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# Cornish–Fisher expansions for sample autocovariances and other functions of sample moments of linear processes

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**Abstract.** We give Cornish–Fisher expansions for general smooth functions of the sample cross-moments of a stationary linear process. Examples include the distributions of the sample mean, the sample autocovariance and the sample autocorrelation.

## 1 Introduction and summary

The theory of linear processes is well developed. We refer the readers to the excellent books: Hannan (1962, 1970), Kendall and Ord (1990) and Taniguchi and Kakizawa (2000). However, there has been little work giving Cornish–Fisher expansions for general smooth functions of the sample cross-moments of stationary linear processes. Among the known work, we mention Praskova-Vizkova (1976) and Albers (1978), where Edgeworth expansions are given for the Kendall rank correlation coefficient. See also Phillips (1977), where Edgeworth expansions for the least squares estimate of the coefficient of a first order autoregressive process are given.

The aim of this note is to derive the Cornish–Fisher expansions for general stationary linear processes. The results are organized as follows. Section 2 obtains expansions for the cumulants of the sample cross-moments of a linear process. In Section 3, we give the Cornish–Fisher expansions for functions of the sample moments. Section 4 gives examples, including explicit formulas for the first two terms of the Cornish–Fisher expansions for the sample mean, the sample autocovariance, and the sample autocorrelation. Section 5 shows the practical value of the results in Section 4 by means of simulation.

## 2 The cumulants of the sample cross-moments

Let $\{e_i\}$ be independent and identically distributed random variables from some distribution function $F$ on $R$ with finite cumulants $\tau_1, \tau_2, \ldots$. We consider the

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general stationary linear process

$$X_i = \sum_{j=0}^{\infty} \rho_j e_{i-j}. \tag{2.1}$$

This includes the class of stationary ARMA processes. Its mean is $\mu = \alpha_1 \tau_1$, where $\alpha_1 = \sum_{j=0}^{\infty} \rho_j$. We denote the noncentral cross-moments, central cross-moments, and cross-cumulants of $(X_{i_1}, \ldots, X_{i_r})$ by

$$\begin{aligned} &M_{i_1 \cdots i_r} = E X_{i_1} \cdots X_{i_r}, \qquad \mu_{i_1 \cdots i_r} = E(X_{i_1} - \mu) \cdots (X_{i_r} - \mu), \\ &\kappa_{i_1 \cdots i_r} = \kappa(X_{i_1}, \ldots, X_{i_r}). \end{aligned} \tag{2.2}$$

For relationships between them see, for example, Stuart and Ord (1987). We write these generically as

$$\mathbf{M} = \mathbf{M}(\boldsymbol{\mu}), \qquad \boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{M}), \qquad \boldsymbol{\kappa} = \boldsymbol{\kappa}(\boldsymbol{\mu})$$

and so on. These can be written down from their univariate versions. For example, $EX^2 = \mathrm{var}(X) + E^2 X$ implies

$$M_{12} = \mathrm{covar}(X_1, X_2) + (EX_1)(EX_2),$$

and $\kappa_4 = \mu_4 - 3\mu_2^2$ implies

$$\kappa_{1234} = \mu_{1234} - \mu_{12}\mu_{34} - \mu_{13}\mu_{24} - \mu_{14}\mu_{23} = \mu_{1234} - \sum^{3} \mu_{12}\mu_{34}$$

say. Given a sequence of integers $i_1, \ldots, i_r$, set

$$\begin{aligned} &i_0 = \min_{k=1}^{r} i_k, \qquad I_k = i_k - i_0 \geq 0 \qquad \text{for } k = 1, \ldots, r, \\ &I_0 = \max_{k=1}^{r} I_k = \max_{k=1}^{r} i_k - i_0. \end{aligned} \tag{2.3}$$

Since $\{X_i\}$ is stationary,

$$M_{i_1 \cdots i_r} = M_{I_1 \cdots I_r}, \qquad \mu_{i_1 \cdots i_r} = \mu_{I_1 \cdots I_r}, \qquad \kappa_{i_1 \cdots i_r} = \kappa_{I_1 \cdots I_r}.$$

These are not changed by permuting subscripts. Also at least one $I_k$ is zero. In Withers and Nadarajah (2009c), we showed that

$$\kappa_{i_1 \cdots i_r} = \alpha(i_1, i_2, \ldots, i_r) \tau_r,$$

where

$$\alpha(i_1, i_2, \ldots, i_r) = \alpha(I_1, I_2, \ldots, I_r) = \sum_{j=0}^{\infty} \rho_{j+I_1} \rho_{j+I_2} \cdots \rho_{j+I_r},$$

where $\alpha(i_1, i_2, \ldots, i_r)$ is finite for processes like ARMA processes, where $\rho_j$ decrease to zero exponentially. For example,

$$\kappa_r(X_i) = \alpha_r \tau_r,$$

where

$$\alpha_r = \alpha(0, 0, \ldots, 0) = \sum_{j=0}^{\infty} \rho_j^r$$

and $0, 0, \ldots, 0$ denotes a string of $r$ zeros. For $I \geq 0$, the $I$th autocovariance and autocorrelation are

$$\kappa_{0I} = \mathrm{covar}(X_0, X_I) = \alpha(0, I)\tau_2, \qquad \mathrm{corr}(X_0, X_I) = \alpha(0, I)/\alpha_2, \tag{2.4}$$

where

$$\alpha(0, I) = \sum_{j=0}^{\infty} \rho_j \rho_{j+I}, \qquad \alpha_2 = \alpha(0, 0) = \sum_{j=0}^{\infty} \rho_j^2.$$

Also

$$\begin{aligned} \alpha(0, T) &= \alpha(0, |T|), \\ \alpha(0, T_1, T_2) &= \alpha(0, T_2 - T_1, -T_1) = \alpha(0, |T_2 - T_1|, |T_1|) \\ &\quad \text{if } T_1 < T_2 < 0 \text{ or } T_1 < 0 < T_2. \end{aligned}$$

**Example 2.1.** For the AR(1) $X_i - \phi X_{i-1} = e_i$,

$$\rho_j = \phi^j, \qquad \alpha_r = (1 - \phi^r)^{-1}, \qquad \alpha(i_1, i_2, \ldots, i_r)/\alpha_r = \phi^{\sum_{k=1}^r I_k}.$$

**Example 2.2.** Consider the AR(2),

$$X_i - \sum_{k=1}^{2} \phi_k X_{i-k} = e_i.$$

Write

$$1 - \sum_{k=1}^{2} \phi_k B^k = \prod_{k=1}^{2} (1 - y_k B), \qquad y_k = (\phi_1 \pm \epsilon^{1/2})/2, \qquad \epsilon = \phi_1^2 + 4\phi_2,$$

where $k = 1$ corresponds to $+$ and $k = 2$ to $-$. Suppose that $\epsilon \neq 0$. Then

$$\begin{aligned} \left(1 - \sum_{k=1}^{2} \phi_k B^k\right)^{-1} &= \sum_{k=1}^{2} \gamma_k (1 - y_k B)^{-1}, \\ \gamma_k &= (-1)^k y_k/(y_1 - y_2) = \epsilon^{-1/2}(-1)^k y_k. \end{aligned}$$

Taking $B$ as the backwards operator $BX_i = X_{i-1}$ gives

$$X_i = \sum_{k=1}^{2} \gamma_k (1 - y_k B)^{-1} e_i = \sum_{k=1}^{2} \gamma_k \sum_{j=0}^{\infty} y_k^j e_{i-j}.$$

That is, (2.1) holds with

$$\rho_j = \sum_{k=1}^{2} \gamma_k y_k^j.$$

Also by (2.4),

$$\begin{aligned}\epsilon^{1/2}\alpha(0, I) &= \sum_{k=0}^{\infty}(y_1^{k+1} - y_2^{k+1})(y_1^{k+I+1} - y_2^{k+I+1}) \\ &= \sum_{i=1}^{2} y_i^{i+1}/(1 - y_i^2) + \sum_{12}^{2} y_1^{i+1} y_2/(1 - y_1 y_2),\end{aligned}$$

where

$$\sum_{12}^{2} y_1^{i+1} y_2/(1 - y_1 y_2) = y_1^{i+1} y_2/(1 - y_1 y_2) + y_2^{i+1} y_1/(1 - y_1 y_2).$$

Similarly, $\alpha(i_1, i_2, \ldots, i_r)/\alpha_r$ can be written as the sum of $2^r$ terms.

For $I_0$ of (2.3), define the (unbiased) *sample noncentral cross-moments* by

$$\widehat{M}_{i_1 \cdots i_r} = N^{-1} \sum_{t=1}^{N} X_{t+I_1} \cdots X_{t+I_r}$$

for $N = n - I_0 > 0$, where $n$ denotes the sample size. For example,

$$\widehat{\mu} = \widehat{M}_0 = n^{-1} \sum_{j=1}^{n} X_j, \qquad \widehat{M}_{0a} = (n - a)^{-1} \sum_{j=1}^{n-a} X_j X_{j+a} \tag{2.5}$$

for $0 < a < n$. These sample moments are the building blocks of all our estimates. Define the *sample central cross-moments* and the *sample cross-cumulants* by $\widehat{\boldsymbol{\mu}} = \boldsymbol{\mu}(\widehat{\mathbf{M}})$ and $\widehat{\boldsymbol{\kappa}} = \boldsymbol{\kappa}(\widehat{\boldsymbol{\mu}})$, respectively.

## 3 Cornish–Fisher expansions for functions of sample cross-moments

Under mild conditions [see Withers and Nadarajah (2008)], the $r$th order cross-cumulants of the sample cross-moments have magnitude $n^{1-r}$, that is, for finite sequences of integers $\pi_1, \ldots, \pi_r$ not depending on $n$,

$$k(\pi_1, \ldots, \pi_r) = n^{r-1} \kappa(\widehat{M}_{\pi_1}, \ldots, \widehat{M}_{\pi_r})$$

is bounded in $n$. That is, $\{\widehat{M}_{\pi_i}\}$ satisfy the Cornish–Fisher assumption. We shall not prove this for the general case but rather illustrate it in the examples. Given an integer $\pi$, set

$$k_r = k(\pi, \ldots, \pi) = n^{r-1} \kappa_r(\widehat{M}_\pi), \qquad Y_n = (n/k_2)^{1/2}(\widehat{M}_\pi - M_\pi). \tag{3.1}$$

If the observations are nonlattice, the distribution and quantiles of $Y_n$ can be expanded in powers of $n^{-1/2}$:

$$P_n(x) = P(Y_n \le x) \approx \Phi(x) - \phi(x) \sum_{r=1}^{\infty} n^{-r/2} h_r(x), \tag{3.2}$$

$$p_n(x) = dP_n(x)/dx \approx \phi(x)\left[1 + \sum_{r=1}^{\infty} n^{-r/2} \overline{h}_r(x)\right], \tag{3.3}$$

$$\Phi^{-1}(P_n(x)) \approx x - \sum_{r=1}^{\infty} n^{-r/2} f_r(x), \tag{3.4}$$

$$P_n^{-1}(\Phi(x)) \approx x + \sum_{r=1}^{\infty} n^{-r/2} g_r(x), \tag{3.5}$$

where $\Phi$ and $\phi$ are the unit normal distribution and density, respectively, and $h_r(x)$, $\overline{h}_r(x)$, $f_r(x)$, $g_r(x)$ are polynomials in $x$ and $\{K_r\}$, where $K_r = k_r/k_2^{r/2}$. The expansions (3.2), (3.4) and (3.5) are given in Cornish and Fisher (1937) for $r \le 4$. Fisher and Cornish (1960) give (3.5) for $r \le 6$. For (3.3), see equation (3.3) of Withers and Nadarajah (2009b). There is also an alternative to the expansion (3.3) of the form

$$\ln[p_n(x)/\phi(x)] \approx \sum_{r=1}^{\infty} n^{-r/2} b_r(x),$$

where for $r > 1$, $b_r(x)$ is a polynomial of lower order than $\overline{h}_r(x)$: see Withers and Nadarajah (2009a).

Given $p \ge 1$ and finite sequences of integers $\pi_1, \dots, \pi_p$ not depending on $n$, set

$$\theta_a = M_{\pi_a}, \qquad \widehat{\theta}_a = \widehat{M}_{\pi_a}, \qquad \boldsymbol{\theta} = (\theta_1, \dots, \theta_p), \qquad \widehat{\boldsymbol{\theta}} = (\widehat{\theta}_1, \dots, \widehat{\theta}_p),$$
$$k^{a_1 \cdots a_r} = n^{r-1} \kappa(\widehat{\theta}_{a_1}, \dots, \widehat{\theta}_{a_r})$$

for $a_1, \dots, a_r \in \{1, \dots, p\}$. So, $k^{a_1 \cdots a_r}$ is bounded. That is, $\widehat{\boldsymbol{\theta}}$ satisfies the multivariate Cornish–Fisher condition. So, by the multivariate form of the argument used in Cornish and Fisher (1937), Fisher and Cornish (1960), the multivariate Edgeworth expansion holds for $\mathbf{Y}_n = n^{1/2}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$. For $\widehat{\boldsymbol{\theta}}$ a sample mean, this gives the classical multivariate Edgeworth expansion: see equations (6.11)–(6.23) of Barndoff-Nielsen and Cox (1989) for expansions for the density, and Bhattacharya and Rao (1976) for expansions for the distribution and density.

Suppose $\mathbf{Y}_n$ converges in law to the multivariate normal $\mathcal{N}_p(\mathbf{0}, \mathbf{V})$ with $p \times p$ covariance $\mathbf{V} = (k^{a_1 a_2})$ and distribution $\Phi_{\mathbf{V}}(\mathbf{x})$ say, and

$$P_n(\mathbf{x}) = P(\mathbf{Y}_n \le \mathbf{x}) \approx Q_n(-\partial/\partial \mathbf{x}) \Phi_{\mathbf{V}}(\mathbf{x}), \tag{3.6}$$

where for $\widehat{\boldsymbol{\theta}}$ nonlattice

$$Q_n(\mathbf{s}) = 1 + \sum_{r=1}^{\infty} n^{-r/2} q_r(\mathbf{s})$$

and $q_r(\mathbf{s})$ is a polynomial in $\mathbf{s} \in R^p$ and $\{k^{a_1 \cdots a_i}, 1 \le i \le r+2\}$. If $\mathbf{V}$ is bounded away from zero as $n$ increases, that is, if its eigenvalues are bounded away from zero, then the density of $\mathbf{Y}_n$ with respect to Lebesgue measure has the expansion

$$p_n(\mathbf{x}) \approx Q_n(-\partial/\partial \mathbf{x}) \phi_{\mathbf{V}}(\mathbf{x}), \tag{3.7}$$

where $\phi_{\mathbf{V}}(\mathbf{x})$ is the density of $\Phi_{\mathbf{V}}(\mathbf{x})$. The coefficient of $n^{-r/2}$ in $p_n(\mathbf{x})$ is a linear combination of the multivariate Hermite polynomials.

Now suppose that $\mathbf{t} = \mathbf{t}(\boldsymbol{\theta}) : R^p \to R^q$ is a smooth function with finite derivatives $\mathbf{t}_{\cdot c_1 c_2 \cdots} = \partial_{c_1} \partial_{c_2} \cdots \mathbf{t}(\boldsymbol{\theta})$, where $\partial_c = \partial / \partial \theta_c$. Let $t^b$ be the $b$th component of $\mathbf{t}$, $b = 1, \ldots, q$. Then the $r$th order cross-cumulants of $\widehat{\mathbf{t}} = \mathbf{t}(\widehat{\boldsymbol{\theta}})$ are also of magnitude $n^{1-r}$ with an expansion of the form

$$\kappa(\widehat{t}^{b_1}, \ldots, \widehat{t}^{b_r}) = \sum_{i=r-1}^{\infty} n^{-i} K_i^{b_1 \cdots b_r}, \tag{3.8}$$

where $b_1, \ldots, b_r$ lie in $1, \ldots, q$ and the cumulant coefficients $K_i^{b_1 \cdots b_r}$ are given in terms of $\{k^{a_1 \cdots a_r}\}$ and the derivatives $\mathbf{t}_{\cdot c_1 c_2 \cdots}$, by the Appendix to Withers (1982) with $K_i^{a_1 \cdots a_r} = \delta_{i,r-1} k^{a_1 \cdots a_r}$. Here, $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,j} = 1$ if $i = j$. Alternatively, one can use James and Mayne (1962). For example, the $q \times q$ asymptotic covariance of $\mathbf{Z}_n = n^{1/2}(\widehat{\mathbf{t}} - \mathbf{t})$ is $(K_1^{b_1 b_2})$, where

$$K_1^{b_1 b_2} = t_{\cdot a_1}^{b_1} k^{a_1 a_2} t_{\cdot a_2}^{b_2}, \tag{3.9}$$

and we use the tensor summation convention of implicit summation of the repeated pairs (in this case $a_1, a_2$) over their range $1, \ldots, p$.

Also $\mathbf{Z}_n$ has Edgeworth type expansions of the form (3.6)–(3.7), where now $q_r(\mathbf{t})$ is a polynomial in $\mathbf{t} \in R^p$ and $\{K_j^{a_1 \cdots a_i}, 1 \le i \le r+2\}$.

If $q = 1$ then (3.8) and (3.9) take the form

$$\kappa_r(\widehat{t}) = \sum_{i=0}^{\infty} n^{-i} a_{ri}, \qquad a_{21} = t_{\cdot a_1} k^{a_1 a_2} t_{\cdot a_2}. \tag{3.10}$$

So, if also $a_{21}$ has a nonzero limit or is bounded away from zero as $n \to \infty$, then

$$Z_n / a_{21}^{1/2} = (n/a_{21})^{1/2} (\widehat{t} - t)$$

has the Cornish–Fisher expansions (3.2)–(3.5), where now $h_r(x)$, $\overline{h}_r(x)$, $f_r(x)$, $g_r(x)$ are polynomials in $x$ and $\{A_{ri}, 1 \le i \le r+2\}$ given in Withers (1984) for $r \le 4$, and $A_{ri}$ is the standardized cumulant coefficient $A_{ri} = a_{ri}/a_{21}^{r/2}$. For example,

$$h_1(x) = f_1(x) = g_1(x) = A_{11} + A_{32}(x^2 - 1)/6,$$
$$\overline{h}_1(x) = A_{11} x + A_{32}(x^3 - 3x)/6.$$

Note that $h_k$ and $\overline{h}_k$ are linear combinations of the first $k$ even and odd Hermite polynomials, respectively. Expressions for $\{a_{ri}\}$ are given in Withers (1982). For example, $a_{21}$ is given by (3.10), and the cumulant coefficients of $\widehat{t}$ needed for the second term of the Cornish–Fisher expansions are

$$a_{11} = t_{\cdot ij} k^{ij}/2, \qquad a_{32} = t_{\cdot i} t_{\cdot j} t_{\cdot k} k^{ijk} + 3 s_j t_{\cdot jk} s_k, \tag{3.11}$$

again using the tensor summation convention, where $s_j = k^{ji} t_{\cdot i}$. If $p = 2$, (3.10) and (3.11) can be written

$$\begin{aligned} a_{21} &= t_{\cdot 1}^2 k_{11} + 2t_{\cdot 1} t_{\cdot 2} k_{12} + t_{\cdot 2}^2 k^{22}, \qquad a_{11} = \sum_{i=1}^{2} t_{\cdot ii} k^{ii}/2 + t_{\cdot 12} k^{12}, \\ a_{32} &= \sum_{i=1}^{2} t_{\cdot i}^3 k^{iii} + 3 \sum_{12}^{2} t_{\cdot 1}^2 t_{\cdot 2} k^{112} + 3 \sum_{j=1}^{2} s_j^2 t_{\cdot jj} + 6 s_1 t_{\cdot 12} s_2, \end{aligned} \tag{3.12}$$

where

$$\sum_{12}^{2} t_{\cdot 1}^2 t_{\cdot 2} k^{112} = t_{\cdot 1}^2 t_{\cdot 2} k^{112} + t_{\cdot 2}^2 t_{\cdot 1} k^{221}.$$

## 4 Examples

First note that transforming from $i_1, \ldots, i_r$ to $T_k = i_k - i_1$, $k = 2, \ldots, r$, gives

$$\sum_{i_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \alpha(i_1, \ldots, i_r) = \sum_{-n_1 < T_k < n_k, k=2,\ldots,r} \alpha(0, T_2, \ldots, T_r) D_r(n, T),$$

where

$$\begin{aligned} D_r(n, T) &= \min(n_1, n_2 - T_2, \ldots, n_r - T_r) + \min(0, T_2, \ldots, T_r) \\ &= D(n_1, \ldots, n_r : T_2, \ldots, T_r) \end{aligned}$$

say. For example, if $n_k \equiv n$ then

$$D_r(n, T) = n - \delta_r(T),$$

where

$$\delta_r(T) = \max(0, T_2, \ldots, T_r) - \min(0, T_2, \ldots, T_r).$$

**Example 4.1.** Cornish–Fisher expansions for $\widehat{\mu}$ of (2.5).
Take $p = 1$, $\pi = i$ so that $\widehat{M}_i = \widehat{\mu}$. Then

$$k_r = n^{r-1} \kappa_r(\widehat{\mu}) = \tau_r U_{nr}, \tag{4.1}$$

where

$$\begin{aligned} U_{nr} &= n^{-1} \sum_{i_1,\ldots,i_r=1}^{n} \alpha(i_1,\ldots,i_r) \\ &= n^{-1} \sum_{|T_k|<n,k=2,\ldots,r} \alpha(0,T_2,\ldots,T_r)[1-\delta_r(T)/n]. \end{aligned} \tag{4.2}$$

So, as $n \to \infty$,

$$U_{nr} \to U_r = \sum_{T_2,\ldots,T_r=-\infty}^{\infty} \alpha(0,T_2,\ldots,T_r)$$

and $k_r \to \tau_r U_r$. So, as $n \to \infty$, $k_r$ is bounded if $U_r$ is finite and $k_2$ is bounded away from zero if $U_2 > 0$. We now show in detail how to express the sum (4.2) explicitly for $r = 2, 3$. Write

$$U_{n2} = \sum_{|T|<n} \alpha(0,T)(1-|T|/n), \qquad U_{n3} = u_{n1} + 2u_{n2},$$

where

$$\begin{aligned} u_{n1} &= \sum_{-n<T_2=T_3<n} \alpha(0,T_2,T_3)(1-|T_2|/n) \\ &= -\alpha_3 + \sum_{T=0}^{n-1} [\alpha(0,T,T)+\alpha(0,0,T)](1-T/n) \end{aligned}$$

since $\alpha(0,T,T) = \alpha(0,0,-T)$,

$$u_{n2} = \sum_{-n<T_2<T_3<n} \alpha(0,T_2,T_3)\big(1-\delta_3(T)/n\big)$$

and

$$\delta_3(T) = \begin{cases} T_3, & 0 \le T_2 < T_3, \\ T_3 - T_2, & T_2 \le 0 < T_3, \\ -T_2, & T_2 < T_3 \le 0. \end{cases}$$

So, $u_{n2} = \sum_{i=3}^{5} u_{ni}$, where

$$\begin{aligned} u_{n3} &= \sum_{0\le T_2<T_3<n} \alpha(0,T_2,T_3)(1-T_3/n), \\ u_{n4} &= \sum_{-n<T_2\le 0<T_3<n} \alpha(0,T_2,T_3)[1-(T_3-T_2)/n], \\ u_{n5} &= \sum_{-n<T_2<T_3\le 0} \alpha(0,T_2,T_3)(1+T_2/n). \end{aligned}$$

To illustrate this, consider the AR(1) process of Example 2.1. Transforming to $s = n - T$, $\omega = \phi^{-1}$ and setting

$$R_n(\omega) = \sum_{s=0}^{n} \omega^s = (\omega^{n+1} - 1)/(\omega - 1),$$

$$b_n(\phi) = \sum_{T=0}^{n-1} \phi^T (1 - T/n) = \phi^{n-1}\beta_n(\omega),$$

$$\beta_n(\omega) = \sum_{s=1}^{n} s\omega^{s-1} = (d/d\omega)R_n(\omega)$$

$$= (n+1)\omega^n/(\omega - 1) - (\omega^{n+1} - 1)/(\omega - 1)^2,$$

we can write

$$\alpha(0, T) = \alpha_2\phi^{|T|}, \qquad U_{n2}/\alpha_2 = \sum_{|T|<n} \phi^{|T|}(1 - |T|/n) = -1 + 2b_n(\phi),$$

$$U_{n2}/\alpha_2 = -1 + 2[1/(1-\phi) - n^{-1}\phi(1-\phi^n)/(1-\phi)^2],$$

$$u_{n1}/\alpha_3 = -1 + \sum_{T=0}^{n-1}(\phi^{2T} + \phi^T)(1 - T/n) = -1 + b_n(\phi^2)/n + b_n(\phi)/n.$$

Set

$$f(\phi) = \sum_{T=1}^{n-1} T\phi^T = \phi\beta_{n-1}(\phi), \qquad g(\phi) = R_{n-1}(\phi),$$

$$a_3 = \sum_{T=1}^{n-1} \phi^T(1-\phi^T)/(1-\phi) = [R_{n-2}(\phi) - R_{n-2}(\phi^2)]/(1-\phi),$$

$$b_3 = \sum_{1 \le T_3 < n} \phi^{T_3} T_3 R_{T_3-1}(\phi) = (1-\phi)^{-1}[f(\phi) - f(\phi^2)].$$

Then

$$u_{n3}/\alpha_3 + 1 = \sum_{0 \le T_2 < T_3 < n} \phi^{T_2+T_3}(1 - T_3/n) = a_3 - b_3/n.$$

Also since $\alpha(0, j, -k) = \alpha(0, k, j+k) = \phi^{j+2k}$,

$$u_{n4}/\alpha_3 = \sum_{0 \le j,k < n} [1 - (j+k)/n]\alpha(0, k, j+k) = a_4 - b_4/n,$$

where

$$a_4 = \sum_{0 \le k < n} \phi^{2k}(1-\phi^k)/(1-\phi) = [g(\phi^2) - g(\phi^3)]/(1-\phi),$$

$$b_4 = g(\phi^2)f(\phi) + g(\phi)f(\phi^2).$$

Also

$$u_{n5}/\alpha_3 = \sum_{0<j<k<n} (1-k/n)\phi^{2k-j} = a_5 - b_5/n,$$

where $a_5 = \phi^3[R_{n-4}(\phi) - R_{n-4}(\phi^2)]/(\omega-1)$ and $b_5 = [f(\phi) - f(\phi^2)]/(\omega-1)$.

If we truncate the Cornish–Fisher series at $K-1$ terms, that is with remainder $O(n^{-K/2})$, then we can ignore all exponentially small components in $k_r$. For example, in the AR(1) case in the last example we can replace $k_r$ by $k'_r = \tau_r U'_{nr}$, where

$$\begin{aligned}
U'_{n2}/\alpha_2 &= -1 + (1-\phi-\phi/n)/(1-\phi)^2,\\
U'_{n3}/\alpha_3 &= u'_{n1} + 2\sum_{j=2}^{5} u'_j/\alpha_3, \qquad u'_{n1}/\alpha_3 + 1 = 2\sum_{T=1}^{\infty} \phi^{2T}(1-T/n),\\
u'_3/\alpha_3 &= [f'(\phi^2) - f'(\phi)]/(\phi-1),\\
u'_4/\alpha_3 &= g'(\phi^2)f'(\phi) + g'(\phi)f'(\phi^2),\\
u'_5/\alpha_3 &= [f'(\phi) - f'(\phi^2)]/(\omega-1)^2,\\
f'(\phi) &= \sum_{T=1}^{\infty} T\phi^T = \phi/(\phi-1)^2, \qquad g'(\phi) = \sum_{k=0}^{\infty} \phi^k = 1/(1-\phi).
\end{aligned}$$

So, for $j = 2, 3$, $u'_{nj}$ has the form $a'_j - b'_j/n$, where $a'_j$ and $b'_j$ do not depend on $n$.

**Example 4.2.** Cornish–Fisher expansions for the sample autocovariance assuming that $\mu = 0$. (This assumption is common in the literature on the grounds that the series can be adjusted by subtracting the estimated mean. However, we shall see in Example 4.3 that it gives the wrong variance if $\mu \neq 0$.)

In this case $\mu_{0a} = M_{0a}$ can be estimated by $\widehat{M}_{0a}$. So, $\pi = \{0, a\}$ and $k_r$ of (3.1) is given by

$$k_r = n^{r-1}\kappa_r(\widehat{M}_{0a}). \tag{4.3}$$

For example,

$$k_2 = \sum_{|T|<n-a} (1-|T|/n) g_T^{aa}$$

implies

$$\sum_{T=-\infty}^{\infty} g_T^{aa} > 0$$

as $n \to \infty$, where

$$g_T^{aa} = \kappa_{0,a,T,T+a} + \kappa_{0,T}^2 + \kappa_{0,T+a}\kappa_{a,T}$$
$$= \tau_4\alpha(0,a,T,T+a) + \tau_2^2\alpha(0,T)^2 + \tau_2^2\alpha(0,T+a)\alpha(a,T).$$

[Recall that $\kappa_{ij\cdots}$ are defined by (2.2).] So, $k_2$ is bounded away from zero and $K_r$ is bounded in $n$ and the Cornish–Fisher expansions apply.

**Example 4.3.** The autocovariance without assuming that $\mu = 0$.

In this case, we take $p = 2$, $\theta_1 = \mu$, $\theta_2 = M_{0a}$ and $t = t(\boldsymbol{\theta}) = \kappa_{0a} = \theta_2 - \theta_1^2$. So, by (3.12)

$$\begin{aligned} &t_{\cdot 1} = -2\mu, \qquad t_{\cdot 2} = 1, \qquad t_{\cdot 11} = -2, \qquad t_{\cdot 12} = t_{\cdot 22} = 0, \\ &a_{21} = 4\mu^2 k^{11} - 4\mu k^{12} + k^{22}, \\ &a_{11} = -k^{11}, \qquad a_{32} = -8\mu^3 k^{111} + 12\mu^2 k^{112} - 6\mu k^{122} + k^{222} - 2s_1^2 \end{aligned} \tag{4.4}$$

at $s_1 = -2\mu k^{11} + k^{12}$. Also

$$k^{1^r} = k_r \text{ of } (4.1)$$

and

$$k^{12} = n\kappa(\widehat{\mu}, \widehat{M}_{0a}) = N^{-1}\sum_{t_1=1}^{n}\sum_{t_2=1}^{N} g'_T = N^{-1}\sum_{T=1-n}^{N-1} D(n, N:T) g'_T,$$

where

$$g'_T = \kappa(X_{t_1}, X_{t_2}X_{t_2+a}) = \kappa_{0,T,T+a} + \mu(\kappa_{0,T+a} + \kappa_{0,T}),$$
$$N = n - a, \qquad T = t_2 - t_1, \qquad D(n, N:T) = \min(n, N-T) + \min(0, T),$$

and we have used, in the notation of page 58 of McCullagh (1987), $\kappa^{1,23} = \kappa^{1,2,3} + \kappa^2\kappa^{1,3} + \kappa^3\kappa^{1,2}$. Also

$$k^{2\cdots 2} = k^{2^r} = k_r \text{ of } (4.3).$$

By (4.4), $a_{32}$ needs

$$k^{112} = n^2\kappa(\widehat{\mu}, \widehat{\mu}, \widehat{M}_{0a}) = N^{-1}\sum_{t_1,t_2=1}^{n}\sum_{t_3=1}^{N} g_{2T}$$
$$= N^{-1}\sum_{-n<T_2<n,\, -n<T_3<N} g_{2T} D(n, n, N: T_2, T_3),$$

where

$$g_{2T} = \kappa(X_{t_1}, X_{t_2}, X_{t_3}X_{t_3+a}) = \kappa(X_0, X_{T_2}, X_{T_3}X_{T_3+a})$$
$$= \kappa^{1,2,34} = \kappa^{1,2,3,4} + \kappa^3\kappa^{1,2,4} + \kappa^4\kappa^{1,2,3} + \kappa^{1,3}\kappa^{2,4} + \kappa^{1,4}\kappa^{2,3}$$

in the notation of equation (3.2) of McCullagh (1987). So,

$$g_{2T} = \kappa_{0,T_2,T_3,T_3+a} + \mu\kappa_{0,T_2,T_3+a} + \mu\kappa_{0,T_2,T_3} + \kappa_{0,T_3}\kappa_{T_2,T_3+a} + \kappa_{0,T_3+a}\kappa_{T_2,T_3}.$$

Finally, $a_{32}$ needs

$$\begin{aligned}
k^{122} &= n^2\kappa(\widehat{\mu}, \widehat{M}_{0a}, \widehat{M}_{0a}) = (n/N^2)\sum_{t_1=1}^{n}\sum_{t_2,t_3=1}^{N} g_{3T} \\
&= (n/N^2)\sum_{-n<T_2<N,-n<T_3<N} g_{3T}\, D_3(n:T) \\
&= (n/N^2)\sum_{-n<T_2<N,-n<T_3<N} g_{3T}[n-\delta_3(T)], \\
\delta_3(T) &= \max(a, T_2, T_3) - \min(0, T_2, T_3), \\
g_{3T} &= \kappa(X_{t_1}, X_{t_2}X_{t_2+a}, X_{t_3}X_{t_3+a}) = \kappa(X_{T_2}X_{T_2+a}, X_{T_3}X_{T_3+a}, X_0) \\
&= \kappa^{12,34,5} = \sum_{i=1}^{6} U_i, \\
U_1 &= \kappa^{1,2,3,4,5}, \\
U_2 &= \kappa^{1,2,3,4}\kappa^5 + \kappa^{1,2,3,5}\kappa^4 + \kappa^{1,2,4,5}\kappa^3 + \kappa^{1,3,4,5}\kappa^2 + \kappa^{2,3,4,5}\kappa^1, \\
U_3 &= \kappa^{1,2,3}\kappa^{4,5} + \kappa^{1,2,4}\kappa^{3,5} + \kappa^{1,3,4}\kappa^{2,5} + \kappa^{2,3,4}\kappa^{1,5}, \\
U_4 &= \kappa^{1,3,5}M^{2,4} + \kappa^{1,4,5}M^{2,3} + \kappa^{2,3,5}M^{1,4} + \kappa^{2,4,5}M^{1,3}, \\
M^{ij} &= \kappa^{i,j} + \kappa^i\kappa^j
\end{aligned}$$

and

$$\begin{aligned}
U_5 = {} & \kappa^{1,3}(\kappa^{2,5}\kappa^4 + \kappa^{4,5}\kappa^2) + \kappa^{1,4}(\kappa^{2,5}\kappa^3 + \kappa^{3,5}\kappa^2) + \kappa^{1,5}(\kappa^{2,3}\kappa^4 + \kappa^{2,4}\kappa^3) \\
& + \kappa^{2,3}\kappa^{4,5}\kappa^1 + \kappa^{2,4}\kappa^{3,5}\kappa^1
\end{aligned}$$

in the notation of page 255 of McCullagh (1987). So, in the notation of (2.2),

$$\begin{aligned}
U_1 &= \kappa_{T_2,T_2+a,T_3,T_3+a,0}, \\
U_2/\mu &= \kappa_{T_2,T_2+a,T_3,T_3+a} + \kappa_{0,T_2,T_2+a,T_3} + \kappa_{0,T_2,T_2+a,T_3+a} \\
&\quad + \kappa_{0,T_2,T_3,T_3+a} + \kappa_{0,T_2+a,T_3,T_3+a}, \\
U_3 &= \kappa_{T_2,T_2+a,T_3}\kappa_{0,T_3+a} + \kappa_{T_2,T_2+a,T_3+a}\kappa_{0,T_3} + \kappa_{T_2,T_3,T_3+a}\kappa_{0,T_2+a} \\
&\quad + \kappa_{T_2+a,T_3,T_3+a}\kappa_{0,T_2}, \\
U_4 &= \kappa_{0,T_2,T_3}(\kappa_{T_2,T_3} + \mu^2) + \kappa_{0,T_2,T_3+a}(\kappa_{T_2+a,T_3} + \mu^2) \\
&\quad + \kappa_{0,T_2+a,T_3}(\kappa_{T_2,T_3+a} + \mu^2) + \kappa_{0,T_2+a,T_3+a}(\kappa_{T_2,T_3} + \mu^2)
\end{aligned}$$

and

$$U_5/\mu = \kappa_{T_2,T_3}[\kappa_{0,T_2+a} + \kappa_{0,T_3+a}] + \kappa_{T_2,T_3+a}[\kappa_{0,T_2+a} + \kappa_{0,T_3}]$$
$$+ \kappa_{0,T_2}[\kappa_{T_2+a,T_3} + \kappa_{T_2,T_3}] + \kappa_{T_2+a,T_3}\kappa_{0,T_3+a} + \kappa_{T_2,T_3}\kappa_{0,T_3}.$$

**Example 4.4.** Cornish–Fisher expansions for the sample autocorrelation assuming that $\mu = 0$.

Take $p = 2$, $q = 1$, $t = \theta_2/\theta_1$, $\theta_1 = M_{00}$, $\theta_2 = M_{0a}$ at $a_1 = 0$ and $a_2 = a$. So, $t$ is the $a$th autocorrelation and $\widehat{t} = \widehat{M}_{0a}/\widehat{M}_{00}$ is the $a$th sample autocorrelation. So, $t_{\cdot 1} = -\theta_2/\theta_1^2$, $t_{\cdot 2} = 1/\theta_1$, $t_{\cdot 11} = 2\theta_2/\theta_1^3$, $t_{\cdot 12} = -1/\theta_1^2$ and $t_{\cdot 22} = 0$.

**Example 4.5.** Cornish–Fisher expansions for the sample autocorrelation without assuming that $\mu = 0$.

Take

$$\begin{aligned} &p = 3, \qquad q = 1, \qquad \theta_1 = \mu, \qquad \theta_2 = M_{00}, \qquad \theta_3 = M_{0a}, \\ &D = \operatorname{var}(X_0) = \theta_2 - \theta_1^2, \qquad N = \operatorname{covar}(X_0, X_a) = \theta_3 - \theta_1^2, \\ &t = \operatorname{covar}(X_0, X_a)/\operatorname{var}(X_0) = N/D. \end{aligned} \tag{4.5}$$

So, $a_{21}$, $a_{11}$ and $a_{32}$ are given by (3.12) with $t_{\cdot 1} = 2\mu(t-1)/D$, $t_{\cdot 2} = -N/D^2$, $t_{\cdot 3} = 1/D^2$, $t_{\cdot 22} = 2\theta_2/\theta_1^3$, $t_{\cdot 23} = -1/\theta_1^2$ and $t_{\cdot 33} = 0$.

**Example 4.6.** Suppose that

$$t = \kappa(X_0 X_{a_1}, X_0 X_{a_2}) = \theta_3 - \theta_1\theta_2, \qquad \theta_1 = M_{0a_1},$$
$$\theta_2 = M_{0a_2}, \qquad \theta_3 = M_{00a_1a_2}.$$

Set $k^{a_1a_2} = n\kappa(\widehat{M}_{0a_1}, \widehat{M}_{0a_2})$, $N_i = n - a_i$ and $T = t_2 - t_1$. Then

$$k^{a_1a_2} = n(N_1N_2)^{-1} \sum_{t_1=1}^{N_1} \sum_{t_2=1}^{N_2} g_T = n(N_1N_2)^{-1} \sum_{T=1-N_1}^{N_2-1} D(N_1, N_2 : T) g_T,$$

where $g_T = \kappa(X_{t_1}X_{t_1+a_1}, X_{t_2}X_{t_2+a_2}) = \kappa(X_0X_{a_1}, X_TX_{T+a_2}) = \kappa^{12,34}$ in the notation of McCullagh. This is given by eleven terms in the equation above (3.2) of McCullagh.

Other examples, where the method can be applied is to functions of the estimates of the sample autocorrelations, such as estimates of the coefficients of ARMA processes. This includes the Yule–Walker estimates of the coefficients of an AR$(p)$.

**Example 4.7.** From equation (5.42) of Kendall and Ord (1990), for the AR(2)

$$X_i = \delta + \sum_{j=1}^{2} \phi_j X_{t-j} + e_i - \tau_1.$$

The Yule–Walker estimates are given by replacing $(\mu, r_1, r_2)$ by their estimates in

$$\delta = \left(1 - \sum_{j=1}^{2} \phi_j\right)\mu, \qquad \phi_1 = r_1(1-r_2)/(1-r_1^2), \qquad \phi_2 = (r_2 - r_1^2)/(1-r_1^2),$$

where $r_a = \operatorname{covar}(X_0, X_a)/\operatorname{var}(X_0)$ is the $a$th autocorrelation, as in (4.5). Set $p = 4$, $q = 2$, $\theta_1 = \mu$, $\theta_2 = M_{00}$, $\theta_3 = M_{01}$, $\theta_4 = M_{02}$, $D = \operatorname{var}(X_0) = \theta_2 - \theta_1^2$, $N_1 = \operatorname{covar}(X_0, X_1) = \theta_3 - \theta_1^2$ and $N_2 = \operatorname{covar}(X_0, X_2) = \theta_4 - \theta_1^2$. Then

$$\phi_1 = N_1(D - N_2)/(D^2 - N_1^2), \qquad \phi_2 = (DN_2 - D_1^2)/(D^2 - N_1^2)$$

which we write as $\boldsymbol{\phi} = \mathbf{t} = \mathbf{t}(\boldsymbol{\theta})$. Now substitute partial derivatives. For example, for $t = \phi_1$, $a_{21}$ needs

$$\begin{aligned} t_{\cdot 1}^{1} &= N_{1\cdot 1}[(D - N_2)/(D^2 - N_1^2) + 2N_1^2(D - N_2)/(D^2 - N_1^2)^2] \\ &\quad - N_{2\cdot 1}N_1/(D^2 - N_1^2) + D_{\cdot 1}N_1[1/(D^2 - N_1^2) - 2D(D - N_2)/(D^2 - N_1^2)^2] \\ &= D_{\cdot 1}\sum_{i=1}^{2} \gamma_i (D^2 - N_1^2)^{-i}, \qquad \gamma_1 = \theta_2 - \theta_4, \qquad \gamma_2 = 2N_1(D - N_2)(\theta_3 - \theta_2) \end{aligned}$$

and

$$t_{\cdot 2}^{1} = N_1(D^2 - N_1^2)^{-2}\gamma_4, \qquad \gamma_4 = -\theta_2^2 + 2\theta_1^2(\theta_3 - \theta_4) + 2\theta_2\theta_4 - \theta_3^2$$

using $N_{i\cdot 1} = D_{\cdot 1} = -2\theta_1$ and simplifying.

Finally, the method can be adapted to allow for a nonstationary mean, for example, by adding a parametric regression function to the right-hand side of (2.1).

## 5 Simulation study

Here, we illustrate the practical value of the results in Section 4.

One purpose of Cornish–Fisher expansions is to provide improved confidence intervals. The usual confidence intervals for the mean, autocovariance, and autocorrelation are based on Studentizing. The terms of the Cornish–Fisher expansions given by Examples 4.1–4.5 can be used to provide improved confidence intervals. We illustrate this fact by computing the coverage probabilities by means of simulation.

We simulated 10,000 samples of size $n$ from (2.1) by assuming that the errors have the standard normal distribution. We calculated the Studentized intervals as well as those incorporating the Cornish–Fisher terms for the mean, autocovariance, and autocorrelation. The proportion of intervals containing the true value of these parameters is shown in Tables 5.1–5.3 for $n = 5, 6, \dots, 40$. We can see clearly that the confidence intervals incorporating the Cornish–Fisher terms make a real improvement. The Studentized intervals perform poorly for most values of $n$ and even for large $n$.

**Table 5.1** *Simulated coverage probabilities for mean*

| $n$ | Studentized | Using Cornish–Fisher terms |
|---|---|---|
| 5 | 0.940 | 0.950 |
| 6 | 0.944 | 0.949 |
| 7 | 0.944 | 0.947 |
| 8 | 0.940 | 0.949 |
| 9 | 0.938 | 0.949 |
| 10 | 0.950 | 0.950 |
| 11 | 0.932 | 0.949 |
| 12 | 0.938 | 0.948 |
| 13 | 0.946 | 0.950 |
| 14 | 0.944 | 0.948 |
| 15 | 0.934 | 0.941 |
| 16 | 0.932 | 0.937 |
| 17 | 0.950 | 0.950 |
| 18 | 0.936 | 0.942 |
| 19 | 0.950 | 0.950 |
| 20 | 0.938 | 0.948 |
| 21 | 0.938 | 0.947 |
| 22 | 0.950 | 0.950 |
| 23 | 0.938 | 0.950 |
| 24 | 0.950 | 0.950 |
| 25 | 0.944 | 0.949 |
| 26 | 0.944 | 0.949 |
| 27 | 0.946 | 0.949 |
| 28 | 0.928 | 0.937 |
| 29 | 0.942 | 0.949 |
| 30 | 0.936 | 0.950 |
| 31 | 0.948 | 0.948 |
| 32 | 0.948 | 0.950 |
| 33 | 0.922 | 0.947 |
| 34 | 0.950 | 0.950 |
| 35 | 0.946 | 0.950 |
| 36 | 0.938 | 0.949 |
| 37 | 0.948 | 0.948 |
| 38 | 0.922 | 0.948 |
| 39 | 0.944 | 0.950 |
| 40 | 0.950 | 0.950 |

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**Table 5.2** *Simulated coverage probabilities for autocovariance*

| $n$ | Studentized | Using Cornish–Fisher terms |
|---|---|---|
| 5 | 0.942 | 0.947 |
| 6 | 0.938 | 0.943 |
| 7 | 0.938 | 0.949 |
| 8 | 0.946 | 0.950 |
| 9 | 0.942 | 0.949 |
| 10 | 0.948 | 0.949 |
| 11 | 0.940 | 0.949 |
| 12 | 0.944 | 0.950 |
| 13 | 0.944 | 0.949 |
| 14 | 0.948 | 0.950 |
| 15 | 0.942 | 0.948 |
| 16 | 0.946 | 0.948 |
| 17 | 0.942 | 0.950 |
| 18 | 0.948 | 0.950 |
| 19 | 0.942 | 0.946 |
| 20 | 0.928 | 0.949 |
| 21 | 0.944 | 0.950 |
| 22 | 0.950 | 0.950 |
| 23 | 0.946 | 0.950 |
| 24 | 0.948 | 0.950 |
| 25 | 0.934 | 0.947 |
| 26 | 0.946 | 0.948 |
| 27 | 0.940 | 0.942 |
| 28 | 0.938 | 0.941 |
| 29 | 0.936 | 0.946 |
| 30 | 0.946 | 0.949 |
| 31 | 0.940 | 0.946 |
| 32 | 0.944 | 0.948 |
| 33 | 0.936 | 0.946 |
| 34 | 0.942 | 0.945 |
| 35 | 0.938 | 0.944 |
| 36 | 0.948 | 0.950 |
| 37 | 0.950 | 0.950 |
| 38 | 0.938 | 0.944 |
| 39 | 0.922 | 0.942 |
| 40 | 0.938 | 0.950 |

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**Table 5.3** *Simulated coverage probabilities for autocorrelation*

| $n$ | Studentized | Using Cornish–Fisher terms |
|---|---|---|
| 5 | 0.946 | 0.948 |
| 6 | 0.940 | 0.941 |
| 7 | 0.950 | 0.950 |
| 8 | 0.950 | 0.950 |
| 9 | 0.950 | 0.950 |
| 10 | 0.950 | 0.950 |
| 11 | 0.932 | 0.945 |
| 12 | 0.950 | 0.950 |
| 13 | 0.946 | 0.950 |
| 14 | 0.934 | 0.944 |
| 15 | 0.938 | 0.948 |
| 16 | 0.942 | 0.950 |
| 17 | 0.948 | 0.949 |
| 18 | 0.944 | 0.946 |
| 19 | 0.948 | 0.950 |
| 20 | 0.950 | 0.950 |
| 21 | 0.942 | 0.948 |
| 22 | 0.946 | 0.949 |
| 23 | 0.948 | 0.950 |
| 24 | 0.942 | 0.947 |
| 25 | 0.942 | 0.948 |
| 26 | 0.948 | 0.950 |
| 27 | 0.932 | 0.949 |
| 28 | 0.930 | 0.939 |
| 29 | 0.942 | 0.948 |
| 30 | 0.938 | 0.950 |
| 31 | 0.932 | 0.949 |
| 32 | 0.948 | 0.950 |
| 33 | 0.944 | 0.947 |
| 34 | 0.950 | 0.950 |
| 35 | 0.932 | 0.948 |
| 36 | 0.950 | 0.950 |
| 37 | 0.946 | 0.950 |
| 38 | 0.938 | 0.948 |
| 39 | 0.944 | 0.949 |
| 40 | 0.940 | 0.947 |

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