The Annals of Probability
2011, Vol. 39, No. 2, 439–470
DOI: 10.1214/10-AOP555
© Institute of Mathematical Statistics, 2011

AN EXTENSION OF THE LÉVY CHARACTERIZATION TO FRACTIONAL BROWNIAN MOTION

BY YULIYA MISHURA[1] AND ESKO VALKEILA[2]

Kiev University and Aalto University

Assume that X is a continuous square integrable process with zero mean, defined on some probability space $(\Omega, \mathrm{F}, \mathrm{P})$. The classical characterization due to P. Lévy says that X is a Brownian motion if and only if X and $X_t^2 - t$, $t \geq 0$, are martingales with respect to the intrinsic filtration F^X. We extend this result to fractional Brownian motion.

1. Introduction. In classical stochastic analysis, Lévy's characterization result for standard Brownian motion is a fundamental result. We extend Lévy's characterization result to fractional Brownian motion, giving three necessary and sufficient properties for the process X to be a fractional Brownian motion. Fractional Brownian motion is a self-similar Gaussian process with stationary increments. However, these two properties are not explicitly present in the three conditions we shall give.

Fractional Brownian motion is a popular model in applied probability, in particular, in teletraffic modeling and, to some extent, in finance. Fractional Brownian motion is not a semimartingale and there has been much research on how to define stochastic integrals with respect to fractional Brownian motion. A large part of the developed theory depends on the fact that fractional Brownian motion is a Gaussian process. Since we want to prove that X is a special Gaussian process, we cannot use this machinery for our proof. Lévy's characterization result is based on Itô calculus. We cannot perform computations using the process X. Instead, we use the representation of the process X with respect to a certain martingale. In this way, we can perform computations using methods from classical stochastic analysis.

Notation and definitions. We use the following notation: $\overset{L^p(\mathrm{P})}{\to}$ means convergence in the space $L^p(\mathrm{P})$, $\overset{\mathrm{P}}{\to}$ (resp., $\overset{\text{a.s.}}{\to}$) means convergence in probability (resp., almost sure convergence) and $B(a,b)$ is the beta integral $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$, defined for $a, b > 0$. The notation $X_n \leq Y + o_{\mathrm{P}}(1)$ means

Received March 2008; revised February 2010.

[1]Supported in part by by the Suomalainen Tiedeakatemia.

[2]Supported by the Academy of Finland Grants 210465 and 212875.

AMS 2000 subject classifications. Primary 60G15; secondary 60E05, 60H99.

Key words and phrases. Fractional Brownian motion, Lévy theorem.

that we can find random variables ϵ_n such that $\epsilon_n = o_{\mathrm{P}}(1)$ and $X_n \le Y + \epsilon_n$. If, in addition, we have $X = \mathrm{P} - \lim X_n$ in such a situation, then $X \le Y$.

If M is a continuous square integrable martingale, then the bracket of M is denoted by $[M]$. Recall that, in this case, we have

$$[M]_t = \mathrm{P} - \lim_{|\pi^n| \to 0} \sum_{k=1}^{n} (M_{t_k} - M_{t_{k-1}})^2.$$

Fractional Brownian motion. A continuous square integrable centered process $X = (X_t)_{t \ge 0}$ with $X_0 = 0$ is a *fractional Brownian motion* with self-similarity index $H \in (0, 1)$ if it is a Gaussian process with zero mean and covariance function

$$\mathrm{E}(X_s X_t) = \tfrac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \qquad s, t \ge 0. \tag{1.1}$$

If X is a continuous Gaussian process with covariance (1.1), then, obviously, X has stationary increments and X is self-similar with index H. Mandelbrot named the Gaussian process X from (1.1) *fractional Brownian motion* and proved an important representation result for fractional Brownian motion in terms of standard Brownian motion in [3]. For results concerning fractional Brownian motion before Mandelbrot, we refer to [5].

Characterization of fractional Brownian motion. Throughout this paper, we work with special partitions. For $t > 0$, we put $t_k := t\frac{k}{n}$, $k = 0, \ldots, n$. Further, let F^X be the filtration generated by the process X. Fix $H \in (0, 1)$. Fractional Brownian motion has the following three properties:

(a) the sample paths of the process X are β-Hölder continuous for any $\beta \in (0, H)$;
(b) for $t > 0$, we have

$$n^{2H-1} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2 \overset{L^1(\mathrm{P})}{\longrightarrow} t^{2H} \tag{1.2}$$

as $n \to \infty$;
(c) the process

$$M_t = \int_0^t s^{1/2-H} (t - s)^{1/2-H} \, dX_s \tag{1.3}$$

is a martingale with respect to the filtration F^X.

If the process X satisfies (a), then we say that it is *Hölder up to H*. The property (b) characterizes the *weighted quadratic variation* of the process X and the process M in (c) is the *fundamental martingale* of X. It is a martingale with the bracket $c_H t^{2-2H}$ for some constant c_H and is actually a time-changed Brownian motion, up to a constant. It follows from property (a) that the integral (1.3) can be understood as a Riemann–Stieltjes integral (see [6] and Section 2.3 for more details).

Fractional Brownian motion satisfies property (a): from (1.1), we have that

$$\mathrm{E}(X_t - X_s)^2 = (t-s)^{2H}.$$

Since the process X is a Gaussian process, we obtain from Kolmogorov's theorem [7], Theorem I.2.1, page 26, that the process X is β-Hölder continuous with $\beta < H$. Fractional Brownian motion also satisfies property (b). The proof of this fact is based on the self-similarity and the ergodicity of the fractional Gaussian noise sequence $Z_k := X_k - X_{k-1}$, $k \geq 1$. The fact that property (c) holds for fractional Brownian motion was established in Molchan [5] and recently rediscovered by several authors (see [6]).

We now summarize our main result.

THEOREM 1.1. *Assume that X is a continuous square integrable centered process with $X_0 = 0$. Then, the following properties are equivalent*:

- *the process X is a fractional Brownian motion with self-similarity index $H \in (0, 1)$*;
- *the process X has properties* (a), (b) *and* (c) *for some $H \in (0, 1)$*.

REMARK 1.1. Theorem 1.1 appears in [4] with a different proof.

Discussion. If $H = \frac{1}{2}$, then assumption (c) means that the process X is a martingale. If X is a martingale, then condition (b) means that $X_t^2 - t$ is a martingale. Hence, we obtain the classical Lévy characterization theorem when $H = \frac{1}{2}$. Note that, in this case, property (a) follows from the fact that X is a standard Brownian motion.

Fractional Brownian motion X also has the following property (see, e.g., [8]): for $t > 0$,

$$\sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|^{1/H} \overset{L^1(\mathrm{P})}{\to} \mathrm{E}|X_1|^{1/H} t \tag{1.4}$$

as $n \to \infty$. To check that (1.4) holds for fractional Brownian motion, similarly to (1.2), one can use self-similarity and ergodicity of the fractional Gaussian noise sequence. This provides another possibility to generalize the quadratic variation property of standard Brownian motion. However, it is difficult to replace condition (b) by the condition (1.4).

REMARK 1.2. In the recent work of Hu et al. [2], condition (b) is replaced by the condition (1.4), with the additional assumption that $[M]$ is absolutely continuous with respect to the Lebesgue measure for $H > \frac{1}{2}$. The authors show that conditions (a), (c) and (1.4) also characterize fractional Brownian motion. In our work, we do not suppose the absolute continuity of $[M]$, but prove it under other assumptions; however, we restrict ourselves to (b).

In the next section we give one auxiliary result. The rest of the paper is devoted to the proof of the main result, first for $H > \frac{1}{2}$ and then for $H < \frac{1}{2}$.

2. Auxiliary result.

2.1. *Martingales and random variables.* In the proof, we will use random variables which are final values of martingales of a special type. All martingales vanish at zero.

Two continuous martingales M, N are (*strongly*) *orthogonal* if $[M, N] = 0$; we write this as $M \perp N$. Integration by parts gives that for such M, N, the product MN is a local martingale and it then has a bracket $[MN]$. We use the notation $N \cdot M$ for the stochastic integral of N with respect to $M : (N \cdot M)_t = \int_0^t N_s \, dM_s$. Let M be a continuous martingale. Put $I_2(M)_t := (M \cdot M)_t = \int_0^t M_s \, dM_s$.

Let $0 < a < b < t$ and suppose that p, q are deterministic continuous functions. Define martingales N and $\tilde{N}$ by $N_s = \int_0^s p_u 1_{(0,a]}(u) \, dM_u$ and $\tilde{N}_s = \int_0^s q_u 1_{(a,b]}(u) \, dM_u$, respectively. The martingales N and $\tilde{N}$ are orthogonal by construction and hence their product is a martingale. Note that $N_s \tilde{N}_s = 0$ whenever $s \le a$ and $N_s \tilde{N}_s = N_a \tilde{N}_s$ for $s > a$. The bracket of the martingale $N\tilde{N}$ is $[N\tilde{N}]_s = 0$ whenever $s \le a$ and $[N\tilde{N}]_s = N_a^2 [\tilde{N}]_s$ for $s > a$.

For orthogonal martingales, we have following lemma, which we will use in our proof.

LEMMA 2.1. *Assume that $(M_t^{n,k})_{t \ge 0}$ is a double array of continuous square integrable martingales with the properties*:

(i) *for n fixed and $k \neq l$, $M^{n,k}$ and $M^{n,l}$ are orthogonal martingales*;

(ii) *for any $t \ge 0$, $\sum_{k=1}^{k_n} [M^{n,k}]_t \le C$, where C is a constant*;

(iii) *for any $t \ge 0$, $\max_k [M^{n,k}]_t \xrightarrow{\mathrm{P}} 0$ as $n \to \infty$*,

where $1 \le k \le k_n$ and $k_n \to \infty$ as $n \to \infty$; then, for any $t \ge 0$,

$$\sum_{k=1}^{k_n} I_2(M^{n,k})_t \xrightarrow{L^2(\mathrm{P})} 0 \tag{2.1}$$

as $n \to \infty$.

PROOF. Since the martingales $M^{n,k}$ are pairwise orthogonal, when n is fixed, the same is true for the iterated integrals $I_2(M^{n,k})$. Recall [1], Theorem 1, page 354, which states that $\mathrm{E}(I_2(M^{n,k})_t)^2 \le B_{2,2} \mathrm{E}[M^{n,k}]_t^2$. Here, $B_{2,2}$ is constant independent of n, t and k, and this, together with property (ii), gives that the iterated integrals $I_2(M^{n,k})$ are square integrable. Hence, by the orthogonality of the iterated integrals, we have

$$\mathrm{E}\left(\sum_{k=1}^{k_n} I_2(M^{n,k})_t\right)^2 = \sum_{k=1}^{k_n} \mathrm{E}(I_2(M^{n,k})_t)^2.$$

However,

$$\sum_{k=1}^{k_n}[M^{n,k}]_t^2 \le \max_k[M^{n,k}]_t \sum_{k=1}^{k_n}[M^{n,k}]_t \xrightarrow{\mathrm{P}} 0$$

as $n \to \infty$. The claim (2.1) now follows since $\max_k[M^{n,k}]_t \le \sum_{k=1}^{k_n}[M^{n,k}]_t$ and this, together with property (ii), gives

$$\sum_{k=1}^{k_n}[M^{n,k}]_t^2 \le \max_k[M^{n,k}]_t \sum_{k=1}^{k_n}[M^{n,k}]_t \le C^2. \qquad \square$$

2.2. *A consequence of* (b). We now fix t and let $\mathcal{R}_t := \{s \in [0,t] : \frac{s}{t} \in \mathrm{Q}\}$. Note that the set $\mathcal{R}_t$ is dense on the interval $[0,t]$. Now, also fix $s \in \mathcal{R}_t$ and let $\tilde{n} = \tilde{n}(s)$ be a subsequence of $n \in \mathrm{N}$ such that $\tilde{n}\frac{s}{t} \in \mathrm{N}$. Put $\Delta X_{t_{k,n}} := X_{t_k} - X_{t_{k-1}}$.

The next lemma opens the way to bound from below and above the bracket $[M]$ on $[0,T]$ for any $T > 0$ and this goal will be achieved in Section 3.4.

LEMMA 2.2. *Fix $t > 0$, $s \in \mathcal{R}_t$ and suppose that $\tilde{n}\frac{s}{t} \in \mathrm{N}$ and $\tilde{n} \to \infty$. Then,*

$$\tilde{n}^{2H-1} \sum_{k=\tilde{n}s/t+1}^{\tilde{n}} (\Delta X_{t_{k,\tilde{n}}})^2 \xrightarrow{L^1(\mathrm{P})} t^{2H-1}(t-s).$$

PROOF. We have that

$$\begin{aligned}
\tilde{n}^{2H-1} &\sum_{k=1}^{\tilde{n}s/t} (\Delta X_{t_{k,\tilde{n}}})^2 \\
&= \tilde{n}^{2H-1} \sum_{k=1}^{\tilde{n}s/t} (\Delta X_{s_{k,\tilde{n}s/t}})^2 \\
&= \left(\frac{t}{s}\right)^{2H-1} \cdot \left(\tilde{n}\frac{s}{t}\right)^{2H-1} \sum_{k=1}^{\tilde{n}s/t} (\Delta X_{s_{k,\tilde{n}s/t}})^2 \xrightarrow{L^1(\mathrm{P})} s^{2H} \cdot \left(\frac{t}{s}\right)^{2H-1} \\
&= st^{2H-1}.
\end{aligned}$$

Since $\tilde{n}^{2H-1}\sum_{k=1}^{\tilde{n}}(\Delta X_{t_{k,\tilde{n}}})^2 \xrightarrow{L^1(\mathrm{P})} t^{2H}$, we obtain the proof. $\square$

In what follows, we shall write n for $\tilde{n}$ and t_k for $t\frac{k}{n}$.

2.3. *Some representation results*. We shall use the following notation. Let $Y_t = \int_0^t s^{1/2-H}\, dX_s$. We then we have $X_t = \int_0^t s^{H-1/2}\, dY_s$ and can write the fundamental martingale M as

$$M_t = \int_0^t (t-s)^{1/2-H}\, dY_s. \tag{2.2}$$

We also work with the martingale $W_t = \int_0^t s^{H-1/2}\,dM_s$. We have $[W]_t = \int_0^t s^{2H-1}\,d[M]_s$ and $[M]_t = \int_0^t s^{1-2H}\,d[W]_s$.

The equation (2.2) is a generalized Abel integral equation and the process Y can be expressed in terms of the process M:

$$Y_t = \frac{1}{\Gamma(H+1/2)\Gamma(3/2-H)} \int_0^t (t-s)^{H-1/2}\,dM_s. \tag{2.3}$$

Note that all of the integrals can be understood as pathwise Riemann–Stieltjes integrals (see [6]).

3. Proof of Theorem 1.1: $H > \frac{1}{2}$.

3.1. *Basic representation.* We shall now prove that M is a martingale with a bracket $c_H t^{2-2H}$ for some constant c_H and this, together with Lemma 3.1, will give that X is a fractional Brownian motion with index H.

We shall use the following modified representation result between X and M.

LEMMA 3.1. *Assume that $H > \frac{1}{2}$ and that properties* (a) *and* (c) *hold. Then, the process X has the representation*

$$X_t = \frac{1}{B_1} \int_0^t \left(\int_u^t s^{H-1/2}(s-u)^{H-3/2}\,ds \right) dM_u \tag{3.1}$$

with $B_1 = B(H-\frac{1}{2}, \frac{3}{2}-H)$.

PROOF. Integration by parts in (2.3) gives

$$Y_t = \frac{1}{B_1} \int_0^t (t-s)^{H-3/2} M_s\,ds.$$

Next, by using integration by parts and Fubini's theorem, we obtain

$$\begin{aligned}
X_t &= \int_0^t s^{H-1/2}\,dY_s \\
&= t^{H-1/2} Y_t - \left(H - \frac{1}{2}\right) \int_0^t s^{H-3/2} Y_s\,ds \\
&= \frac{t^{H-1/2}}{B_1} \int_0^t (t-s)^{H-3/2} M_s\,ds \\
&\quad - \frac{H-1/2}{B_1} \int_0^t s^{H-3/2} \int_0^s (s-u)^{H-3/2} M_u\,du\,ds \\
&= \frac{t^{H-1/2}}{(H-1/2)B_1} \int_0^t (t-s)^{H-1/2}\,dM_s \\
&\quad - \frac{1}{B_1} \int_0^t s^{H-3/2} \int_0^s (s-u)^{H-1/2}\,dM_u\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{t^{H-1/2}}{(H-1/2)B_1}\int_0^t (t-s)^{H-1/2}\,dM_s \\
&\quad - \frac{1}{B_1}\int_0^t \left[\int_u^t s^{H-3/2}(s-u)^{H-1/2}\,ds\right] dM_u \\
&= \frac{1}{B_1}\int_0^t \left[\frac{t^{H-1/2}}{H-1/2}(t-u)^{H-1/2}\right. \\
&\qquad\qquad \left. - \int_u^t s^{H-3/2}(s-u)^{H-1/2}\,ds\right] dM_u \\
&= \frac{1}{B_1}\int_0^t \left[\int_u^t s^{H-1/2}(s-u)^{H-3/2}\,ds\right] dM_u.
\end{aligned}
$$

This proves claim (3.1). □

Our plan is now as follows: we will attempt to prove that M is a martingale with the bracket $C_H t^{2-2H}$ and this, together with Lemma 3.1, will give that X is a fractional Brownian motion with parameter H.

3.2. *The basic estimation.* We can assume that the processes M, W, $[M]$ and $[W]$ are bounded with a deterministic constant L. If this is not the case, then consider a stopping time τ,

$$\tau = \inf\{s : |M_s| \geq L \text{ or } |W_s| \geq L \text{ or } [M]_s \geq L \text{ or } [W]_s \geq L\}.$$

Note that τ is independent of the partition (t_k^n), $k = 0, \dots, n$, and hence we have

$$1_{\{\tau \geq t\}} n^{2H-1} \sum_{k=1}^{n} (\Delta X_{t_{k,n}})^2 \xrightarrow{\text{P}} 1_{\{\tau \geq t\}} t^{2H}.$$

Given $\epsilon > 0$, take L big enough such that $\text{P}(\tau < t) < \epsilon$. Since our asymptotic results concern convergence in probability, it is enough to prove them only in the set $\{\tau \geq t\}$. We do not write the stopping time τ or the indicator $1_{\{\tau \geq t\}}$ explicitly in the proof below.

We want to use the expression

$$n^{2H-1} \sum_{k=ns/t+1}^{n} (\Delta X_{t_{k,n}})^2$$

to obtain estimates for the increment of the bracket $[M]$, with the help of (3.1).

Use (3.1) to obtain

$$\Delta X_{t_{k,n}} = \frac{1}{B_1}\left(\int_0^{t_{k-1}} f_k^t(s)\,dM_s + \int_{t_{k-1}}^{t_k} g_k^t(s)\,dM_s\right), \tag{3.2}$$

where we have used the notation

$$f_k^t(s) := \int_{t_{k-1}}^{t_k} u^{H-1/2}(u-s)^{H-3/2}\,du \tag{3.3}$$

and

$$g_k^t(s) := \int_{s}^{t_k} u^{H-1/2}(u-s)^{H-3/2}\,du.$$

Rewrite the increment of X as

$$\begin{aligned}\Delta X_{t_{k,n}} &= \frac{1}{B_1}(I_k^{n,1} + I_k^{n,2} + I_k^{n,3})\\ &:= \frac{1}{B_1}\biggl(\int_0^{t_{k-2}} f_k^t(s)\,dM_s + \int_{t_{k-2}}^{t_{k-1}} f_k^t(s)\,dM_s + \int_{t_{k-1}}^{t_k} g_k^t(s)\,dM_s\biggr).\end{aligned} \tag{3.4}$$

We need such a decomposition because the behavior of the kernels in the integrands is different for different arguments. Now, we intend to use this decomposition and to show that the sequence $n^{2H-1}\sum_{k=ns/t+1}^{n}(\Delta X_{t_{k,n}})^2$ verifies relation (e) from Section 3.4. In order to do this, we use Lemma 3.1, decompose the increment $\Delta X_{t_{k,n}}$ according to (3.4) into several terms and apply Itô's formula to the square of the increments. We then try to find asymptotically nontrivial terms and terms of order $o_{\mathrm{P}}(1)$, and nontrivial terms must be of the form that will be appropriate for finding the bounds for $[M]$. Even at this point, we can note that the nontrivial terms will appear when we consider sums of the form $n^{2H-1}\sum_{k=ns/t+2}^{n}\int_0^{t_{k-2}}(f_k^t(u))^2\,d[M]_u$, etc. So, at first, we estimate the sums with such a form and only then consider the remainder terms.

We note that the random variables $I_k^{n,j}$ are the final values at moment t of the martingales $\int_0^{t_{k-2}\wedge v} f_k^t(u)\,dM_u$, $\int_{t_{k-2}\wedge v}^{t_{k-1}\wedge v} f_k^t(u)\,dM_u$ and $\int_{t_{k-1}\wedge v}^{t_k\wedge v} g_k^t(u)\,dM_u$, $0 \le v \le t$, respectively. By construction, these martingales are orthogonal.

Next, the following upper bound holds for the functions f_k^t:

$$f_k^t(s) \le t_k^{H-1/2}(t_{k-1}-s)^{H-3/2}\frac{t}{n}; \tag{3.5}$$

note that this estimate is finite (not bounded) for $s \in [0, t_{k-1})$ and bounded for $s \in [0, t_{k-2}]$. Further, we need the following technical result.

LEMMA 3.2. *For $u < s$, we have*

$$\begin{aligned}\sum_{k=ns/t+2}^{n}(t_{k-1}-u)^{2H-3} &\le \Bigl(s+\frac{t}{n}-u\Bigr)^{2H-3}\\ &\qquad + \frac{n}{(2-2H)t}\Bigl(s+\frac{t}{n}-u\Bigr)^{2H-2}\end{aligned} \tag{3.6}$$

and for $u \le t_i$, we have

$$\sum_{k=i+2}^{n} (t_{k-1} - u)^{2H-3} \le (t_{i+1} - u)^{2H-3} + \frac{n}{(2-2H)t}(t_{i+1} - u)^{2H-2}. \tag{3.7}$$

PROOF. For $u < s$, we have

$$\begin{aligned}
&\sum_{k=ns/t+2}^{n} (t_{k-1} - u)^{2H-3} \\
&\quad = \left(s + \frac{t}{n} - u\right)^{2H-3} + \frac{n}{t} \sum_{k=ns/t+3}^{n} (t_{k-1} - u)^{2H-3} \frac{t}{n} \\
&\quad \le \left(s + \frac{t}{n} - u\right)^{2H-3} + \frac{n}{(2-2H)t}\left(s + \frac{t}{n} - u\right)^{2H-2}
\end{aligned}$$

by estimating the second sum in the first line from above by the integral. This proves (3.6). Inequality (3.7) is proved in the same way. □

We can now give two-sided bounds for the brackets of the martingales in (3.4). As was mentioned before, these brackets give rise to nontrivial terms in our estimates.

LEMMA 3.3. *Fix $t > 0$ and $s \in \mathcal{R}_t$, and let $\tilde{n}$ be such that $\tilde{n}\frac{s}{t} \in \mathrm{N}$ and $\tilde{n} \to \infty$ (we write n instead of $\tilde{n}$ in what follows). Then, there exist two constants, C_1, $C_2 > 0$, such that*

$$\begin{aligned}
C_1 t^{2H-1} \int_{s-t/n}^{t-2t/n} u^{2H-1}\, d[M]_u &\le n^{2H-1} \sum_{k=ns/t+2}^{n} \int_0^{t_{k-2}} (f_k^t(u))^2\, d[M]_u \\
&\le C_2 t^{4H-2}([M]_t - [M]_s) + o_{\mathrm{P}}(1).
\end{aligned} \tag{3.8}$$

PROOF. We will not write the constants explicitly.

Upper bound in (3.8). First, we estimate

$$i^n := n^{2H-1} \sum_{k=ns/t+2}^{n} \int_0^{t_{k-2}} (f_k^t(u))^2\, d[M]_u$$

from above. From (3.5), we obtain the following estimate for i^n:

$$i^n \le n^{2H-3} t^{2H+1} \sum_{k=ns/t+2}^{n} \int_0^{t_{k-2}} (t_{k-1} - u)^{2H-3}\, d[M]_u. \tag{3.9}$$

We can assume that $0 < s < t$ and $2 \le n\frac{s}{t} \le n-4$, and rewrite the estimate in (3.9) as

$$
\begin{aligned}
i^n &\le n^{2H-3} t^{2H+1} \Biggl(\sum_{i=1}^{ns/t} \sum_{k=ns/t+2}^{n} + \sum_{i=ns/t+1}^{n-2} \sum_{k=i+2}^{n} \Biggr) \\
&\qquad \times \int_{t_{i-1}}^{t_i} (t_{k-1}-u)^{2H-3}\, d[M]_u \\
&= n^{2H-3} t^{2H+1} \sum_{i=1}^{ns/t} \int_{t_{i-1}}^{t_i} \Biggl(\sum_{k=ns/t+2}^{n} (t_{k-1}-u)^{2H-3} \Biggr) d[M]_u \\
&\quad + n^{2H-3} t^{2H+1} \sum_{i=ns/t+1}^{n-2} \int_{t_{i-1}}^{t_i} \Biggl(\sum_{k=i+2}^{n} (t_{k-1}-u)^{2H-3} \Biggr) d[M]_u.
\end{aligned} \tag{3.10}
$$

We estimate the first term in the last equation in (3.10) by (3.6):

$$
\begin{aligned}
R_n^t &:= n^{2H-3} t^{2H+1} \sum_{i=1}^{ns/t} \int_{t_{i-1}}^{t_i} \sum_{k=ns/t+2}^{n} (t_{k-1}-u)^{2H-3}\, d[M]_u \\
&\le t^{2H-1} \int_0^s \Biggl(t^2 (ns+t-nu)^{2H-3} \\
&\qquad\qquad + \frac{t}{2-2H} (ns+t-u)^{2H-2} \Biggr) d[M]_u.
\end{aligned} \tag{3.11}
$$

Note that $(ns+t-nu)^{2H-3}$ and $(ns+t-nu)^{2H-2}$ are bounded and both converge to 0 as $n \to \infty$. So, $R_n^t = o_{\mathrm{P}}(1)$, by the dominated convergence theorem.

For the second term in the last equation of (3.10), we obtain, from (3.7), using the estimate $(t_{i+1}-u)^{H-1/2} \le (\frac{t}{n})^{H-1/2}$ and summing,

$$
\begin{aligned}
&n^{2H-3} t^{2H+1} \sum_{i=ns/t+1}^{n} \int_{t_{i-1}}^{t_i} \biggl[(t_{i+1}-u)^{2H-3} + \frac{n}{(2-2H)t} (t_{i+1}-u)^{2H-2} \biggr] d[M]_u \\
&\quad \le c_H t^{4H-2} ([M]_t - [M]_s).
\end{aligned}
$$

Hence, we have proven the upper bound (3.8) and have

$$
i^n \le c_H t^{4H-2} ([M]_t - [M]_s) + o_{\mathrm{P}}(1).
$$

Lower bound in (3.8). We complete the proof of Lemma 3.3 by giving the lower bound. From the definition of i^n, we easily obtain a lower estimate:

$$
i^n \ge n^{2H-1} \sum_{k=ns/t+2}^{n} \int_{t_{k-3}}^{t_{k-2}} (f_k^t(u))^2\, d[M]_u. \tag{3.12}
$$

Further, for $u \in (t_{k-3}, t_{k-2})$, $v \in (t_{k-1}, t_k)$, we have $v - u \le \frac{3}{n} t$, $u < v$ and we get the estimate

$$(f_k^t(u))^2 \ge 3^{2H-3} t^{2H-1} n^{1-2H} u^{2H-1}. \tag{3.13}$$

We use (3.13) in the lower bound (3.12) to obtain

$$\begin{aligned} i^n &\ge 3^{2H-3} t^{2H-1} \sum_{k=ns/t+2}^{n} \int_{t_{k-3}}^{t_{k-2}} u^{2H-1}\, d[M]_u \\ &= 3^{2H-3} t^{2H-1} \int_{s-t/n}^{t-2t/n} u^{2H-1}\, d[M]_u \end{aligned}$$

and this gives the lower bound in (3.8). The proof of Lemma 3.3 is now complete. □

REMARK 3.1. Clearly, we can rewrite i^n similarly to (3.10) as

$$\begin{aligned} i^n &= n^{2H-3} t^{2H+1} \left(\sum_{i=1}^{ns/t} \sum_{k=ns/t+2}^{n} + \sum_{i=ns/t+1}^{n-2} \sum_{k=i+2}^{n} \right) \int_{t_{i-1}}^{t_i} (f_k^t(u))^2\, d[M]_u \\ &= n^{2H-3} t^{2H+1} \sum_{i=1}^{ns/t} \int_{t_{i-1}}^{t_i} \sum_{k=ns/t+2}^{n} (f_k^t(u))^2\, d[M]_u \\ &\quad + n^{2H-3} t^{2H+1} \sum_{i=ns/t+1}^{n-2} \int_{t_{i-1}}^{t_i} \sum_{k=i+2}^{n} (f_k^t(u))^2\, d[M]_u \end{aligned} \tag{3.14}$$

and obtain from (3.5), and similarly to (3.11), that

$$n^{2H-3} t^{2H+1} \sum_{i=1}^{ns/t} \int_{t_{i-1}}^{t_i} \sum_{k=ns/t+2}^{n} (f_k^t(u))^2\, d[M]_u \xrightarrow{\mathrm{P}} 0, \tag{3.15}$$

change summation indices for further convenience and deduce from (3.14), (3.15) that

$$\begin{aligned} &\mathrm{P} - \lim_{n\to\infty} i^n \\ &\quad = \mathrm{P} - \lim_{n\to\infty} n^{2H-3} t^{2H+1} \sum_{k=ns/t+1}^{n-2} \int_{t_{k-1}}^{t_k} \sum_{i=k+2}^{n} (f_i^t(u))^2\, d[M]_u. \end{aligned} \tag{3.16}$$

We now return to (3.4), take the bracket of the next term and so estimate the term

$$\int_{t_{k-2}}^{t_{k-1}} (f_k^t(s))^2\, d[M]_s.$$

LEMMA 3.4. *There exists a constant $C_3 > 0$ such that*

$$n^{2H-1} \sum_{k=ns/t+2}^{n} \int_{t_{k-2}}^{t_{k-1}} (f_k^t(u))^2 \, d[M]_u \le C_3 t^{4H-2}([M]_t - [M]_s). \tag{3.17}$$

PROOF. We have the following upper estimate for the function f_k^t:

$$\begin{aligned} f_k^t(u) &\le t_k^{H-1/2} \int_{t_{k-1}}^{t_k} (v-u)^{H-3/2} \, dv \\ &= \frac{1}{H-1/2} t_k^{H-1/2} \big((t_k - u)^{H-1/2} - (t_{k-1} - u)^{H-1/2}\big) \\ &\le \frac{1}{H-1/2} t^{H-1/2} \left(\frac{t}{n}\right)^{H-1/2}. \end{aligned}$$

This gives the claim (3.17). □

The last estimate for nontrivial terms in (3.4) concerns the terms of the form

$$\int_{t_{k-1}}^{t_k} (g_k^t(s))^2 \, d[M]_s.$$

LEMMA 3.5. *There exists a constant C_4 such that*

$$n^{2H-1} \sum_{k=ns/t+1}^{n} \int_{t_{k-1}}^{t_k} (g_k^t(u))^2 \, d[M]_u \le C_4 t^{4H-2}([M]_t - [M]_s). \tag{3.18}$$

PROOF. We have that

$$\begin{aligned} g_k^t(z) &= \int_z^{t_k} v^{H-1/2}(v-z)^{H-3/2} \, dv \le (t_k)^{H-1/2} \frac{(t_k - z)^{H-1/2}}{H-1/2} \\ &\le C(t_k)^{H-1/2} \left(\frac{t}{n}\right)^{H-1/2} \le C t^{2H-1} \left(\frac{1}{n}\right)^{H-1/2}. \end{aligned}$$

This gives the claim (3.18). □

3.3. *The $o_P(1)$ terms.* We shall now prove that after the decomposition of the increment $\Delta X_{t_{k,n}}$ according to (3.4), taking the square of this increment and applying Itô's formula to the decomposition, all the terms except the three brackets of the martingales become asymptotically trivial. In this order, we take the terms of the form $(I_k^{n,j})^2$, $j = 1, 2, 3$, decompose them by Itô's formula on the bracket and martingale part and also prove that the terms containing the cross products $I_k^{n,i} I_k^{n,j}$, $i \ne j$, are asymptotically trivial. More exactly, Itô's formula implies that

$$(I_k^{n,1})^2 = \int_0^{t_{k-2}} (f_k^t(v))^2 \, d[M]_v + 2 \int_0^{t_{k-2}} f_k^t(u) \left(\int_0^u f_k^t(v) \, dM_v \right) dM_u.$$

We shall show that

$$n^{2H-1} \sum_{k=ns/t+2}^{n} \int_0^{t_{k-2}} f_k^t(u) \left(\int_0^u f_k^t(v)\, dM_v \right) dM_u \xrightarrow{\mathrm{P}} 0 \tag{3.19}$$

as $n \to \infty$. Clearly, it is sufficient to consider the sums of the form

$$S^n = n^{2H-1} \sum_{k=3}^{n} \int_0^{t_{k-2}} \left(\int_0^u f_k^t(s)\, dM_s \right) f_k^t(u)\, dM_u,$$

(note that $n\frac{s}{t} \geq 1$) since the sums

$$\sum_{k=3}^{ns/t+1} \int_0^{t_{k-2}} \left(\int_0^u f_k^t(s)\, dM_s \right) f_k^t(u)\, dM_u$$

for $n\frac{s}{t} \geq 2$ can be considered in a similar way. We rewrite S^n as

$$\begin{aligned} S^n &= n^{2H-1} \sum_{i=1}^{n-2} \int_{t_{i-1}}^{t_i} \left(\sum_{k=i+2}^{n} f_k^t(u) \int_0^u f_k^t(s)\, dM_s \right) dM_u \\ &= n^{2H-1} \int_0^{t_{n-2}} \Upsilon_{u,n}^M\, dM_u, \end{aligned}$$

where

$$\Upsilon_{u,n}^M = \sum_{k=i+2}^{n} f_k^t(u) \int_0^u f_k^t(s)\, dM_s, \qquad u \in [t_{i-1}, t_i).$$

We use the following version of the Lenglart inequality: if N is a locally square integrable continuous martingale, then, for any $\varepsilon > 0$, $t > 0$ and $A > 0$,

$$\mathrm{P}\Big\{ \sup_{0 \leq s \leq t} |N(s)| \geq \varepsilon \Big\} \leq \frac{A}{\varepsilon^2} + \mathrm{P}\{[N]_t \geq A\}. \tag{3.20}$$

It follows from inequality (3.20) that it is sufficient to prove the relation

$$n^{4H-2} \int_0^{t_{n-2}} (\Upsilon_{u,n}^M)^2\, d[M]u \xrightarrow{\mathrm{P}} 0, \qquad n \to \infty. \tag{3.21}$$

First, using integration by parts, we estimate the function

$$\Upsilon_{u,n}^M = \sum_{k=i+2}^{n} f_k^t(u) \left[f_k^t(u) M_u - \int_0^u M_s (f_k^t(s))'_s\, ds \right], \qquad u \in [t_{i-1}, t_i).$$

Clearly,

$$(f_k^t(u))'_u = \left(\frac{3}{2} - H \right) \int_{t_{k-1}}^{t_k} v^{H-1/2} (v-u)^{H-5/2}\, dv.$$

Therefore,

$$|\Upsilon_{u,n}^M| \le L \sum_{k=i+2}^{n} (f_k^t(u))^2 + L\left(\frac{3}{2} - H\right) \sum_{k=i+2}^{n} f_k^t(u) \int_0^u \int_{t_{k-1}}^{t_k} v^{H-1/2}(v-s)^{H-5/2}\, dv\, ds, \quad u \in [t_{i-1}, t_i).$$

We estimate the terms separately: since $f_k^t(u) \le \frac{t^{H+1/2}}{n}(t_{k-1}-u)^{H-3/2}$, we have that, for $u \in [t_{i-1}, t_i)$,

$$\begin{aligned}\sum_{k=i+2}^{n} (f_k^t(u))^2 &\le \frac{t^{2H+1}}{n^2} \sum_{k=i+2}^{n} (t_{k-1} - u)^{2H-3} \\ &\le \frac{t^{2H+1}}{n^2}(t_{i+1} - u)^{2H-3} + \frac{t^{2H+1}}{n} \int_{t_{i+1}}^{1} (tx - u)^{2H-3}\, dx \\ &\le \frac{t^{4H-2}}{n^{2H-1}} + \frac{t^{2H}}{n} \frac{(t_{i+1} - u)^{2H-2}}{2-2H} \\ &\le Cn^{1-2H}\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=i+2}^{n} f_k^t(u) \int_0^u \int_{t_{k-1}}^{t_k} v^{H-1/2}(v-s)^{H-5/2}\, dv\, ds \\ &\quad\le C \sum_{k=i+2}^{n} f_k^t(u) \int_{t_{k-1}}^{t_k} v^{H-1/2}(v-u)^{H-3/2}\, dv \\ &\quad\le C \sum_{k=i+2}^{n} (f_k^t(u))^2 \le Cn^{1-2H}.\end{aligned}$$

From these estimates, it follows that $n^{4H-2}(\Upsilon_{u,n}^M)^2 \le C$. Therefore, the bounded majorant in (3.21) exists. So, in order to establish (3.19), it is sufficient to prove that $\Upsilon_{u,n}^M n^{2H-1} \xrightarrow{\mathrm{P}} 0$, $0 < u < t$. We have that

$$\mathrm{E}(\Upsilon_{u,n}^M n^{2H-1})^2 = n^{4H-2}\mathrm{E} \int_0^u \left(\sum_{k=i+2}^{n} f_k^t(u) f_k^t(s) \right)^2 d[M]_s, \quad u \in [t_{i-1}, t_i). \tag{3.22}$$

Similarly to previous estimates, we obtain that

$$\begin{aligned}
&n^{4H-2}\biggl(\sum_{k=i+2}^{n} f_k^t(u) f_k^t(s)\biggr)^2\\
&\quad\le Cn^{4H-2}\biggl(\sum_{k=i+2}^{n}\frac{1}{n^2}(t_{k-1}-u)^{H-3/2}(t_{k-1}-s)^{H-3/2}\biggr)^2\\
&\quad\le Cn^{4H-4}\biggl(\frac{1}{n}\sum_{k=i+2}^{n}(t_{k-1}-u)^{2H-3}\biggr)^2\\
&\quad\le Cn^{4H-4}\biggl(\frac{n^{3-2H}}{n}+n^{2-2H}\biggr)^2\le C \qquad \text{for some } C>0.
\end{aligned}$$

This means that the bounded dominant in (3.22) exists. Moreover,

$$\begin{aligned}
&n^{2H-1}\sum_{k=i+2}^{n} f_k^t(u) f_k^t(s)\\
&\quad\le Cn^{2H-1}\sum_{k=i+2}^{n} f_k^t(u)\cdot\frac{1}{n}(u-s)^{H-3/2}\\
&\quad\le Cn^{2H-1}\cdot\frac{1}{n}\int_{(i+1)/n}^{1} v^{H-1/2}(v-u)^{H-3/2}\,dv\cdot(u-s)^{H-3/2}\to 0
\end{aligned}$$

for any $s<u$. Putting together, this means that $\Upsilon_{u,n}^{M} n^{2H-1}\xrightarrow{\mathrm{P}} 0$, $0<u<1$, whence $S^n\xrightarrow{\mathrm{P}} 0$ and, consequently, (3.19) holds. Next, consider the sums

$$n^{2H-1}\sum_{k=ns/t+2}^{n}\int_{t_{k-2}}^{t_{k-1}} f_k^t(u)\int_{t_{k-2}}^{u} f_k^t(v)\,dM_v\,dM_u$$

and

$$n^{2H-1}\sum_{k=ns/t+1}^{n}\int_{t_{k-1}}^{t_k} g_k^t(u)\int_{t_{k-1}}^{u} g_k^t(v)\,dM_v\,dM_u.$$

The assumptions of Lemma 2.1 are satisfied with martingales

$$N_v^{n,k}:=n^{H-1/2}\int_{t_{k-2}\wedge v}^{t_{k-1}\wedge v} f_k^t(u)\,dM_u$$

and

$$\tilde N_v^{n,k}:=n^{H-1/2}\int_{t_{k-1}\wedge v}^{t_k\wedge v} g_k^t(u)\,dM_u.$$

Indeed, property (ii) follows from (3.17) and (3.18), and property (iii) can be easily checked. Hence, both sums are of the order $o_{\mathrm{P}}(1)$. The next statement is an immediate consequence of Lemma 3.3 and (3.19). There exist two constants $C_1 > 0$, $C_2 > 0$ such that

$$\begin{aligned} C_1 t^{2H-1} \int_s^t u^{2H-1}\, d[M]_u &\le \mathrm{P} - \lim_{n\to\infty} n^{2H-1} \sum_{k=ns/t+1}^{n} (I_k^{n,1})^2 \\ &\le C_2 t^{4H-2}([M]_t - [M]_s). \end{aligned} \tag{3.23}$$

Similarly, one can show that the cross product sums with $i \neq j$ satisfy $n^{2H-1} \times \sum_k I_k^{n,i} I_k^{n,j} = o_{\mathrm{P}}(1)$. Indeed, let $i = 1$ and $j = 2$; other cases can be considered similarly. We have that, in this case,

$$n^{4H-2}\mathrm{E}\left(\sum_{k=1}^{n} I_k^{n,1} I_k^{n,2}\right)^2 = n^{4H-2}\mathrm{E}\sum_{k=1}^{n} (I_k^{n,1})^2 J_k^{n,2},$$

where $J_k^{n,2} = \int_{t_{k-2}}^{t_{k-1}} (f_k^t(s))^2\, d[M]_s$, since $I_k^{n,1}$, $I_k^{n,2}$, $I_k^{n,3}$ are pairwise orthogonal. Moreover, the product sum $n^{2H-1}\sum_k I_k^{n,i} I_k^{n,j}$ can be considered as a final value of a square integrable martingale with quadratic characteristic $\sum_{k=1}^n (I_k^{n,1})^2 J_k^{n,2}$. So, it follows from the Lenglart inequality that it is sufficient to prove the relation

$$n^{4H-2}\sum_{k=1}^{n} (I_k^{n,1})^2 J_k^{n,2} \xrightarrow{\mathrm{P}} 0. \tag{3.24}$$

According to (3.23), we have that

$$\mathrm{P} - \lim_{n\to\infty} n^{2H-1}\sum_{k=1}^{n} (I_k^{n,1})^2 \le C_2 t^{4H-2}[M]_t$$

and, also,

$$n^{2H-1} \max_{1\le k\le n} \int_{t_{k-2}}^{t_{k-1}} (f_k^t(s))^2\, d[M]_s \le \left(H - \frac{1}{2}\right)^{-2} \max_{1\le k\le n} ([M]_{t_{k-1}} - [M]_{t_{k-2}}) \xrightarrow{\mathrm{P}} 0,$$

whence (3.24) follows.

We are now ready to finish the proof of Theorem 1.1 in the case $H > \frac{1}{2}$.

3.4. *Completion of the proof for the case* $H > \frac{1}{2}$. Suppose, for the moment, that we consider the fixed interval $[0, t]$. By using our estimates, we can conclude that for rational s, consequently for any $s < t$, the following claims hold:

(d) there exist two constants, $C_1 > 0$ and $C_2 > 0$, such that

$$C_1 \int_s^t u^{2H-1}\, d[M]_u \le t - s \le C_2 t^{2H-1}([M]_t - [M]_s);$$

this estimate can be rewritten in terms of W and $[W]$ (recall that $W_t = \int_0^t s^{H-1/2}\,dM_s$) as

$$C_1([W]_t - [W]_s) \le t - s \le C_2 t^{2H-1} \int_s^t u^{1-2H}\,d[W]_u;$$

(e)

$$\mathrm{P} - \lim_{n\to\infty} n^{2H-1} \sum_{k=ns/t+1}^{n} (\triangle X_{t_k})^2 = \mathrm{P} - \lim_{n\to\infty} \int_s^t \varphi_n^t(u)\,d[M]_u,$$

where we can take $\varphi_n^t(u)$ from (3.16), (3.17) and (3.18), and they equal

$$\varphi_n^t(u) = \left(n^{2H-3} t^{2H+1} \sum_{i=k+2}^{n} (f_i^t(u))^2 + n^{2H-1}(g_k^t(u))^2\right) 1_{\{u\in[t_{k-1},t_k)\}}$$
$$+ n^{2H-1}(f_k^t(u))^2 1_{\{u\in[t_{k-2},t_{k-1})\}}.$$

Clearly, $\varphi_n^t(u)$ are positive, bounded, nonrandom functions and it follows from (3.13) that they are separated from 0 by some constant multiplied by u^{2H-1}.

From the left-hand side of (d), it follows that $[W]_t$ is absolutely continuous with respect to the Lebesgue measure, so $[W]_t = \int_0^t \theta_s\,ds$, where θ_s is a bounded, possibly random, variable. From the right-hand side of (d), it follows that

$$\int_s^t u^{1-2H}\theta_u\,du \ge \frac{1}{C_2}(t^{2-2H} - st^{1-2H}) \ge C_3(t^{2-2H} - s^{2-2H}) = C_3 \int_s^t u^{1-2H}\,du.$$

This means that

$$\int_s^t u^{1-2H}(\theta_u - C_3)\,du \ge 0,$$

whence we immediately obtain that $\theta_u(\omega) > C_3 > 0$ for almost all u, ω, concluding that $[W]$ is equivalent to the Lebesgue measure and so $W_t = \int_0^t \theta_s^{1/2}\,dV_s$, where $\{V_s, \mathrm{F}_s, s \ge 0\}$ is some Wiener process.

Now, if we perform all of the same calculations as before, but for "true" fractional Brownian motion B_t^H, we obtain that

$$\mathrm{P} - \lim_{n\to\infty} n^{2H-1} \sum_{k=ns/t+1}^{n} (\triangle B_{t_{k,n}}^H)^2 = \mathrm{P} - \lim_{n\to\infty} \int_s^t \varphi_s^n s^{2H-1}\,ds$$
$$= t^{2H-1}(t-s).$$

(It is sufficient to take $s = 0$.) Therefore, $\mathrm{P} - \lim_{n\to\infty} \int_s^t \psi_u^n\,du = 0$, where $\psi_u^n = u^{2H-1}\varphi_u^n(\theta_u - 1)$.

From this, we obtain that $\theta_u \equiv 1$ [otherwise, consider the set $D = \{(\omega, u) : \theta_u > 1 + \alpha,\ \text{or } \theta_u < 1 - \alpha\}$ for $\alpha > 0$; clearly, it has zero measure].

4. Proof of Theorem 1.1: $H < \frac{1}{2}$. For $H < \frac{1}{2}$, we use, in general, principally the same ideas. However, technical details are different. Indeed, it is well known (see, e.g., [6]) that the kernel $z(t,s)$ participating in the representation of X via M or W [see (4.2)] is more complicated in the case $H < \frac{1}{2}$. The brackets of the martingales that are to be estimated as before also have an additional singularity because the power $2H-1$, or any other power of such a form, is now negative. Therefore, the proofs are more technical and the reasons for this will be mentioned below in all relevant places.

4.1. *Starting point.* At first, consider the Hölder properties of the processes involved. We can note the following: since $H < \frac{1}{2}$, it is very simple to prove, using integration by parts, that the process Y has the same Hölder properties as X, that is, it is Hölder up to order H. Further, it follows from Lemma 2.1 [6] that M is Hölder up to order $\frac{1}{2}$. Therefore, for any $0 < s_0 \le s < t \le T$ and $\beta < \frac{1}{2}$, there exists a constant $K = K_{s_0,\beta}$ such that $|W_t - W_s| \le K_{s_0,\beta}(t-s)^{\beta}$. Now, it is more convenient to consider W instead of M. We shall show the inequality

$$C_1([W]_t - [W]_s) \le t - s \le C_2([W]_t - [W]_s) \tag{4.1}$$

first for arbitrary $t > 0$ and $s \in \mathcal{R}_t$, $s < t$. Recall that we can assume the processes W and $[W]$ to be bounded, as in Section 3.2.

For $H < \frac{1}{2}$, we use the following representation result, which can be proven as [6], Theorem 5.2.

LEMMA 4.1. *Assume that $H < \frac{1}{2}$ and that properties* (a) *and* (c) *hold. The process X then has the representation*

$$X_t = \int_0^t z(t,s)\,dW_s \tag{4.2}$$

with the kernel

$$z(t,s) = \left(\frac{s}{t}\right)^{1/2-H}(t-s)^{H-1/2} - (H-1/2)s^{1/2-H}\int_s^t u^{H-3/2}(u-s)^{H-1/2}\,du.$$

Put

$$p_k^t(z) = \int_{t_{k-1}}^{t_k}\left(\frac{z}{u}\right)^{1/2-H}(u-z)^{H-3/2}\,du$$

for $z < t_{k-1}$.

Using Lemma 4.1 and integration by parts, we can now write the increment of X as

$$
\begin{aligned}
X_{t_k} - X_{t_{k-1}} = {} & \left(\frac{1}{2} - H\right) \int_0^{t_{k-2}} p_k^t(s)\, dW_s \\
& + \left(\frac{1}{2} - H\right) \int_{t_{k-2}}^{t_{k-1}} p_k^t(s)\, dW_s \\
& + \int_{t_{k-1}}^{t_k} \left(\frac{s}{t_k}\right)^{1/2-H} (t_k - s)^{H-1/2}\, dW_s \\
& + \left(\frac{1}{2} - H\right) \int_{t_{k-1}}^{t_k} s^{1/2-H} \int_s^{t_k} u^{H-3/2}(u-s)^{H-1/2}\, du\, dW_s \\
=: {} & J_k^{n,1} + J_k^{n,2} + J_k^{n,3} + J_k^{n,4}.
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
& \lim_{n\to\infty} n^{2H-1} \sum_{k=ns/t+2}^{n} (\Delta X_{t_k})^2 \\
& \quad = \lim_{n\to\infty} n^{2H-1} \Bigg(\sum_{k=ns/t+2}^{n} (J_k^{n,1})^2 + \sum_{k=ns/t+2}^{n} (J_k^{n,2} + J_k^{n,3} + J_k^{n,4})^2 \\
& \qquad\qquad + 2 \sum_{k=ns/t+2}^{n} J_k^{n,1} (J_k^{n,2} + J_k^{n,3} + J_k^{n,4}) \Bigg).
\end{aligned}
$$

As before,

$$
\lim_{n\to\infty} n^{2H-1} \sum_{k=ns/t+2}^{n} (\Delta X_{t_{k,n}})^2 \overset{L^1(\mathrm{P})}{\to} t^{2H-1}(t-s). \tag{4.3}
$$

First, estimate

$$
\lim_{n\to\infty} n^{2H-1} \sum_{k=ns/t+2}^{n} (J_k^{n,1})^2
$$

from below and above. We start with the analog of Lemma 3.3.

4.2. *Two-sided estimates for the sums* $n^{2H-1} \sum_{k=ns/t+2}^{n} \int_0^{t_{k-2}} (p_k^t(z))^2\, d[W]_z$ *and* $n^{2H-1} \sum_{k=ns/t+2}^{n} (J_k^{n,1})^2$. Put

$$
j^{n,1} = n^{2H-1} \sum_{k=ns/t+2}^{n} \int_0^{t_{k-2}} (p_k^t(z))^2\, d[W]_z.
$$

We decompose this sum as in the case of the proof for $H > \frac{1}{2}$ [see (3.10) and (3.14)]:

$$j^{n,1} := n^{2H-1}\left(\sum_{i=1}^{ns/t}\sum_{k=ns/t+2}^{n} + \sum_{i=ns/t+1}^{n-2}\sum_{k=i+2}^{n}\right)\int_{t_{i-1}}^{t_i}(p_k^t(u))^2\,d[W]_u.$$

Clearly, for $s \le t_{k-2}$,

$$p_k^t(s) \le \left((t_{k-1}-s)^{H-3/2}\frac{t}{n}\right) \wedge \left(\frac{1}{1/2-H}\left(\frac{t}{n}\right)^{H-1/2}\right).$$

Therefore, for n such that $n\frac{s}{t} \in \mathrm{N}$, we have that

$$\begin{aligned} j^{n,1} &\le n^{2H-2}t\int_0^s\left(s+\frac{t}{n}-u\right)^{2H-2}d[W]_u \\ &\quad + n^{2H-3}t^2\int_0^s\left(s+\frac{t}{n}-u\right)^{2H-3}d[W]_u \\ &\quad + \frac{t}{2-2H}\left(\frac{t}{n}\right)^{2H-2}n^{2H-2}\sum_{i=ns/t+1}^{n-2}\int_{t_{i-1}}^{t_i}d[W]_u \\ &\quad + t^2n^{2H-3}\sum_{i=ns/t+1}^{n-2}\int_{t_{i-1}}^{t_i}d[W]_u\left(\frac{t}{n}\right)^{2H-3}. \end{aligned} \tag{4.4}$$

We divide the integral $\int_0^s(s+\frac{t}{n}-u)^{2H-2}\,d[W]_u$ into two parts, $\int_0^{s/2}(s+\frac{t}{n}-u)^{2H-2}\,d[W]_u$ and $\int_{s/2}^s(s+\frac{t}{n}-u)^{2H-2}\,d[W]_u$. The first integral can be estimated as

$$\int_0^{s/2}\left(s+\frac{t}{n}-u\right)^{2H-2}d[W]_u \le \left(\frac{s}{2}+\frac{t}{n}\right)^{2H-2}[W]_{s/2},$$

whence $n^{2H-2}t\int_0^{s/2}(s+\frac{t}{n}-u)^{2H-2}\,d[W]_u \to 0$ as $n\to\infty$ a.s. As for the second part, we apply the following inequality from [6]: let the function $f:[a,b]\to R$ be Hölder on $[a,b]$ of order β, $|f(t)-f(s)| \le K|t-s|^\beta$. Then, for any $\rho > -1+\beta$ and $b < v$, we have that

$$\left|\int_a^b(v-u)^\rho\,df(u)\right| \le K\left(1+\left|\frac{\rho}{\rho+\beta}\right|\right)\left((v-b)^{\rho+\beta}+(v-a)^{\rho+\beta}\right). \tag{4.5}$$

According to the Hölder properties of W mentioned above, we can take any $0 < \beta < \frac{1}{2}$ and define, for any $r \in (\frac{s}{2}, t]$, the random variable

$$K_r(\omega) = \sup_{s/2\le u<v\le r}\frac{|W_v-W_u|}{(v-u)^\beta}.$$

Clearly, $P\{K_t(\omega) \geq N\} \to 0$ as $N \to \infty$. Therefore, it is enough to prove that $\int_{s/2}^{s\wedge\tau_N}(s+\frac{t}{n}-u)^{2H-2}\,d[W]_u n^{2H-2} \xrightarrow{P} 0$ as $N \to \infty$ for any $N > 1$, where $\tau_N = \inf\{r \geq \frac{s}{2} : K_r \geq N\} \wedge t$. According to the Burkholder–Gundy inequality and (4.5),

$$\begin{aligned}
&n^{2H-2}\mathrm{E}\biggl(\int_{s/2}^{s\wedge\tau_N}\Bigl(s+\frac{t}{n}-u\Bigr)^{2H-2}\,d[W]_u\biggr)\\
&\quad\leq Cn^{2H-2}\mathrm{E}\biggl(\int_{s/2}^{s\wedge\tau_N}\Bigl(s+\frac{t}{n}-u\Bigr)^{H-1}\,dW_u\biggr)^2\\
&\quad\leq CN^2n^{2H-2}\biggl(\frac{\beta}{H+\beta-1}\biggr)\\
&\qquad\times\biggl(\Bigl(\frac{t}{n}\Bigr)^{H+\beta-1}+\Bigl(\frac{s}{2}+\frac{t}{n}\Bigr)^{H+\beta-1}\biggr)^2 \to 0\\
&\qquad\qquad\text{as } n\to\infty.
\end{aligned}$$

Finally, we obtain that $n^{2H-2}\int_0^s(s+\frac{t}{n}-u)^{2H-2}\,d[W]_u \xrightarrow{P} 0$ as $n \to \infty$.

The same is true for

$$\int_0^s\Bigl(s+\frac{t}{n}-u\Bigr)^{2H-3}\,d[W]_u\cdot n^{2H-3}.$$

The last two integrals from (4.4) admit the obvious estimate $t^{2H-1}C_2([W]_t-[W]_s)$.

The "remainder" term for $\sum(J_1^k)^2$, that is, the difference between $\sum(J_1^k)^2$ and $j^{n,1}$, equals

$$\begin{aligned}
R_n := n^{2H-1}\sum_{k=ns/t+2}^{n}\int_0^{t_{k-2}}\biggl(\int_0^z p_k^t(v)\,dW_v\biggr)&\\
\times\, p_k^t(u)\,dW_u.&
\end{aligned}$$

For technical simplicity, it is enough to consider $\sum_{k=3}^{nr}$ for any $r \in \mathrm{N}$, instead of $\sum_{k=ns/t+2}^{n} = -\sum_{k=3}^{ns/t+1}+\sum_{k=3}^{n}$. We obtain that

$$\begin{aligned}
\mathrm{E}(R_n)^2 &= n^{4H-2}\mathrm{E}\biggl(\sum_{k=3}^{nr}\sum_{i=1}^{k-2}\int_{t_{i-1}}^{t_i}\int_0^u p_k^t(v)\,dW_v\cdot p_k^t(u)\,dW_u\biggr)^2\\
&= n^{4H-2}\mathrm{E}\biggl(\sum_{i=1}^{nr-2}\sum_{k=i+3}^{nr}\int_{t_{i-1}}^{t_i}\int_0^u p_k^t(v)\,dW_v\cdot p_k^t(u)\,dW_u\biggr)^2\\
&= n^{4H-2}\sum_{i=1}^{nr-2}\mathrm{E}\int_{t_{i-1}}^{t_i}\biggl(\sum_{k=i+3}^{nr}\int_0^u p_k^t(v)\,dW_v\cdot p_k^t(u)\biggr)^2\,d[W]_u.
\end{aligned}$$

Let us estimate

$$\left|\int_0^u p_k^t(v)\,dW_v\right| = \left|p_k^t(u)W_u - \int_0^u W_v(p_k^t(v))_v'\,dv\right| \\ \le L|p_k^t(u)| + L\left|\int_0^u (p_k^t(v))_v'\,dv\right|.$$

We have that

$$\left|\int_0^u (p_k^t(v))_v'\,dv\right| = |p_k^t(u) - p_k^t(0)| \le C\left(\frac{t}{n}\right)^{H-1/2} \qquad \text{for some } C > 0.$$

Moreover,

$$n^{2H-1}\left(\sum_{k=i+3}^{nr} p_k^t(u)\right)^2 \le n^{2H-1}\left(\int_{t_{i+1}}^{tr} (v-u)^{H-3/2}\,dv\right)^2 \\ = Cn^{2H-1}[-(tr-u)^{H-1/2} + (t_{i+1}-u)^{H-1/2}]^2 \\ \le C$$

and the integrand

$$n^{4H-2}\left(\sum_{k=i+2}^{nr} \int_0^u p_k^t(v)\,dW_v \cdot p_k^t(u)\right)^2 \le C,$$

that is, the integrable dominant exists. Therefore, it is sufficient to establish that for any u,

$$n^{2H-1}\sum_{k=i+3}^{nr} \int_0^u p_k^t(v)\,dW_v \cdot p_k^t(u) \xrightarrow{\mathrm{P}} 0.$$

We take the mathematical expectation in the left-hand side and obtain that

$$n^{4H-2}\mathrm{E}\int_0^u \left(\sum_{k=i+3}^{nr} p_k^t(v)p_k^t(u)\right)^2 d[W]_v.$$

Also, here, the bounded dominant exists. Indeed,

$$n^{4H-2}\left(\sum_{k=i+3}^{nr} p_k^t(v)p_k^t(u)\right)^2 \le n^{2H-1}\left(\sum_{k=i+2}^{nr} p_k^t(v)\right)^2 \le C,$$

as before. Further, we must prove that

$$n^{2H-1}\sum_{k=i+3}^{nr} p_k^t(v)p_k^t(u) \to 0$$

for all fixed $0 < v < u$. We have that

$$
\begin{aligned}
&n^{2H-1}\sum_{k=i+2}^{nr} p_k^t(v)p_k^t(u)\\
&\quad\le n^{2H-1}\sum_{k=i+3}^{nr}\int_{t_{k-1}}^{t_k}(s-u)^{H-3/2}\,ds\int_{t_{k-1}}^{t_k}(s-v)^{H-3/2}\,ds\\
&\quad\le n^{2H-1}\sum_{k=i+3}^{nr}(t_{k-1}-u)^{H-3/2}\frac{1}{n}\int_{t_{k-1}}^{t_k}(s-v)^{H-3/2}\,ds\\
&\quad\le n^{2H-2}(t_{i+2}-u)^{H-3/2}\int_{t_{i+2}}^{tr}(s-v)^{H-3/2}\,ds\\
&\quad\le Cn^{H-3/2}(u-v)^{H-3/2}\to 0\\
&\qquad\qquad\qquad\qquad\text{as } n\to\infty \text{ for any } 0<v<u.
\end{aligned}
$$

From all of these estimates, the remainder term $R_n \stackrel{\mathrm{P}}{\to} 0$.

For the lower bounds, we return to $[M]$ instead of $[W]$:

$$
\begin{aligned}
j^{n,1} &= n^{2H-1}\sum_{k=ns/t+2}^{n}\int_0^{t_{k-2}}(f_t^k(u))^2\,d[M]_u\\
&\ge t^2n^{2H-3}\sum_{k=ns/t}^{n}(t_k)^{2H-1}\int_0^{t_{k-2}}(t_k-u)^{2H-3}\,d[M]_u\\
&\ge t^2n^{2H-3}\left(\sum_{i=1}^{ns/t-1}\sum_{k=ns/t+2}^{n}+\sum_{i=ns/t+1}^{n-2}\sum_{k=i+2}^{n}\right)(t_k)^{2H-1}\\
&\qquad\times\int_{t_{i-1}}^{t_i}(t_k-u)^{2H-3}\,d[M]_u\\
&= Ct^{2H+1}n^{2H-2}\sum_{i=ns/t+1}^{n-2}\int_{t_{i-1}}^{t_i}\frac{1}{t}\big((t_{i+2}-u)^{2H-2}-(t-u)^{2H-2}\big)\,d[M]_u.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&n^{2H-2}\sum_{i=ns/t+1}^{n-2}\int_{t_{i-1}}^{t_i}(t-u)^{2H-2}\,d[M]_u\\
&\qquad\sim\left(t-t+\frac{2}{n}\right)^{2H-2+\beta}\cdot n^{2H-2}\to 0\qquad\text{as } n\to\infty.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} n^{2H-1} \sum_{k=ns/t+1}^{n} (J_1^k)^2 \\
&\quad \ge Ct^{2H} n^{2H-2} \sum_{i=ns/t+1}^{n-2} \int_{t_{i-1}}^{t_i} (t_{i+2}-u)^{2H-2}\, d[M]_u \\
&\quad \ge Ct^{2H} n^{2H-2} \sum_{i=ns/t+1}^{n-2} (t_{i+2}-t_{i-1})^{2H-2} \int_{t_{i-1}}^{t_i} d[M]_u.
\end{aligned}$$

Combining this with the upper estimate and taking into account the estimate of the remainder term, we have

$$\begin{aligned}
C_1 t^{4H-2}([M]_t-[M]_s) &\le \lim_{n\to\infty} n^{2H-1} \sum_{k=ns/t+2}^{n} (J_1^k)^2 \\
&\le C_2 t^{2H-1}([W]_t-[W]_s).
\end{aligned} \tag{4.6}$$

[Note that, for $H \in (1/2, 1)$, we have obtained opposite estimates.] Also, note that we cannot immediately estimate $\sum (J_i^k)^2$, $i > 1$, from above. Indeed, the integrand of the form $(t\frac{l}{n} - u)^{H-1/2}$ that admits the estimate $< (\frac{1}{n})^{H-1/2} \to 0$ for $H \in (1/2, 1)$, now, for $H \in (0, 1/2)$, tends to ∞. So, we can mention that $\sum_{k=ns/t+2}^{n} (J_2^k + J_3^k + J_4^k)^2 \ge 0$, intend to prove that $\sum J_1^k (J_2^k + J_3^k + J_4^k) \to 0$, and, from this, condition (b) [or (4.3)] and (4.6), obtain the following estimate from above:

$$C_1 t^{2H-1}([M]_t - [M]_s) \le (t-s).$$

In the sequel, we realize this plan.

4.3. *Auxiliary estimates for "mixed" terms.* We will show that as $n \to \infty$, we have

$$n^{2H-1} \sum_k J_k^{n,1}(J_k^{n,2} + J_k^{n,3} + J_k^{n,4}) \xrightarrow{\mathrm{P}} 0. \tag{4.7}$$

It is sufficient to estimate the sums from $k = 2$ up to $k = n$. By applying the Lenglart inequality to $n^{2H-1} \sum_{k=2}^{n} J_k^{n,1} J_k^{n,2}$ as well as to the final value of corresponding martingale, we obtain that it is sufficient to prove that

$$\begin{aligned}
&n^{4H-2} \sum_{k=2}^{n} \biggl(\int_0^{t_{k-2}} \int_{t_{k-1}}^{t_k} \Bigl(\frac{s}{u}\Bigr)^{1/2-H} (u-s)^{H-3/2}\, du\, dW_s \biggr)^2 \\
&\quad \times \biggl(\int_{t_{k-2}}^{t_{k-1}} \biggl(\int_{t_{k-1}}^{t_k} \Bigl(\frac{s}{u}\Bigr)^{1/2-H} (u-s)^{H-3/2}\, du \biggr)^2 d[W]_s \biggr) \\
&\qquad \le Cn^{4H-2} \sum_{k=2}^{n} \biggl(\int_0^{t_{k-2}} p_k^t(s)\, dW_s \biggr)^2 \int_{t_{k-2}}^{t_{k-1}} (t_{k-1}-s)^{2H-1}\, d[W]_s \xrightarrow{\mathrm{P}} 0.
\end{aligned}$$

Integrate the last integral by parts:

$$\begin{aligned}\int_{t_{k-2}}^{t_{k-1}} &(t_{k-1}-s)^{2H-1} d[W]_s \\ &= (t_{k-1}-t_{k-2})^{2H-1}([W]_{t_{k-1}} - [W]_{t_{k-2}}) \\ &\quad - (2H-1)\int_{t_{k-2}}^{t_{k-1}} (t_{k-1}-s)^{2H-2}([W]_{t_{k-1}} - [W]_s)\, ds \\ &\le Cn^{1-2H}\Delta[W]_{t_{k-1}} + C\int_{t_{k-2}}^{t_{k-1}} (t_{k-1}-s)^{2H-2}([W]_{t_{k-1}} - [W]_s)\, ds.\end{aligned}$$

Now, recall that

$$\begin{aligned}\left(\int_0^{t_{k-2}} p_k^t(s)\, dW_s\right)^2 &= \int_0^{t_{k-2}} (p_k^t(s))^2\, d[W]_s \\ &\quad + 2\int_0^{t_{k-2}}\int_0^s p_k^t(v)\, dW_v \cdot p_k^t(s)\, dW_s.\end{aligned}$$

Clearly,

$$\sigma^{n,1} := n^{2H-1}\sum_{k=2}^n \int_0^{t_{k-2}} (p_k^t(s))^2\, d[W]_s \le j^{n,1},$$

so, it is bounded in probability and, similarly to R_n,

$$\sigma^{n,2} := n^{2H-1}\sum_{k=2}^n \int_0^{t_{k-2}}\int_0^s p_k^t(v)\, dW_v \cdot p_k^t(s)\, dW_s \xrightarrow{\mathrm{P}} 0 \qquad \text{as } n\to\infty.$$

Therefore,

$$\begin{aligned}n^{4H-2}&\sum_{k=2}^n \left(\int_0^{t_{k-2}} p_k^t(s)\, dW_s\right)^2 \cdot Cn^{1-2H}\Delta[W]_{t_{k-1}} \\ &\le C\sigma^{n,1}\cdot \max_k \Delta[W]_{t_{k-1}} + C\sigma^{n,2}\cdot\max_k \Delta[W]_{t_{k-1}} \xrightarrow{\mathrm{P}} 0, \qquad n\to\infty.\end{aligned}$$

Also,

$$\begin{aligned}n^{4H-2}\sum_{k=2}^n &\left(\int_0^{t_{k-2}} p_k^t(s)\, dW_s\right)^2 \cdot \int_{t_{k-2}}^{t_{k-1}} (t_{k-1}-s)^{2H-2} \\ &\qquad\qquad \times ([W]_{t_{k-1}} - [W]_s)\, ds \\ &\le C(\omega)(\sigma^{n,1}+\sigma^{n,2})n^{2H-1}\int_{t_{k-2}}^{t_{k-1}} (t_{k-1}-s)^{2H-1-\varepsilon}\, ds \\ &\le C(\omega)(\sigma^{n,1}+\sigma^{n,2})n^{2H-1}(t_{k-1}-t_{k-2})^{2H-\varepsilon} \\ &\sim \left(\frac{1}{n}\right)^{1-\varepsilon} \to 0 \qquad \text{as } n\to\infty.\end{aligned}$$

This means that we have proven one of the necessary relations: $n^{2H-1}\sum_{k=2}^{n} J_k^{n,1} \times J_k^{n,2} \xrightarrow{\text{P}} 0$ as $n \to \infty$.

Consider

$$
\begin{aligned}
n^{2H-1} \sum_{k=2}^{n} J_k^{n,1} J_k^{n,2} \\
&= n^{2H-1} \sum_{k=2}^{n} \int_0^{t_{k-2}} p_k^t(s)\, dW_s \\
&\qquad \times \int_{t_{k-1}}^{t_k} \left(\frac{s}{t_k}\right)^{1/2-H} (t_k - s)^{H-1/2}\, dW_s.
\end{aligned}
$$

As before, it is sufficient to prove that

$$
n^{4H-2} \sum_{k=2}^{n} \left(\int_0^{t_{k-2}} p_k^t(s)\, dW_s\right)^2 \cdot \int_{t_{k-1}}^{t_k} \left(\frac{s}{t_k}\right)^{1-2H} (t_k - s)^{2H-1}\, d[W]_s \xrightarrow{\text{P}} 0 \qquad \text{as } n \to \infty
$$

or, equivalently,

$$
n^{2H-1} \max_k \int_{t_{k-1}}^{t_k} (t_k - s)^{2H-1}\, d[W]_s \cdot (\sigma^{n,1} + \sigma^{n,2}) \xrightarrow{\text{P}} 0. \tag{4.8}
$$

Note that by [6] and due to the Hölder properties of $[W]$,

$$
\int_{t_{k-1}}^{t_k} (t_k - s)^{2H-1}\, d[W]_s \le C(\omega)(t_k - t_{k-1})^{2H-\varepsilon} \sim \left(\frac{1}{n}\right)^{2H-\varepsilon},
$$

whence we obtain (4.8).

Now, consider $n^{2H-1}\sum J_k^{n,1} J_k^{n,4}$; other sums can be estimated similarly. After some transformations, we obtain

$$
\begin{aligned}
& n^{4H-2} \sum_{k=2}^{n} \left(\int_0^{t_{k-2}} p_k^t(v)\, dW_u\right)^2 \\
&\qquad \times \int_{t_{k-1}}^{t_k} s^{1-2H} \left(\int_s^{t_k} u^{H-3/2}(u-s)^{H-1/2}\, du\right)^2 d[W]_s \\
&\quad \le n^{2H-1} \max_k \int_{t_{k-1}}^{t_k} \left(\int_s^{t_k} u^{H-3/2}(u-s)^{H-1/2}\, du\right)^2 d[W]_s (\sigma^{n,1} + \sigma^{n,2}) \\
&\quad \le n^{2H-1} \max_k \int_{t_{k-1}}^{t_k} \left(\int_s^{t_k} u^{2H-3}\, du \int_s^{t_k} (u-s)^{2H-1}\, du\right) d[W]_s \\
&\qquad \times (\sigma^{n,1} + \sigma^{n,2})
\end{aligned}
$$

$$\leq Cn^{2H-1} \max_k \int_{t_{k-1}}^{t_k} s^{2H-2}(t_k - s)^{2H}\, d[W]_s \cdot (\sigma^{n,1} + \sigma^{n,2})$$

$$\leq Cn \cdot \frac{1}{n} \max_k \int_{t_{k-1}}^{t_k} (t_k - s)^{2H-1}\, d[W]_s \cdot (\sigma^{n,1} + \sigma^{n,2})$$

$$\leq C \max_k (t_k - t_{k-1})^{2H-\varepsilon} \cdot (\sigma^{n,1} + \sigma^{n,2}) \to 0 \qquad \text{as } n \to \infty.$$

4.4. *Upper bounds for* $[M]$ *and* $[W]$. Due to all previous estimates, we can realize our plan and conclude that

$$t^{2H-1}(t-s) = \lim_{n\to\infty} n^{2H-1} \sum_{k=ns/t+2}^{n} (\Delta X_{t_k})^2 \geq C_1 t^{4H-2}([M]_t - [M]_s),$$

that is,

$$\begin{aligned}[M]_t - [M]_s &\leq C_2 t^{1-2H}(t-s) = C_2(t^{-2H} - st^{1-2H}) \\ &\leq C_2(t^{-2H} - s^{-2H})\end{aligned}$$

or

$$\int_s^t u^{1-2H}\, d[W]_u \leq C_2 \int_s^t u^{1-2H}\, du.$$

As before, it follows that $[W]_t$ is absolutely continuous with respect to Lebesgue measure,

$$[W]_t = \int_0^t \theta_s\, ds, \tag{4.9}$$

$0 \leq \theta_s \leq C$, where C is some constant and θ_s is possibly random. Of course, this is not our final goal, but we can now proceed with the above estimates for $n^{2H-1}\sum_{k=ns/t+2}^{n}(J^{n,i})^2$, $i > 1$, and this, together with condition (b) [or (4.3)] and (4.6), will give us the possibility to obtain a lower bound for $[W]_t - [W]_s$, that is, to obtain (4.1).

4.5. *Lower bound for* $[W]_t - [W]_s$. We can continue estimating from above: for example, if we take, for simplicity, the sums over $k = 2$ up to $k = n$, then

$$\begin{aligned} n^{2H-1} &\sum_{k=2}^{n} (J_k^{n,2})^2 \\ &= \widetilde{\sigma}^{n,1} + \widetilde{\sigma}^{n,2} \\ &:= Cn^{2H-1} \sum_{k=1}^{n} \int_{t_{k-2}}^{t_{k-1}} \left(\int_{t_{k-1}}^{t_k} \left(\frac{s}{u}\right)^{1/2-H} (u-s)^{H-3/2}\, du \right)^2 d[W]_s \\ &\quad + Cn^{2H-1} \sum_{k=1}^{n} \int_{t_{k-2}}^{t_{k-1}} \left(\int_{t_{k-2}}^{u} p_k^t(v)\, dW_v \right) p_k^t(u)\, dW_u \end{aligned}$$

and we now need an estimate $p_k^t(s) \le (t_{k-1} - s)^{H-1/2} C$.

Therefore,

$$\widetilde{\sigma}^{n,1} \le C n^{2H-1} \sum_{k=1}^{n} \int_{t_{k-2}}^{t_{k-1}} (t_{k-1} - s)^{2H-1}\, d[W]_s.$$

We cannot now continue to estimate the last expression directly (because of the singularity at the upper point t_{k-1}). So, we take an indirect route: for some $A > 0$,

$$\begin{aligned}
&\int_{t_{k-2}}^{t_{k-1}} (t_{k-1} - s)^{2H-1}\, d[W]_s \\
&\quad \le \int_{t_{k-2}}^{t_{k-1} - t/(nA)} + \int_{t_{k-1} - t/(nA)}^{t_{k-1}} \\
&\quad \le \big(t_{k-1} - \big(t_{k-1} - t/(nA)\big)\big)^{2H-1} \cdot \Delta[W]_{t_k} \\
&\qquad + [\text{thanks to (4.9)}]\; C \int_{t_{k-1} - t/(nA)}^{t_{k-1}} (t_{k-1} - s)^{2H-1}\, ds \\
&\quad \le \left(\frac{t}{nA}\right)^{2H-1} \Delta[W]_{t_k} + C \left(\frac{t}{nA}\right)^{2H}.
\end{aligned}$$

Taking the sum, we obtain

$$\begin{aligned}
\widetilde{\sigma}^{n,1} &\le C n^{2H-1} \sum_{k=1}^{n} \left(\frac{t}{nA}\right)^{2H-1} \Delta[W]_{t_k} + C n^{2H-1} n \left(\frac{t}{nA}\right)^{2H} \\
&\le C A^{1-2H} t^{2H-1} [W]_t + C \frac{1}{A^{2H}} t^{2H}.
\end{aligned}$$

If we estimate the sum from $k = n\frac{s}{t} + 1$ to $k = n$, then

$$\begin{aligned}
\widetilde{\sigma}^{n,1} &\le C A^{1-2H} t^{2H-1} ([W]_t - [W]_s) + C \frac{1}{A^{2H}} t^{2H} \left(1 - \frac{s}{t}\right) \\
&= C A^{1-2H} t^{2H-1} ([W]_t - [W]_s) + C \frac{1}{A^{2H}} t^{2H-1} (t - s).
\end{aligned}$$

We now want to prove that $\widetilde{\sigma}^{n,2} \xrightarrow{\mathrm{P}} 0$ as $n \to \infty$. As usual, it is enough to establish that

$$n^{4H-2} \sum_{k=1}^{n} \int_{t_{k-2}}^{t_{k-1}} \left(\int_{t_{k-2}}^{u} p_k^t(v)\, dW_v\right)^2 (p_k^t(u))^2\, d[W]_u \xrightarrow{\mathrm{P}} 0.$$

We can now bound $[W]_u$ by $C\,du$, take the mathematical expectation and note that $(p_k^t(u))^2 \le C n^{1-2H}$. Therefore, it is sufficient to prove that

$$n^{4H-2} \sum_{k=1}^{n} \int_{t_{k-2}}^{t_{k-1}} \int_{t_{k-2}}^{u} (p_k^t(v))^2\, d[W]_v (p_k^t(u))^2\, du \xrightarrow{\mathrm{P}} 0.$$

Since $C\,dv$ bounds $d[W]_v$, we have that this value can be bounded by

$$Cn^{4H-2}\sum_{k=1}^{n}\int_{t_{k-2}}^{t_{k-1}}\left(\int_{t_{k-2}}^{u}(p_k^t(v))^2\,dv\right)(p_k^t(u))^2\,du$$

$$\le C\sum_{k=1}^{n}\int_{t_{k-2}}^{t_{k-1}}\left(\int_{t_{k-2}}^{u}dv\right)du\le\frac{1}{n}C\to 0\qquad\text{as } n\to\infty.$$

Finally,

$$n^{2H-1}\sum_{k=ns/t+2}^{n}(J_k^{n,2})^2\le CA^{1-2H}t^{2H-1}([W]_t-[W]_s)+C\frac{1}{A^{2H}}t^{2H-1}(t-s).$$

Now, proceed with $J_k^{n,3}$:

$$n^{2H-1}\sum_{k=1}^{n}(J_k^{n,3})^2$$

$$=n^{2H-1}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\left(\frac{s}{t_k}\right)^{1/2-H}(t_k-s)^{H-1/2}\right)^2d[W]_s$$

$$+n^{2H-1}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^{u}\left(\frac{s}{t_k}\right)^{1/2-H}(t_k-s)^{H-1/2}\,dW_s\right)$$

$$\times\left(\frac{u}{t_k}\right)^{1/2-H}(t_k-u)^{H-1/2}\,dW_u.$$

The first term can be estimated as

$$n^{2H-1}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(t_k-s)^{2H-1}\,d[W]_s$$

$$\le C\left(\frac{t}{A}\right)^{2H-1}([W]_t-[W]_s)+\frac{C}{A^{2H}}t^{2H-1}(t-s)$$

as before.

And, with the bound $d[W]_s\le C\,ds$, the second term can be estimated as $n^{4H-2}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{u}(t_k-s)^{2H-1}\,ds\cdot(t_k-u)^{2H-1}\,du\le Cn^{-2}\to 0$. Therefore, for $\sum(J_k^{n,3})^2$, we have the same estimate as for $\sum(J_k^{n,2})^2$. Finally, estimate

$$n^{2H-1}\sum_{k=1}^{n}(J_k^{n,4})^2$$

$$=Cn^{2H-1}\sum_{k=1}^{n}\left(\int_{t_{k-1}}^{t_k}s^{1/2-H}\int_s^{t_k}u^{H-3/2}(u-s)^{H-1/2}\,du\,dW_s\right)^2$$

$$
\begin{aligned}
&= Cn^{2H-1}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} s^{1-2H}\left(\int_s^{t_k} u^{H-3/2}(u-s)^{H-1/2}\,du\right)^2 d[W]_s\\
&\quad + Cn^{2H-1}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{u} s^{1/2-H}\int_s^{t_k} v^{H-3/2}(v-s)^{H-1/2}\,dv\,dW_s\\
&\qquad\qquad \times u^{1/2-H}\int_u^{t_k} v^{H-3/2}(v-u)^{H-1/2}\,dv\,dW_u.
\end{aligned}
$$

The first term can be estimated with the help of (4.9) as

$$
\begin{aligned}
&n^{2H-1}t^{1-2H}\sum_{k=2}^{n}\int_{t_{k-1}}^{t_k}\left(\int_s^{t_k} u^{H-3/2}(u-s)^{H-1/2}\,du\right)^2 d[W]_s\\
&\qquad \le Cn^{-2H}\to 0 \qquad \text{as } n\to\infty.
\end{aligned}
$$

If $k=1$, then, for $\frac{1}{p}+\frac{1}{q}=1,\ p,q>1$,

$$
\begin{aligned}
&n^{2H-1}t^{1-2H}\int_0^{t/n}\left(\int_s^{t/n} u^{H-3/2}(u-s)^{H-1/2}\,du\right)^2 ds\\
&\quad \le n^{2H-1}t^{1-2H}\int_0^{t/n}\left(\int_s^{t/n} u^{p(H-3/2)}\,du\right)^{2/p}\\
&\qquad\qquad \times\left(\int_s^{t/n}(u-s)^{(H-1/2)q}\,du\right)^{2/q} ds\\
&\quad \le n^{2H-1}t^{1-2H}\int_0^{t/n} s^{(pH-3p/2+1)2/p}\left(\frac{t}{n}-s\right)^{(Hq-q/2+1)2/q} ds\\
&\quad = n^{2H-1}t^{1-2H}\int_0^{t/n} s^{2H-3+2/p}\left(\frac{t}{n}-s\right)^{2H-1+2/q} ds\\
&\quad \sim n^{2H-1}t^{1-2H}\left(\frac{t}{n}\right)^{4H-1}\to 0,
\end{aligned}
$$

that is, the "main term" of $n^{2H-1}\sum_{k=1}^{n}(J_k^{n,4})^2$ tends to 0. For the remainder term of $n^{2H-1}\sum_{k=1}^{n}(J_k^{n,4})^2$, it is sufficient to prove that for any $\varepsilon>0$,

$$
\begin{aligned}
\widetilde{\sigma}^{n,3} := n^{4H-2}\sum_{k=n\varepsilon/t}^{n}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{u}&\left(s^{1/2-H}\int_s^{t_k} v^{H-3/2}(v-s)^{H-1/2}\,dv\right)^2 ds\\
&\times u^{1-2H}\left(\int_u^{t_k} v^{H-3/2}(v-u)^{H-1/2}\,dv\right)^2 du\to 0\\
&\qquad \text{as } n\to\infty.
\end{aligned}
$$

However,

$$\widetilde{\sigma}^{n,3} \le n^{4H-2} \sum_{k=n\varepsilon/t}^{n} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{u} \left(\int_s^{t_k} v^{H-3/2}(v-s)^{H-1/2}\, dv \right)^2 ds \times \left(\int_u^{t_k} v^{H-3/2}(v-u)^{H-1/2}\, dv \right)^2 du$$

$$\le n^{-6} \sum_{k=n\varepsilon/t}^{n} (t_{k-1})^{-4} \sim n^{-2} \to 0 \qquad \text{as } n \to \infty.$$

After all estimates, for $s > 0$,

$$\lim_{n\to\infty} n^{2H-1} \sum_{k=ns/t+2}^{n} (\Delta X_{t_{k,n}})^2 \le C_2 A^{1-2H} t^{2H-1}([W]_t - [W]_s) + C_2 \frac{1}{A^{2H}} t^{2H-1}(t-s).$$

We have the opposite estimate,

$$C_1 t^{2H-1}(t-s) \le \lim_{n\to\infty} n^{2H-1} \sum_{k=ns/t+2}^{n} (\Delta X_{t_{k,n}})^2 \le C_2 A^{1-2H} t^{2H-1}([W]_t - [W]_s) + C_2 \frac{1}{A^{2H}} t^{2H-1}(t-s).$$

So, for A sufficiently large, $C_3 := C_1 - C_2 \frac{1}{A^{2H}} > 0$, and we obtain that

$$C_3 t^{2H-1}(t-s) \le C_2 A^{1-2H} t^{2H-1}([W]_t - [W]_s),$$

whence $[W]_t - [W]_s \ge \frac{C_3}{C_2} A^{2H-1}(t-s)$, where the constants do not depend on s and t. Therefore, if we write $[W]_t = \int_0^t p_s\, ds$, then $\varepsilon_1 \le p_s \le \varepsilon_2$, $\varepsilon_i > 0$ and $W_t = \int_0^t p_s^{1/2}\, dV_s$ for some Wiener process V. We can then complete the proof of the theorem using the same arguments as for $H \in (1/2, 1)$.

Acknowledgments. The authors are grateful to the anonymous referees for careful and constructive reading, and for useful suggestions.

REFERENCES

[1] CARLEN, E. and KRÉE, P. (1991). L^p estimates on iterated stochastic integrals. *Ann. Probab.* **19** 354–368. MR1085341

[2] HU, Y., NUALART, D. and SONG, J. (2009). Fractional martingales and characterization of the fractional Brownian motion. *Ann. Probab.* **37** 2404–2430. MR2573562

[3] MANDELBROT, B. B. and VAN NESS, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10** 422–437. MR0242239

[4] MISHURA, Y. S. (2008). *Stochastic Calculus for Fractional Brownian Motion and Related Processes*. *Lecture Notes in Math*. **1929**. Springer, Berlin. MR2378138

[5] MOLCHAN, G. M. (2002). Linear problems for fractional Brownian motion: A group approach. *Teor. Veroyatnost*. *i Primenen*. **47** 59–70. MR1978695

[6] NORROS, I., VALKEILA, E. and VIRTAMO, J. (1999). An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli* **5** 571–587. MR1704556

[7] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357

[8] ROGERS, L. C. G. (1997). Arbitrage with fractional Brownian motion. *Math*. *Finance* **7** 95–105. MR1434408

DEPARTMENT OF MATHEMATICS
KIEV UNIVERSITY
VOLODOMIRSKA STREET 64
01033 KIEV
UKRAINE
E-MAIL: myus@univ.kiev.ua

DEPARTMENT OF MATHEMATICS
AND SYSTEMS ANALYSIS
AALTO UNIVERSITY
P.O. BOX 11100
FI-00076 AALTO
FINLAND
E-MAIL: esko.valkeila@tkk.fi