\begin{aligned} &P\{Y_j(t) \geq ct \text{ for } 1 \leq j \leq K_t\} \\ &\quad = E\left[\prod_{j=1}^{K_t} P\{Y_j(t) \geq ct\}\right] \\ &\quad \leq E\left[\left(\tfrac{1}{2} + \delta\right)^{K_t}\right] \leq P(K_t \leq m) + \left(\tfrac{1}{2} + \delta\right)^m \leq \varepsilon. \end{aligned}$$

□

Fix b and q_2 so that the lemma is satisfied. Let

$$I_t = \{-[bt^\beta] < i \leq 0 : p_i \leq c + q_2 t^{-1/2}\}.$$

Once the rates p_i have been chosen according to distribution F and I_t determined, give each index $i \in I_t$ an independent Poisson process $N_i(\cdot)$ of rate $c + q_2 t^{-1/2}$. Thin $N_i(\cdot)$ appropriately to get the correct rate p_i. These thinned processes are the Poisson clocks for indices $i \in I_t$. Meanwhile, give the other indices their independent Poisson clocks. This way we can claim that, for each $i \in I_t$, the number of jump attempts experienced by particle ξ_i during $(0, t]$ is bounded above by the mean $ct + q_2 t^{1/2}$ Poisson variable $N_i(t)$ that is independent of the rates p_i.

Suppose $\xi_{-[bt^\beta]+1}(t) \geq ct$. By the particle ordering, $\xi_i(t) \geq ct$ for all $i \in I_t$, which implies that $N_i(t) \geq ct$ for all $i \in I_t$. By the lemma above, this event has probability less than ε for large t. To summarize, we have shown that, for an arbitrary $\varepsilon > 0$, b can be chosen so that

$$P\{X_t \geq bt^\beta\} < \varepsilon$$

for large enough t.

7.2. *Proof of the lower bound of Theorem* 4. For an exclusion process with constant rates r, for any $a > 0$ and $0 < \gamma < 1$,

$$\lim_{t\to\infty} \frac{\xi_{-[t^\gamma a]}(t) - rt}{(rt)^{(1+\gamma)/2}} = -2\sqrt{a} \qquad \text{in probability.} \tag{7.1}$$

This statement is a consequence of a limit proved by Glynn and Whitt [5] and the explicit computation of the value 2 on the right-hand side first done in [10]. See Lemma 4.1 in [10] for the derivation of (7.1) from [5]. (But note that the process ξ in [10] is not the same as ξ in the present paper.)

Let $\beta = (3+\nu)^{-1}$. Let $0 < a < \infty$ and $q = 2\sqrt{a} + 2$. Use assumption (2.1) exactly as in the proof of Lemma 1 to show that, given $\varepsilon > 0$, if a is small enough, then for large enough t,

$$P\{p_i \ge c + qt^{-\beta} \text{ for } -[at^{\beta(1+\nu)}] \le i \le 0\} \ge 1 - \varepsilon/2.$$

On this event $\xi_{-[at^{\beta(1+\nu)}]}(t)$ is bounded below by $\tilde{\xi}_{-[at^{\beta(1+\nu)}]}(t)$, where $\tilde{\xi}(t)$ is a process whose clocks ring at constant rate $c + qt^{-\beta}$. For $\tilde{\xi}(t)$, (7.1) gives the following bound: for large t with probability at least $1 - \varepsilon/2$,

$$\begin{aligned}\tilde{\xi}_{-[at^{\beta(1+\nu)}]}(t) &\ge ct + qt^{1-\beta} - 2\sqrt{a}(ct + qt^{1-\beta})^{(1+\beta(1+\nu))/2} - t^{(1+\beta(1+\nu))/2} \\ &> ct.\end{aligned}$$

The last lower bound by ct followed from $1 - \beta = (1 + \beta(1+\nu))/2$ and the choice of q.

We have shown that, given $\varepsilon > 0$ and a small enough $a > 0$, then for large enough t, the inequality $\xi_{-[at^{\beta(1+\nu)}]}(t) > ct$ holds with probability at least $1 - \varepsilon$. This inequality implies $X_t \ge at^{\beta(1+\nu)}$.

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