

T. E. HARRIS'S CONTRIBUTIONS TO RECURRENT MARKOV PROCESSES AND STOCHASTIC FLOWS

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This is a brief survey of T. E. Harris's work on recurrent Markov processes and on stochastic flows, and of some more recent work in these fields.

1. Introduction. I was a colleague of Ted Harris at USC, first as a sabbatical visitor for the academic year 1982–1983 and later as a regular faculty member from 1988 until his retirement. During this time, I spent many hours in discussions with Ted, and appreciated his insight into all areas of probability theory. I feel deeply grateful for the opportunity to have learned so much from him.

This paper covers two areas of Harris's work. Early in his career he wrote his seminal paper [16] on the existence and uniqueness of stationary measures for Markov processes satisfying a certain recurrence condition. Nowadays, this is called Harris recurrence, although Ted was far too modest a person to ever use the term himself. Section 2 contains a brief account of [16] together with some indication of later developments based on his idea.

Later in his career Harris became interested in stochastic flows. These can be regarded as random mappings of an entire state space into itself. Section 3 starts with some details of Harris's construction of a stochastic flow as a limit of random stirring processes. (Here, we see a transition from Poisson point processes and percolation theory into processes more frequently described by stochastic differential equations.) The rest of this section describes Harris's work on isotropic stochastic flows and coalescing stochastic flows, together with some more recent developments in these areas.

2. Harris recurrence. Consider a (time homogeneous) discrete time Markov process $\{X_n : n \geq 0\}$ taking values in a measurable space (S, \mathcal{B}) . For simplicity here, we will assume \mathcal{B} is separable, although many results are valid in a more general setting. Let $P(x, \cdot)$ denote the transition probability function, and let \mathbb{P}^x denote the law of the process with initial condition $X_0 = x$. A nonzero measure μ on (S, \mathcal{B}) is a *stationary measure* for the Markov process if

$$\mu(A) = \int_S \mathbb{P}(x, A) \mu(dx) \quad \text{for all } A \in \mathcal{B}.$$

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DEFINITION. For any set $A \in \mathcal{B}$, define the hitting time $T_A = \inf\{n \geq 1 : X_n \in A\}$. Let m be a σ -finite measure on (S, \mathcal{B}) . The Markov process is said to be m -recurrent if $\mathbb{P}^x(T_A < \infty) = 1$ for all $x \in S$ whenever $m(A) > 0$.

Nowadays, we say the Markov process is *Harris recurrent* if it is m -recurrent for some nonzero m .

THEOREM 2.1 (Harris [16]). *Assume the Markov process $\{X_n : n \geq 0\}$ is m -recurrent for some nonzero σ -finite measure m . Then there is a stationary measure μ for the Markov process. Moreover, μ is unique up to a constant multiplier, and m is absolutely continuous with respect to μ .*

In the case of a discrete state space S , this result was already known; see Derman [13] and Chung [7]. The usual definition of recurrence to a point $a \in S$ corresponds to m -recurrence with m taken to be the Dirac measure δ_a at the point a , and then the measure

$$\mu(B) = \sum_{n=1}^{\infty} \mathbb{P}^a(X_n \in B, T_{\{a\}} \geq n), \quad B \in \mathcal{B},$$

is stationary. In this setting, the times of visits of X_n to the recurrent point a are renewal times for the Markov process. However, with a nondiscrete state space S it will typically be necessary to look for recurrence to larger sets A than just singleton sets, and then the locations of the visits within A are of interest.

For fixed $A \in \mathcal{B}$ with $m(A) > 0$, denote by $T_A^{(k)}$ the times of visits to A , so that $T_A^{(1)} = T_A$ and $T_A^{(k+1)} = \inf\{n > T_A^{(k)} : X_n \in A\}$. The Markov process $Z_m = X_{T_A^{(m)}}$ with values in A is called the *process on A* . The transition probabilities for the process on A will be denoted $\mathbb{P}_A(x, \cdot)$. The following result, combining several lemmas in Harris [16], gives a characterization of the stationary measure showing how the discrete space result is extended.

PROPOSITION 2.1. *Assume the Markov process $\{X_n : n \geq 0\}$ is m -recurrent for some nonzero σ -finite measure m , and that $m(A) > 0$. Suppose that μ_A is a stationary probability measure for the process on A and that m is absolutely continuous with respect to μ_A on A . Define μ on (S, \mathcal{B}) by*

$$(2.1) \quad \mu(B) = \int_A \left(\mathbb{E}^x \sum_{n=1}^{T_A} 1_B(X_n) \right) \mu_A(dx), \quad B \in \mathcal{B}.$$

Then μ is a stationary measure for the Markov process $\{X_n : n \geq 0\}$ on S and m is absolutely continuous with respect to μ . Moreover, any other stationary measure is a multiple of μ .

It remains to consider the existence of the stationary probability μ_A for the process on A . At this point, we see the conflicting requirements for the set A : typically it has to be larger than a singleton so that $m(A) > 0$, but it should be chosen small enough so that the process on A has some good behavior.

Let $p^n(x, y)$ denote the absolutely continuous part of the n -step transition probability $P^n(x, \cdot)$ with respect to m . The following technical lemma of Harris is based in the idea that m -recurrence implies that for each x , the set $T(x) = \{y \in S : p^n(x, y) = 0 \text{ for all } n \geq 1\}$ must satisfy $m(T(x)) = 0$.

LEMMA 2.1. *Assume the Markov process $\{X_n : n \geq 0\}$ is m -recurrent for some nonzero σ -finite measure m . For any $r \in (0, 1)$, there exist $A \in \mathcal{B}$ with $0 < m(A) < \infty$, a positive integer k and a positive constant s such that for all $x \in A$,*

$$m\{y \in A : p^1(x, y) + \dots + p^k(x, y) > s\} > rm(A).$$

This enables Harris to obtain a Doeblin-like condition on the transition operator $R(x, \cdot) = (P_A(x, \cdot) + \dots + P_A^k(x, \cdot))/k$ and the existence of the stationary probability μ_A follows directly.

2.1. *Small sets.*

DEFINITION. A set $A \in \mathcal{B}$ is a *small set* if there exist a positive integer k , a probability measure ν and a constant $\beta > 0$ such that

$$(2.2) \quad \mathbb{P}^k(x, \cdot) \geq \beta \nu(\cdot) \quad \text{for all } x \in A.$$

The following result is a strengthening of Harris’s Lemma 2.1.

PROPOSITION 2.2 (Orey [40]). *Assume the Markov process $\{X_n : n \geq 0\}$ is m -recurrent for some nonzero σ -finite measure m . Every set $E \in \mathcal{B}$ such that $m(E) > 0$ contains a set $A \in \mathcal{B}$ such that $0 < m(A) < \infty$ and*

$$\inf\{p^k(x, y) : x, y \in A\} > 0$$

for some positive integer k .

COROLLARY 2.1. *The Markov process $\{X_n : n \geq 0\}$ is m -recurrent for some nonzero σ -finite measure m if and only if there exists a small set $A \in \mathcal{B}$ such that $\mathbb{P}^x(T_A < \infty) = 1$ for all $x \in S$.*

PROOF. Any set A with the property described in Proposition 2.2 is a small set, with $\nu(B) = m(A \cap B)/m(A)$. Conversely, suppose A has the properties described in Corollary 2.1, with k , ν and β as in (2.2). If $\nu(B) > 0$, then $\mathbb{P}^x(X_{n+k} \in B) \geq \beta \nu(B) \mathbb{P}^x(X_n \in A)$. Since $\mathbb{P}^x(T_A < \infty) = 1$ for all $x \in S$, then

$\mathbb{P}^x(X_n \in A \text{ infinitely often}) = 1$ and so $\mathbb{P}^x(X_n \in B \text{ infinitely often}) = 1$. It follows that $\{X_n : n \geq 0\}$ is ν -recurrent. \square

This result shows that in some sense the study of Harris recurrence is equivalent to the study of small sets with almost surely finite hitting times. The property that A is small can be used in two distinct but related ways. For convenience assume here that A is a small set with $k = 1$. The general case can be handled by applying the methods described below to the k -step process $\{X_{nk} : n \geq 0\}$.

2.1.1. *Coupling.* Two copies $\{X_n : n \geq 0\}$ and $\{X'_n : n \geq 0\}$ of the Markov process can be coupled so that $\mathbb{P}(X_{n+1} = X'_{n+1} | X_n, X'_n) \geq \beta$ whenever $(X_n, X'_n) \in A \times A$. This implies that two copies $\{Z_m : m \geq 0\}$ and $\{Z'_m : m \geq 0\}$ of the process on A can be coupled so that $\mathbb{P}(Z_{m+1} = Z'_{m+1} | Z_m, Z'_m) \geq \beta$. It follows easily that the process on A has a unique stationary probability μ_A , say, with

$$\|\mathbb{P}_A^m(x, \cdot) - \mu_A\| \leq 2(1 - \beta)^m$$

for all $m \geq 0$ and $x \in A$. Details of this coupling argument may be found in Lindvall [29].

2.1.2. *The split chain.* A set C is said to be an *atom* for the Markov chain $\{X_n : n \geq 0\}$ if $\mathbb{P}(x, \cdot) = \mathbb{P}(y, \cdot)$ for all $x, y \in C$. If C is an atom, then the times of visits to C are renewal times for the Markov chain. The excursions away from C will be independent and identically distributed, and many questions about the large time behavior of the Markov chain may be resolved using this fact.

A singleton set $C = \{a\}$ is clearly an atom. Nummelin [38] showed how to use a small set A to build a Markov chain with an atom. [More generally, Nummelin assumes a minorization condition $\mathbb{P}(x, C) \geq h(x)\nu(C)$ with some nontrivial nonnegative function h ; here, we specialize to $h(x) = \beta 1_A(x)$.] Consider the *split chain* $\{(X_n, Y_n) : n \geq 0\}$ with state space $S \times \{0, 1\}$ and transition probabilities given by

$$P\{Y_n = 1 | \mathcal{F}_n^X \vee \mathcal{F}_{n-1}^Y\} = \beta 1_C(X_n),$$

$$P\{X_{n+1} \in A | \mathcal{F}_n^X \vee \mathcal{F}_n^Y\} = \begin{cases} \nu(A), & \text{if } Y_n = 1, \\ \frac{P(X_n, A) - \beta 1_C(X_n)\nu(A)}{1 - \beta 1_C(X_n)}, & \text{if } Y_n = 0. \end{cases}$$

Here, $\mathcal{F}_n^X = \sigma\{X_r : 0 \leq r \leq n\}$ and $\mathcal{F}_n^Y = \sigma\{Y_r : 0 \leq r \leq n\}$. Thus, the split chain evolves as follows. Given X_n , choose Y_n so that $P(Y_n = 1) = \beta 1_C(X_n)$. If $Y_n = 1$ then X_{n+1} has distribution ν , whereas if $Y_n = 0$, then X_{n+1} has distribution $(P(X_n, \cdot) - \beta 1_C(X_n)\nu)/(1 - \beta 1_C(X_n))$. The split chain $\{(X_n, Y_n) : n \geq 0\}$ is designed so that it has an atom $S \times \{1\}$, and so that its first component $\{X_n : n \geq 0\}$ is a copy of the original Markov chain.

Thus, renewal theory can be used to describe the large time and stationary behavior of the split chain $\{(X_n, Y_n) : n \geq 0\}$, and hence of its first component $\{X_n : n \geq 0\}$.

Much more information about small sets, and more generally about the ergodic theory of Harris recurrent Markov chains, may be found in the books of Revuz [42], Nummelin [39] and Meyn and Tweedie [36] and the references therein.

2.2. *Positive recurrence and rates of convergence.* A Harris recurrent Markov process $\{X_n : n \geq 0\}$ is to be *positive Harris recurrent* if the stationary measure μ can be normalized to be a probability measure on (S, \mathcal{B}) .

Suppose that A is a small set with $0 < m(A) < \infty$, and recall that μ_A denotes the stationary probability measure for the process on A . The stationary measure μ given by (2.1) has total mass

$$\mu(S) = \int_A \mathbb{E}^x(T_A)\mu_A(dx).$$

Thus, the issue of positive recurrence depends on estimates of the expected hitting times $\mathbb{E}^x(T_A)$ for $x \in A$.

PROPOSITION 2.3 (Tweedie [43]). *Assume A is a small set. Suppose there exist a measurable function $V \geq 0$ and constants $c > 0, k$ such that $PV(x) \leq V(x) - c$ for $x \notin A$ and $PV(x) \leq k$ for $x \in A$. Then $\sup_{x \in A} \mathbb{E}^x(T_A) \leq 1 + k/c$ and so the Markov process is positive Harris recurrent.*

The proof is based on the fact that the first inequality assumed for the Lyapunov–Foster function V implies that when $X_0 \notin A$ the process $V(X_n) + cn$ stopped at time T_A is a supermartingale. With stronger assumptions on the function V , together with conditions to ensure aperiodicity, results may be obtained concerning the rate of convergence of $P^n(x, \cdot)$ to the stationary probability measure π , say. Several such conditions are given by Meyn and Tweedie [32, 36]. The following result includes also stronger conditions on the small set A so as to ensure aperiodicity.

PROPOSITION 2.4 (Meyn and Tweedie [32]). *Assume the set $A \in \mathcal{B}$ satisfies $P(x, \cdot) \geq \beta v(\cdot)$ for all $x \in A$, where $\beta > 0$ and $v(A) = 1$. Assume also there exist a measurable function $V \geq 1$ and positive constants $\lambda < 1$ and k such that $PV(x) \leq \lambda V(x)$ for $x \notin A$ and $PV(x) \leq k$ for $x \in A$. Then the Markov process has a unique stationary probability measure π , say. Moreover, there exist positive constants $\gamma < 1$ and M with the property that*

$$(2.3) \quad \left| \int_S f(y)P^n(x, dy) - \int f(y)\pi(dy) \right| \leq MV(x)\gamma^n$$

for all $x \in S$ and $n \geq 0$ whenever $f : S \rightarrow \mathbb{R}$ is a measurable function such that $|f(y)| \leq V(y)$ for all $y \in S$.

More details about estimates of the form (2.3) can be found in Chapter 16: V -uniform ergodicity of [36]. In Meyn and Tweedie [37] it is shown that the constants M and γ can be chosen depending only on λ, k and β . See also Baxendale [4].

2.3. *Continuous time Markov processes.*

2.3.1. *Sampled chains.* Suppose that $\{X_t : t \geq 0\}$ is a time homogeneous Markov process, and that $\{T(n) : n \geq 0\}$ is an independent undelayed renewal process with increment distribution a , for some probability distribution on $(0, \infty)$. Then the sampled chain $\{Y_n : n \geq 0\}$ defined by $Y_n = X_{T(n)}$ is a time homogeneous Markov chain. If $P_t(x, \cdot)$ denotes the time t transition probability function for $\{X_t : t \geq 0\}$, then $\{Y_n : n \geq 0\}$ has transition probability function $K_a(x, \cdot) = \int P_t(x, \cdot) a(dt)$.

The Δ -skeleton chain $Y_n = X_{n\Delta}$ corresponds to the deterministic $a = \delta_\Delta$. Alternatively, the resolvent chain observes the process X at the times of a rate 1 Poisson process and has transition probability

$$R(x, \cdot) = \int P_t(x, \cdot) e^{-t} dt,$$

which is the resolvent of the original continuous time process. The advantages and disadvantages of various choices for a , and the connections between the theories of Harris recurrence for continuous time and discrete time Markov processes are discussed in the papers [33, 34] of Meyn and Tweedie. Results analogous to those in Propositions 2.3 and 2.4, involving the action of the infinitesimal generator L on V , are given in Meyn and Tweedie [35].

2.3.2. *Stopping times.* An alternative approach to recurrence for continuous time Markov processes was developed independently by Maruyama and Tanaka [30] and Khas'minskii [21]. Assume that $\{X_t : t \geq 0\}$ is a strong Markov process with right continuous paths and left limits on a separable metric space S . Suppose that D_1 and D_2 are open sets with disjoint closures \overline{D}_1 and \overline{D}_2 with the property that the stopping times $T_{D_1} = \inf\{t \geq 0 : X_t \in D_1\}$ and $T_{D_2} = \inf\{t \geq 0 : X_t \in D_2\}$ are both \mathbb{P}^x -almost surely finite for all $x \in S$. Define inductively sequences σ_n and τ_n of stopping times by $\sigma_0 = \inf\{t \geq 0 : X_t \in D_1\}$, $\tau_n = \inf\{t \geq \sigma_n : X_t \in D_2\}$ for $n \geq 0$, and $\sigma_n = \inf\{t \geq \tau_{n-1} : X_t \in \overline{D}_1\}$ for $n \geq 1$. Then $\{Y_n \equiv X_{\sigma_n} : n \geq 0\}$ is a time homogeneous Markov chain on \overline{D}_1 .

Assume for the moment that the process $\{Y_n : n \geq 0\}$ on \overline{D}_1 has a stationary probability measure $\mu_{\overline{D}_1}$. Then it is shown in [30] and [21] that the measure μ on (S, \mathcal{B}) defined by

$$(2.4) \quad \mu(B) = \int_{\overline{D}_1} \left(\mathbb{E}^x \int_0^{\sigma_1} 1_B(X_s) ds \right) \mu_{\overline{D}_1}(dx), \quad B \in \mathcal{B},$$

is a stationary measure for the original continuous time Markov process $\{X_t : t \geq 0\}$.

So far this is a very natural extension to continuous time of Harris’s result in Proposition 2.1. It remains to show the existence of the stationary probability measure $\mu_{\overline{D}_1}$, and this is where both [30] and [21] impose extra conditions on the process $\{X_t : t \geq 0\}$. In particular, they both use additional properties of the hitting distribution $\mathbb{P}^x(X_{T_{D_1}} \in B)$ for $x \in \overline{D}_2$ and $B \subset \overline{D}_1$ which ensure that the process $\{Y_n : n \geq 0\}$ satisfies Doeblin’s condition (D).

The representation (2.4) can be used to convert occupation time estimates for the process $\{X_t : t \geq 0\}$ in a very direct way into estimates on the stationary measure μ , see, for example, Baxendale and Stroock [6] and Baxendale [3].

3. Stirring processes and stochastic flows.

3.1. *Random stirring in \mathbb{R}^d .* Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a homeomorphism of \mathbb{R}^d onto itself such that $\phi(x) = x$ whenever $\|x\| \geq K$, for some K . The mapping ϕ can be thought of as a stirring of \mathbb{R}^d centered at the origin $0 \in \mathbb{R}^d$. (Harris used the term “stirring” originally in the case where the homeomorphism ϕ is volume preserving, but it is convenient to keep the term in this more general setting.) For $a \in \mathbb{R}^d$, the translated mapping $\phi^a(x) = a + \phi(x - a)$ represents stirring centered at a .

Consider a Poisson point process on $\mathbb{R}^d \times (0, \infty)$ with intensity $\lambda dx dt$. For each atom (a, t) of the point process, apply the mapping ϕ^a at time t . Then for $0 \leq s \leq t < \infty$ the value X_{st} of the stirring process is the random mapping of \mathbb{R}^d to itself obtained as the composition of the stirrings ϕ^a at times u for all the atoms (a, u) with $s < u \leq t$. A percolation argument, using the fact that $\phi(x) - x$ has bounded support, can be used to show that for sufficiently small $t - s$ the restriction of the mapping X_{st} to any bounded set is almost surely given by the composition of a finite number of stirrings. It follows that the process $\{X_{st} : 0 \leq s \leq t < \infty$ is well defined.

The essence of this construction can be seen in the paper by Harris [17] on the construction of an exclusion process with nearest neighbor rates. In [17], he considers a point process on the set of bonds of the integer lattice \mathbb{Z}^d and the corresponding stirring switches the two ends of the bond. The construction in [17] allows the rates to depend on the local configuration, but in the simplest case of constant rate it fits into the setting above. Random stirring on the real line is studied in the paper [28] by Harris’s student W. C. Lee.

3.1.1. *Convergence to a stochastic flow.* Consider the effect of letting the magnitude of the displacement involved in each stirring ϕ tend to zero while letting the intensity of the Poisson process tend to infinity. More precisely, fix a compactly supported vector field V on \mathbb{R}^d and let ϕ_n denote the time $1/\sqrt{n}$ flow along V . Let $\{X_{st}^n : 0 \leq s \leq t < \infty\}$ be the random stirring process obtained using the stirring function ϕ_n together with a Poisson process with rate $n\lambda dx dt$. Assume the

centering condition

$$\int_{\mathbb{R}^d} V(x) dx = 0.$$

Then under appropriate smoothness conditions on the vector field V the processes $\{X_{st}^n : 0 \leq s \leq t < \infty\}$ converge weakly to a process $\{X_{st} : 0 \leq s \leq t < \infty\}$ with values in the group of homeomorphisms of \mathbb{R}^d . This is proved in Harris [18] for the case $d = 2$ when V is divergence free and rotationally symmetric. A more general form of result (although on a compact manifold) is given in Matsumoto and Shigekawa [31].

The process $\{X_{st} : 0 \leq s \leq t < \infty\}$ has the properties:

- (i) for each $x \in \mathbb{R}^d$ and $s \geq 0$ the mapping $t \rightarrow X_{st}(x)$ is continuous;
- (ii) $X_{tu} \circ X_{st} = X_{su}$ whenever $s \leq t \leq u$;
- (iii) if $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \dots \leq s_n \leq t_n$ the mappings $X_{s_i t_i}$, $1 \leq i \leq n$ are independent;
- (iv) the distribution of X_{st} depends only on $t - s$.

Any process with these properties will be called a (time-homogeneous) *stochastic flow* on \mathbb{R}^d . Much information about stochastic flows may be found in the books of Kunita [22] and Arnold [1].

The law of a stochastic flow is determined by the laws of its k -point motions

$$t \rightarrow (X_{0t}(x_1), X_{0t}(x_2), \dots, X_{0t}(x_k)) \in \mathbb{R}^{dk}$$

for all $k \geq 1$ and $x_1, x_2, \dots, x_k \in \mathbb{R}^d$. It is easy to see that each k -point motion is a Markov process on \mathbb{R}^{dk} . Under suitable regularity conditions, the k -point motions are diffusion processes and the infinitesimal generator for the k point motion can be explicitly written in terms of the generator L , say, for the one-point motion and a covariance matrix $B(x, y)$ for the two-point motion. In particular for $f \in C^2(\mathbb{R}^d)$ with compact support,

$$Lf(x) = \lim_{t \searrow 0} \frac{\mathbb{E}f(X_{0t}(x)) - f(x)}{t}, \quad x \in \mathbb{R}^d,$$

and

$$B^{pq}(x, y) = \lim_{t \searrow 0} \frac{\mathbb{E}[(X_{0t}^p(x) - x^p)(X_{0t}^q(y) - y^q)]}{t}, \quad x, y \in \mathbb{R}^d.$$

The operator L and the matrix function B are related by the fact that $B(x, x)$ is the symbol of the operator L . Together, L and B are called the local characteristics of the flow; see Le Jan and Watanabe [27]. For the stochastic flow constructed above as the limit of random stirring processes, the operator L has constant coefficients and $B(x, y)$ depends only $x - y$. This implies that the law of the stochastic flow is homogeneous in space as well as time. For the example on \mathbb{R}^2 considered by Harris in [18], the rotational invariance of the vector field V implies that law of the stochastic flow is invariant under rigid motions of \mathbb{R}^2 , and in particular the one-point motion is Brownian motion (up to a scaling factor).

3.2. *Isotropic stochastic flows.* A stochastic flow $\{X_{st} : 0 \leq s \leq t < \infty\}$ is *isotropic* if its law is invariant under rigid motions of \mathbb{R}^d . Harris [18] studied incompressible isotropic stochastic flows on \mathbb{R}^2 , and Baxendale and Harris [5] and Le Jan [23] studied the general d -dimensional case.

For an isotropic stochastic flow, the generator L for the one-point motion is a multiple of the Laplace operator Δ , and the law of the flow is determined by the covariance matrix B . Invariance under translations implies $B(x, y) = B(x - y, 0)$, and then invariance under rotations implies $B(x) \equiv B(x, 0)$ satisfies $B(x) = G^*B(Gx)G$ for all real orthogonal matrices G . This condition gives a representation of B using Bessel functions; see Yaglom [44] and Itô [20]. A corresponding representation for isotropic stochastic flows on a sphere \mathbb{S}^d appears in Raimond [41].

The isotropy condition implies that certain geometric properties of the flow can be calculated explicitly. For example, the length $\|v_t\|$ of a tangent vector $v_t = DX_{0t}(x)(v)$ is a geometric Brownian motion and the top Lyapunov exponent $\lambda_1 = \lim_{t \rightarrow \infty} t^{-1} \log \|v_t\|$ can be calculated explicitly in terms of B . Other local geometric properties such as the curvature of a submanifold of \mathbb{R}^d have been calculated; see Le Jan [24] and Cranston and Le Jan [8].

Of more interest are results involving the joint behavior of infinitely many points. A result of Baxendale and Harris on the length of a small curve in the case $\lambda_1 < 0$ has recently been sharpened by Dimitroff [14]. Results of Cranston, Scheutzow and Steinsaltz [10, 11] show that, while $\|X_{0t}(x)\|$ grows like \sqrt{t} for each fixed x , if D is a nonsingleton connected set in \mathbb{R}^d for $d \geq 2$ and the isotropic stochastic flow has $\lambda_1 > 0$ then $\sup\{\|X_{0t}(y)\| : y \in D\}$ grows almost surely linearly as $t \rightarrow \infty$.

For any measure (distribution of mass) ν , let ν_t denote the induced random measure $\nu \circ X_{0t}^{-1}$. Zirbel [45] contains estimates on the first two moments of ν_t . Recently Cranston and Le Jan [9] and Dimitroff and Scheutzow [15] have proved asymptotic normality of the rescaled random measure $A \rightarrow \nu_t(\sqrt{t}A)$.

3.3. *Coalescing flows.* For vector fields V_0, V_1, V_2, \dots on \mathbb{R}^d and independent scalar Brownian motions $\{W_t^1 : t \geq 0\}, \{W_t^2 : t \geq 0\}, \dots$, consider the stochastic differential equation

$$(3.1) \quad dx_t = V_0(x_t) dt + \sum_{\alpha \geq 1} V_\alpha(x_t) dW_t^\alpha.$$

Under suitable regularity and growth conditions on the vector fields V_0, V_1, V_2, \dots , the strong solutions of (3.1) for different initial conditions can be pieced together to give a stochastic flow $\{X_{st} : 0 \leq s \leq t < \infty\}$ of homeomorphisms \mathbb{R}^d ; see, for example, Kunita [22]. The local characteristics of the flow are the operator

$$Lf(x) = \sum_{p=1}^d V_0^p(x) \frac{\partial f}{\partial x^p}(x) + \frac{1}{2} \sum_{p,q=1}^d \sum_{\alpha \geq 1} V_\alpha^p(x) V_\alpha^q(x) \frac{\partial^2 f}{\partial x^p \partial x^q}(x)$$

and the covariance function

$$(3.2) \quad B^{pq}(x, y) = \sum_{\alpha \geq 1} V_{\alpha}^p(x) V_{\alpha}^q(y).$$

Conversely, any stochastic flow in which L and B have sufficiently smooth coefficients arises as the solution of a stochastic differential equation (taking the V_{α} to be an orthonormal basis of the reproducing kernel Hilbert space of B) and the flow consists of homeomorphisms.

Harris [19] introduced the study of coalescing stochastic flows. These are ones where the mappings X_{st} may be many to one. Harris studied the case $d = 1$ with continuous homogenous (in space) covariance function B , and obtained conditions for coalescence in terms of the modulus of continuity of B at 0. (In contrast, the ‘‘Arratia flow’’ of independent coalescing Brownian motions; see [2], has discontinuous $B = 1_{\{0\}}$.)

The issue of the existence of nonhomeomorphic stochastic flows in dimensions $d \geq 2$ was addressed by Darling [12]. More recently, Le Jan and Raimond [25, 26] have developed new techniques to interpret the stochastic differential equation (3.1) when the covariance function B given by (3.2) is non-Lipschitz. In this more general setting, there is not only the possibility of coalescence; there is also the possibility that the solution of (3.1) has to be interpreted as a flow of probability kernels. The flow of probability kernels, rather than a flow of mappings, corresponds to the lack of uniqueness in the solutions of (3.1). Examples of such flows include flows on Euclidean space \mathbb{R}^d and spheres \mathbb{S}^d with isotropic, but non-Lipschitz, covariance functions B .

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