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A note on the Berman condition

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Abstract. It is established that if a time series satisfies the Berman condition, and another related (summability) condition, the result of filtering that series through a certain type of filter also satisfies the two conditions. In particular it follows that if X_t satisfies the two conditions and if X_t and a_t are related by an invertible ARMA model, then the a_t satisfy the two conditions.

1 Introduction

The condition (on the autocovariances γ_k of a stationary time series)

$$\lim_{k\to\infty} |\gamma_k| \ln k = 0 \tag{1.1}$$

was introduced by Berman (1964, Theorem 3.1, p. 510). It appears to have been adopted as a fundamental sufficient condition in proving results about extreme value distributions for correlated data. It is cited for instance in Leadbetter et al. (1983, equation 2.5.1, p. 444), Lindgren and Rootzén (1987, equation 5.1, p. 248), Leadbetter and Rootzén (1988, equation 4.1.1, p. 80), Galambos (1978, Theorem 3.8.2, p. 169; see also p. 198), and in Embrechts et al. (1999, Theorem 4.4.8, p. 217), where it is described as being "very weak." It appears to be effectively the weakest condition that one can assume and still obtain positive results in this context.

In Chareka, Matarise and Turner (2006) the authors found it necessary to assume, in addition to the Berman condition, another condition

$$\sum_{k=1}^{\infty} \frac{|\gamma_k|}{k^{\varepsilon}} < \infty \qquad \text{for some } \varepsilon < 1. \tag{1.2}$$

This is given as condition (7) on page 598 of Chareka et al. (2006). In that paper the authors find it expedient to deal with the residuals from fitting an ARMA model to the time series X_t under consideration. They are thereby concerned with the innovation terms of such a model. Suppose that X_t and a_t are related via an ARMA model in which the a_t play the role of the innovations. Chareka et al. remark that

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if the time series X_t is a fractionally integrated ARMA ("FARIMA") time series (whence it satisfies the two conditions of interest (1.1) and (1.2)) then the innovations a_t also form a FARIMA series provided that the model is invertible. Hence the a_t satisfy the two conditions of interest as well.

Chareka et al. assert that more is true: if X_t is *any* stationary time series satisfying (1.1) and (1.2) and if X_t and a series of innovations a_t are related by an invertible ARMA model, then the a_t will also satisfy these conditions. In this note we present the proof of that claim.

We now remark that interest is focussed on the a_t and these quantities are thought of as being the output of a filter, with the X_t being the input. However the phrasing of the claim, with the a_t being the innovations of an ARMA model, makes it appear as if the a_t are the *input* to a filter, which is rather confusing. The required condition of invertibility of the ARMA model is also somewhat disconcerting. Finally, it turns out that a slightly stronger claim may be established. We therefore rephrase the assertion to be proven, in a stronger and less confusing form, and state the original claim as a corollary of the rephrased assertion.

2 The main result

We state the result to be proven as follows:

Theorem. *Suppose that X_t is a stationary time series with autocovariances γ_k satisfying conditions* (1.1) *and* (1.2) *and that the series Y_t is the output of a linear filter with input X_t given as follows*:

$$Y_t = \sum_{n=0}^{\infty} \psi_n X_{t-n}.$$

Suppose that the ψ_n are summable (*whence the Y_t form a stationary time series*). *Furthermore suppose that the ψ_n satisfy the condition*

$$|\gamma_k^W| \leq Cr^{|k|} \qquad \text{for all } k \tag{2.1}$$

for some constants C and r, $0 < r < 1$, where

$$\gamma_k^W = \sum_{n=-\infty}^{\infty} \psi_n \psi_{n+k}$$

and where we set $\psi_n = 0$ for $n < 0$ (*to simplify the notation*). *Then the autocovariances γ_k^Y of the series Y_t satisfy* (1.1) *and* (1.2) *as well.*

Proof. We remark that the γ_k^W are in fact the autocovariances of a time series W_t defined by

$$W_t = \sum_{n=0}^{\infty} \psi_n b_{t-n},$$

where b_t is white noise with variance 1.

Observe that

$$\gamma_k^Y = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \psi_n \psi_m \gamma_{m-n+k}$$
$$= \sum_{h=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \psi_n \psi_{n+h} \gamma_{k+h}$$
$$= \sum_{h=-\infty}^{\infty} \gamma_h^W \gamma_{k+h}.$$

To show that the γ_k^Y satisfy condition (1.1) we write

$$\gamma_k^Y = \sum_{h=-\infty}^{-k-1} \gamma_h^W \gamma_{k+h} + \sum_{h=-k}^{-1} \gamma_h^W \gamma_{k+h} + \sum_{h=0}^{\infty} \gamma_h^W \gamma_{k+h}$$
$$= \sum_{j=1}^{\infty} \gamma_{k+j}^W \gamma_j + \sum_{j=0}^{k-1} \gamma_{k-j}^W \gamma_j + \sum_{j=0}^{\infty} \gamma_j^W \gamma_{k+j}$$
$$= \xi_1(k) + \xi_2(k) + \xi_3(k) \qquad \text{(say)}.$$

To deal with $\xi_1(k)$ we observe that

$$|\xi_1(k)| \ln k \le \sum_{j=1}^{\infty} |\gamma_{k+j}^W||\gamma_j| \ln k \le C\gamma_0 r^k \ln k \frac{r}{1-r}$$

using (2.1). This quantity $\to 0$ as $k \to \infty$ since $r < 1$.

Similarly

$$|\xi_3(k)| \ln k \le \sum_{j=0}^{\infty} |\gamma_j^W||\gamma_{k+j}| \ln(k+j) \le C \sum_{j=0}^{\infty} r^j |\gamma_{k+j}| \ln(k+j).$$

Take $\delta > 0$; for sufficiently large k, $|\gamma_{k+j}| \ln(k+j) \le \delta$ for all $j \ge 0$. Hence

$$|\xi_3(k)| \ln k \le \frac{\delta \times C}{1-r}$$

for sufficiently large k, and since δ is arbitrary, $|\xi_3(k)| \ln k \to 0$ as $k \to \infty$.

To deal with the middle term $\xi_2(k)$ we note that

$$|\xi_2(k)| \le C \sum_{j=0}^{k-1} |\gamma_j| r^{k-j}$$
$$= C\left[\sum_{j=0}^{[k/2]} |\gamma_j| r^{k-j} + \sum_{j=[k/2]+1}^{k-1} |\gamma_j| r^{k-j}\right]$$

$$\le C\sum_{j=0}^{[k/2]}\gamma_0 r^{k-j} + C\sum_{j=[k/2]+1}^{k-1}|\gamma_j| r^{k-j}$$
$$\le C\gamma_0\frac{r^{k/2}}{1-r} + C\gamma_{j^*(k)}\frac{r}{1-r},$$

where $j^*(k) = \text{argmax}\{|\gamma_j| : [k/2]+1 \le j \le k-1\}$.

Hence

$$\ln k \times |\xi_2(k)| \le C\Big[\gamma_0 \ln k \frac{r^{k/2}}{1-r} + (\ln k/2 + \ln 2)\Big(\gamma_{j^*(k)}\frac{r}{1-r}\Big)\Big]$$
$$\le C\Big[\gamma_0 \ln k \frac{r^{k/2}}{1-r} + \big(\ln j^*(k) + \ln 2\big)\Big(\gamma_{j^*(k)}\frac{r}{1-r}\Big)\Big]$$

which $\to 0$ as $k \to \infty$.

We have thus established that the autocovariances γ_k^Y satisfy (1.1). We now proceed to show that condition (1.2) is satisfied:

$$\sum_{k=1}^{\infty}\frac{|\gamma_k^Y|}{k^\varepsilon} = \sum_{k=1}^{\infty}\Big|\sum_{h=-\infty}^{\infty}\gamma_h^W\frac{\gamma_{k+h}}{k^\varepsilon}\Big| \le \sum_{h=-\infty}^{\infty}|\gamma_h^W|\sum_{k=1}^{\infty}\frac{|\gamma_{k+h}|}{k^\varepsilon}$$
$$= \sum_{h=-\infty}^{-1}|\gamma_h^W|\sum_{k=1}^{\infty}\frac{|\gamma_{k+h}|}{k^\varepsilon} + \sum_{h=0}^{\infty}|\gamma_h^W|\sum_{k=1}^{\infty}\frac{|\gamma_{k+h}|}{k^\varepsilon}$$
$$= \sum_{j=1}^{\infty}|\gamma_j^W|\sum_{k=1}^{\infty}\frac{|\gamma_{k-j}|}{k^\varepsilon} + \sum_{h=0}^{\infty}|\gamma_h^W|\sum_{k=1}^{\infty}\frac{|\gamma_{k+h}|}{k^\varepsilon}$$
$$\le \sum_{j=1}^{\infty}|\gamma_j^W|\sum_{k=1}^{\infty}\frac{|\gamma_{k-j}|}{k^\varepsilon} + \sum_{h=0}^{\infty}|\gamma_h^W|\sum_{k=1}^{\infty}\frac{|\gamma_{k+h}|}{(k+h)^\varepsilon}\Big(\frac{k+h}{k}\Big)^\varepsilon$$
$$\le \sum_{j=1}^{\infty}|\gamma_j^W|\sum_{k=1}^{\infty}\frac{|\gamma_{k-j}|}{k^\varepsilon} + \zeta\sum_{h=0}^{\infty}|\gamma_h^W|(1+h), \qquad \text{where } \zeta = \sum_{k=1}^{\infty}\frac{|\gamma_k|}{k^\varepsilon}$$
$$= \sum_{j=1}^{\infty}|\gamma_j^W|\Big[\sum_{k=1}^{j}\frac{|\gamma_{k-j}|}{k^\varepsilon} + \sum_{k=j+1}^{\infty}\frac{|\gamma_{k-j}|}{k^\varepsilon}\Big] + \zeta\sum_{h=0}^{\infty}|\gamma_h^W|(1+h)$$
$$\le \sum_{j=1}^{\infty}Cr^j\Big[\gamma_0\sum_{k=1}^{j}\frac{1}{k^\varepsilon} + \sum_{\ell=1}^{\infty}\frac{|\gamma_\ell|}{\ell^\varepsilon}\Big(\frac{\ell}{\ell+j}\Big)^\varepsilon\Big] + \zeta\sum_{h=0}^{\infty}Cr^h(1+h)$$
$$\le C\sum_{j=1}^{\infty}r^j\Big[j\gamma_0 + \sum_{\ell=1}^{\infty}\frac{|\gamma_\ell|}{\ell^\varepsilon}\Big] + \zeta C\sum_{h=0}^{\infty}r^h(1+h)$$

$$\leq C \sum_{j=1}^{\infty} r^j [j\gamma_0 + \zeta] + \zeta C \sum_{h=0}^{\infty} r^h (1+h)$$

$$= C\left[\gamma_0 \sum_{j=1}^{\infty} jr^j + \zeta \sum_{j=1}^{\infty} r^j + \zeta \sum_{j=0}^{\infty} r^j + \zeta \sum_{j=1}^{\infty} jr^j\right]$$

$$= C\left[(\gamma_0 + \zeta) \sum_{j=1}^{\infty} jr^j + \zeta \frac{1+r}{1-r}\right] < \infty$$

since $\sum_{j=1}^{\infty} jr^j$ converges. (The radius of convergence of this power series is 1, and by assumption $0 < r < 1$.) □

3 The original claim

We state the original claim as:

Corollary. *Suppose that X_t satisfies conditions (1.1) and (1.2) and that X_t and a_t are related by the invertible ARMA model*

$$\phi(B)X_t = \theta(B)a_t,$$

where $\phi(z)$ and $\theta(z)$ are polynomials and B is the "backshift" operator. Then the a_t satisfy conditions (1.1) and (1.2) as well.

Proof. The invertibility of the model tells us that a_t can be expressed as

$$a_t = \sum_{n=0}^{\infty} \psi_n X_{t-n},$$

where

$$\frac{\phi(z)}{\theta(z)} = \psi(x) = \sum_{n=0}^{\infty} \psi_n z^n$$

with the coefficients ψ_n being summable. (It is more usual in such a context to denote the coefficients of the series expansion as "π_n" rather than "ψ_n," but to make clear the relationship to the main result we eschew the π_n notation.)

Now if we set

$$W_t = \sum_{n=0}^{\infty} \psi_n b_{t-n},$$

where b_t is white noise (with variance 1) then basic results about ARMA time series (see, e.g., Brockwell and Davis (1991, Chapter 3, problem 3.11)) tell us that the autocovariances γ_k^W of W_t satisfy (2.1). Hence the claim follows by the theorem proven in Section 2. □

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