

# INFINITELY DIVISIBLE CENTRAL PROBABILITY MEASURES ON COMPACT LIE GROUPS—REGULARITY, SEMIGROUPS AND TRANSITION KERNELS

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We introduce a class of central symmetric infinitely divisible probability measures on compact Lie groups by lifting the characteristic exponent from the real line via the Casimir operator. The class includes Gauss, Laplace and stable-type measures. We find conditions for such a measure to have a smooth density and give examples. The Hunt semigroup and generator of convolution semigroups of measures are represented as pseudo-differential operators. For sufficiently regular convolution semigroups, the transition kernel has a tractable Fourier expansion and the density at the neutral element may be expressed as the trace of the Hunt semigroup. We compute the short time asymptotics of the density at the neutral element for the Cauchy distribution on the  $d$ -torus, on  $SU(2)$  and on  $SO(3)$ , where we find markedly different behaviour than is the case for the usual heat kernel.

**1. Introduction.** The heat kernel on a compact Riemannian manifold has been the subject of extensive investigations by analysts, geometers and probabilists. One reason for this is that its small and large time asymptotic expansions contain important topological and geometric information (see, e.g., [32]). Another reason is that it is the transition density of manifold-valued Brownian motion which is a stochastic process of intrinsic interest (see, e.g., [10]). If the manifold is a Lie group, then the heat kernel is naturally related to Dedekind’s eta function via Macdonald’s identities (see [12]). In this paper, we will mainly be concerned with compact Lie groups. In this case, the heat kernel is naturally associated to a vaguely (or equivalently, weakly) continuous convolution semigroup of probability measures which we’ll refer to as the “heat semigroup” in the sequel.

The study of the entire class of such convolution semigroups has had a long development (see, e.g., [5, 19, 20]). From a probabilistic point of view, they correspond to Lévy processes, that is, stochastic processes with stationary and independent increments. Compared to Brownian motion which has continuous sample paths (with probability one), the paths of the generic Lévy process are only right continuous and have jump discontinuities of arbitrary size occurring at random times.

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The purpose of this paper is to study a class of convolution semigroups which on the one hand, are sufficiently close in structure to the heat semigroup to enable us to do some interesting analysis and on the other hand, are sufficiently broad as to display all the interesting features that one finds with a generic Lévy process. The first observation is that measures comprising the heat semigroup are central and so we focus on this class. It is worth pointing out that that central measures as a class have also received some attention from analysts (see, e.g., [16, 31]). Second, we remark that if  $(\mu_t, t \geq 0)$  is the heat semigroup then its noncommutative Fourier transform (see [17, 36] for background on this concept) takes the form  $\hat{\mu}_t(\pi) = e^{-(t/2)\kappa_\pi} I_\pi$  at the irreducible representation  $\pi$  where  $-\kappa_\pi I_\pi$  is the Casimir operator. But  $u \rightarrow u^2/2$  is the negative-definite function on the real line associated with the standard Gaussian measure. The generalization that we make here is to consider a class of semigroups that are given by the prescription  $\hat{\mu}_t(\pi) = e^{-t\eta(\kappa_\pi^{1/2})} I_\pi$  where  $\eta$  is a symmetric negative definite function. Other examples of measures subsumed within this class include the Laplace distribution on a Lie group, which has been utilized in recent statistical work on the problem of deconvolution (see, e.g., [24, 26]) and analogues of stable laws. Indeed any semigroup of probability measures that is obtained by subordinating the heat semigroup belongs to this class. We study these measures both from the static perspective, where the emphasis is on a single infinitely divisible measure, and the dynamic perspective where we focus on an entire semigroup.

The organization of this paper is as follows. In Section 2, we study central probability measures, introduce our main class and examine some examples. In Section 3, we use Sobolev spaces to find conditions on our induced measures which enable them to have a smooth density. In Section 4, we turn our attention to convolution semigroups and the associated semigroup of operators (the *Hunt semigroup*) on the  $L^2$  space of normalized Haar measure. When  $G$  is a Euclidean space, it is known (see [22], Chapter 3 of [4]) that these operators, and their generators, can be realized as pseudo-differential operators. Using Peter–Weyl theory, Ruzhansky and Turunen [33] have developed an intrinsic theory of pseudo-differential operators on compact groups. We adapt this theory to our needs and show that the Hunt semigroup and its generator are pseudo-differential operators in the sense of Ruzhansky and Turunen. This part of the work is carried out in full generality. In the case of our induced class, we show that the generator has the same Sobolev regularity as the Laplacian. Although these results have an analytic flavor, they are important for probabilists as they indicate a route to investigate general classes of Feller–Markov processes on compact Lie groups using the symbol of the generator as the key tool (see [23] for an account of this theory in the case where  $G$  is Euclidean space).

In Section 5, we investigate the transition kernel for convolution semigroups of central measures. We remark that the first investigation of densities for such measures (under a hypo-ellipticity condition that we do not require here) were

made by Liao in [28] (see also Theorem 4.4 in [29], page 96). A necessary and sufficient condition for the semigroup to be trace-class for any positive time is that the corresponding probability measure has a square-integrable density [5]. We compute the trace in both the main  $L^2$ -space and the subspace of central functions. In the former case, the coordinate functions form a complete set of eigenvectors for the Hunt semigroup. Comparing the traces in these two spaces, leads to an interesting inequality for transition kernels which appears to be new even in the heat kernel case. Finally, in Section 6, we study the small time asymptotics of the transition kernel in the case of the Cauchy distribution on the  $d$ -torus, on  $SU(2)$  and on  $SO(3)$  and show that it blows up much faster than the heat kernel.

**2. Infinite divisibility of central measures.** Let  $G$  be a compact group with neutral element  $e$  and let  $\mathcal{M}(G)$  be the set of all probability measures defined on  $(G, \mathcal{B}(G))$  where  $\mathcal{B}(G)$  is the Borel  $\sigma$ -algebra of  $G$ . We say that  $\mu \in \mathcal{M}(G)$  is *central* (or *conjugate-invariant*) if  $\mu(\sigma A \sigma^{-1}) = \mu(A)$  for all  $\sigma \in G, A \in \mathcal{B}(G)$  and  $\mu$  is said to be *symmetric* if  $\mu(A^{-1}) = \mu(A)$  for all  $A \in \mathcal{B}(G)$ . Let  $\mathcal{M}_c(G)$  ( $\mathcal{M}_s(G)$ ) be the subsets of  $\mathcal{M}(G)$  comprising central (symmetric) measures (resp.) and define  $\mathcal{M}_{c,s}(G) := \mathcal{M}_c(G) \cap \mathcal{M}_s(G)$ . Normalized Haar measure on  $G$  will always be denoted  $d\sigma$  when integrating functions of  $\sigma \in G$ .

Let  $\widehat{G}$  be the set of all equivalence classes of irreducible representations of  $G$ . We will, without further comment, frequently identify equivalence classes with a particular representative element when there is no loss of generality. The trivial representation will always be denoted by  $\delta$ . Each  $\pi \in \widehat{G}$  acts as a  $d_\pi \times d_\pi$  unitary matrix on a complex linear space  $V_\pi$  having dimension  $d_\pi$ . We define the Fourier transform of each  $\mu \in \mathcal{M}(G)$  to be the Bochner integral

$$\widehat{\mu}(\pi) = \int_G \pi(\sigma)\mu(d\sigma),$$

where  $\pi \in \widehat{G}$ . We will frequently use the well-known and easily verified fact that

$$\widehat{\mu * \nu}(\pi) = \widehat{\mu}(\pi)\widehat{\nu}(\pi)$$

for all  $\mu, \nu \in \mathcal{M}(G), \pi \in \widehat{G}$ , where  $*$  denotes convolution of measures.

Suppose we are given  $\mu \in \mathcal{M}(G)$ . It is shown in [34] that  $\mu \in \mathcal{M}_c(G)$  if and only if for each  $\pi \in \widehat{G}$  there exists  $c_\pi \in \mathbb{C}$  such that

$$(2.1) \quad \widehat{\mu}(\pi) = c_\pi I_\pi,$$

where  $I_\pi$  is the identity matrix acting on  $V_\pi$ . Indeed this is a straightforward consequence of Schur’s lemma. Moreover, one has the formula

$$(2.2) \quad c_\pi = \frac{1}{d_\pi} \int_G \chi_\pi(\sigma)\mu(d\sigma),$$

where  $\chi_\pi(\cdot) := \text{tr}(\pi(\cdot))$  is the group character.

It is well known (and easily verified) that  $\mu \in \mathcal{M}_s(G)$  if and only if  $\hat{\mu}(\pi)$  is self-adjoint for all  $\pi \in \widehat{G}$ . Consequently,  $\mu \in \mathcal{M}_{cs}(G)$  if and only if  $\hat{\mu}(\pi) = c_\pi I_\pi$  with  $c_\pi \in \mathbb{R}$  for all  $\pi \in \widehat{G}$ .

A probability measure  $\mu$  is *infinitely divisible* if for each  $n \in \mathbb{N}$  there exists  $\nu_n \in \mathcal{M}(G)$  such that  $\nu_n^{*n} = \mu$ . In this case, we write  $\mu_{1/n} := \nu_n$ .

**PROPOSITION 2.1.** *If  $G$  is a compact Lie group and  $\mu \in \mathcal{M}_{cs}(G)$  is infinitely divisible, then for each  $\pi \in \widehat{G}$  there exists  $\alpha_\pi \leq 0$  such that  $\hat{\mu}(\pi) = e^{\alpha_\pi} I_\pi$ .*

**PROOF.** By the results on pages 220–221 of [19],  $\mu$  may be embedded as  $\mu_1$  into a vaguely continuous convolution semigroup of probability measures  $(\mu_t, t \geq 0)$  where  $\mu_0$  is normalized Haar measure on a closed subgroup  $H$  of  $G$ . It follows (see [2, 30]) that for each  $\pi \in \widehat{G}$ ,  $(\hat{\mu}_t(\pi), t \geq 0)$  is a strongly continuous contraction semigroup of matrices acting on  $V_\pi$  and so we may write  $\hat{\mu}_t(\pi) = \hat{\mu}_0(\pi)e^{tA_\pi}$  for all  $t \geq 0$  where  $A_\pi$  is a  $d_\pi \times d_\pi$  matrix. Now since  $\mu_1 \in \mathcal{M}_{cs}(G)$ , there exists  $\lambda_\pi \in \mathbb{R}$  such that

$$(*) \quad \hat{\mu}_1(\pi) = \hat{\mu}_0(\pi)e^{A_\pi} = \lambda_\pi I_\pi \dots$$

If  $\lambda_\pi = 0$ , the required result holds with  $\alpha_\pi = -\infty$  so assume that  $\lambda_\pi \neq 0$ . Since  $\mu_1 = \mu_1 * \mu_0$ , we have

$$\hat{\mu}_0(\pi)e^{A_\pi} \hat{\mu}_0(\pi) = \lambda_\pi I_\pi.$$

On the other hand, post-multiplying both sides of (\*) by  $\hat{\mu}_0(\pi)$  yields

$$\hat{\mu}_0(\pi)e^{A_\pi} \hat{\mu}_0(\pi) = \lambda_\pi \hat{\mu}_0(\pi).$$

It follows that  $\hat{\mu}_0(\pi) = I_\pi$  and hence  $H = \{e\}$ . We then have  $A_\pi = \alpha_\pi I_\pi$  where  $\alpha_\pi \in \mathbb{R}$  and  $\lambda_\pi = e^{\alpha_\pi}$ . But  $\hat{\mu}_1(\pi)$  is a contraction on  $V_\pi$  and hence  $\alpha_\pi \leq 0$ .  $\square$

**EXAMPLE** (The compound Poisson distribution). Consider the probability measure  $\mu_{\lambda,\gamma}$  where  $\gamma$  is a given probability measure on  $G$  and  $\lambda > 0$ . This is defined by

$$\mu_{\lambda,\gamma} := e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \gamma^{*n}.$$

It is well known (see, e.g., [34]) that for all  $\pi \in \widehat{G}$ ,

$$\hat{\mu}_{\lambda,\gamma}(\pi) = \exp \{ \lambda(\hat{\gamma}(\pi) - I_\pi) \}.$$

**PROPOSITION 2.2.**

- (1) *The measure  $\mu_{\lambda,\gamma}$  is central if and only if  $\gamma$  is.*
- (2) *The measure  $\mu_{\lambda,\gamma}$  is symmetric if and only if  $\gamma$  is.*

PROOF.

(1) The if part is straightforward and is established in Proposition 4 of [34]. Conversely, if  $\mu_{\lambda,\gamma}$  is central then for all  $g \in G, \pi \in \widehat{G}$

$$\pi(g)\hat{\mu}_{\lambda,\gamma}(\pi)\pi(g^{-1}) = \hat{\mu}_{\lambda,\gamma}(\pi),$$

and so

$$\hat{\mu}_{\lambda,\gamma}(\pi) = \exp \{ \lambda(\pi(g)\hat{\gamma}(\pi)\pi(g^{-1}) - I_\pi) \}.$$

Now by uniqueness of Fourier transforms and injectivity of the exponential map on matrices, we have

$$\pi(g)\hat{\gamma}(\pi)\pi(g^{-1}) = \hat{\gamma}(\pi)$$

for all  $g \in G, \pi \in \widehat{G}$  and the result follows.

(2) This result is proved similarly using the fact that a probability measure is symmetric if and only if its Fourier transform comprises self-adjoint matrices.  $\square$

It follows that a central probability measure  $\mu$  is a compound Poisson distribution if and only if there exists  $\lambda > 0$  and a central probability measure  $\gamma$  with  $\hat{\gamma}(\pi) = b_\pi I_\pi$  for all  $\pi \in \widehat{G}$  such that

$$(2.3) \quad \hat{\mu}(\pi) = \exp\{\lambda(b_\pi - 1)I_\pi\}.$$

We now introduce a class of central symmetric measures which are key to this paper. For this part, we assume that  $G$  is a compact Lie group. Let  $\rho$  be a symmetric infinitely divisible probability measure on  $\mathbb{R}$ . Then we have the Lévy–Khintchine formula

$$\int_{\mathbb{R}} e^{iux} \rho(dx) = e^{-\eta(u)},$$

where

$$(2.4) \quad \eta(u) = \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}-\{0\}} (1 - \cos(uy))\nu(dy),$$

where  $\sigma \geq 0$  and  $\nu$  is a symmetric Lévy measure on  $\mathbb{R} - \{0\}$ , that is, a  $\sigma$ -finite Borel symmetric measure for which  $\int_{\mathbb{R}-\{0\}} \min\{1, |x|^2\}\nu(dx) < \infty$  (see, e.g., [6]).

For each  $\pi \in \widehat{G}$ , let  $K_\pi$  be the Casimir operator acting in  $V_\pi$ . Then  $K_\pi = -\kappa_\pi I_\pi$  where  $\kappa_\pi \geq 0$  with  $\kappa_\pi = 0$  if and only if  $\pi = \delta$ . If  $\mu$  is a probability measure on  $G$  for which

$$(2.5) \quad \hat{\mu}(\pi) = e^{-\eta(\kappa_\pi^{1/2})} I_\pi,$$

we say that  $\mu$  is a central symmetric probability measure on  $G$  induced by an infinitely divisible probability measure on  $\mathbb{R}$  and we write  $\mu \in \text{CID}_{\mathbb{R}}(G)$ .

The following two examples have been applied to statistical inference on groups (see, e.g., [24, 26]).

EXAMPLE 1 (Gaussian measure). Here we take  $\nu = 0$  and so  $c_\pi = \exp\{-\frac{1}{2} \times \sigma^2 \kappa_\pi\}$ . Gaussian measure is embeddable into the Brownian motion or heat semigroup of measures for which  $\hat{\mu}_t(\pi) = \exp\{-\frac{t}{2} \sigma^2 \kappa_\pi\}$  for  $t \geq 0$  which has been extensively studied by both analysts and probabilists.

EXAMPLE 2 (The Laplace distribution on  $G$ ). Here we take  $\sigma = 0, \nu(dx) = \frac{\exp\{-|x|/\beta\}}{|x|} dx$  (with  $\beta > 0$ ) and  $c_\pi = (1 + \beta^2 \kappa_\pi)^{-1}$  (see [35], page 98 for a discussion of the underlying distribution on  $\mathbb{R}$ ).

Now consider a central symmetric compound Poisson distribution  $\mu_{\lambda,\gamma}$ . We consider conditions under which  $\mu_{\lambda,\gamma} \in \text{CID}_{\mathbb{R}}(G)$ . First, take  $\sigma = 0$  and  $\nu$  to be a finite symmetric measure in (2.4) and rewrite

$$\eta(u) = \lambda \int_{\mathbb{R}-\{0\}} (1 - \cos(uy)) \tilde{\nu}(dy),$$

where  $\lambda := \nu(\mathbb{R} - \{0\})$  and  $\tilde{\nu}(\cdot) := \frac{1}{\lambda} \nu(\cdot)$ . For  $\mu_{\lambda,\gamma} \in \text{CID}_{\mathbb{R}}(G)$  with this value of  $\lambda$ , we require that  $b_\pi = g(\kappa_\pi^{1/2})$  in (2.3) where  $g(u) = \int_{\mathbb{R}} \cos(ux) \tilde{\nu}(dx)$ . For example, if we take  $\nu$  to be a constant multiple of a centred Gaussian measure with variance  $\sigma^2$  on  $\mathbb{R}$  then  $b_\pi = \exp\{-\frac{1}{2} \sigma^2 \kappa_\pi\}$ .

We now consider an important subclass of measures in  $\text{CID}_{\mathbb{R}}(G)$ . Let  $(\rho_t^f, t \geq 0)$  be the law of a subordinator with associated Bernstein function  $f : (0, \infty) \rightarrow [0, \infty)$  so that  $(\rho_t^f, t \geq 0)$  is a vaguely continuous convolution semigroup of probability measures on  $[0, \infty)$  and for each  $t \geq 0, u > 0$ ,

$$(2.6) \quad \int_0^\infty e^{-us} \rho_t^f(ds) = e^{-tf(u)},$$

and  $f$  has the generic form

$$f(u) = au + \int_{(0,\infty)} (1 - e^{-uy}) \lambda(dy),$$

where  $a \geq 0$  and  $\int_{(0,\infty)} \min\{1, y\} \lambda(dy) < \infty$  (see, e.g., [35], Section 30 and [4], Section 1.3.2 for details). It is straightforward to verify that if  $(\mu_t, t \geq 0)$  is a vaguely continuous convolution semigroups of measures on  $G$  and  $(\rho_t^f, t \geq 0)$  is a subordinator as above then we get another vaguely continuous convolution semigroups of measures on  $G$  which we denote  $(\mu_t^f, t \geq 0)$  via the vague integral

$$\mu_t^f(A) = \int_0^\infty \mu_s(A) \rho_t^f(ds)$$

for  $A \in \mathcal{B}(G)$ . Now let  $(\mu_t, t \geq 0)$  be the Brownian motion semigroup with  $\sigma = \sqrt{2}$ . Then for each  $\pi \in \widehat{G}, t \geq 0$  we have

$$\begin{aligned} \hat{\mu}_t^f(\pi) &= \int_0^\infty \int_G \pi(\sigma)\mu_s(d\sigma)\rho_t^f(ds) \\ &= \left( \int_0^\infty e^{-s\kappa_\pi} \rho_t^f(ds) \right) I_\pi \\ &= e^{-tf(\kappa_\pi)} I_\pi, \end{aligned}$$

and so  $\mu_1^f \in \text{CID}_{\mathbb{R}}(G)$  with  $\eta(\kappa_\pi^{1/2}) = f(\kappa_\pi)$ .

Note that the Laplace distribution (as described above) is obtained in this way with  $f(u) = \log(1 + \beta^2 u)$ . It is worth pointing out that it also arises as  $\beta^{-2} V \beta^{-2}$  where for  $c > 0, V^c$  is the potential measure of the Brownian motion semigroup defined by the vague integral  $V^c(\cdot) = \int_0^\infty e^{-ct} \mu_t(\cdot) dt$  (see [35], pages 203–205 for the case in  $\mathbb{R}^d$ ).

Other examples of measures in  $\text{CID}_{\mathbb{R}}(G)$  which are obtained by subordination include stable-type distributions where  $\sigma = 0$  in (2.4) and  $\nu(dx) = \frac{b^\alpha}{|x|^{1+\alpha}} dx$  where  $b > 0$  and  $0 < \alpha < 2$ . In this case, we have  $f(u) = b^\alpha u^{\alpha/2}$  and  $c_\pi = \exp\{-b^\alpha \kappa_\pi^{\alpha/2}\}$ . We may also consider the relativistic Schrödinger distribution for  $m > 0$  where  $f(u) = \sqrt{u + m^2} - m$  and  $c_\pi = e^{-(\sqrt{m^2 + \kappa_\pi} - m)}$ . It again has  $\sigma = 0$  in (2.4). The precise form of  $\nu$  is complicated and as we do not require it here we refer the reader to [21].

It is an interesting problem to determine the class of all  $\eta$  in (2.4) which give rise to a probability measure on  $G$  of the form (2.5).

**3. Regularity of densities.** In this section we will assume that  $G$  is a compact semi-simple Lie group having Lie algebra  $\mathfrak{g}$ . We say that  $\mu \in \mathcal{M}(G)$  has a density  $k \in L^1(G, \mathbb{R})$  if  $\mu$  is absolutely continuous with respect to normalized Haar measure on  $G$ . We then define  $k$  to be the Radon–Nikodým derivative  $\frac{d\mu}{d\sigma}$ .

If a density  $k$  exists for  $\mu \in \mathcal{M}_c(G)$  with  $\hat{\mu}(\pi) = c_\pi I_\pi$  and  $k \in L^2(G, \mathbb{R})$  then it has the form

$$(3.1) \quad k(\sigma) = \sum_{\pi \in \widehat{G}} d_\pi \overline{c_\pi} \chi_\pi(\sigma)$$

for almost all  $\sigma \in G$  (see equation (3.4) in [3]).

Before we investigate densities in greater detail, we need some preliminaries.

**3.1. Dominant weights.** Fix a maximal torus  $\mathbb{T}$  in  $G$ . Let  $\mathbb{T}$  be its Lie algebra and  $\mathbb{T}^*$  be the dual vector space to  $\mathbb{T}$ . Let  $P$  be the lattice of weights in  $\mathbb{T}^*$  and  $D \subset \mathbb{T}^*$  be the dominant chamber. The celebrated theorem of the highest weight asserts that there is a one-to-one correspondence between elements of  $\widehat{G}$  and the highest weights which are precisely the members of  $P \cap D$ . For details, see, for

example, Chapters 8 and 9 in [14]. In the following, the inner product  $\langle \cdot, \cdot \rangle$  and norm  $|\cdot|$  on  $\mathfrak{T}^*$  are that induced by the Killing form via duality.

Let  $\lambda_\pi$  be the dominant weight for the representation  $\pi$ . Then we know from Sugira [37] [page 39, equation (1.17)] that

$$(3.2) \quad d_\pi \leq N|\lambda_\pi|^m,$$

where  $N$  is a universal constant and

$$(3.3) \quad m = \frac{1}{2}(\dim(G) - r),$$

where  $r$  is the rank of  $G$ , that is, the dimension of any maximal torus. It is also well known that (see, e.g., [37], Lemma 1.1 or [25], Proposition 5.28)

$$(3.4) \quad \kappa_\pi = |\lambda_\pi + \rho|^2 - |\rho|^2 = \langle \lambda_\pi, \lambda_\pi + 2\rho \rangle,$$

where  $\rho$  is half the sum of positive roots. It follows easily that

$$(3.5) \quad |\lambda_\pi|^2 \leq \kappa_\pi \leq |\lambda_\pi|^2 + 2|\lambda_\pi||\rho| \leq C(1 + |\lambda_\pi|^2),$$

where  $C > 1$  is a constant.

We also need the fact (which is implicit in the proof of Lemma 1.3 in [37]) that there exists  $C_1, C_2 > 0$  such that for all  $\lambda \in P \cap D$ , there exists  $n = (n_1, \dots, n_r) \in \mathbb{Z}^r$  such that

$$(3.6) \quad C_1 \|n\| \leq |\lambda| \leq C_2 \|n\|,$$

where  $\|n\|^2 := n_1^2 + \dots + n_r^2$ .

The final result we need from Sugiura [37] is Lemma 1.3 therein that

$$(3.7) \quad \zeta(s) := \sum_{\lambda \in P \cap D - \{0\}} \langle \lambda, \lambda \rangle^{-s}$$

converges if  $2s > r$ .

**3.2. Sobolev spaces.** Let  $\{X_1, \dots, X_d\}$  be a basis for the Lie algebra  $\mathfrak{g}$  of left-invariant vector fields. We define the Sobolev space  $\mathcal{H}_p(G)$  by the prescription

$$\mathcal{H}_p(G) := \{f \in L^2(G); X_{i_1} \cdots X_{i_k} f \in L^2(G); 1 \leq k \leq p, i_1, \dots, i_k = 1, \dots, d\}.$$

It is a complex separable Hilbert space with associated norm

$$\| \| f \| \|^2_p = \| f \|^2 + \sum_{i_1, \dots, i_k} \| X_{i_1} \cdots X_{i_k} f \|^2.$$

It is not difficult to show that an equivalent norm is given by

$$(3.8) \quad \| \| f \| \|^2_p = \sum_{\pi \in \hat{G}} d_\pi (1 + \kappa_\pi)^p \operatorname{tr}(\hat{f}(\pi) \hat{f}(\pi)^*),$$



where  $\hat{f}(\pi) := \int_G \pi(\sigma^{-1})f(\sigma) d\sigma$  is the Fourier transform<sup>1</sup> (and we are abusing notation by using  $\|\cdot\|$  in each case).

As is pointed out in [33], Section 10.3.1,  $\mathcal{H}_p(G)$  coincides with the usual Sobolev space on a manifold constructed using partitions of unity. So in particular, the Sobolev embedding theorem extends to this context and hence

$$C^\infty(G) \supseteq \bigcap_{k \in \mathbb{N}} \mathcal{H}_k(G).$$

3.3. *A regularity result.* We summarize the results we need on regularity in the following.

PROPOSITION 3.1. *Let  $\mu \in \mathcal{M}_c(G)$  with  $\hat{\mu}(\pi) = c_\pi I_\pi$  for all  $\pi \in \hat{G}$ .*

(1) *The measure  $\mu$  has a square-integrable density if and only if*

$$(3.9) \quad \sum_{\pi \in \hat{G}} d_\pi^2 |c_\pi|^2 < \infty.$$

(2) *The measure  $\mu$  has a continuous density if*

$$(3.10) \quad \sum_{\pi \in \hat{G}} d_\pi^2 |c_\pi| < \infty.$$

(3) *The measure  $\mu$  has a  $C^k$  density if*

$$(3.11) \quad \sum_{\pi \in \hat{G}} d_\pi^2 (1 + \kappa_\pi)^p |c_\pi|^2 < \infty,$$

where  $p > k + \frac{d}{2}$ .

PROOF. (1) follows from Theorem 3.1 in [3] and (2) from Proposition 6.6.1 in [11]. (3) is a straightforward consequence of the Sobolev embedding theorem. □

3.4. *Examples.* Now we consider different families of measures and apply Proposition 3.1. In all cases, we take  $\mu \in \text{CID}_{\mathbb{R}}(G)$  so that  $c_\pi = e^{-\eta(\kappa_\pi^{1/2})}$ .

3.4.1. *The case where there is a nontrivial Gaussian component.* We say that  $\mu$  has a *nontrivial Gaussian component* if  $\eta$  is such that  $\sigma > 0$  in (2.4). We can obtain many examples of such measures by defining  $\mu = \mu_1 * \mu_2$  where  $\mu_1$  is Gaussian and  $\mu_2$  is of compound Poisson type or is obtained by subordination as in Section 2. We show that  $\mu$  has a  $C^\infty$ -density for all  $\sigma > 0$ . To prove this, we

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<sup>1</sup>Note that we are here using the analyst’s convention for Fourier transforms of functions which, as usual, is not quite consistent with the probabilist’s convention for Fourier transforms of measures.

use (3.11), (3.2), (3.5) and (3.6) and the fact that  $\eta(u) \geq \frac{1}{2}\sigma^2 u^2$  for all  $u \in \mathbb{R}$  to see that for all  $k \in \mathbb{N}$

$$\begin{aligned} \sum_{\pi \in \widehat{G}} d_\pi^2 (1 + \kappa_\pi)^k c_\pi^2 &\leq \sum_{\pi \in \widehat{G}} d_\pi^2 (1 + \kappa_\pi)^k \exp\{-\sigma^2 \kappa_\pi\} \\ &\leq M \sum_{\lambda \in P \cap D} |\lambda|^{2m} (1 + |\lambda|^2)^k \exp\{-\sigma^2 |\lambda|^2\} \\ &\leq K_1 \sum_{n \in \mathbb{Z}^r} \|n\|^{2m} (1 + \|n\|^2)^k \exp\{-K_2 \|n\|^2\} \\ &= K_1 \sum_{j=0}^\infty a(j) j^m (1 + j)^k \exp\{-K_2 j\} \\ &\leq K_1 \sum_{j=0}^\infty j^m (2\sqrt{j} + 1)^r (1 + j)^k \exp\{-K_2 j\} < \infty, \end{aligned}$$

where  $M, K_1, K_2 > 0$ ,  $a(j) := \#\{n \in \mathbb{Z}^r; \|n\|^2 = j\}$  and we use the fact that  $a(j) \leq (2\sqrt{j} + 1)^r$  for all  $j \in \mathbb{N}$ .<sup>2</sup>

3.4.2. *Stable-type densities.* Take  $c_\pi = \exp\{-b^\alpha \kappa_\pi^{\alpha/2}\}$  with  $0 < \alpha < 2$ . Again we show that the densities are  $C^\infty$ . Indeed arguing as above we have

$$\begin{aligned} \sum_{\pi \in \widehat{G}} d_\pi^2 (1 + \kappa_\pi)^k c_\pi^2 &\leq M \sum_{\lambda \in P \cap D} |\lambda|^{2m} (1 + |\lambda|^2)^k \exp\{-2b^\alpha |\lambda|^\alpha\} \\ &\leq K_3 \sum_{j=0}^\infty j^m (2\sqrt{j} + 1)^r (1 + j)^k \exp\{-K_4 j^{\alpha/2}\}, \end{aligned}$$

where  $K_3, K_4 > 0$ . To see that the series converges, it is sufficient to show that  $\sum_{n=1}^\infty n^\kappa e^{-n^\beta}$  converges for all  $\kappa \geq 0$  where  $0 < \beta < 1$ . This follows by comparison with  $\sum_{n=1}^\infty \frac{1}{n^2}$  since  $\lim_{n \rightarrow \infty} n^{\kappa+2} e^{-n^\beta} = \lim_{x \rightarrow \infty} x^{(\kappa+2)/\beta} e^{-x} = 0$ .

3.4.3. *Relativistic Schrödinger density.* Here we have

$$\begin{aligned} \sum_{\pi \in \widehat{G}} d_\pi^2 (1 + \kappa_\pi)^k c_\pi^2 &\leq e^{2m} \sum_{\lambda \in P \cap D} |\lambda|^{2m} |1 + |\lambda|^2|^k e^{-2\sqrt{m^2 + |\lambda|^2}} \\ &\leq e^{2m} \sum_{\lambda \in P \cap D} |\lambda|^{2m} |1 + |\lambda|^2|^k e^{-2|\lambda|} < \infty, \end{aligned}$$

so by the result of Section 3.4.2 (with  $\alpha = 1$ ) this case also yields a  $C^\infty$  density.

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<sup>2</sup>Of course in the pure Gaussian case, smoothness of the density is well known and can be proved using pde techniques.

3.4.4. *Laplace density.* In this case, we take  $c_\pi = (1 + \beta^2 \kappa_\pi)^{-1}$ . We restrict ourselves to seeking an  $L^2$ -density. Applying (3.9), we use (3.2), (3.5) and (3.7) to obtain

$$\sum_{\pi \in \widehat{G - \{\delta\}}} \frac{d_\pi^2}{(1 + \beta^2 \kappa_\pi)^2} \leq \frac{N}{\beta^2} \sum_{\lambda \in P \cap D - \{0\}} \frac{|\lambda|^{2m}}{|\lambda|^4} = \frac{N}{\beta^2} \zeta(2 - m),$$

where  $N > 0$ .

By Sugiura’s convergence result for  $\zeta(s)$ , we see that a sufficient condition for convergence is  $m < 2 - \frac{1}{2}$ . Hence by (3.3),  $\dim(G) \in \{1, 2, 3\}$ . So for example, the Laplace distribution has a square-integrable density on the groups  $SO(3)$ ,  $SU(2)$  and  $Sp(1)$ , each of which has dimension 3 and rank 1.

**4. Pseudo-differential operator representations.** In this section,  $G$  is an arbitrary compact group. Let  $(\mu_t, t \geq 0)$  be a vaguely continuous convolution semigroup of probability measures on  $G$  wherein  $\mu_0 = \delta_e$ . It then follows that  $\mu_t$  is infinitely divisible for each  $t \geq 0$ . We let  $(T_t, t \geq 0)$  be the associated  $C_0$  semigroup on  $C(G)$  (*Hunt semigroup*) defined by

$$T_t f(\sigma) = \int_G f(\sigma \tau) \mu_t(d\tau)$$

for all  $t \geq 0$ . Necessary and sufficient conditions for a densely defined linear operator to extend to the infinitesimal generator of  $(T_t, t \geq 0)$  were found by Hunt [20] (see [29] for a modern treatment) in the case of a Lie group and generalized by Born [9] to locally compact groups.

$(T_t, t \geq 0)$  extends to a positivity-preserving contraction semigroup on  $L^2(G) := L^2(G, \mathbb{C})$  and from now on we will always work with this extended action. Our aim in this section is to represent the semigroup and its generator as pseudo-differential operators using Peter–Weyl theory (cf. [33]). If  $A \in M_n(\mathbb{C})$ , we define its Hilbert–Schmidt norm by  $\|A\|_{\text{hs}} := \text{tr}(AA^*)^{1/2}$ .

The celebrated Peter–Weyl theorem asserts that  $f \in L^2(G)$  has the associated Fourier series  $\sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\hat{f}(\pi)\pi)$  and we will make frequent use of Plancherel’s theorem in this context which tells us that

$$\|f\|^2 = \sum_{\pi \in \widehat{G}} d_\pi \|\hat{f}(\pi)\|_{\text{hs}}^2.$$

We also need the corresponding Parseval identity:

$$\langle f, g \rangle = \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\hat{f}(\pi)\hat{g}(\pi)^*)$$

for  $f, g \in L^2(G)$ .

We say that a densely defined linear operator  $S$  on  $L^2(G)$  has a (*simple*) *pseudo-differential operator representation* if for each  $\pi \in \widehat{G}$  there exists a  $d_\pi \times d_\pi$  matrix  $\sigma_S(\pi)$  such that

$$\widehat{Sf}(\pi) = \sigma_S(\pi) \hat{f}(\pi)$$

for all  $f \in \text{Dom}(S)$  and all  $\pi \in \widehat{G}$ .

We call  $\sigma_S$  the *symbol* of the operator  $S$ . The word “simple” is included as we do not require the symbol be a function defined on  $G \times \widehat{G}$  as in [33]. Indeed such a more complicated class of symbols will be associated with representations of more general Feller–Markov semigroups (see [23] for the case where  $G$  is the real numbers).

**THEOREM 4.1.** *For each  $t \geq 0$ ,  $T_t$  is a pseudo-differential operator with symbol  $\widehat{\mu}_t(\pi)$  at  $\pi \in \widehat{G}$ .*

**PROOF.** For each  $\rho \in G$  let  $R_\rho$  denote right translation so that  $R_\rho f(\sigma) = f(\sigma\rho)$  for each  $f \in L^2(G), \sigma \in G$ . We will need the fact that  $\widehat{R_\rho f}(\pi) = \pi(\rho)\widehat{f}(\pi)$  for each  $\pi \in \widehat{G}$ .

By Fubini’s theorem and the Parseval identity,

$$\begin{aligned} \|T_t f\|^2 &= \int_G \int_G \int_G f(\sigma\tau)\overline{f(\sigma\rho)}\mu_t(d\tau)\mu_t(d\rho) d\sigma \\ &= \int_G \int_G \int_G R_\tau f(\sigma)\overline{R_\rho f(\sigma)} d\sigma \mu_t(d\tau)\mu_t(d\rho) \\ &= \int_G \int_G \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\pi(\tau)\widehat{f}(\pi)\widehat{f}(\pi)^*\pi(\rho)^*)\mu_t(d\tau)\mu_t(d\rho). \end{aligned}$$

We can use Fubini’s theorem to interchange summation and integration since by the contraction property of  $T_t$ ,

$$\int_G \int_G \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\pi(\tau)\widehat{f}(\pi)\widehat{f}(\pi)^*\pi(\rho)^*)\mu_t(d\tau)\mu_t(d\rho) \leq \|f\|^2.$$

Hence, we have

$$\begin{aligned} \|T_t f\|^2 &= \sum_{\pi \in \widehat{G}} d_\pi \text{tr}\left(\int_G \pi(\tau)\mu_t(d\tau)\widehat{f}(\pi)\widehat{f}(\pi)^* \int_G \pi(\rho)^*\mu_t(d\rho)\right) \\ &= \sum_{\pi \in \widehat{G}} d_\pi \|\widehat{\mu}_t(\pi)\widehat{f}(\pi)\|_{\text{hs}}^2, \end{aligned}$$

and the result follows.  $\square$

Let  $\mathcal{A}$  be the infinitesimal generator of  $(T_t, t \geq 0)$ . We here use the fact that for each  $t \geq 0, \pi \in \widehat{G}, \widehat{\mu}_t(\pi) = e^{t\mathcal{L}_\pi}$  where  $\mathcal{L}_\pi$  is a  $d_\pi \times d_\pi$  matrix (see [2, 18, 30] where an explicit “Lévy–Khintchine type” representation of  $\mathcal{L}_\pi$  can be found when  $G$  is a Lie group).

**THEOREM 4.2.**  *$\mathcal{A}$  is a pseudo-differential operator with symbol  $\mathcal{L}_\pi$  at  $\pi \in \widehat{G}$ .*

PROOF. For each  $f \in \text{Dom}(\mathcal{A})$ ,  $g \in L^2(G)$ , we have by Parseval’s identity and Theorem 4.1

$$\begin{aligned} \langle \mathcal{A}f, g \rangle &= \lim_{t \rightarrow 0} \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr} \left( \frac{(\hat{\mu}_t(\pi) - I_\pi)}{t} \hat{f}(\pi) \hat{g}(\pi)^* \right) \\ &= \lim_{t \rightarrow 0} \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr} \left( \frac{(e^{t\mathcal{L}_\pi} - I_\pi)}{t} \hat{f}(\pi) \hat{g}(\pi)^* \right). \end{aligned}$$

Now fix  $\pi' \in \widehat{G}$  and let  $g \in \mathcal{M}_{\pi'}$  where  $\mathcal{M}_{\pi'}$  is the subspace of  $L^2(G)$  generated by mappings of the form  $\sigma \rightarrow \langle \pi'(\sigma)u, v \rangle$  for  $u, v \in V_{\pi'}$ . It follows from the Peter–Weyl theorem that  $\hat{g}(\pi) = 0$  if  $\pi \neq \pi'$  and so

$$\begin{aligned} \langle \mathcal{A}f, g \rangle &= d_{\pi'} \lim_{t \rightarrow 0} \operatorname{tr} \left( \frac{(e^{t\mathcal{L}_{\pi'}} - I_{\pi'})}{t} \hat{f}(\pi') \hat{g}(\pi')^* \right) \\ &= d_{\pi'} \operatorname{tr}(\mathcal{L}_{\pi'} \hat{f}(\pi') \hat{g}(\pi')^*) \\ &= \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(\mathcal{L}_\pi \hat{f}(\pi) \hat{g}(\pi)^*). \end{aligned}$$

The required result follows from the Parseval identity since (by the Peter–Weyl theorem)  $L^2(G)$  is the closure of  $\bigoplus_{\pi \in \widehat{G}} \mathcal{M}_\pi$  (see, e.g., [11], page 108).  $\square$

For completeness, we will also give the pseudo-differential operator representation of the resolvent  $R_\lambda := (\lambda I - \mathcal{A})^{-1}$  for  $\lambda > 0$ .

PROPOSITION 4.1. *For each  $\lambda > 0$ ,  $R_\lambda$  is a pseudo-differential operator having symbol  $(\lambda I_\pi - \mathcal{L}_\pi)^{-1}$  at  $\pi \in \widehat{G}$ .*

PROOF. First, note that  $(\lambda I_\pi - \mathcal{L}_\pi)^{-1}$  always exists since the eigenvalues of the matrix  $\mathcal{L}_\pi$  have nonpositive real parts. We use the fact that for all  $\lambda > 0$ ,  $R_\lambda = \int_0^\infty e^{-\lambda t} T_t dt$ . Then for all  $f, g \in L^2(G)$ , by Theorem 4.1

$$\langle R_\lambda f, g \rangle = \sum_{\pi \in \widehat{G}} d_\pi \int_0^\infty e^{-\lambda t} \operatorname{tr}(e^{t\mathcal{L}_\pi} \hat{f}(\pi) \hat{g}(\pi)^*) dt.$$

The result follows from Fubini’s theorem using the estimate

$$\sum_{\pi \in \widehat{G}} d_\pi \int_0^\infty e^{-\lambda t} |\operatorname{tr}(e^{t\mathcal{L}_\pi} \hat{f}(\pi) \hat{g}(\pi)^*)| dt \leq \frac{1}{\lambda} \|f\| \|g\|,$$

which is obtained by routine computations.  $\square$

Now we assume that  $\mu_1 \in \text{CID}_\mathbb{R}(G)$ . It follows that  $\mu_t \in \text{CID}_\mathbb{R}(G)$  for all  $t \geq 0$  and that  $\hat{\mu}_t(\pi) = e^{-t\eta(\kappa_\pi^{1/2})} I_\pi$  for each  $\pi \in \widehat{G}$  for some negative definite function  $\eta$  and  $\mathcal{A}$  has symbol whose value at  $\pi \in \widehat{G}$  is  $-\eta(\kappa_\pi^{1/2}) I_\pi$ .

**THEOREM 4.3.** *If  $G$  is a compact Lie group and  $\mu_1 \in \text{CID}_{\mathbb{R}}(G)$  then for all  $p \geq 2$ ,  $\mathcal{H}_p(G) \subseteq \text{Dom}(\mathcal{A})$  and  $\mathcal{A}$  is a bounded linear operator from  $\mathcal{H}_p(G)$  to  $\mathcal{H}_{p-2}(G)$ .*

**PROOF** (Cf. [33], Theorem 10.81, pages 571–572). We will make use of the fact that there exists  $K > 0$  such that  $|\eta(u)| \leq K(1 + |u|^2)$  for all  $u \in \mathbb{R}$  (see, e.g., [4], page 31). By Theorem 4.2 and (3.8) for each  $f \in \mathcal{H}_p(G)$

$$\begin{aligned} \|\mathcal{A}f\|_{p-2}^2 &= \sum_{\pi \in \widehat{G}} d_{\pi}(1 + \kappa_{\pi})^{p-2} \|\mathcal{L}_{\pi} \hat{f}(\pi)\|_{\text{hs}}^2 \\ &= \sum_{\pi \in \widehat{G}} d_{\pi}(1 + \kappa_{\pi})^{p-2} |\eta(\kappa_{\pi}^{1/2})|^2 \|\hat{f}(\pi)\|_{\text{hs}}^2 \\ &\leq K \sum_{\pi \in \widehat{G}} d_{\pi}(1 + \kappa_{\pi})^p \|\hat{f}(\pi)\|_{\text{hs}}^2 \\ &= K \|f\|_p^2. \end{aligned}$$

In particular, it follows that  $\|\mathcal{A}f\|^2 < \infty$  and so  $f \in \text{Dom}(\mathcal{A})$ .  $\square$

**5. Transition densities for convolution semigroups.** In this section, we continue to work with the Hunt semigroup  $(T_t, t \geq 0)$  acting on the space  $L^2(G)$  that is associated to the convolution semigroup of measures  $(\mu_t, t \geq 0)$  on a compact group  $G$ . Let  $L_c^2(G) := \{f \in L^2(G), f(g\sigma g^{-1}) = f(\sigma) \text{ for all } \sigma, g \in G\}$ . It is well known that  $\{\chi_{\pi}, \pi \in \widehat{G}\}$  is a complete orthonormal basis for  $L_c^2(G)$  (see, e.g., [11], Proposition 6.5.3, page 117).

**PROPOSITION 5.1.** *If  $\mu_t \in M_c(G)$  for some  $t \geq 0$ , then  $T_t(L_c^2(G)) \subseteq L_c^2(G)$ .*

**PROOF.** For all  $\sigma, g \in G, f \in L_c^2(G)$ , we have

$$\begin{aligned} T_t f(g\sigma g^{-1}) &= \int_G f(g\sigma g^{-1}\tau) \mu_t(d\tau) \\ &= \int_G f(g\sigma g^{-1}\tau) \mu_t(dg^{-1}\tau g) \\ &= \int_G f(g\sigma \tau g^{-1}) \mu_t(d\tau) \\ &= T_t f(\sigma). \end{aligned} \quad \square$$

Now suppose that  $\mu_t \in M_{cs}(G)$  for all  $t \geq 0$ . This implies in particular that  $(T_t, t \geq 0)$  is self-adjoint in  $L^2(G)$  and hence in  $L_c^2(G)$  (see [5, 27]). By Proposition 2.1, we also have that there exists  $\alpha_{\pi} \leq 0$  for each  $\pi \in \widehat{G}$  such that  $\hat{\mu}_t(\pi) = e^{t\alpha_{\pi}} I_{\pi}$ .

**THEOREM 5.1.** *If  $\mu_t \in M_{cs}(G)$  for all  $t \geq 0$ , then  $\{\chi_\pi, \pi \in \widehat{G}\}$  is a complete set of eigenvectors for the action of  $T_t$  on  $L_c^2(G)$  and*

$$(5.1) \quad T_t \chi_\pi = e^{t\alpha_\pi} \chi_\pi$$

for all  $\pi \in \widehat{G}, t \geq 0$ .

**PROOF.** For all  $\sigma \in G, t \geq 0$ ,

$$\begin{aligned} T_t \chi_\pi(\sigma) &= \int_G \chi_\pi(\sigma\tau) \mu_t(d\tau) \\ &= \int_G \text{tr}(\pi(\sigma)\pi(\tau)) \mu_t(d\tau) \\ &= \text{tr}\left(\pi(\sigma) \int_G \pi(\tau) \mu_t(d\tau)\right) \\ &= e^{t\alpha_\pi} \chi_\pi(\sigma). \end{aligned} \quad \square$$

It is shown in [5] that  $T_t$  is trace-class for  $t > 0$  if and only if  $\mu_t$  has a square-integrable density. In this case, we have

$$(5.2) \quad \text{tr}(T_t) = \sum_{\pi \in \widehat{G}} e^{t\alpha_\pi}$$

for  $t > 0$ , where  $\text{tr}$  denote the trace in the Hilbert space  $L_c^2(G)$ .

From now on, we assume that for  $t > 0, \mu_t \in \mathcal{M}_{cs}(G)$  has a density  $k_t \in L_c^2(G, \mathbb{R})$ . We define the *transition density*  $h_t \in L^2(G \times G, \mathbb{R})$  by

$$h_t(\sigma, \tau) := k_t(\sigma^{-1}\tau)$$

for each  $t > 0, \sigma, \tau \in G$ . Indeed  $h_t$  is precisely the transition probability density at time  $t$  of a  $G$ -valued Lévy process whose law at time  $t$  is  $k_t$ .

Note that for each  $\sigma \in G, (t, \rho) \rightarrow h_t(\sigma, \rho)$  satisfies the backward equation (in the distributional sense)

$$\frac{\partial h_t}{\partial t}(\sigma, \rho) = \mathcal{A}h_t(\sigma, \rho),$$

with  $h_t(\sigma, \rho) \rightarrow \delta_\sigma(\rho)$  as  $t \rightarrow 0$ . For example, if  $(\mu_t, t \geq 0)$  is the Brownian motion semigroup which is characterized by  $\hat{\mu}_t(\pi) = e^{-(t/2)\kappa_\pi} I_\pi$  for each  $\pi \in \widehat{G}$ , then  $h_t$  is the well-known heat kernel and for this reason we will sometimes refer to our more general  $h_t$  as the *transition kernel*.

**THEOREM 5.2.** *For each  $t > 0$ :*

(1)

$$(5.3) \quad \int_G h_t(g^{-1}\sigma, \rho g^{-1}) dg = \sum_{\pi \in \widehat{G}} e^{t\alpha_\pi} \overline{\chi_\pi(\sigma)} \chi_\pi(\rho)$$

for all  $\sigma, \rho \in G$ .

(2)

$$(5.4) \quad \text{tr}(T_t) = \int_G \int_G k_t(\rho^{-1}g\rho g^{-1}) dg d\rho.$$

PROOF.

(1) By (3.1)

$$(5.5) \quad h_t(\sigma, \rho) = k_t(\sigma^{-1}\rho) = \sum_{\pi \in \widehat{G}} d_\pi e^{t\alpha_\pi} \chi_\pi(\sigma^{-1}\rho),$$

and so

$$\begin{aligned} \int_G h_t(g^{-1}\sigma, \rho g^{-1}) dg &= \sum_{\pi \in \widehat{G}} d_\pi e^{t\alpha_\pi} \int_G \chi_\pi(\sigma^{-1}g\rho g^{-1}) dg \\ &= \sum_{\pi \in \widehat{G}} e^{t\alpha_\pi} \overline{\chi_\pi(\sigma)} \chi_\pi(\rho), \end{aligned}$$

by Proposition 6.5.2 in [11] (page 116). The interchange of integral and sum is justified by Fubini’s theorem since

$$\sum_{\pi \in \widehat{G}} d_\pi e^{t\alpha_\pi} \int_G |\chi_\pi(\sigma^{-1}g\rho g^{-1})| dg \leq \sum_{\pi \in \widehat{G}} d_\pi^2 e^{t\alpha_\pi} < \infty$$

by (3.9) since  $k_t \in L^2(G)$  for each  $t > 0$ . Here, we have used the crude estimate  $\sup_{\sigma \in G} |\chi_\pi(\sigma)| \leq d_\pi$ .

(2) Put  $\rho = \sigma$  in (5.3) and then integrate both sides with respect to  $\sigma$  using the fact that  $\int_G |\chi_\pi(\sigma)|^2 d\sigma = 1$ . The result then follows from (5.2). Note that the interchange of integral and summation is justified by Fubini’s theorem using a similar argument to that presented in (1).  $\square$

COROLLARY 5.1. *If  $k_t$  is continuous for each  $t > 0$ ,*

$$(5.6) \quad k_t(e) = \sum_{\pi \in \widehat{G}} d_\pi^2 e^{t\alpha_\pi}.$$

PROOF. Put  $\sigma = \rho$  in (5.5) [or argue directly from (3.1)].  $\square$

We now work in the Hilbert space  $L^2(G)$  and we use  $\text{Tr}$  to denote the trace in this Hilbert space. By the Peter–Weyl theorem,  $\{d_\pi^{1/2} \pi_{ij}, 1 \leq i, j \leq d_\pi, \pi \in \widehat{G}\}$  is a complete orthonormal basis for  $L^2(G)$  where  $\pi_{ij}$  denotes the coordinate function  $\pi_{ij}(\sigma) := \pi(\sigma)_{ij}$ , for each  $\sigma \in G$ .

The following two results are well known for the heat kernel (see, e.g., Chapter 12 of [14]). Here we extend them to more general Hunt semigroups.



THEOREM 5.3.

(1) For each  $t \geq 0$ , the set  $\{\pi_{ij}, 1 \leq i, j \leq d_\pi, \pi \in \widehat{G}\}$  is a complete orthogonal set of eigenvectors for  $T_t$  and

$$(5.7) \quad T_t \pi_{ij} = e^{t\alpha_\pi} \pi_{ij}$$

for each  $1 \leq i, j \leq d_\pi, \pi \in \widehat{G}$ .

(2) If  $k_t$  is continuous for each  $t > 0$ ,

$$(5.8) \quad k_t(e) = \text{Tr}(T_t).$$

PROOF.

(1) For all  $\sigma \in G$

$$\begin{aligned} T_t \pi_{ij}(\sigma) &= \int_G \pi_{ij}(\sigma\tau) \mu_t(d\tau) \\ &= \sum_{k=1}^{d_\pi} \pi_{ik}(\sigma) \int_G \pi_{kj}(\tau) \mu_t(d\tau) \\ &= \sum_{k=1}^{d_\pi} \pi_{ik}(\sigma) \hat{\mu}_t(\pi)_{kj} \\ &= e^{t\alpha_\pi} \pi_{ij}(\sigma). \end{aligned}$$

(2) Since each eigenvalue  $e^{t\alpha_\pi}$  has multiplicity  $d_\pi^2$  on the closed subspace of  $L^2(G)$  spanned by  $\{\pi_{ij}, 1 \leq i, j \leq d_\pi\}$  it is clear that

$$\text{Tr}(T_t) = \sum_{\pi \in \widehat{G}} d_\pi^2 e^{t\alpha_\pi},$$

and the result follows from (5.6).  $\square$

The next result can be deduced directly from (3.1). We give a direct proof to make the paper more self-contained. Note that results of this type are well known for Markov processes taking values in compact metric spaces and having suitably square-integrable transition probabilities (see Theorem 6.4 in [15]).

THEOREM 5.4. If  $k_t$  is continuous for each  $t > 0$ ,

$$(5.9) \quad h_t(\sigma, \rho) = \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} d_\pi e^{t\alpha_\pi} \overline{\pi_{ij}(\sigma)} \pi_{ij}(\rho)$$

for all  $\sigma, \rho \in G$ .

PROOF. For each  $\sigma \in G$  let  $L_\sigma$  denote left translation so that  $L_\sigma f(\rho) = f(\sigma^{-1}\rho)$  for each  $f \in L^2(G)$ ,  $\rho \in G$ . By Fourier expansion and using (5.7),

$$\begin{aligned} L_\sigma k_t &= \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} d_\pi \langle L_\sigma k_t, \pi_{ij} \rangle \pi_{ij} \\ &= \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} d_\pi \left( \int_G k_t(\sigma^{-1}\tau) \overline{\pi_{ij}(\tau)} d\tau \right) \pi_{ij} \\ &= \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} d_\pi \left( \int_G k_t(\tau) \overline{\pi_{ij}(\sigma\tau)} d\tau \right) \pi_{ij} \\ &= \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} d_\pi T_t \overline{\pi_{ij}(\sigma)} \pi_{ij} \\ &= \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} d_\pi e^{t\alpha_\pi} \overline{\pi_{ij}(\sigma)} \pi_{ij}. \end{aligned}$$

Since  $\sup_{\sigma, \rho \in G} |\sum_{i,j=1}^{d_\pi} \overline{\pi_{ij}(\sigma)} \pi_{ij}(\rho)| = \sup_{\sigma, \rho \in G} |\text{tr}(\pi(\sigma^{-1}\rho))| \leq d_\pi$ , we deduce uniform convergence of the series from (3.9) and so for all  $\rho \in G$

$$h_t(\sigma, \rho) = L_\sigma k_t(\rho) = \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} d_\pi e^{t\alpha_\pi} \overline{\pi_{ij}(\sigma)} \pi_{ij}(\rho).$$

□

COROLLARY 5.2. *If  $k_t$  is continuous for each  $t > 0$ ,*

$$k_t(e) \geq \int_G \int_G k_t(\rho^{-1}g\rho g^{-1}) dg d\rho$$

*with equality if and only if  $G$  is Abelian.*

PROOF. The inequality follows from (5.2), (5.4), (5.6) and (5.8). If  $G$  is Abelian, then equality is obvious. If  $G$  is non-Abelian, we must have  $d_\pi > 1$  for at least one  $\pi \in \widehat{G}$  and then it is clear that strict inequality holds. □

**6. Small time asymptotics of densities.** Assume that  $G$  is a compact semi-simple Lie group. We would like to obtain an asymptotic expansion for  $k_t(e)$  as  $t \rightarrow 0$ . We assume that  $\mu_t \in \text{CID}_{\mathbb{R}}(G)$  for each  $t \geq 0$ . In this case, if  $k_t$  is continuous for  $t > 0$ , we may follow the arguments on page 106 of [14] to obtain

$$\begin{aligned} k_t(e) &= \text{Tr}(T_t) = \sum_{\lambda \in P \cap D} d_{\lambda+\rho}^2 \exp\{-t\eta((|\lambda + \rho|^2 - |\rho|^2)^{1/2})\} \\ (6.1) \quad &= \sum_{\lambda \in P \cap D} d_\lambda^2 \exp\{-t\eta((|\lambda|^2 - |\rho|^2)^{1/2})\} \end{aligned}$$

$$= \frac{1}{|W|} \sum_{\lambda \in P} d_\lambda^2 \exp\{-t\eta((|\lambda|^2 - |\rho|^2)^{1/2})\},$$

where  $d_\lambda$  denotes the dimension of the representation space with highest weight  $\lambda$  and  $|W|$  is the order of the Weyl group of  $G$ .

When  $\eta(u) = \frac{|u|^2}{2}$ ,  $k_t$  is the density generating the heat kernel and it is known that as  $t \rightarrow 0$ ,

$$(6.2) \quad k_t(e) \sim Ct^{-(\dim(G))/2} e^{t|\rho|^2}$$

(see, e.g., [14], page 109), where  $C > 0$ .

We will examine the case where  $\eta$  is the characteristic exponent of a symmetric Cauchy distribution so that  $\eta(u) = \sigma|u|$  for all  $u \in \mathbb{R}$ , where  $\sigma > 0$ .

EXAMPLE 1.  $G = \Pi^d$  where  $\Pi := \mathbb{R}/\mathbb{Z}$ . In this case  $\widehat{G} = \mathbb{Z}^d$ , each  $d_\pi = 1$  and for each  $\pi = n \in \mathbb{Z}^d$ ,  $\kappa_\pi = 4\pi^2|n|^2$  where  $n^2 = n_1^2 + \dots + n_d^2$  for  $n = (n_1, \dots, n_d)$ . The equation (6.1) then takes the form

$$k_t(e) = \sum_{n \in \mathbb{Z}^d} e^{-2\pi t\sigma|n|}.$$

When  $d = 1$ , we easily calculate

$$\begin{aligned} k_t(e) &= 1 + 2 \sum_{n=1}^{\infty} e^{-2\pi t\sigma n} \\ &= \coth(\pi\sigma t) \sim \frac{1}{\pi\sigma t} \quad \text{as } t \rightarrow 0. \end{aligned}$$

When  $d > 1$ , we apply the Poisson summation formula to obtain

$$\begin{aligned} k_t(e) &= \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \sum_{m \in \mathbb{Z}^d} \frac{\sigma t}{(\sigma^2 t^2 + |m|^2)^{(d+1)/2}} \\ &\sim \frac{\Gamma((d+1)/2)}{\sigma^d \pi^{(d+1)/2}} \frac{1}{t^d} \quad \text{as } t \rightarrow 0. \end{aligned}$$

EXAMPLE 2.  $G = \text{SU}(2)$ . In this case,  $\widehat{G} \cong \mathbb{Z}_+$  with  $d_n = n + 1$  and for each  $n \in \mathbb{Z}_+$ ,  $\kappa_n = n(n + 2)$ . Hence,

$$\begin{aligned} k_t(e) &= \sum_{n=0}^{\infty} (n+1)^2 e^{-t\sigma\sqrt{n(n+2)}} \\ &= \sum_{n=0}^{\infty} (n+1)^2 e^{-t\sigma\sqrt{(n+1)^2-1}} \\ &= \sum_{m=1}^{\infty} m^2 e^{-t\sigma\sqrt{m^2-1}}. \end{aligned}$$

From this, we get the easy estimate

$$(6.3) \quad e^{-t\sigma} \sum_{m=1}^{\infty} m^2 e^{-t\sigma m} \leq k_t(e) \leq e^{t\sigma} \sum_{m=1}^{\infty} m^2 e^{-t\sigma m}.$$

Now define  $g(t) := \sum_{m=1}^{\infty} e^{-t\sigma m}$  for  $t \in (0, \infty)$ . The function  $g$  is  $C^\infty$  and we have

$$\begin{aligned} \sum_{m=1}^{\infty} m^2 e^{-t\sigma m} &= \frac{1}{\sigma^2} \frac{d^2}{dt^2} g(t) \\ &= \frac{1}{\sigma^2} \frac{d^2}{dt^2} \left( \frac{e^{-t\sigma}}{1 - e^{-t\sigma}} \right) \\ &= \frac{e^{-t\sigma}}{(1 - e^{-t\sigma})^2} \coth\left(\frac{\sigma t}{2}\right), \end{aligned}$$

and hence we conclude that

$$k_t(e) \sim \frac{2}{\sigma^3 t^3} \quad \text{as } t \rightarrow 0.$$

This should be compared with the usual heat kernel where the following very precise asymptotic expansion is known (see [13], Proposition 2.3, page 662):

$$k_t(e) \sim 32\sqrt{2}\pi^2(4\pi\sigma t)^{-3/2}e^{(\sigma t)/8},$$

so the leading term has the slower decay

$$k_t(e) \sim 32\sqrt{2}\pi^2(4\pi\sigma)^{-3/2}t^{-3/2}.$$

It is not difficult to verify that the relativistic Schrödinger semigroup on  $SU(2)$  with mass parameter  $m$ , for which  $\eta(u) = (m^2 + u^2)^{1/2} - m$  for  $u \in \mathbb{R}$ , has exactly the same short time asymptotics as the Cauchy semigroup.

EXAMPLE 3.  $G = S0(3)$ . Here we have  $\widehat{G} \cong \mathbb{Z}_+$  with  $d_n = 2n + 1$  and for each  $n \in \mathbb{Z}_+$ ,  $\kappa_n = n(n + 1)$ . So we obtain

$$\begin{aligned} k_t(e) &= \sum_{n=0}^{\infty} (2n + 1)^2 e^{-t\sigma\sqrt{n(n+1)}} \\ &= \sum_{m \text{ odd}} m^2 e^{-t(\sigma/2)\sqrt{m^2-1}} \quad \text{as } t \rightarrow 0. \end{aligned}$$

Using the results obtained in Example 2, we find that

$$k_t(e) \sim \frac{8}{\sigma^3 t^3} \quad \text{as } t \rightarrow 0.$$

Based on these calculations, we conjecture that  $k_t(e) \sim Ct^{-d}$  for the Cauchy process on an arbitrary compact semisimple Lie group of dimension  $d$ . It is also

tempting to further conjecture that if  $k_t$  is associated to an arbitrary  $\alpha$ -stable process so that  $\eta(u) = |u|^\alpha$  where  $0 < \alpha \leq 2$ , then  $k_t(e) \sim Ct^{-d/\alpha}$  as  $t \rightarrow 0$ . This is consistent with the known behavior of densities of symmetric stable processes in Euclidean space (see, e.g., [8]) where it essentially follows by scaling arguments. We remind the reader that this technology is not available on compact groups (see, e.g., Theorem 2.2. in [1], page 117).

Now let  $N(\lambda)$  denote the number of eigenvalues of  $-\mathcal{A}$  that do not exceed  $\lambda$  and note that for all  $t > 0$ ,

$$\mathrm{Tr}(T_t) = \int_0^\infty e^{-t\lambda} dN(\lambda).$$

If the above conjecture holds then by Karamata's Tauberian theorem we have

$$N(\lambda) \sim \frac{C\lambda^{d/\alpha}}{\Gamma(1 + \frac{d}{\alpha})} \quad \text{as } \lambda \rightarrow \infty$$

(cf. Theorem 2.3 in [7]).

So far we know that this eigenvalue asymptotics is valid when  $\alpha = 2$ , and when  $\alpha = 1$  with  $G = \Pi^d$ ,  $G = \mathrm{SU}(2)$  and  $G = \mathrm{SO}(3)$ .

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