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THE ALGEBRAIC DIFFERENCE OF TWO RANDOM CANTOR SETS: THE LARSSON FAMILY

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In this paper, we consider a family of random Cantor sets on the line and consider the question of whether the condition that the sum of the Hausdorff dimensions is larger than one implies the existence of interior points in the difference set of two independent copies. We give a new and complete proof that this is the case for the random Cantor sets introduced by Per Larsson.

1. Introduction. Algebraic differences of Cantor sets occur naturally in the context of the dynamical behavior of diffeomorphisms. From these studies originated a conjecture by Palis and Takens [8], relating the size of the arithmetic difference

$$C_2 - C_1 = \{y - x : x \in C_1, y \in C_2\}$$

to the Hausdorff dimensions of the two Cantor sets C_1 and C_2: if

$$\dim_{\mathrm{H}} C_1 + \dim_{\mathrm{H}} C_2 > 1, \tag{1}$$

then, *generically*, it should be true that

$$C_2 - C_1 \text{ contains an interval.}$$

For generic dynamically generated *nonlinear* Cantor sets, this was proven in 2001 by de Moreira and Yoccoz [1]. The problem is open for generic linear Cantor sets. The problem was put into a probabilistic context by Per Larsson in his thesis [5] (see also [6]). He considers a two-parameter family of random Cantor sets $C_{a,b}$, and claims to prove that the Palis conjecture holds for all relevant choices of the parameters a and b. Although the main idea of Larsson's argument is brilliant, unfortunately, the proof contains significant gaps and incorrect reasoning. The aim of the present paper is to give a correct proof of this theorem. The most important error made by Larsson is as follows: during the construction, a multitype branching

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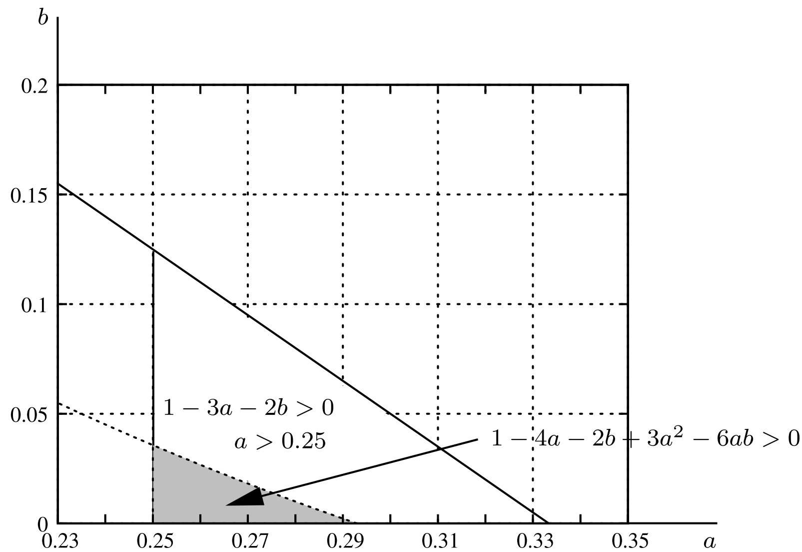

FIG. 1. *Regions described by equations* (3) *and* (4).

process with uncountably many types appears naturally. The number of individuals in the nth generation having types which fall into the set A is denoted $\mathcal{Z}_n(A)$ and the probability measure describing the branching process starting with a single type-x individual is denoted by $\mathbb{P}_x$. The argument presented in Larsson's paper requires that for some positive δ, q, $\rho > 1$ and for a set A of which the interior contains 0, we have that, *uniformly*, both in x and in n, the following holds:

$$\mathbb{P}_x(\mathcal{Z}_n(A) > \delta \cdot \rho^n) > q. \tag{2}$$

However, the main result in the theory of general multitype branching processes [4], Theorem 14.1, invoked by Larsson implies (2) without any uniformity.

Further (as shown in [3]), the idea presented in Larsson's paper works only in the region (see also Figure 1) where

$$1 - 4a - 2b + 3a^2 - 6ab > 0. \tag{3}$$

Although we use a different setup, the main idea presented here follows the line of Larsson's proof.

We remark that for linear Cantor sets of a different nature, the first two authors investigated the same problem in [2]. Further developments in this direction in [7] lead us to conjecture that in the critical case, that is, $\dim_H(C_{a,b}) = 1/2$, the difference set will a.s. contain no interval.

1.1. *Larsson's random Cantor sets.* It is assumed throughout this paper that

$$a > \tfrac{1}{4} \quad \text{and} \quad 3a + 2b < 1. \tag{4}$$

FIG. 2. *The construction of the Cantor set $C_{a,b}$. The figure shows $C^1_{a,b}, \dots, C^4_{a,b}$.*

The first condition is a growth condition and since

$$\dim_{\mathrm{H}} C_{a,b} = -\frac{\log 2}{\log a},$$

this condition is equivalent to $\dim_{\mathrm{H}} C_{a,b} > 1/2$, which is equivalent to (1). The second condition is a geometric condition: Larsson's Cantor set is a natural randomization of the classical Cantor set; see Figure 2. In the first step of the construction, intervals of length a are put into the intervals $[b, \frac{1}{2} - \frac{a}{2}]$ and $[\frac{1}{2} + \frac{a}{2}, 1 - b]$. Dismissing the trivial case $3a + 2b = 1$, this obviously requires $3a + 2b < 1$. We remark that it is useful to force a forbidden zone of length at least a in the middle since otherwise the Newhouse thickness of the Cantor set would be larger than 1, which yields an interval in the difference set by Newhouse's theorem (see [8], page 63). The two intervals of length a each have room to move in an interval of length $\frac{1}{2} - \frac{a}{2} - b$, that is, there is a free space of size $\frac{1}{2} - \frac{a}{2} - b - a$ and we denote this gap by $\mathbb{g}$:

$$\mathbb{g} := \frac{1 - 3a - 2b}{2}.$$

The construction is as follows: first, remove the middle a part, then the b parts from both the beginning and the end of the unit interval. Then, place intervals of length a according to a uniform distribution in the remaining two open spaces $[b, \frac{1}{2} - \frac{a}{2}]$ and $[\frac{1}{2} + \frac{a}{2}, 1 - b]$. These two randomly chosen intervals of length a are called the *level-one intervals* of the random Cantor set $C_{a,b}$. We write $C^1_{a,b}$ for their union. In both of the two level-one intervals, we repeat the same construction independently of each other and of the previous step. In this way, we obtain four disjoint intervals of length a^2. We emphasize that, because of independence, the relative positions of these second level intervals in the first level ones are, in general, completely different. Similarly, we construct the 2^n level-n intervals of length a^n. We call their union $C^n_{a,b}$. Larsson's random Cantor set is then defined by

$$C_{a,b} := \bigcap_{n=1}^{\infty} C^n_{a,b}.$$

See Figure 2.

The next theorem was stated by P. Larsson.

THEOREM 1. *Let C_1, C_2 be independent random Cantor sets having the same distribution as $C_{a,b}$ defined above. Then, the algebraic difference $C_2 - C_1$ almost surely contains an interval.*

This paper is organized as follows. In the next section, we give an elementary proof of the fact that the probability that $C_2 - C_1$ contains an interval is either 0 or 1. For the main part of the proof, our starting point is the observation that $C_2 - C_1$ can be viewed as a 45° projection of the product set $C_1 \times C_2$. This leads, in Section 3.1, to the introduction of the level-n squares formed as the product of level-n intervals of the Cantor sets C_1, C_2. We remark that Larsson does not use these squares at all. Then, based on the family of these squares we will construct the intrinsic branching process and state our Main Lemma, which will replace (2). In Section 4, we prove Theorem 1, assuming the Main Lemma. In Sections 5–10, we give a proof of the Main Lemma.

2. A 0–1 law. Undoubtedly, Larsson introduced his Cantor sets as a natural randomization of the classical triadic Cantor set. Actually, these sets can also be considered as very simple examples of statistically self-similar sets, which permits us to give a simple proof of the 0–1 law for the interval property. A set C is *statistically self-similar* if there is a collection of m random functions $\{\varphi_1, \dots, \varphi_m\}$ such that

$$C = \bigcup_{i=1}^{m} \varphi_i(C_i),$$

where the C_i are independent random sets with the same distribution as C. For Larsson's sets, $m = 2$ and the random functions are the affine functions

$$\varphi_1(x) = ax + b + U_1 \quad \text{and} \quad \varphi_2(x) = ax + (1+a)/2 + U_2,$$

where U_1 and U_2 are independent random variables, both uniformly distributed over $[0, \mathbb{g}]$.

PROPOSITION 1. $\mathbb{P}(C_2 - C_1 \supset I) = 0$ *or* 1.

PROOF. For $1 \le i, j \le 2$, let $C_{i,j}$ be independent copies of $C = C_{a,b}$ and let

$$C_1 = \varphi_1(C_{1,1}) \cup \varphi_2(C_{1,2}), \qquad C_2 = \varphi_1(C_{2,1}) \cup \varphi_2(C_{2,2})$$

be the self-similarity equations for C_1 and C_2. We will also write "$C_2 - C_1$ contains an interval" equivalently as "$C_2 - C_1$ has nonempty interior."

Using the facts that for arbitrary subsets A, B, C and D of $\mathbb{R}$,

$$(A \cup B) - (C \cup D) \supset (A - C) \cup (B - D),$$

that $\varphi(A-B)=\varphi(A)-\varphi(B)$ for affine functions $\varphi:\mathbb{R}\to\mathbb{R}$ and that affine functions are continuous, we can set up the following chain of (in)equalities:

$$\begin{aligned}
p &:= \mathbb{P}(C_2 - C_1 \supset I)\\
&= 1-\mathbb{P}\big(\mathrm{Int}(C_2-C_1)=\varnothing\big)\\
&\geq 1-\mathbb{P}\big(\mathrm{Int}\big(\varphi_1(C_{2,1})-\varphi_1(C_{1,1})\big)=\varnothing, \mathrm{Int}\big(\varphi_2(C_{2,2})-\varphi_2(C_{1,2})\big)=\varnothing\big)\\
&= 1-\mathbb{P}\big(\mathrm{Int}\big(\varphi_1(C_{2,1})-\varphi_1(C_{1,1})\big)=\varnothing\big)\mathbb{P}\big(\mathrm{Int}\big(\varphi_2(C_{2,2})-\varphi_2(C_{1,2})\big)=\varnothing\big)\\
&= 1-\mathbb{P}\big(\mathrm{Int}\big(\varphi_1(C_{2,1}-C_{1,1})\big)=\varnothing\big)\mathbb{P}\big(\mathrm{Int}\big(\varphi_2(C_{2,2}-C_{1,2})\big)=\varnothing\big)\\
&= 1-\mathbb{P}\big(\mathrm{Int}\big(C_{2,1}-C_{1,1}\big)=\varnothing\big)\mathbb{P}\big(\mathrm{Int}\big(C_{2,2}-C_{1,2}\big)=\varnothing\big)\\
&= 1-(1-p)^2.
\end{aligned}$$

This implies that $p\leq p^2$ and hence $p=0$ or 1. $\square$

3. Notation and the Main Lemma. In the remainder of the paper, we fix a pair (a,b) satisfying condition (4) and always deal with Larsson's Cantor sets, so we will suppress the labels a, b.

3.1. *The geometry of the algebraic difference* $C_2 - C_1$. The 45° projection of a point $(x_1,x_2)\in\mathbb{R}^2$ onto the x_2-axis is denoted by Proj_{45°. That is,

$$\mathrm{Proj}_{45^\circ}(x_1,x_2) := x_2 - x_1.$$

The following trivial fact is the motivation for constructing our branching process of labeled squares:

$$x\in\mathrm{Proj}_{45^\circ}(C_1\times C_2)\quad\text{if and only if}\quad x\in C_2-C_1.$$

So,

$$C_2-C_1=\bigcap_{n=0}^{\infty}\mathrm{Proj}_{45^\circ}(C_1^n\times C_2^n).$$

We can naturally label the squares in $C_1^n\times C_2^n$ as follows: we call the upper-left first level square Q_1 and continue labeling the first level squares Q_2, Q_3, Q_4 in the clockwise direction; then, within each of these squares, we continue in this way; see Figure 3.

For an $x\in[-1,1]$, we write $e(x)$ for that line with slope 1 which intersects the vertical axis at x. As we observed above

$$x\in C_2-C_1\quad\text{if and only if}\quad e(x)\cap(C_1\times C_2)\neq\varnothing. \tag{5}$$

Fix x and an arbitrary n. Let $\mathcal{S}_n$ be the set of all $a^n\times a^n$ squares contained in $[0,1]^2$. Note that for every $Q\in\mathcal{S}_n$, by the statistical self-similarity of the construction, the probability of the event $e(x)\cap(Q\cap(C_1\times C_2))\neq\varnothing$ conditional

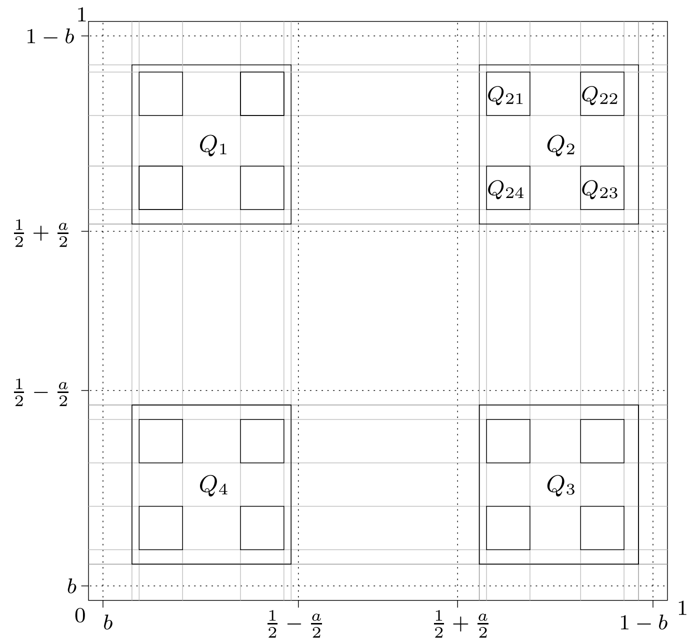

FIG. 3. *The first level squares* $Q_1, \dots, Q_4$ *and four second level squares* Q_{21}, Q_{22}, Q_{23}, Q_{24}.

on $Q \subset C_1^n \times C_2^n$ is equal to the probability of the event $e(\Phi) \cap (C_1 \times C_2) \neq \emptyset$, where we construct $\Phi = \Phi(Q, x)$ as follows: we rescale the square Q (which is an $a^n \times a^n$ square) by the factor $1/a^n$, then we choose Φ such that the line segment $e(\Phi) \cap [0, 1]^2$ is the rescaled copy of $e(x) \cap Q$; see Figure 4. More precisely, if (u, v) is the lower-left corner of Q, that is, $Q = [u, u + a^n] \times [v, v + a^n]$, then we define

$$(6) \qquad \Phi(Q, x) := \begin{cases} \dfrac{u - v + x}{a^n}, & \text{if } e(x) \text{ intersects } Q, \\ \Theta, & \text{otherwise,} \end{cases}$$

where Θ is a symbol representing the emptiness of the intersection. Observe that $\Phi(Q, x) > 0$ if and only if the center of Q is located below the line $e(x)$ and $e(x)$ meets Q. Further, $\Phi(Q, x) = 1$ if $e(x)$ intersects Q at the upper-left corner and $\Phi(Q, x) = -1$ if $e(x)$ intersects Q at the lower-right corner.

3.2. *The probability space.* We write $\mathcal{T} := \bigcup_{n=0}^{\infty} \{1, 2\}^n$ for the dyadic tree, with nodes $\underline{i}_n = i_1 i_2 \dots i_n$, where i_k is 1 or 2, and root Λ. For the construction of Larsson's Cantor set, the probability space is $\Omega_1 = [0, \mathbb{g}]^{\mathcal{T}}$ [recall that $\mathbb{g} = (1 - 3a - 2b)/2$]. An element of Ω_1 is denoted by U, that is, the value at the node $i_1 i_2 \dots i_n$ is $U_{i_1 i_2 \dots i_n}$. The corresponding σ-algebra is $\mathcal{B}_1 := \prod_{\mathcal{T}} \mathcal{B}[0, \mathbb{g}]$. Finally,

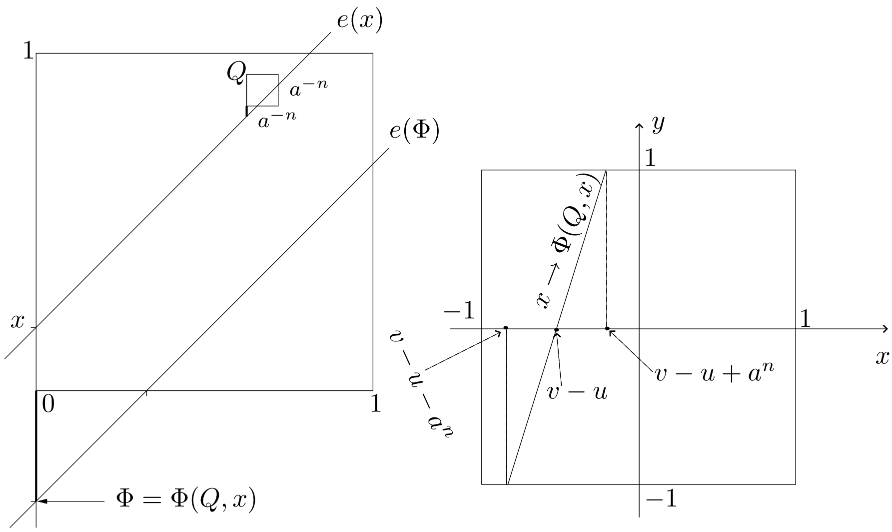

FIG. 4. *A level-n square Q and its rescaled type $\Phi(Q, x)$.*

the probability measure for Larsson's Cantor set is

$$\mathbb{P}_1 := \delta_0 \times \prod_{\mathcal{T}\setminus\{\Lambda\}} \text{Uniform}[0, \mathbb{g}],$$

where δ_0 is the Dirac mass at 0 associated with the mass at the root Λ. Note that the randomness starts at level 1. So, the probability space for $C_1 \times C_2$ is as follows:

$$\Omega := \Omega_1 \times \Omega_1, \qquad \mathcal{B} := \mathcal{B}_1 \times \mathcal{B}_1, \qquad \mathbb{P} := \mathbb{P}_1 \times \mathbb{P}_1. \tag{7}$$

An element of Ω is a pair of labeled binary trees. The 4^n level-n pairs of indices $(i_1 i_2 \dots i_n, j_1 j_2 \dots j_n)$ are naturally associated with level-n squares $Q'_{(i_1 i_2 \dots i_n, j_1 j_2 \dots j_n)}$ of size $a^n \times a^n$ whose relative positions are given by $U_{i_1 i_2 \dots i_n}$ and $U_{j_1 j_2 \dots j_n}$. Note, however, that (to simplify the notation) we have given new indices to these squares and positions: $Q_1 := Q'_{1,2}$, $Q_2 := Q'_{2,2}$, $Q_3 := Q'_{2,1}$, $Q_4 := Q'_{1,1}$ and similarly for higher order squares and their positions (see Figure 3).

3.3. *The branching process.* On the probability space Ω, we define a multitype branching process $\mathcal{Z} = (\mathcal{Z}_n)_{n=0}^{\infty}$. For a Borel set A, the natural number $\mathcal{Z}_n(A)$ represents the number of objects in generation n whose type falls into the set A. The type space T is a subset of $[-1, 1]$, but for the moment we can think of $T = [-1, 1]$. The objects of the nth generation are squares $Q \in \mathcal{S}_n$ and, given a fixed $x \in [-1, 1]$, their type is $\Phi(Q, x)$, as defined in (6). Note that although we speak of Θ as a type, it is *not* an element of T.

The process $(\mathcal{Z}_n)$ is a Markov chain whose states are collections of squares labeled by their types. The transition mechanism is as described in Section 3.1.

The initial condition of the chain is the square $[0, 1] \times [0, 1]$, with type x (also called the *ancestor* of the branching process). As usual, we then write, for $n \geq 1$,

$$\begin{aligned}
&\mathbb{P}_x\big(\mathcal{Z}_n(A_1) = r_1, \ldots, \mathcal{Z}_n(A_k) = r_k\big)\\
&\qquad = \mathbb{P}\big(\mathcal{Z}_n(A_1) = r_1, \ldots, \mathcal{Z}_n(A_k) = r_k | \mathcal{Z}_0(\{x\}) = 1\big)
\end{aligned}$$

for all $k \geq 1$, $A_1, \ldots, A_k \subset T$ and nonnegative integers $r_1, \ldots, r_k$.

A collection of squares all with type Θ is an absorbing state: it only generates squares with type Θ. This is obvious from the definition of $\Phi(Q, x)$, but we will extend this property to the case of smaller type spaces T, where, by definition, a square has type Θ if its type is not in T (this will be further explained in Section 6.1).

A major role in our analysis is played by the expectations $\mathbb{E}_x[\mathcal{Z}_n(A)]$ for $A \subset T$, $n \geq 1$. Let us define, for $i = 1, 2, 3, 4$,

$$\mathcal{Z}_1^i(A) = \begin{cases} 1, & \text{if } \Phi(Q_i, x) \in A, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Then, $\mathcal{Z}_1(A) = \mathcal{Z}_1^1(A) + \cdots + \mathcal{Z}_1^4(A)$ and so

$$\begin{aligned}
\mathbb{E}_x[\mathcal{Z}_1(A)] &= \int_\Omega \mathcal{Z}_1(A)\, \mathrm{d}\mathbb{P}_x = \int_\Omega \sum_{i=1}^4 \mathcal{Z}_1^i(A)\, \mathrm{d}\mathbb{P}_x\\
&= \sum_{i=1}^4 \mathbb{P}_x\big(\Phi(Q_i, x) \in A\big) = \sum_{i=1}^4 \int_A f_{x,i}(y)\, \mathrm{d}y,
\end{aligned}$$

where the $f_{x,i}$ are the densities of the random variables $\Phi(Q_i, x)$ (apart from an atom in Θ). In Section 5.2, these densities will be determined explicitly. It follows that for $n = 1$,

$$M_n(x, A) := \mathbb{E}_x[\mathcal{Z}_n(A)]$$

has a density $m_1(x, y)$, called the *kernel* of the branching process, given by

$$m(x, y) := m_1(x, y) = \sum_{i=1}^4 f_{x,i}(y). \tag{9}$$

We remark that if M_1 has a density, then M_n also has a density. Let us write $m_n(x, \cdot)$ for the density of $M_n(x, \cdot)$. The branching structure of $\mathcal{Z}$ yields (see [4], page 67)

$$m_{n+1}(x, y) = \int_T m_n(x, z) m_1(z, y)\, \mathrm{d}z. \tag{10}$$

The main problem to be solved is that the natural choice of $T = [-1, 1]$ as type space does not work because of condition (C) below and because we need the uniformity alluded to in equation (2).

Since the definition of T is complicated, we postpone it to Section 6. However, here we collect the most important properties of T:

(A) T is the disjoint union of finitely many closed intervals;

(B) there exists a $K > 0$ such that $[-K, K] \subset T$;

(C) the kernel $m_n(x, y)$ defined in (10) is uniformly positive on $T \times T$ [see condition (C1) below] and it has Perron–Frobenius eigenvalue greater than 1 [see condition (C2) below].

3.4. *The asymptotic behavior of the branching process* $\mathcal{Z}$. We will prove in Sections 6, 7 and 8 that there exists an integer n_0 such that m_{n_0} is a uniformly bounded function, that is, there exist $0 < a_{\min} < a_{\max}$ such that for all $x, y \in T$, we have

$$0 < a_{\min} \leq m_{n_0}(x, y) \leq a_{\max} < \infty. \tag{C1}$$

In the next step, we consider the following two operators:

$$g(x) \mapsto \int_{\mathbb{R}} m_1(x, y) \cdot g(y)\,\mathrm{d}y, \qquad h(y) \mapsto \int_{\mathbb{R}} h(x) \cdot m_1(x, y)\,\mathrm{d}x. \tag{11}$$

We cite the following theorem from [4], Theorem 10.1.

THEOREM 2 (Harris). *It follows from* (C1) *that the operators in* (11) *have a common dominant eigenvalue* ρ. *Let* $\mu(x)$ *and* $\nu(y)$ *be the corresponding eigenfunctions of the first and second operator in* (11), *respectively. Then, the functions* $\mu(x)$ *and* $\nu(y)$ *are bounded and uniformly positive. Moreover, apart from a scaling,* μ *and* ν *are the only nonnegative eigenfunctions of these operators. Further, if we normalize* μ *and* ν *so that* $\int \mu(x)\nu(x)\,\mathrm{d}x = 1$, *which will be henceforth assumed, then, for all* $x, y \in T$, *as* $n \to \infty$,

$$\left|\frac{m_n(x, y)}{\rho^n} - \mu(x)\nu(y)\right| \leq C_1 \mu(x)\nu(y)\Delta^n,$$

where the bound $\Delta < 1$ *can be taken independently of* x *and* y, *and the constant* C_1 *is independent of* x, y *and* n.

Later in this paper, we will prove that in our case, this Perron–Frobenius eigenvalue is greater than one:

$$\rho > 1. \tag{C2}$$

Using Theorem 2, Harris proves that $\mathcal{Z}_n(A)$ in fact grows exponentially with rate ρ. Introducing

$$W_n(A) := \frac{\mathcal{Z}_n(A)}{\rho^n},$$

he obtains (see [4], Theorem 14.1) the following result.

THEOREM 3 (Harris). *If*

$$\sup_{x\in T} \mathbb{E}_x[\mathcal{Z}_1(T)^2] < \infty, \tag{C3}$$

then it follows from (C1) *and* (C2) *that for all* $x \in T$,

$$\mathbb{P}_x\Big(\lim_{n\to\infty} W_n(A) =: W(A)\Big) = 1. \tag{12}$$

Further, for every Borel measurable $A \subset T$ *with* $\mathcal{L}\mathbf{eb}_1(A) > 0$, *we have*

$$\mathbb{P}_x\big(W(A) > 0\big) > 0. \tag{13}$$

Moreover, let A *and* B *be subsets of* T *such that their Lebesgue measures are positive. Then, the relation*

$$W(B) = \frac{\int_B \nu(y)\,\mathrm{d}y}{\int_A \nu(y)\,\mathrm{d}y} W(A)$$

holds $\mathbb{P}_x$ *almost surely for any* $x \in T$.

We are going to use this theorem to prove our Main Lemma, which summarizes everything we need concerning our branching process. Roughly speaking, the Main Lemma says that for the branching process associated to Larsson's Cantor set, the statement in Theorem 3 holds uniformly both in n and x for an appropriately chosen small interval of x's.

MAIN LEMMA. *There exist positive numbers* δ *and* q, *an* $N \in \mathbb{N}$ *and a small interval* $[-K, K] \subset T$ *centered at the origin such that the following inequality holds*:

$$\inf_{n\geq N} \inf_{x\in[-K,K]} \mathbb{P}_x\big(\mathcal{Z}_n([-K,0]) > \delta\rho^n, \mathcal{Z}_n([0,K]) > \delta\rho^n\big) \geq q. \tag{14}$$

4. The proof of Theorem 1. In Section 3.1, we defined the type of a square Q by means of its intersection with a line $e(x)$. Here, we will elaborate on this intersection.

4.1. *Nice intersection of a square with a line* $e(x)$. We say that a square Q has a *nice intersection* with $e(x)$ if

$$\Phi(Q, x) \in [-K, K],$$

where K comes from Main Lemma. For small K, this means that the center of Q is close to the line $e(x)$.

Let $\mathcal{A}^0 = \{[0,1]^2\}$, $\mathcal{A}^n$ be the set $\{Q \in \mathcal{S}_n : Q \subset C_1^n \times C_2^n\}$ and $\mathcal{A}_x^n$ be the set of squares from $\mathcal{A}^n$ having nice intersection with $e(x)$. That is, for $x \in T$ and $n \geq 1$, we define

$$\mathcal{A}_x^n := \{Q \in \mathcal{A}^n : |\Phi(Q, x)| \leq K\}.$$

Moreover, for $m \geq 0$ and a square $Q \in \mathcal{A}_x^m$, we write $l_n^+(Q,x)$ and $(l_n^-(Q,x))$ for the numbers of level-$(m+n)$ squares contained in Q which have nice intersection with $e(x)$ with center below and above the line $e(x)$, respectively. That is, for a $Q = Q_{i_1 \dots i_m}$, let

$$l_n^+(Q,x) = \#\{Q_{i_1\dots i_m j_1 \dots j_n} \in \mathcal{A}^{m+n} : 0 \leq \Phi(Q_{i_1\dots i_m j_1\dots j_n}, x) \leq K\}.$$

Similarly, let

$$l_n^-(Q,x) = \#\{Q_{i_1\dots i_m j_1 \dots j_n} \in \mathcal{A}^{m+n} : -K \leq \Phi(Q_{i_1\dots i_m j_1\dots j_n}, x) \leq 0\}.$$

Finally, for every $n \geq 1$, $x \in T$ and $Q \in \mathcal{A}_x^m$, we define the event

$$A_n(Q,x) := \{l_n^-(Q,x) > \delta\rho^n, l_n^+(Q,x) > \delta\rho^n\},$$

where δ comes from the Main Lemma. Note that the self-similarity of the construction of the squares and the Main Lemma for the underlying branching process imply the following: for $n \geq N$ and a square $Q \in \mathcal{S}_m$, we have

$$\begin{aligned} &\mathbb{P}(A_n(Q,x) | Q \in \mathcal{A}_x^m) \\ &\quad = \mathbb{P}_{\Phi(Q,x)}\big(\mathcal{Z}_n([-K,0]) > \delta\rho^n, \mathcal{Z}_n([0,K]) > \delta\rho^n\big) \geq q. \end{aligned} \tag{15}$$

4.2. *The difference set $C_2 - C_1$ contains an interval with positive $\mathbb{P}$ probability.* We introduce the interval

$$I := [-Ka^N, Ka^N]$$

with N and K from the Main Lemma. Note that $|I| := \mathcal{L}\mathbf{eb}_1(I) = 2Ka^N$.

Our goal is to prove that

$$\mathbb{P}(C_2 - C_1 \supset I) > 0.$$

First, we divide the interval I into 4^{2N} intervals I_{i_1} of equal length with indices $\pm 1, \dots, \pm\frac{1}{2}4^{2N}$. Then, we divide all of these intervals into 4^{3N} intervals $I_{i_1 i_2}$ of equal length. If we have already defined the $(k-1)$th level intervals, then we define the kth level intervals $I_{i_1\dots i_k}$ by subdividing each $(k-1)$th level interval $I_{i_1\dots i_{k-1}}$ into $4^{(k+1)N}$ intervals of equal length with indices $\pm 1, \dots, \pm\frac{1}{2}4^{(k+1)N}$. We denote the center of $I_{i_1\dots i_k}$ by $z_{i_1\dots i_k}$. That is,

$$I_{i_1\dots i_k} = \big[z_{i_1\dots i_k} - Ka^N 4^{-[2+\dots+(k+1)]N}, z_{i_1\dots i_k} + Ka^N 4^{-[2+\dots+(k+1)]N}\big],$$

where the $z_{i_1\dots i_k}$ are equally spaced in $I_{i_1\dots i_{k-1}}$.

Note that the interval $I_{i_1\dots i_k}$ has length

$$|I_{i_1\dots i_k}| = 2Ka^N 4^{-[2+\dots+(k+1)]N} < 2Ka^{g_k}, \tag{16}$$

where we put

$$g_k := (1 + \dots + (k+1))N = \tfrac{1}{2}(k+1)(k+2)N.$$

In the following, we will go from generation g_{k-1} to generation g_k.

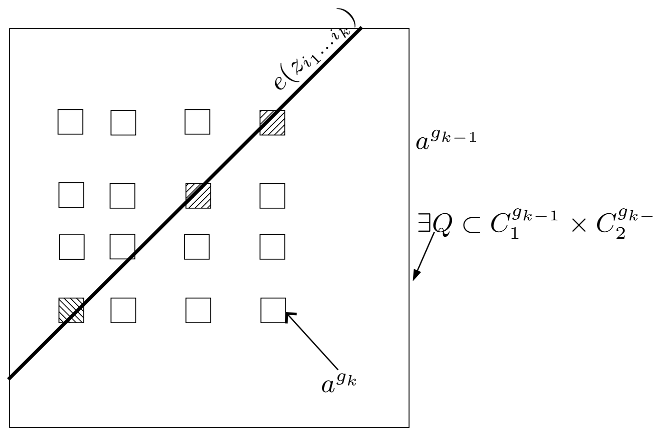

FIG. 5. *Event $B_k(z_{i_1\dots i_k})$: there is a level-g_{k-1} square Q in which the number of striped level-g_k squares (the nicely intersecting ones) is at least $\delta\rho^{N(k+1)}$, both for the squares with center above and the squares with center below the line $e(z_{i_1\dots i_k})$.*

DEFINITION 1. We say that the event $B_k(z_{i_1\dots i_k})$ occurs if there exists some square $Q \in \mathcal{A}^{g_{k-1}}$, itself having nice intersection with $e(z_{i_1\dots i_k})$, such that $A_{(k+1)N}(Q, z_{i_1\dots i_k})$ holds—cf. Figure 5. In formulae,

$$B_k(z_{i_1\dots i_k}) = \bigcup_{Q \in \mathcal{A}^{g_{k-1}}_{z_{i_1\dots i_k}}} A_{(k+1)N}(Q, z_{i_1\dots i_k}). \tag{17}$$

The following lemma is one of the key statements of the argument.

LEMMA 1. *Assume that $B_k(z_{i_1\dots i_k})$ occurs with the square Q. Let $\mathcal{Q}^+$ and $\mathcal{Q}^-$ be the collections of level-g_k squares within Q having nice intersection with $e(z_{i_1\dots i_k})$ with center below and above the line $e(z_{i_1\dots i_k})$, respectively. Then,*

(1)

$$\mathrm{Proj}_{45^\circ}\Big(\bigcup_{\widetilde{Q} \in \mathcal{Q}^+} \widetilde{Q}\Big) \supset I_{i_1\dots i_k}, \qquad \mathrm{Proj}_{45^\circ}\Big(\bigcup_{\widetilde{Q} \in \mathcal{Q}^-} \widetilde{Q}\Big) \supset I_{i_1\dots i_k}.$$

(2) *For every $i_{k+1} = \pm 1, \dots, \pm\frac{1}{2}4^{(k+2)N}$, the line $e(z_{i_1\dots i_k i_{k+1}})$ has nice intersection with all squares from either $\mathcal{Q}^+$ or $\mathcal{Q}^-$. Thus, the line $e(z_{i_1\dots i_k i_{k+1}})$ has nice intersection with at least $\delta\rho^{(k+1)N}$ squares contained in Q such that either all have center below the line $e(z_{i_1\dots i_k})$ or all have center above the line $e(z_{i_1\dots i_k})$.*

PROOF. Choose an arbitrary $y \in I_{i_1\dots i_k}$. Without loss of generality, we may assume that $y \le z_{i_1\dots i_k}$. Then, to show both (1) and (2), it is enough to prove that $e(y)$ has nice intersection with all squares from $\mathcal{Q}^+$.

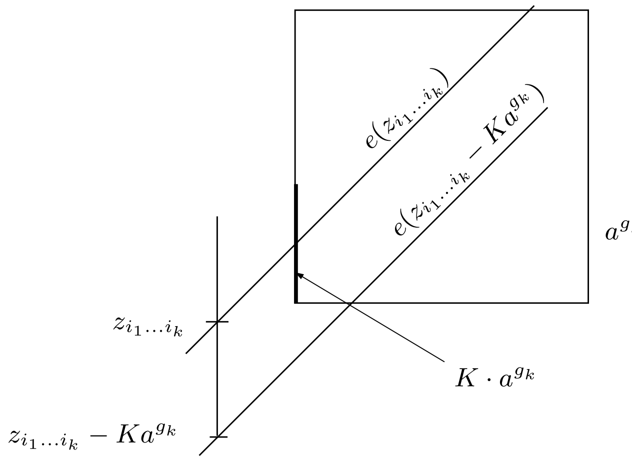

FIG. 6. *Nice intersections.*

Fix an arbitrary $Q \in \mathcal{Q}^+$. By the definition of $\mathcal{Q}^+$, the square Q is a level-g_k square such that its lower-left corner is in between the parallel lines $e(z_{i_1\ldots i_k})$ and $e(z_{i_1\ldots i_k} - Ka^{g_k})$. So, for every point $y^* \in [z_{i_1\ldots i_k} - Ka^{g_k}, z_{i_1\ldots i_k}]$, the line $e(y^*)$ has nice intersection with Q; see Figure 6.

To show that for any $y \in I_{i_1\ldots i_k} \cap (-\infty, z_{i_1\ldots i_k}]$, $e(y)$ has nice intersection with all squares from $\mathcal{Q}^+$, it is enough to prove that

$$I_{i_1\ldots i_k} \cap (-\infty, z_{i_1\ldots i_k}] \subset [z_{i_1\ldots i_k} - Ka^{g_k}, z_{i_1\ldots i_k}],$$

based on the previous paragraph. However, since

$$|I_{i_1\ldots i_k} \cap (-\infty, z_{i_1\ldots i_k}]| = \tfrac{1}{2}|I_{i_1\ldots i_k}| < Ka^{g_k},$$

this follows using (16). $\square$

DEFINITION 2. Let $E_0 := A_N([0,1]^2, 0)$ and let $E_k := \bigcap_{i_1\ldots i_k} B_k(z_{i_1\ldots i_k})$.

LEMMA 2. *The following inequality holds*:

$$\mathbb{P}(C_2 - C_1 \supset I) \geq q \prod_{k \geq 1} \mathbb{P}(E_k | E_{k-1}). \tag{18}$$

PROOF. Using the fact that $I = [-Ka^N, Ka^N] = \bigcup_{i_1\ldots i_k} I_{i_1\ldots i_k}$, it follows immediately from Lemma 1 that if the event E_k holds, then the event

$$S_k := \{\mathrm{Proj}_{45^\circ}(C_1^{g_k} \times C_2^{g_k}) \supset I\}$$

will hold. Therefore, $E_k \subset S_k$. Since the sets $C_1^{g_k} \times C_2^{g_k}$ are decreasing, we obtain that $S_k \supset S_{k+1}$. Thus,

$$\mathbb{P}(C_2 - C_1 \supset I) = \mathbb{P}\Big(\bigcap_{k \geq 1} S_k\Big) = \lim_{k\to\infty} \mathbb{P}(S_k) \geq \inf_{k\geq 1} \mathbb{P}(E_k)$$
$$\geq \mathbb{P}(E_0) \prod_{k\geq 1} \mathbb{P}(E_k | E_{k-1}).$$

The last inequality holds since

$$\mathbb{P}(E_0) \prod_{i\geq 1} \mathbb{P}(E_i | E_{i-1}) \leq \mathbb{P}(E_0)\mathbb{P}(E_1|E_0) \cdots \mathbb{P}(E_k|E_{k-1})$$
$$= p\mathbb{P}(E_k E_{k-1}) \leq \mathbb{P}(E_k),$$

where

$$p = \frac{\mathbb{P}(E_0)}{\mathbb{P}(E_0)} \frac{\mathbb{P}(E_1 E_0)}{\mathbb{P}(E_1)} \cdots \frac{\mathbb{P}(E_{k-1}E_{k-2})}{\mathbb{P}(E_{k-1})} \leq 1.$$

Since the Main Lemma yields $\mathbb{P}(E_0) \geq q$, one obtains the statement of the lemma. □

In Lemma 3, we give a lower bound for $\mathbb{P}(E_k|E_{k-1})$ for every k.

LEMMA 3. *For any* $k \geq 1$, *we have*

$$\mathbb{P}(E_k|E_{k-1}) \geq 1 - 4^{2N+\cdots+(k+1)N}(1-q)^{\delta\rho^{kN}}.$$

PROOF. We recall that E_k was defined as

$$E_k := \bigcap_{i_1 \dots i_k} B_k(z_{i_1 \dots i_k}).$$

Therefore, we have to prove that

$$\mathbb{P}\Big(\bigcup_{i_1\dots i_k} B_k^c(z_{i_1\dots i_k}) \Big| E_{k-1}\Big) \leq 4^{2N+\cdots+(k+1)N}(1-q)^{\delta\rho^{kN}}.$$

Note that the number of indices $i_1 \dots i_k$ on the left-hand side is equal to $4^{2N+\cdots+(k+1)N}$. Therefore, it is enough to show that for each index $i_1 \dots i_k$, we have

$$\mathbb{P}(B_k^c(z_{i_1\dots i_k})|E_{k-1}) \leq (1-q)^{\delta\rho^{kN}}.$$

By Definition 1, to see this, we have to prove that

$$\mathbb{P}\Big(\bigcap_{Q \in \mathcal{A}_{z_{i_1\dots i_k}}^{g_{k-1}}} A_{(k+1)N}^c(Q, z_{i_1\dots i_k}) \Big| E_{k-1}\Big) \leq (1-q)^{\delta\rho^{kN}}. \tag{19}$$

We assume E_{k-1}, so, in particular, we know that $B_{k-1}(z_{i_1\dots i_{k-1}})$ holds. That is, there exists a level-g_{k-2} square Q_{big} such that the event $A_{kN}(Q_{\text{big}}, z_{i_1\dots i_{k-1}})$ holds. By definition, this means that we can find at least $[\delta\rho^{kN}]+1$ squares in Q_{big} in $\mathcal{A}^{g_{k-1}}_{z_{i_1\dots i_{k-1}}}$ having center below, and at least as many squares having center above, the line $e(z_{i_1\dots i_{k-1}})$. Using the second part of Lemma 1 (for k instead of $k+1$), we obtain that the line $e(z_{i_1\dots i_k})$ has nice intersection with either all the squares above or with all the squares below the line $e(z_{i_1\dots i_{k-1}})$. Without loss of generality, we may assume the former.

However, for all these squares Q, the events $A^c_{(k+1)N}(Q, z_{i_1\dots i_k})$ are (conditionally) independent, so, to obtain (19), it is enough to show that

$$\mathbb{P}\big(A^c_{(k+1)N}(Q, z_{i_1\dots i_k}) | Q \in \mathcal{A}^{g_{k-1}}_{z_{i_1\dots i_k}}\big) \le 1-q \tag{20}$$

and this follows directly from equation (15). $\square$

LEMMA 4. *For all $n \ge 1$, we have*

$$\prod_{j=1}^{\infty}\big(1 - 4^{[2+\cdots+(j+1)]n}(1-q)^{\delta\rho^{jn}}\big) > 0. \tag{21}$$

PROOF. We have to show that $\sum_{j=1}^{\infty} a_j$ converges, where

$$a_j = 4^{(1/2)j(j+1)n}(1-q)^{\delta\rho^{jn}}.$$

It is therefore sufficient that $a_j \le e^{-j}$ for all large j. This is true since

$$\frac{1}{j}\log a_j = \frac{1}{2}(j+1)n\log 4 + \frac{1}{j}\delta(\rho^n)^j \log(1-q) \le -1,$$

which holds for j large enough since $\rho^n > 1$ and $\log(1-q) < 0$. $\square$

Therefore, using Lemmas 2, 3 and 4, we obtain that

$$\mathbb{P}(C_2 - C_1 \supset I) \ge q \prod_{k=1}^{\infty}\big(1 - 4^{[2+\cdots+(k+1)]N}(1-q)^{\delta\rho^{kN}}\big) > 0.$$

Combining this with Proposition 1 from Section 2, this completes the proof of Theorem 1.

In the next six sections, we prove our Main Lemma.

5. Distribution of types. In this section, the density function of $\Phi(Q, x)$ will be determined for the four squares Q from $\mathcal{S}_1$.

5.1. *The distribution of* $\Phi(Q, x)$. Let U_1, U_2, U_3, U_4 be four independent Uniform($[0, \mathbb{g}]$)-distributed random variables. The left corners of the two level-one intervals of the random Cantor set C_i are determined by U_{2i-1}, U_{2i} for $i = 1, 2$. Let (u_i, v_i) be the lower-left corner of the squares Q_i, $i = 1, \dots, 4$ (see Figure 7). Then,

$$
\begin{aligned}
(u_1, v_1) &= \left(b + U_1, \frac{1}{2} + \frac{a}{2} + U_4\right), \\
(u_2, v_2) &= \left(\frac{1}{2} + \frac{a}{2} + U_2, \frac{1}{2} + \frac{a}{2} + U_4\right), \\
(u_3, v_3) &= \left(\frac{1}{2} + \frac{a}{2} + U_2, b + U_3\right), \\
(u_4, v_4) &= (b + U_1, b + U_3).
\end{aligned}
$$

For an $x \in [-1, 1]$, we define $\Phi_i(x) := \Phi(Q_i, x)$. From (6), simple computations yield

$$
\Phi_1(x) = \begin{cases} \frac{1}{a}\left(-\frac{1}{2} - \frac{a}{2} + b + U_1 - U_4 + x\right), & \\ \qquad \text{if } \frac{1}{a}\left(-\frac{1}{2} - \frac{a}{2} + b + U_1 - U_4 + x\right) \in [-1, 1], & \\ \Theta, \qquad \text{otherwise}, & \end{cases}
\tag{22}
$$
$$
\Phi_2(x) = \begin{cases} \frac{1}{a}(U_2 - U_4 + x), & \text{if } \frac{1}{a}(U_2 - U_4 + x) \in [-1, 1], \\ \Theta, & \text{otherwise} \end{cases}
$$

and, similarly,

$$
\Phi_3(x) = \begin{cases} \frac{1}{a}\left(\frac{1}{2} + \frac{a}{2} - b + U_2 - U_3 + x\right), & \\ \qquad \text{if } \frac{1}{a}\left(\frac{1}{2} + \frac{a}{2} - b + U_2 - U_3 + x\right) \in [-1, 1], & \\ \Theta, \qquad \text{otherwise}, & \end{cases}
\tag{23}
$$
$$
\Phi_4(x) = \begin{cases} \frac{1}{a}(U_1 - U_3 + x), & \text{if } \frac{1}{a}(U_1 - U_3 + x) \in [-1, 1], \\ \Theta, & \text{otherwise}. \end{cases}
$$

To get a better geometric understanding of the distribution of the random variables $\Phi_i(x)$, we define the three slanted stripes S_k, $k = 1, 2, 3$ (see Figure 8), in such a way that $S_k \subset [-1, 1]^2$ is bounded by the lines ℓ_{2k-1}, ℓ_{2k}, where

$$
\begin{aligned}
&\ell_1(x) = \frac{1}{a}x + \frac{1}{a}(1 - a - 2b), \qquad \ell_2(x) = \frac{1}{a}x + 2, \qquad \ell_3(x) = \frac{1}{a}x + \frac{\mathbb{g}}{a}, \\
&\ell_4(x) = \frac{1}{a}x - \frac{\mathbb{g}}{a}, \qquad \ell_5(x) = \frac{1}{a}x - 2, \qquad \ell_6(x) = \frac{1}{a}x - \frac{1}{a}(1 - a - 2b).
\end{aligned}
\tag{24}
$$

An immediate calculation shows that the following result holds.

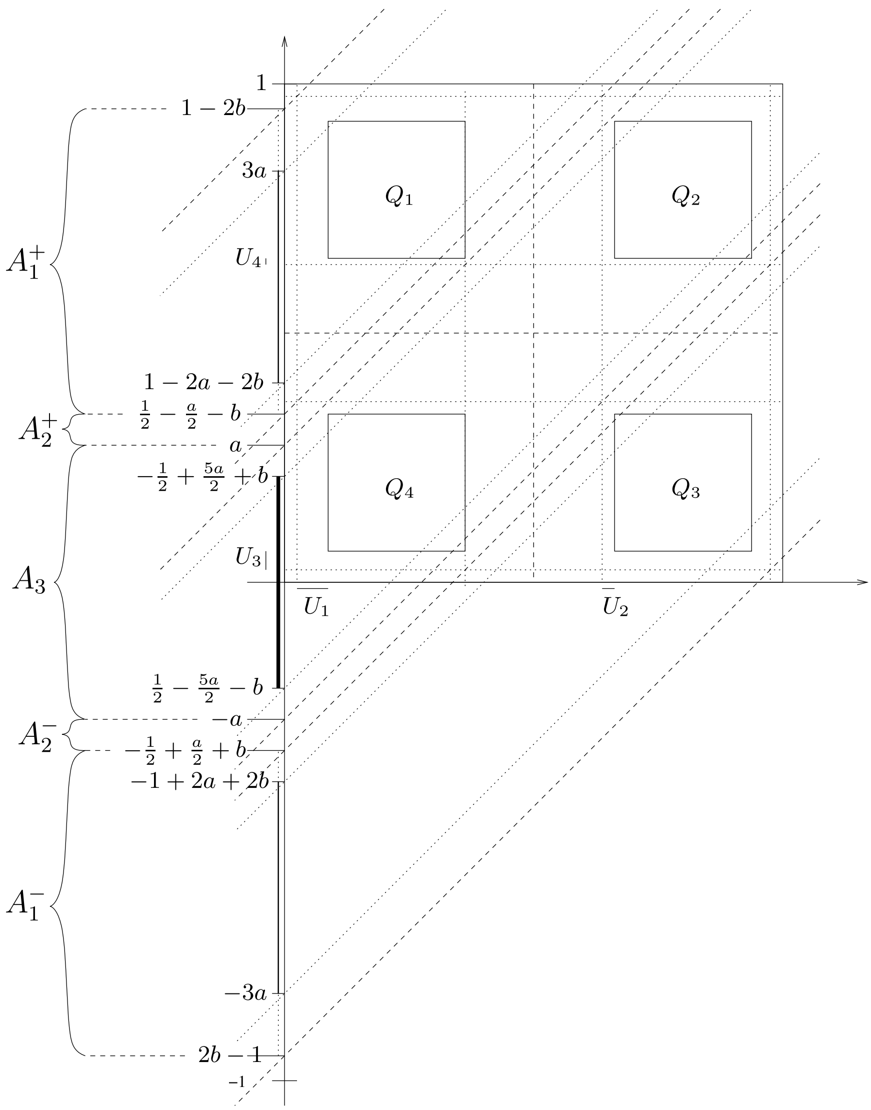

FIG. 7. *If x is an element of the bold vertical line, then the line $e(x)$ intersects exactly two squares. If x is an element of one of the two plain vertical lines, then $e(x)$ intersects one square. If x is an element of one of the four dotted vertical lines, then $e(x)$ intersects at most one square. If x is such that $a \le x \le 1-2a-2b$ or $-1+2a+2b \le x \le -a$, then $e(x)$ intersects at most two squares with probability one. If x is such that $-\frac{1}{2}+\frac{5a}{2}+b \le x \le a$ or $-a \le x \le \frac{1}{2}-\frac{5a}{2}-b$, then $e(x)$ intersects exactly two squares.*

LEMMA 5. *For every $x \in [-1, 1]$ and every $i = 1, \dots, 4$, if $\Phi_i(x) \neq \Theta$, then*

$$(x, \Phi_i(x)) \in S_1 \cup S_2 \cup S_3.$$

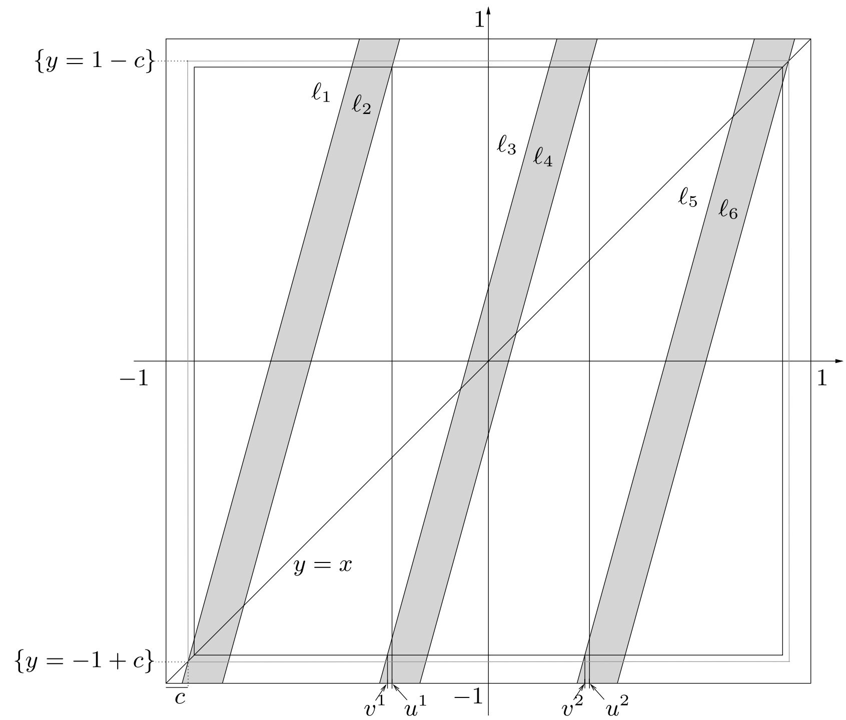

FIG. 8. *The support of the density functions in the simple case.*

Let us call ℓ_j the graph of the function $\ell_j(x)$. Observe that the reflection in the origin of ℓ_j is ℓ_{7-j} for $j = 1, \ldots, 6$. For a point $(x_1, x_2) \in \mathbb{R}^2$, we write $\pi_m(x_1, x_2) := x_m$, $m = 1, 2$. We then define $c > 0$ by

$$-1 + c := \pi_1(\ell_1 \cap \{y = x\})$$

and obtain $c = \frac{2b}{1-a}$. By symmetry, it follows that

$$1 - c = \pi_1(\ell_6 \cap \{y = x\}).$$

Using the fact that $-1 + 2b = \pi_1(\ell_1 \cap \{y = -1\})$, it follows from the symmetry mentioned above that

$$\begin{aligned} &x \notin (-1 + 2b, 1 - 2b) \\ &\qquad \Longrightarrow \quad e(x) \text{ does not intersect any level-one square.} \end{aligned} \tag{25}$$

The functions $\ell_1(x)$, $\ell_6(x)$ have repelling fixed point $-1 + c$, $1 - c$, respectively. Therefore,

$$\begin{aligned} &x \in [-1, -1 + c) \cup (1 - c, 1] \\ &\qquad \Longrightarrow \quad \exists n \text{ such that } (x) \cap Q = \varnothing \text{ for all } Q \in \mathcal{S}_n. \end{aligned} \tag{26}$$

With probability 1, no line $e(x)$ can intersect more than two descendants, in fact, $[-1+2b, 1-2b]$ can be partitioned into five sets, according to which descendants can be produced, given by (see also Figure 7)

$$\begin{aligned} & A_1^- = \left[-1+2b, -\frac{1}{2}+\frac{a}{2}+b\right), \qquad A_1^+ = \left(\frac{1}{2}-\frac{a}{2}-b, 1-2b\right], \\ & A_2^- = \left[-\frac{1}{2}+\frac{a}{2}+b, -a\right), \qquad A_2^+ = \left(a, \frac{1}{2}-\frac{a}{2}-b\right], \\ & A_3 = \left[-a, a\right]. \end{aligned} \tag{27}$$

LEMMA 6. *If $x \in A_3$, then x can only produce descendants with type $\Phi_2(x)$ and/or $\Phi_4(x)$. If $x \in A_1^+$ (resp. $x \in A_1^-$), then x can produce at most one descendant with type $\Phi_1(x)$ [resp. $\Phi_3(x)$]. If $x \in A_2^+$, then there are two possibilities. First, if x produces $\Phi_1(x)$, then $\Phi_2(x)$ and $\Phi_4(x)$ cannot be born. Second, if x produces any of $\Phi_2(x)$ and $\Phi_4(x)$, then $\Phi_1(x)$ cannot be born. If $x \in A_2^-$, then there are two similar possibilities.*

PROOF. In Figure 7, observe that $\mathrm{Proj}_{45^\circ}(Q_1) \cap \mathrm{Proj}_{45^\circ}(Q_4) \neq \varnothing$ can happen only in the extreme situation if the bottom of the square Q_1 is the same as the bottom of the dotted square which contains Q_1 on Figure 3. This means that $U_4 = 0$, which happens with probability zero. Similarly, $\mathrm{Proj}_{45^\circ}(Q_3) \cap \mathrm{Proj}_{45^\circ}(Q_4) \neq \varnothing$ happens only if $U_2 = 0$, which also has probability zero. $\mathrm{Proj}_{45^\circ}(Q_1) \cap \mathrm{Proj}_{45^\circ}(Q_3) = \varnothing$ always holds, which completes the proof of our lemma. $\square$

5.2. *The density functions.* In this subsection, we will determine the density functions $f_{\Phi_i(x)}(y)$ of the random variables $\Phi_i(x)$, $i = 1, 2, 3, 4$, given explicitly by (22) and (23). We do not call them *probability* density functions since the $\Phi_i(x)$ may be equal to Θ with positive probability for some x. The probability density function of the difference of two independent Uniform($[0, \mathbb{g}]$)-distributed random variables is the triangular distribution given by $f_\triangle(z) = 0$ if $|z| > \mathbb{g}$ and for $0 \leq |z| \leq \mathbb{g}$ by

$$f_\triangle(z) = \frac{1}{\mathbb{g}^2}(\mathbb{g} - |z|). \tag{28}$$

To get $f_{\Phi_i(x)}(y)$, we apply simple transformations to $f_\triangle(z)$ and find

$$f_{\Phi_i(x)}(y) = a f_\triangle(ay + c_i - x)\mathbf{1}_{[-1,1]}(y) \tag{29}$$

with $c_1 = -c_3 = \frac{1}{2} + \frac{a}{2} - b$ and $c_2 = c_4 = 0$.

From the definition,

$$\mathbb{P}\big(\Phi_i(x) = \Theta\big) = 1 - \int_{[-1,1]} f_{\Phi_i(x)}(y)\,\mathrm{d}y.$$

6. A uniformly positive kernel. Here, and in the next two sections, we are going to define the type space T of the branching process introduced in Section 3.3. In order to ensure that conditions (C1), (C2), (C3) of Section 3.4 hold, we introduce a type space T which also satisfies properties (A), (B), (C) of Section 3.3. It follows from (26) that we must choose our type space $T \subset [-1+c, 1-c]$.

Unfortunately, the construction of the type space T satisfying the above conditions is quite involved and technical for those values of the parameters a, b which do not satisfy (3). Therefore, we split the presentation into two parts. In this section, we present the construction of T across three lemmas: Lemmas 7A, 8A and 9A. In the next section, we present the general case with the corresponding Lemmas 7, 8 and 9. The main difference between these lemmas lies in the proofs of Lemmas 7 and 7A. Lemma 8 is almost the same as Lemma 8A. Finally, the proof of Lemma 9 follows the same line as the proof of Lemma 9A, but is more technical.

6.1. *Descendant distributions and the kernel of the branching process.* We introduce the random variables $X_1(x), X_2(x), X_3(x), X_4(x)$ for $1 \le i \le 4$ by

$$X_i(x) = \begin{cases} \Phi_i(x), & \text{if } \Phi_i(x) \in T, \\ \Theta, & \text{otherwise.} \end{cases} \tag{30}$$

So, the density of $X_i(x)$ is

$$f_{x,i}(y) := f_{\Phi_i(x)}(y)\mathbf{1}_T(y) \tag{31}$$

for $i = 1, \dots, 4$. In general, $X_i(x)$ also has an atom: $\mathbb{P}(X_i(x) = \Theta) = 1 - \int_T f_{x,i}(y)\,\mathrm{d}y$.

Recall [see equation (9)] that the kernel of the branching process can be expressed as the sum of the density functions of the random variables $X_i(x)$, $i = 1, \dots, 4$:

$$m(x, y) = f_{x,1}(y) + f_{x,2}(y) + f_{x,3}(y) + f_{x,4}(y).$$

The structure of the support of this kernel is very important for the sequel. Since the functions $f_{x,i}(y)$ ($i = 1, 2, 3, 4$) are piecewise continuous on $[-1, 1]$, $m(\cdot, \cdot)$ is piecewise continuous on $[-1, 1] \times [-1, 1]$. The support of $m(\cdot, \cdot)$ is a subset of the three slanting stripes S_k, $k = 1, 2, 3$, introduced earlier; see also Figure 8.

6.2. *The possible holes in the support of the kernel of $\mathcal{Z}$.* We have seen in (26) that the branching process with ancestor type in the set $[-1, -1+c]$ or $[1-c, 1]$ dies out in a finite number of generations almost surely. Therefore, it is reasonable to restrict the type space to $[-1+c+\varepsilon, 1-c-\varepsilon]$ for some small positive ε. However, in some cases, we have to make further restrictions. Namely, for $i = 1, 2$, we define

$$u^i := \pi_1(\ell_{2i} \cap \{y = 1-c\}), \qquad v^i := \pi_1(\ell_{2i+1} \cap \{y = -1+c\}); \tag{32}$$

see Figure 8. Clearly, $u^1 - v^1 = u^2 - v^2$ and an easy calculation shows that

$$v^1 < u^1 \quad \Longleftrightarrow \quad c < \frac{\mathbb{g}}{2a}. \tag{33}$$

We remark that this condition is equivalent to the condition in equation (3) (see also Figure 1). On the other hand, if $u^i < v^i$, $i = 1, 2$, holds, then, for $x \in [u^i, v^i]$, the set

$$E_1(x) := \{y : m(x, y) > 0\} \tag{34}$$

is contained in $[-1, -1 + c] \cup [1 - c, 1]$. This implies that the process dies out in finitely many steps for $x \in [u^i, v^i]$ (see Figure 9). Therefore, if the condition stated in (33) does not hold, then we have to make more restrictions on our type space $[-1 + c + \varepsilon, 1 - c - \varepsilon]$. This is what we are going to do in Section 8. For

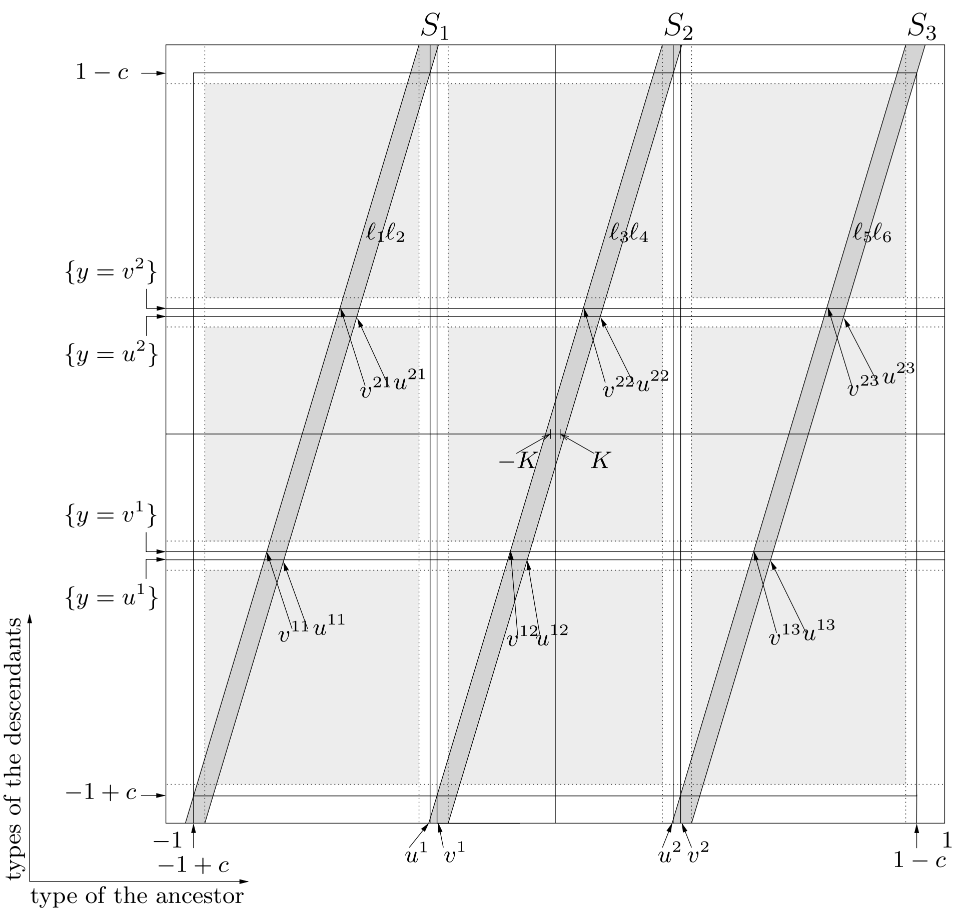

FIG. 9. *Some points and lines related to the kernel $m(x, y)$ if $l = 1$.*

the convenience of the reader, in Section 7, we treat the simpler case when (33) holds.

7. A uniformly positive kernel in the simple case. In the remainder of this section, we will prove that if (33) holds, that is, $v^1 < u^1$, then we can choose a sufficiently small $\varepsilon_0 > 0$ such that

$$T = [-1 + c + \varepsilon_0, 1 - c - \varepsilon_0]$$

satisfies conditions (C1), (C2) and (C3) [and also properties (A), (B), (C)]. The kernel in the simple case is illustrated in Figure 8.

LEMMA 7A. *Assume that $v^1 < u^1$. Fix an $\varepsilon > 0$ satisfying*

$$\varepsilon < \frac{\mathfrak{g}}{2a} - c. \tag{35}$$

Further, in this simpler case, let

$$T = T(\varepsilon) = [-1 + c + \varepsilon, 1 - c - \varepsilon]. \tag{36}$$

Then, the kernel $m(x, y)$ of the branching process $\mathcal{Z}$ has the following property:

(37) $\exists \kappa > 0$ *such that* $\forall x \in T$, *the set* $E_1(x)$ *contains an interval of length* κ.

PROOF. There are two possibilities for the shape of $E_1(x)$ [defined in (34)]:

(1) $E_1(x)$ *consists of two intervals*: $[-1+c+\varepsilon, \ell_{2k+1}(x)) \cup (\ell_{2k}(x), 1-c-\varepsilon]$ (for $k = 1$ or $k = 2$). The length of one of these intervals is at least half of $\ell_3(u^1) - (-1 + c + \varepsilon)$, that is, $\kappa_1 = \frac{1}{2} \cdot (\frac{\mathfrak{g}}{a} - 2c)$.

(2) $E_1(x) = (\ell_{2k-1}(x), \ell_{2k}(x))$ (*for some* $1 \le k \le 3$) *is an open interval with length* $\kappa_2 = \frac{4}{a}\mathfrak{g}$.

Summarizing these cases, define $\kappa = \min\{\kappa_1, \kappa_2\}$. $\square$

LEMMA 8A. *Let m^ε be the kernel in Lemma 7A with type space $T = T(\varepsilon)$, as in (36). One can choose $\varepsilon > 0$ which satisfies (35) such that the largest eigenvalue of m^ε is larger than 1. From now on, we fix such an ε and call it ε_0.*

PROOF. Let $T(0) := [-1 + c, 1 - c]$, with corresponding kernel m^0. Define [as in (11)] the operator $\mathcal{T}_\varepsilon$ for all $\varepsilon \ge 0$ by

$$\mathcal{T}_\varepsilon h(y) = \int_{\mathbb{R}} h(x) m^\varepsilon(x, y)\, \mathrm{d}x$$

for functions with $\mathrm{supp}(h) \subset T(\varepsilon)$.

We shall prove that $4a$ is an eigenvalue of the operator $\mathcal{T}_0$ with eigenfunction $h(x) = \mathbf{1}_{T(0)}(x)$:

$$\begin{aligned}
\mathcal{T}_0 h(y) &= \int_{\mathbb{R}} h(x) m^0(x, y)\, \mathrm{d}x \\
&= \int_{\mathbb{R}} h(x) \left(\sum_{i=1}^{4} f_{x,i}(y) \right) \mathbf{1}_{T(0)}(y)\, \mathrm{d}x \\
&= 4a h(y) \int_{T(0)} \sum_{i=1}^{4} f_{\triangle}(ay + c_i - x)\, \mathrm{d}x \\
&= 4a h(y),
\end{aligned}$$

provided we show that for all $i = 1, 2, 3, 4$,

$$\int_{[-1+c, 1-c]} f_{\triangle}(ay + c_i - x)\, \mathrm{d}x = 1.$$

Since $f_{\triangle}$ is a probability density with support lying in $[-\mathbb{g}, \mathbb{g}]$, it then suffices to show that for all $y \in [-1 + c, 1 - c]$ and for $i = 1, 2, 3, 4$, we have

$$ay + c_i - 1 + c \le -\mathbb{g} \quad \text{and} \quad ay + c_i + 1 - c \ge \mathbb{g}.$$

Taking the worst case for y, this boils down to showing

$$a(1 - c) + c_i - 1 + c \le -\mathbb{g} \quad \text{and} \quad a(-1 + c) + c_i + 1 - c \ge \mathbb{g}.$$

For $i = 1$, we have $c_1 = (a + 1)/2 - b$, so there we have to check that

$$(1 - c)(a - 1) + \frac{a + 1}{2} - b \le -\mathbb{g} \quad \text{and} \quad (1 - c)(1 - a) + \frac{a + 1}{2} - b \ge \mathbb{g}.$$

First, note that since $c_3 = -c_1$, the case $i = 3$ is covered by the case $i = 1$. Further, note that the left inequality implies the right one since $a + 1 > 2b$ always holds. Moreover, $a + 1 > 2b$ also gives that the left inequality will imply both inequalities for $i = 2, 4$. The calculation is then completed by substituting $c = 2b/(1 - a)$ in the left inequality, which turns out to be an equality.

The conclusion of the lemma follows from a simple fact noted by Larsson [6]: if the two kernels m^0 and m^ε are close to each other in L^2-sense, then the eigenvalues of the operators $\mathcal{T}_0$ and $\mathcal{T}_\varepsilon$ are close to each other. $\square$

LEMMA 9A. *Let T be as in Lemma 8A. Then there exists an index n such that for all $x \in T$, $\{y : m_n(x, y) > 0\} = T$.*

Since the function $m_n(\cdot, \cdot)$ is piecewise continuous on the compact set T, Lemma 9A implies that there exists an $a_{\min} > 0$ such that $m(x, y) \ge a_{\min}$ for any $x, y \in T$. Further, using the fact that $m(x, \cdot)$ is bounded, we immediately obtain that $a_{\max} := \sup_{x \in T} \mathbb{E}_x \mathcal{Z}_1^2(T)$ is finite. Therefore, we have the following result.

COROLLARY 1. *Let T be as in Lemma 8A. The branching process $\mathcal{Z}$ with type space T satisfies conditions* (C1) *and* (C3).

PROOF OF LEMMA 9A. Basically, we will prove that if (37) holds, then Lemma 9A also holds since the slope of the lines ℓ_i is equal to $\frac{1}{a}$, which is bigger than one. Let $E_n(x) = \{y : m_n(x, y) > 0\}$. We will prove that in both cases of the proof of Lemma 7A, the sequence $(E_n(x))$ reaches the whole type space in a finite number of steps, uniformly in n and $x \in T$.

We can derive $E_{n+1}(x)$ from $E_n(x)$ by means of the equation

$$m_{n+1}(x, y) = \int_T m_n(x, z) m_1(z, y)\,\mathrm{d}z,$$

which implies that

$$E_{n+1}(x) = \bigcup_{y \in E_n(x)} E_1(y). \tag{38}$$

In the proof of Lemma 7A, we treated two separate cases. We continue this proof according to those two cases:

(1) $E_1(x)$ *consists of two intervals.* Take the longer one, so its length is at least $\kappa_1 = \frac{1}{2} \cdot (\frac{\mathfrak{g}}{4a} - 2c)$. The following two facts hold. This interval contains either $-1 + c + \varepsilon$ or $1 - c - \varepsilon$, and if $E_n(x)$ contains one of these points, then $E_{n+1}(x)$ also contains the same point because of (38). Therefore, if $E_n(x) \neq T$ and is of the form, for example, $[-1 + c + \varepsilon, -1 + c + \varepsilon + s)$ for some positive s, then $E_{n+1}(x) \supset [-1 + c + \varepsilon, -1 + c + \varepsilon + \frac{1}{a}s)$ or $E_{n+1}(x) = T$. Hence, if $E_1(x) = [-1 + c + \varepsilon, -1 + c + \varepsilon + s)$, then in

$$n_1(x) = \left\lceil \log_{1/a}\left(\frac{2(1 - c - \varepsilon)}{s}\right) \right\rceil$$

steps, $E_n(x)$ reaches T, that is, $E_{n_1(x)}(x) = T$. $s \geq \kappa_1$ implies that $n_1(x) \leq \lceil \log_{1/a}(\frac{2(1-c-\varepsilon)}{\kappa_1}) \rceil = n_1^*$.

(2) $E_1(x) = (\ell_{2k-1}(x), \ell_{2k}(x))$ *(for some $1 \leq k \leq 3$) is an open interval with length $\kappa_2 = \frac{4}{a}\mathfrak{g}$.* If, for some n, $E_n(x)$ does not contain either $-1 + c + \varepsilon$ or $1 - c - \varepsilon$, then we have three possibilities for $E_{n+1}(x)$: (i) it does not contain any of these two points; (ii) it contains one of them; (iii) it equals T. In case (iii) we obtained what we wanted. In case (i), the length of $E_{n+1}(x)$ equals $\frac{1}{a}|E_n(x)| + \frac{2\mathfrak{g}}{a}$; in case (ii), we have $E_{n+n_1^*}(x) = T$ by (1) above, so we estimate the number of necessary iterations from below if we suppose that case (i) happens in each step then case (ii) in n_1^* number of steps. As in (1), we have a uniform bound for the number of iterations in (2): $n_2^* = \lceil \log_{1/a}(\frac{2(1-c-\varepsilon)}{\kappa_2}) \rceil$. Therefore, in this case, we have $E_{n_1^*+n_2^*}(x) = T$ for any x.

Summarizing these considerations, one obtains that for $n \geq n_1^* + n_2^*$, one has $E_n(x) = T$. $\square$

8. A uniformly positive kernel in the general case. The construction of T consists of two steps. We will call any open subset of $[-1, 1]$ a *pre-type space*. First, we inductively construct a sequence of pre-type spaces $T^0 \supset T^1 \supset \cdots \supset T^l$ and prove that T^r, $r = 0, \dots, l$, consists of 3^r disjoint open intervals of equal length. Those elements of T^l which are "far" from the endpoints of the components of T^l satisfy (39). Unfortunately, the same does not hold for the points close to the the boundary of the components of T^l. So, as a second step of the construction of T, we remove a small neighborhood of the boundary of T^l from T^l.

LEMMA 7. *There exists a restriction of the pre-type space* $(-1 + c, 1 - c)$ *to a closed set* T *such that the kernel* m *of the branching process* $\mathcal{Z}$ *with type space* T *satisfies*

(39) $\quad \exists \kappa > 0$ *such that* $\forall x \in T$, *the set* $E_1(x)$ *contains an interval of length* κ.

Further, T *consists of* 3^l *disjoint closed intervals of equal length for some* $l \in \mathbb{N}$. *Moreover,* 0 *is contained in the interior of* T.

PROOF. We recall that u^1, v^1 were defined in (32) and we take the pre-type space $T^0 := (-1 + c, 1 - c)$. If $v^k < u^k$, then we define $l := 0$ and the proof of (39) was achieved in Lemma 7A. So, we can assume that $u^k \le v^k$, $k = 1, 2$. To ensure that (39) holds, we need to remove the intervals $[u^1, v^1]$ and $[u^2, v^2]$ from the pre-type space T^0 (see Figure 9). So, we restrict ourselves to the next pre-type space: $T^1 = T^0 \setminus \{[u^1, v^1] \cup [u^2, v^2]\}$. The size of each of the intervals removed is $\varrho_1 := v^1 - u^1 = v^2 - u^2$. We define the second generation endpoints $u^{i_1 k}$ and $v^{i_1 k}$ as follows:

$$u^{i_1 k} = \pi_1(\{y = u^{i_1}\} \cap \ell_{2k}) \quad \text{and} \quad v^{i_1 k} = \pi_1(\{y = v^{i_1}\} \cap \ell_{2k-1}),$$

where $i_1 = 1, 2$ and $k = 1, 2, 3$; see Figure 9. If $v^{i_1 k} < u^{i_1 k}$, then we define $l := 1$. Otherwise, we continue defining the sets T^r and the endpoints of the subtracted intervals $v^{i_1 \dots i_r}$ and $u^{i_1 \dots i_r}$ ($i_1 = 1, 2, i_2, \dots, i_r = 1, 2, 3$) as follows: assuming that $u^{i_1 \dots i_{r-1}} \le v^{i_1 \dots i_{r-1}}$, we define the level-$r$ endpoints as

$$\begin{aligned} u^{i_1 \dots i_{r-1} k} &= \pi_1(\{y = u^{i_1 \dots i_{r-1}}\} \cap \ell_{2k}) \quad \text{and} \\ v^{i_1 \dots i_{r-1} k} &= \pi_1(\{y = v^{i_1 \dots i_{r-1}}\} \cap \ell_{2k-1}) \end{aligned} \tag{40}$$

for $i_1 = 1, 2$ and $i_2, \dots, i_{r-1}, k = 1, 2, 3$. Put

$$T_r = T_{r-1} \setminus \{[u^{i_1 i_2 \dots i_r}, v^{i_1 i_2 \dots i_r}], i_1 = 1, 2, i_2, \dots, i_r = 1, 2, 3\}. \tag{41}$$

The size of each of the intervals removed is $\varrho_r := v^{i_1 i_2 \dots i_r} - u^{i_1 i_2 \dots i_r}$. Using $\ell_{2k}(x) - \ell_{2k-1}(x) = 2\mathfrak{g}/a$ (see also the left-hand side of Figure 10), one can easily

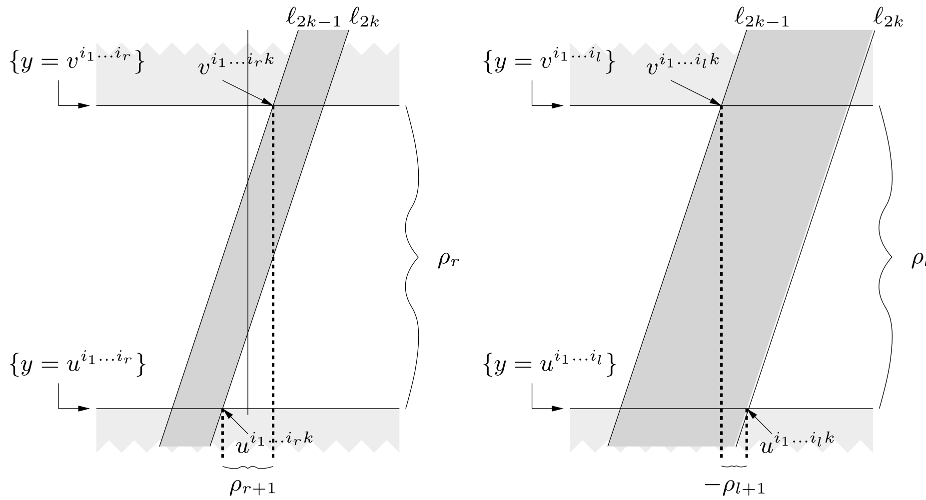

FIG. 10. *The recursion of* $\{\rho_r\}_r$. *On the left-hand side*, $r \le l-1$.

check that

$$\forall r \ge 1, \qquad \rho_{r+1} = a\rho_r - 2\mathbb{g} \quad \text{and} \quad \rho_1 = v^1 - u^1. \tag{42}$$

Consider the smallest $r \ge 1$ for which $v^{i_1 \ldots i_{r+1}} < u^{i_1 \ldots i_{r+1}}$ or, equivalently, $\rho_{r+1} < 0$. We then set $l = r$ and the recursion ends. The fact that l is finite is immediate from (42).

We can represent T^{l-1} and T^l as follows:

$$T^{l-1} = \bigcup_{j=1}^{3^{l-1}} (\gamma_j, \delta_j), \qquad T^l = \bigcup_{i=1}^{3^l} (\alpha_i, \theta_i).$$

Using (40), it follows from elementary geometry (see Figure 10) that

$$\begin{aligned} \forall i, \exists j, \exists k: \qquad \alpha_i &= \pi_1\big(\{(x,y) : y = \gamma_j\} \cap \ell_{2k-1}\big), \\ \theta_i &= \pi_1\big(\{(x,y) : y = \delta_j\} \cap \ell_{2k}\big). \end{aligned} \tag{43}$$

We need further restrictions because condition (39) is not satisfied around the endpoints α_i, β_i. Therefore, we remove sufficiently small intervals from both ends of each of the 3^l intervals of T^l. Namely, we define the type space of the process by

$$T(\varepsilon) := \bigcup_{i=1}^{3^l} [\alpha_i + \varepsilon, \beta_i - \varepsilon], \tag{44}$$

where

$$0 < \varepsilon < \frac{\mathbb{g}}{a} - \frac{1}{2}\rho_l. \tag{45}$$

This bound will be used in part (c) at the end of this proof. For any $j \in \{1, \dots, 3^{l-1}\}$, we can find $i' \in \{1, \dots, 3^l\}$ such that

$$[\gamma_j + \varepsilon, \delta_j - \varepsilon] = \bigcup_{m=0}^{2} [\alpha_{i'+m} + \varepsilon, \beta_{i'+m} - \varepsilon] \cup \bigcup_{h=1}^{2} R_h^{(j)}, \tag{46}$$

where $R_h^{(j)}$, $h = 1, 2$, are intervals of length $\rho_l + 2\varepsilon$; see Figure 10. Further, for every $1 \le i \le 3^l$, $1 \le j \le 3^{l-1}$, the set $(\alpha_i + \varepsilon, \beta_i - \varepsilon) \times (\gamma_j + \varepsilon, \delta_j - \varepsilon) \cap T(\varepsilon) \times T(\varepsilon)$ consists of three congruent squares aligned on top of each other, of side-length

$$s := \beta_i - \alpha_i - 2\varepsilon.$$

The distance between two neighboring squares is $\rho_l + 2\varepsilon$.

We now prove that (39) holds. That is, we want to estimate the length of the longest interval in $E_1(x)$ from below. The argument uses only elementary geometry.

For any $x \in T(\varepsilon)$, there is a unique $k \in \{1, 2, 3\}$ such that $E_1(x) \subseteq (\ell_{2k}(x), \ell_{2k-1}(x))$ holds. Using (24), one can immediately see that the length of the interval $(\ell_{2k}(x), \ell_{2k-1}(x))$ is $\frac{2\mathfrak{g}}{a}$. Geometrically, this means that the vertical line through x intersects the stripe S_k in a (vertical) interval of length $\frac{2\mathfrak{g}}{a}$.

Since there are many holes in $T(\varepsilon)$, for some $x \in T(\varepsilon)$, the set $E_1(x)$ consists of at most three subintervals of $(\ell_{2k}(x), \ell_{2k-1}(x))$; see Figure 10. We prove that the maximum length of these intervals is uniformly bounded away from zero.

Fix a component $[\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset T(\varepsilon)$ and let $x \in [\alpha_i + \varepsilon, \beta_i - \varepsilon]$. For this i, we choose j and k according to the formula (43). We now distinguish three possibilities for $x \in T(\varepsilon)$:

(a) first we assume that the intersection of the vertical line through x with the stripe S_k is not contained in the rectangle $[\alpha_i + \varepsilon, \beta_i - \varepsilon] \times [\gamma_j + \varepsilon, \delta_j - \varepsilon]$ [see Figure 10], then, using the fact that the slope of the lines ℓ_m, $m = 1, \dots, 6$, is $1/a > 3$, by elementary geometry, we obtain that the set $E_1(x)$ contains an interval of length $\kappa := \frac{1}{a}\varepsilon - \varepsilon > 2\varepsilon > 0$ (see Figure 10B);

(b) next, we assume that there exists $m \in \{0, 1, 2\}$ such that the intersection of the vertical line through x with the stripe S_k is contained in the square $[\alpha_i + \varepsilon, \beta_i - \varepsilon] \times [\alpha_{i'+m} + \varepsilon, \beta_{i'+m} - \varepsilon]$, where i' is defined as in (46)—in this case, the set $E_1(x) = (\ell_{2k}(x), \ell_{2k-1}(x))$ and then the assertion holds with the choice of $\kappa := \frac{2\mathfrak{g}}{a} > 0$ [see (45)];

(c) finally, we assume that the intersection of the vertical line through x with the stripe S_k has a nonempty intersection with one of the rectangles $[\alpha_i + \varepsilon, \beta_i - \varepsilon] \times R_h^{(j)}$, $h = 1, 2$—in this case, by elementary geometry (see Figure 10A), $E_1(x)$

contains an interval of length at least

$$\begin{aligned}\kappa &:= \min\left\{s, \frac{1}{2}\cdot\left(\ell_{2k-1}(x)-\ell_{2k}(x)\right)-(\rho_l+2\varepsilon)\right\}\\ &= \min\left\{s, \frac{1}{2}\left(\frac{2\mathfrak{g}}{a}-(\rho_l+2\varepsilon)\right)\right\}.\end{aligned}$$

It follows from (45) that $\kappa > 0$. □

We will now deal with the problem of still having a kernel with largest eigenvalue larger than 1.

LEMMA 8. *Let m^ε be the kernel in Lemma 7 with type space $T = T(\varepsilon)$. One can choose ε so small that the largest eigenvalue of m^ε is larger than* 1.

PROOF. Changing T^0 to T^l in the proof of Lemma 8A, we obtain the proof of Lemma 8. More precisely, it is enough to prove that $4a$ is an eigenvalue of the operator $\mathcal{T}_l$ with eigenfunction $h(x) = \mathbf{1}_{T^l}(x)$, where T^l is defined in the proof of Lemma 7:

$$\begin{aligned}\mathcal{T}_l h(y) &= \int_{\mathbb{R}} h(x)m(x,y)\,\mathrm{d}x\\ &= \int_{\mathbb{R}} h(x)\left(\sum_{i=1}^{4} f_{x,i}(y)\right)\mathbf{1}_{T^l}(y)\,\mathrm{d}x\\ &= 4ah(y)\int_{T^l}\sum_{i=1}^{4} f_{\triangle}(ay+c_i-x)\,\mathrm{d}x\\ &= 4ah(y),\end{aligned}$$

provided we show that for all $i = 1, 2, 3, 4$ and for all $y \in T^l$,

$$\int_{T^l} f_{\triangle}(ay+c_i-x)\,\mathrm{d}x = 1.$$

So, we have to show that for all $y \in T^l$ and for $i = 1, 2, 3, 4$, we have

$$\{x : f_{\triangle}(ay+c_i-x) > 0\} \subset T^l. \tag{47}$$

This holds since we have constructed the intermediate type space T^l so that this property is satisfied; see the left figure in Figure 11. We have subtracted intervals of the form $(u^{i_1\ldots i_r k}, v^{i_1\ldots i_r k})$ in (41) during the construction of successive intermediate type spaces T^{r+1}, $r = 0, \ldots, l-1$. If $y \in T^{r+1}$, then each interval of the form $(u^{i_1\ldots i_r k}, v^{i_1\ldots i_r k})$ is disjoint from $[\ell_{2k-1}^{-1}(y), \ell_{2k}^{-1}(y)]$ for all $y \in T^l$ and $k = 1, 2, 3$. Therefore, for any $y \in T^l$, we have $[\ell_{2k-1}^{-1}(y), \ell_{2k}^{-1}(y)] \subset T^l$. Further,

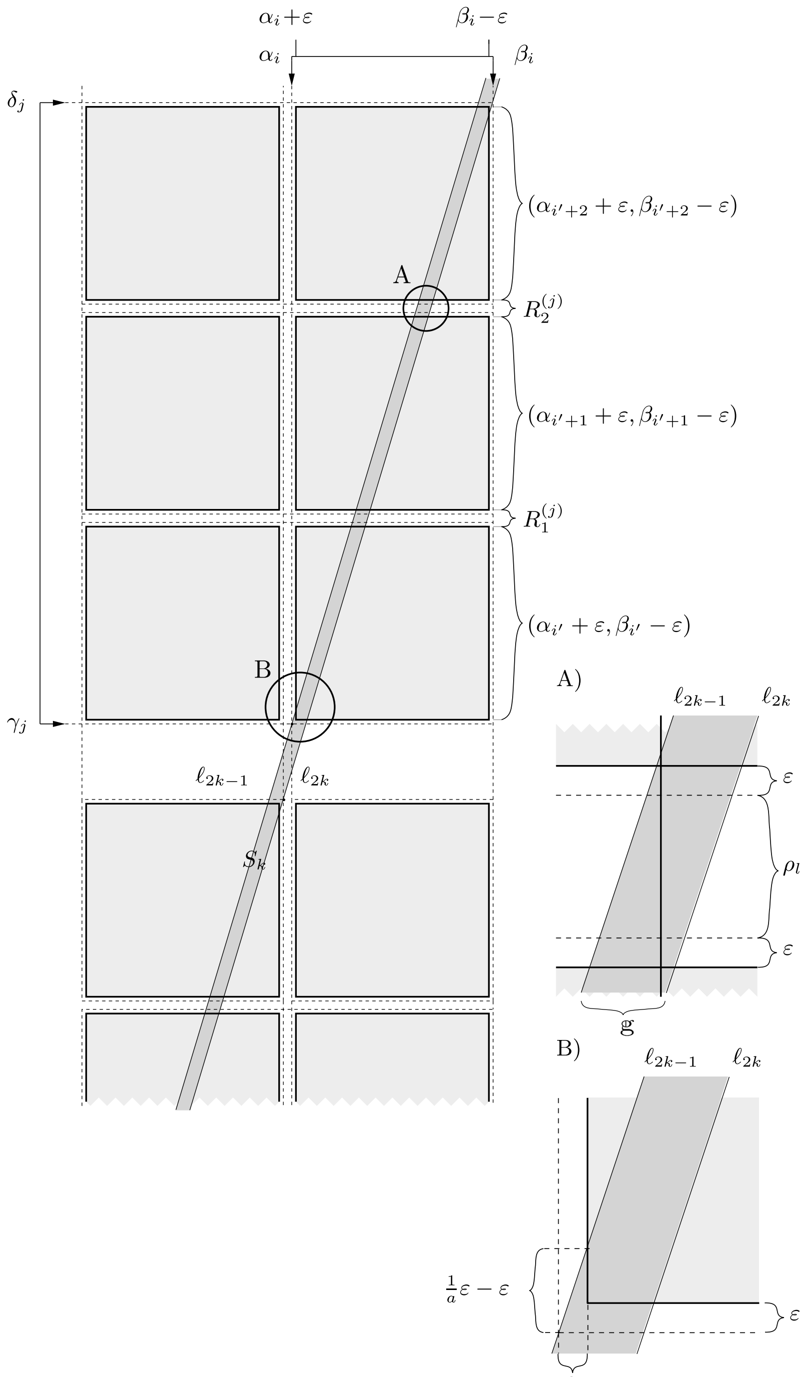

FIG. 11. *Stripe S_k and level-l squares.*

for any $i = 1, 2, 3, 4$, there exists a positive integer k_i ($k_1 = 1, k_2 = k_4 = 2, k_3 = 3$) such that

$$\{x : f_\triangle(ay + c_i - x) > 0\} = (\ell_{2k_i-1}^{-1}(y), \ell_{2k_i}^{-1}(y)).$$

Hence, (47) holds.

The proof is now completed analogously to the proof of Lemma 8A. $\square$

LEMMA 9. *Let T be as in Lemma 8. There then exists an n such that for all $x \in T$, $\{y : m_n(x, y) > 0\} = T$.*

Since the function $m_n(\cdot, \cdot)$ is piecewise continuous on the compact set T, Lemma 9 implies that there exists an $a_{\min} > 0$ such that $m(x, y) \geq a_{\min}$ for any $x, y \in T$. Further, using the fact that $m(x, \cdot)$ is bounded, we immediately obtain that $a_{\max} := \sup_{x\in T} \mathbb{E}_x[\mathcal{Z}_1^2(T)]$ is finite. Therefore, we have the following result.

COROLLARY 2. *Let T be as in Lemma 8. The branching process $\mathcal{Z}$ with type space T satisfies the conditions* (C1) *and* (C3).

PROOF OF LEMMA 9. We will prove the lemma in two steps. Recall the definition of $E_n(x)$: $E_n(x) = \{y : m_n(x, y) > 0\}$.

STEP 1. *$\forall x \in T, \exists i, n$ such that $[\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset E_n(x)$ implies that $E_{n+l}(x) = T$.*

STEP 2. *There exists an N such that for every $x \in T$, we can find a positive integer $n(x) \leq N$ such that the following holds*:

$$\exists i, \qquad [\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset E_{n(x)}(x).$$

As a corollary of these two statements, we obtain that the assertion of the lemma holds with the choice $n = N + l$. Namely, for any $x \in T$, we have $E_{N+l}(x) = T$.

PROOF OF STEP 1. To verify Step 1, we first observe that by (38), we have

$$\begin{aligned} E_{n+1}(x) &= \bigcup_{y\in E_n(x)} E_1(y) \\ &= \bigcup_{y\in E_n(x)} \bigl((\ell_2(y), \ell_1(y)) \cup (\ell_4(y), \ell_3(y)) \cup (\ell_6(y), \ell_5(y))\bigr) \cap T. \end{aligned} \tag{48}$$

Fix an $i \in \{1, \dots, 3^l\}$. First, we define $\alpha_{i,l-r}$ and $\beta_{i,l-r}$ for $r = 0, \dots, l$, inductively. For $r = 0$, let $(\alpha_{i,l}, \beta_{i,l}) := (\alpha_i, \beta_i)$. Assume that we have already defined

$(\alpha_{i,l-r}, \beta_{i,l-r})$. Using (40), we define $\alpha_{i,l-(r+1)}$ and $\beta_{i,l-(r+1)}$ as the unique numbers satisfying

$$\alpha_{i,l-r} = \pi_1\bigl(\{(x,y) : y = \alpha_{i,l-(r+1)}\} \cap \ell_{2k(r)-1}\bigr), \\ \beta_{i,l-r} = \pi_1\bigl(\{(x,y) : y = \beta_{i,l-(r+1)}\} \cap \ell_{2k(r)}\bigr), \tag{49}$$

where $k(r) = 1, 2, 3$. Then, by the construction, we have $(\alpha_{i,0}, \beta_{i,0}) = (-1+c, 1-c)$. Let $x \in T$. According to the assumption of Step 1, we can find i, n such that

$$[\alpha_i + \varepsilon, \beta_i - \varepsilon] = (\alpha_i, \beta_i) \cap T \subset E_n(x) \tag{50}$$

holds. Using induction, we prove that

$$E_{n+r}(x) \supset (\alpha_{i,l-r}, \beta_{i,l-r}) \cap T \qquad \text{for } 0 \le r \le l. \tag{51}$$

Namely, for $r = 0$, the assertion in the induction is identical to (50). We now suppose that (51) holds for $r < l$. By (48) and (49), we have

$$\begin{aligned} E_{n+r+1}(x) &= \bigcup_{y \in E_{n+r}(x)} E_1(y) \\ &\supset \bigcup_{y \in (\alpha_{i,l-r}, \beta_{i,l-r}) \cap T} \bigl(\ell_{2k(r)}(y), \ell_{2k(r)-1}(y)\bigr) \cap T \\ &= \bigl(\alpha_{i,l-(r+1)}, \beta_{i,l-(r+1)}\bigr) \cap T, \end{aligned}$$

which completes the proof of (51). We apply (51) for $r = l$. This yields that $E_{n+l} = (-1+c, 1-c) \cap T = T$ holds. $\square$

PROOF OF STEP 2. First, observe that the largest interval in $E_1(x)$ either has an endpoint that is an endpoint of a connected component of T [this happens in case (a) and (c) in the end of the proof of Lemma 7] or $E_1(x) = (\ell_{2k_1}(x), \ell_{2k_1-1}(x))$ [which is case (b) in the same proof]. However, in the last case, using (48), after N_1 steps, where N_1 is the smallest solution of the inequality $(\frac{2}{a})^{N_1} \cdot \frac{2\mathfrak{g}}{a} > s$, we obtain that the largest interval contained in $E_{N_1}(x)$ has an endpoint of a connected component of T (see Figure 10) and its length is greater than κ. In this way, because of the symmetry between the endpoints of the connected components of T, from now on, we may assume that $[\alpha_i + \varepsilon, \alpha_i + \varepsilon + z_1) \subset E_1(x)$, where $z_1 \ge \kappa$. Using (48), we can write

$$\begin{aligned} E_2(x) &\supset \bigcup_{y \in [\alpha_i + \varepsilon, \alpha_i + \varepsilon + z_1)} (\ell_{2k_1}(y), \ell_{2k_1-1}(y)) \cap T \\ &= \bigl(\ell_{2k_1}(\alpha_i + \varepsilon), \ell_{2k_1-1}(\alpha_i + \varepsilon + z_1)\bigr) \cap T \\ &= \bigl[\alpha^{(2)} + \varepsilon, \alpha^{(2)} + \varepsilon + z_2\bigr) \cap T \end{aligned} \tag{52}$$

for some $k_1 \in \{1, 2, 3\}$, left endpoint $\alpha^{(2)} \in T$ and $z_2 > \frac{1}{a} z_1 \ge \frac{1}{a}\kappa$. If $z_2 < s$, then the largest connected component of $E_2(x)$ has a left endpoint of one of the connected components of T, $\alpha^{(2)}$, but the other endpoint is in the interior of the same connected component of T. If $z_2 \geq s$, then $E_2(x)$ clearly contains a connected component of T. For $E_n(x)$, $n \geq 3$, we can inductively define k_n, left endpoint $\alpha^{(n)}$ and length z_n in the same way as above. Observe that $z_n > (\frac{1}{a})^{n-1}\kappa$ for any $n \geq 2$. Let N_2 the smallest solution of the inequality $(\frac{1}{a})^{N_2-1}\kappa > s$. Then, $E_{N_2}(x)$ contains a connected component of T.

Let $N = N_1 + N_2$. Then, $E_N(x)$ contains a connected component of T. □

9. Uniform exponential growth. In this section, we want to prove an extension of Theorem 3 stating that the population can grow uniformly exponentially starting from any element of a special interval. For the precise statement, see Lemma 12.

First, we will determine the density of the measure $\mathbb{P}_x(\mathcal{Z}_1(A) \in \cdot)$. We use the notation of Lemma 6 and define, for $x_1, x_2 \in T$,

$$\mathbb{P}_{x_1,x_2} := \mathbb{P}_{x_1} \otimes \mathbb{P}_{x_2},$$

the convolution of the measures $\mathbb{P}_{x_1}$ and $\mathbb{P}_{x_2}$. Recalling the definitions of A_1^+, A_2^+, A_3, A_2^-, A_1^- in equation (27) and the definition of $f_{x,i}$, $i = 1, 2, 3, 4$, in equation (31), we can state the following lemma.

LEMMA 10. *For $x \in T$, $A \subset T$ and a natural number L, we have the following equation for any $n \geq 1$:*

$$\begin{aligned} \mathbb{P}_x(\mathcal{Z}_{n+1}(A) = L) &= \int_T \mathbb{P}_z(\mathcal{Z}_n(A) = L) h_1(x, z)\,\mathrm{d}z \\ &\quad + \int_T \int_T \mathbb{P}_{z_1,z_2}(\mathcal{Z}_n(A) = L) h_2(x, z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2, \end{aligned} \tag{53}$$

where $h_1(x, z) : T \times T \to \mathbb{R}_+$ and $h_2(x, z_1, z_2) : T \times T \times T \to \mathbb{R}_+$ are defined as

$$h_1(x,z) = \begin{cases} f_{x,1}(z), & \text{if } x \in A_1^+ \cap T, \\ f_{x,1}(z) + 2f_{x,2}(z)\Bigl(1 - \int_T f_{x,4}(y)\,\mathrm{d}y\Bigr), & \text{if } x \in A_2^+ \cap T, \\ 2f_{x,2}(z)\Bigl(1 - \int_T f_{x,4}(y)\,\mathrm{d}y\Bigr), & \text{if } x \in A_3 \cap T, \\ f_{x,3}(z) + 2f_{x,2}(z)\Bigl(1 - \int_T f_{x,4}(y)\,\mathrm{d}y\Bigr), & \text{if } x \in A_2^- \cap T, \\ f_{x,3}(z), & \text{if } x \in A_1^- \cap T \end{cases}$$

and

$$h_2(x, z_1, z_2) = \begin{cases} 2f_{x,2}(z_1) f_{x,4}(z_2), & \text{if } x \in (A_3 \cup A_2^+ \cup A_2^-) \cap T, \\ 0, & \text{otherwise.} \end{cases}$$

Both are bounded and piecewise uniformly continuous functions in x on T for any fixed $z, z_1, z_2 \in T$.

PROOF. The decomposition (53) is obtained from the Chapman–Kolmogorov equation, that is, by conditioning on the first generation. In the corresponding formula (54), we use one of the conclusions of Lemma 6, that is, that exactly two squares in generation 1 can only be generated by Q_2 and Q_4:

$$\begin{aligned}\mathbb{P}_x\big(\mathcal{Z}_{n+1}(A)=L\big) &= \int_T \mathbb{P}_z\big(\mathcal{Z}_n(A)=L\big)\mathbb{P}_x\big(\mathcal{Z}_1(\mathrm{d}z)=1\big)\\ &\quad + \int_T\int_T \mathbb{P}_{z_1,z_2}\big(\mathcal{Z}_n(A)=L\big)\mathbb{P}_x\big(\mathcal{Z}_1^2(\mathrm{d}z_1)=1, \mathcal{Z}_1^4(\mathrm{d}z_2)=1\big).\end{aligned}\tag{54}$$

We have to determine the density function $h_1(x,z)$ of exactly one descendant with type $\mathrm{d}z$ and the density function $h_2(x,z_1,z_2)$ of exactly two descendants with type $\mathrm{d}z_1\,\mathrm{d}z_2$. To perform the computation, we note that the statement of Lemma 6 remains valid if we replace $\Phi_i(x)$ by $X_i(x)$ because of the definition of $X_i(x)$ in equation (30). One can decompose the probability of having exactly one descendant such that the type of this descendant falls into the set $(-\infty,z]$ (for any real z) as follows:

$$\mathbb{P}_x\big(\mathcal{Z}_1((-\infty,z])=1\big)=\sum_{i=1}^4 \mathbb{P}\big(X_i(x)\in(-\infty,z], X_j(x)=\Theta, \forall j\neq i\big).$$

The decomposition in Lemma 6, together with the remark in the first paragraph of this proof, implies that $\{X_2(x)\neq\Theta\}\cup\{X_4(x)\neq\Theta\}$, $\{X_1(x)\neq\Theta\}$ and $\{X_3(x)\neq\Theta\}$ are disjoint events for any $x\in T$. Therefore, one obtains

$$\begin{aligned}\mathbb{P}_x\big(\mathcal{Z}_1((-\infty,z])=1\big) &= \mathbb{P}\big(X_1(x)\in(-\infty,z]\big)\\ &\quad + 2\mathbb{P}\big(X_2(x)\in(-\infty,z]\big)\mathbb{P}\big(X_4(x)=\Theta\big)\\ &\quad + \mathbb{P}\big(X_3(x)\in(-\infty,z]\big),\end{aligned}$$

using the fact that $X_2(x)$ and $X_4(x)$ are independent and identically distributed. Since $X_i(x)$ has density $f_{x,i}$, one gets that this equals

$$\begin{aligned}&\int_{(-\infty,z]} f_{x,1}(y)\,\mathrm{d}y\cdot\mathbf{1}_{(A_1^+\cup A_2^+)\cap T}(x)\\ &\quad + 2\int_{(-\infty,z]} f_{x,2}(y)\,\mathrm{d}y\left(1-\int_T f_{x,4}(y)\,\mathrm{d}y\right)\cdot\mathbf{1}_{(A_3\cup A_2^+\cup A_2^-)\cap T}(x)\\ &\quad + \int_{(-\infty,z]} f_{x,3}(y)\,\mathrm{d}y\cdot\mathbf{1}_{(A_1^-\cup A_2^-)\cap T}(x)\\ &\quad = \int_{(-\infty,z]} h_1(x,y)\,\mathrm{d}y.\end{aligned}$$

Let us next deal with exactly two descendants with types falling into $(-\infty,z_1]$ (resp. $(-\infty,z_2]$). This probability equals

$$2\mathbb{P}\big(X_2(x)\in(-\infty,z_1], X_4(x)\in(-\infty,z_2]\big).$$

Since $X_2(x)$ and $X_4(x)$ are independent and identically distributed, one obtains that this equals

$$\begin{aligned}2\int_{(-\infty,z_1]} f_{x,2}(y)\,\mathrm{d}y \int_{(-\infty,z_2]} f_{x,4}(y)\,\mathrm{d}y\cdot \mathbf{1}_{(A_3\cup A_2^+\cup A_2^-)\cap T}(x)\\ = \int_{(-\infty,z_1]}\int_{(-\infty,z_2]} h_2(x,y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2.\end{aligned}$$

Summarizing these considerations, one obtains (53).

The piecewise continuity of $h_1(x,z)$ and $h_2(x,z_1,z_2)$ in x follows from the definitions of h_1 and h_2, respectively. Since they have compact support, h_1 and h_2 are piecewise uniformly continuous in x. $\square$

Let $A\subset T$ such that the Lebesgue measure of A is positive. Let $W_n(A)=\mathcal{Z}_n(A)\rho^{-n}$ and $W(A)=\lim_{n\to\infty}W_n(A)$, which almost surely exists by Theorem 3. We need a stronger result: the random variable $W(A)$ is strictly separated from 0 with uniformly positive probability for some neighborhood of the initial type 0. This is shown in the next lemma.

LEMMA 11. *For some neighborhood $J\subset T$ of* 0 *and positive numbers y and r, we have*

$$\inf_{x\in J}\mathbb{P}_x\big(W(A)>y\big)\geq r. \tag{55}$$

PROOF. Lemma 10 implies that

$$\begin{aligned}\mathbb{P}_x\big(W_{n+1}(A)\leq y\big) &= \mathbb{P}_x\big(\mathcal{Z}_{n+1}(A)\leq \rho^{n+1}y\big)\\ &= \int_T \mathbb{P}_z\big(W_n(A)\leq\rho y\big)h_1(x,z)\,\mathrm{d}z\\ &\quad+\int_T\int_T \mathbb{P}_{z_1,z_2}\big(W_n(A)\leq \rho y\big)h_2(x,z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2.\end{aligned}\tag{56}$$

We will investigate the convergence of the last two terms in (56).

Theorem 3 implies that we have, for all $z\in T$,

$$\lim_{n\to\infty}\mathbb{P}_z\big(W_n(A)\leq y\big)=\mathbb{P}_z\big(W(A)\leq y\big) \tag{57}$$

if $y\in\mathrm{Cont}(\mathbb{P}_{z,A})$, where $\mathrm{Cont}(\mathbb{P}_{z,A})$ denotes the set of continuity points of the distribution function on the right-hand side of (57).

Next, we seek the weak convergence of the measure $\mathbb{P}_{z_1,z_2}(W_n(A)\in\cdot)$, which is the convolution of the measures $\mathbb{P}_{z_1}(W_n(A)\in\cdot)$ and $\mathbb{P}_{z_2}(W_n(A)\in\cdot)$. Since they are weakly convergent, the convolution is also weakly convergent. So,

$$\lim_{n\to\infty}\mathbb{P}_{z_1,z_2}\big(W_n(A)\leq y\big)=\mathbb{P}_{z_1,z_2}\big(W(A)\leq y\big) \tag{58}$$

if $y\in\mathrm{Cont}(\mathbb{P}_{z_1,z_2,A})$.

Let, for $z, z_1, z_2 \in T$, $y > 0$ and ε a small positive number (to be chosen later), $t_y := t(z, z_1, z_2; y, \varepsilon)$ be a real number such that

$$y \le t_y < y + \varepsilon \quad \text{and} \quad \rho t_y \in \mathrm{Cont}(\mathbb{P}_{z,A}) \cap \mathrm{Cont}(\mathbb{P}_{z_1,z_2,A}),$$

and let us define the following two functions:

$$\begin{aligned}
\theta_{n+1}(x, y, A) &= \int_T \mathbb{P}_z\bigl(W_n(A) \le \rho t_y\bigr) h_1(x, z)\,\mathrm{d}z \\
&\quad + \int_T \int_T \mathbb{P}_{z_1,z_2}\bigl(W_n(A) \le \rho t_y\bigr) h_2(x, z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2, \\
\theta(x, y, A) &= \int_T \mathbb{P}_z\bigl(W(A) \le \rho t_y\bigr) h_1(x, z)\,\mathrm{d}z \\
&\quad + \int_T \int_T \mathbb{P}_{z_1,z_2}\bigl(W(A) \le \rho t_y\bigr) h_2(x, z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2.
\end{aligned}$$

Using the decomposition (56), the definition of t_y and the right-continuity of distribution functions, we can derive the following bounds:

$$\mathbb{P}_x\bigl(W_{n+1}(A) \le y\bigr) \le \theta_{n+1}(x, y, A) \le \mathbb{P}_x\bigl(W_{n+1}(A) \le y + \varepsilon\bigr).$$

By using (57), (58) and the bounded convergence theorem, we get that $\theta_n(x, y, A)$ converges as $n \to \infty$, so

$$\mathbb{P}_x\bigl(W(A) \le y\bigr) \le \theta(x, y, A) \le \mathbb{P}_x\bigl(W(A) \le y + \varepsilon\bigr). \tag{59}$$

Using the piecewise continuity of h_1 and h_2 in x (Lemma 10) and bounded convergence, one can see that $\theta_n(x, y, A)$ and $\theta(x, y, A)$ are piecewise continuous on T in x.

Using inequality (13) in Theorem 3 and the right-continuity of distribution functions, we can find two positive numbers r, u such that $\mathbb{P}_0(W(A) > u) > 2r$ or, equivalently, $\mathbb{P}_0(W(A) \le u) \le 1 - 2r$. Let $y = u - \varepsilon$ for some positive $\varepsilon < u$. Using the second inequality of (59), one gets $\theta(0, y, A) \le \mathbb{P}_0(W(A) \le y + \varepsilon) \le 1 - 2r$. Since $\theta(x, y, A)$ is piecewise continuous on T, there exist an interval $J \subset T$ which is a neighborhood of 0 such that the bound $\theta(x, y, A)$ is uniformly smaller than 1 on this interval, that is, $\sup_{x \in J} \theta(x, y, A) \le 1 - r$. The first inequality of (59) implies that $\sup_{x \in J} \mathbb{P}_x(W(A) \le y) \le \sup_{x \in J} \theta(x, y, A) \le 1 - r$, which yields the required bound in (55). □

LEMMA 12. *There exist two positive numbers η, r, an integer N and a number K with $0 < K < \frac{1}{8}$ such that*

$$\inf_{n \ge N} \inf_{x \in [-K, K]} \mathbb{P}_x\bigl(\mathcal{Z}_n([-K, K]) > \eta \rho^n\bigr) > \frac{r}{2}.$$

PROOF. We apply Lemma 11 with $A = T$ and obtain the numbers y, r and the set J. Let K be a positive number such that $K < \frac{1}{8}$ and $[-K, K] \subset J$. We then have

$$\inf_{x \in [-K,K]} \mathbb{P}_x\big(W(T) > y\big) \geq r.$$

Using Theorem 3, we get that

$$W([-K, K]) = \gamma W(T)$$

holds $\mathbb{P}_x$ almost surely for any $x \in T$, where

$$\gamma = \frac{\int_{[-K,K]} \nu(z)\,\mathrm{d}z}{\int_T \nu(z)\,\mathrm{d}z}.$$

Hence, we have the bound

$$\inf_{x \in [-K,K]} \mathbb{P}_x\big(W([-K, K]) > \eta + \varepsilon\big) > r,$$

where $\eta + \varepsilon = \gamma y$ for some positive η and ε. This and the second inequality of (59) together imply that $\theta(x, \eta, [-K, K])$ is uniformly smaller than 1:

$$\sup_{x \in [-K,K]} \theta(x, \eta, [-K, K]) \leq \sup_{x \in [-K,K]} \mathbb{P}_x\big(W([-K, K]) \leq \eta + \varepsilon\big) \leq 1 - r. \tag{60}$$

We will show that $\theta_n(x, \eta, [-K, K])$ converges uniformly to $\theta(x, \eta, [-K, K])$ on $[-K, K]$ as n tends to infinity. Writing

$$E_n := W_n([-K, K]) \leq \rho\eta_t \quad \text{and} \quad E := W([-K, K]) \leq \rho\eta_t,$$

using trivial estimations, one gets the following chain of inequalities:

$$\begin{aligned}
&\sup_{x \in [-K,K]} |\theta_{n+1}(x, \eta, [-K, K]) - \theta(x, \eta, [-K, K])| \\
&\quad \leq \sup_{x \in [-K,K]} \int_T |\mathbb{P}_z(E_n)\mathbb{P}_z(E)| h_1(x, z)\,\mathrm{d}z \\
&\qquad + \sup_{x \in [-K,K]} \int_T \int_T |\mathbb{P}_{z_1,z_2}(E_n) - \mathbb{P}_{z_1,z_2}(E)| h_2(x, z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 \\
&\quad \leq \sup_{x,z \in T} h_1(x, z) \cdot \int_T |\mathbb{P}_z(E_n) - \mathbb{P}_z(E)|\,\mathrm{d}z \\
&\qquad + \sup_{x,z_1,z_2 \in T} h_2(x, z_1, z_2) \cdot \int_T \int_T |\mathbb{P}_{z_1,z_2}(E_n) - \mathbb{P}_{z_1,z_2}(E)|\,\mathrm{d}z_1\,\mathrm{d}z_2.
\end{aligned}$$

By bounded convergence, both integrals in the last expression converge to 0. The suprema are finite since h_1 and h_2 are bounded (see Lemma 11). So,

$\theta_n(x, \eta, [-K, K])$ uniformly converges to $\theta(x, \eta, [-K, K])$ on $[-K, K]$. Therefore, there exists an index N such that for $n \geq N$,

$$\sup_{x\in[-K,K]} |\theta_n(x, \eta, [-K, K]) - \theta(x, \eta, [-K, K])| \leq \frac{r}{2}.$$

Using the first inequality of (59), the triangular inequality, (60) and Lemma 11, one can write

$$\begin{aligned}\sup_{x\in[-K,K]} \mathbb{P}_x\big(W_n([-K, K]) \leq \eta\big) &\leq \sup_{x\in[-K,K]} \theta_n(x, \eta, [-K, K]) \\ &\leq \sup_{x\in[-K,K]} \theta(x, \eta, [-K, K]) \\ &\quad + \sup_{x\in[-K,K]} |\theta_n(x, \eta, [-K, K]) \\ &\qquad\qquad - \theta(x, \eta, [-K, K])| \\ &\leq 1 - r + \frac{r}{2} = 1 - \frac{r}{2}\end{aligned}$$

for $n \geq N$. This gives the conclusion of the lemma. $\square$

10. The proof of the Main Lemma. We first repeat the Main Lemma.

MAIN LEMMA. *There exist three positive numbers δ, q, K and an index N such that*

$$\inf_{n>N} \inf_{x\in[-K,K]} \mathbb{P}_x\big(\mathcal{Z}_n([0, K]) > \delta\rho^n \,\&\, \mathcal{Z}_n([-K, 0]) > \delta\rho^n\big) > q.$$

PROOF. Take K as defined in Lemma 12. Since $[-K, K] = [-K, 0] \cup [0, K]$ and type 0 has probability 0 to occur, it follows directly from Lemma 12 that one of $\mathbb{P}_x(\mathcal{Z}_n([0, K]) > \delta\rho^n)$ and $\mathbb{P}_x(Z_n([-K, 0]) > \delta\rho^n)$ is larger than $r/4$ for all $x \in [-K, K]$ and $n > N$. But, then, by symmetry, *both* of these probabilities are larger than $r/4$.

Now, take any $x \in [-K, K]$. Since $K < \frac{1}{8}$, it follows that with a positive probability denoted by $p_{2,4}$, in the first generation, the squares Q_2 and Q_4—with respective types x_2 and x_4 from a subinterval of $[-K, K]$—will be present. But, by the above, these two squares will, independently of each other and with probability at least $r/4$, generate more than $\delta\rho^n$ squares with type in $[0, K]$ (resp. $[-K, 0]$) in generation $n + 1$. Thus, for all $x \in [-K, K]$ and $n > N$,

$$\mathbb{P}_x\big(\mathcal{Z}_{n+1}([0, K]) > \delta\rho^n \,\&\, \mathcal{Z}_{n+1}([-K, 0]) > \delta\rho^n\big) > p_{2,4} \cdot \frac{r}{4} \cdot \frac{r}{4}.$$

So, replacing δ by δ/ρ, N by $N + 1$ and defining $q = p_{2,4}r^2/16$, this proves the Main Lemma. $\square$

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