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QUANTILE CLOCKS

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Quantile clocks are defined as convolutions of subordinators L, with quantile functions of positive random variables. We show that quantile clocks can be chosen to be strictly increasing and continuous and discuss their practical modeling advantages as *business activity times* in models for asset prices. We show that the marginal distributions of a quantile clock, at each fixed time, equate with the marginal distribution of a single subordinator. Moreover, we show that there are many quantile clocks where one can specify L, such that their marginal distributions have a desired law in the class of generalized s-self decomposable distributions, and in particular the class of self-decomposable distributions. The development of these results involves elements of distribution theory for specific classes of infinitely divisible random variables and also decompositions of a gamma subordinator, that is of independent interest. As applications, we construct many price models that have continuous trajectories, exhibit volatility clustering and have marginal distributions that are equivalent to those of quite general exponential Lévy price models. In particular, we provide explicit details for continuous processes whose marginals equate with the popular VG, CGMY and NIG price models. We also show how to perfectly sample the marginal distributions of more general classes of convoluted subordinators when L is in a sub-class of generalized gamma convolutions, which is relevant for pricing of European style options.

1. Introduction. Let $Q_R(u) = \inf\{t : F_R(t) \geq u\}$, $0 < u < 1$ denote the quantile function of a nonnegative continuous random variable R with strictly increasing cumulative distribution function (c.d.f.) F_R, and finite first moment $\mathbb{E}[R]$. In this paper, we introduce and describe detailed distributional properties of a class of random time changes $T_R := (T_R(t), t \geq 0)$, which we call *quantile clocks*. These processes are defined as,

$$T_R(t) = \int_0^t Q_R\big((1 - s/t)_+\big)L(ds), \qquad t \geq 0, \tag{1.1}$$

where L is a subordinator, and $(a)_+ := \max(0, a)$. While applicable in many settings, we follow the framework in [4] and discuss the modeling advantages of quantile clocks as *business activity times* in time changed models for asset prices.

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The quantile clocks, may be written as special cases of *convoluted subordinators*, which are processes described in Bender and Marquardt [4]. That is to say, processes defined as $T(t) = \int_0^t k(t,s)L(ds)$, $t \geq 0$, for $k(t,s)$ a known kernel. The authors [4], Proposition 1, provide mild conditions on $k(t,s)$ and L such that a process $T := (T(t), t \geq 0)$ has almost surely strictly continuous and increasing sample paths. In terms of applications, [4] argue that one can use $W(T(t))$, where W is a brownian motion with drift, as time changed models for the log price of assets that possess continuous trajectories, where $T(t)$ is now interpreted as *business activity time*. Furthermore, such models can correct deficiencies in Black–Scholes type price models. In particular, it is known that (i) the log returns of asset prices have nonnormal distributions, and often exhibit semi-heavy or heavier tail behavior, (ii) the volatility or variance is dependent on time, (iii) asset prices exhibit volatility *clustering* or *persistence.* Reference [4] also describe a general formula for European style option prices that depend on the marginal distribution of $T(\tau)$ for some fixed time to maturity $\tau > 0$. For other applications of processes representable as convoluted subordinators, see, for instance, [18, 30, 31, 37] and references therein.

In the literature, exponential Lévy price models, defined as $\mathrm{e}^{-\chi(t)}$ for a Lévy process χ on $\mathbb{R}$, have been quite successful in terms of their ability to capture some of the stylistic features of asset prices (i) and (ii) listed above. In addition, there are many choices of χ where one can easily calibrate pricing models to the options market, capturing volatility smiles and skews, via Monte Carlo methods or perhaps more generally by the fast Fourier transform (FFT) methods outlined in Carr and Madan [10]. Many Lévy processes χ, can be expressed as $W(\zeta(t))$ for some subordinator ζ. However, the precise ζ that is associated with a χ is not always known explicitly, and χ is often modeled via its Lévy density. Arguably, the most popular models of this type include the variance gamma process (VG) by [28], where ζ is a gamma subordinator, the Carr–Geman–Madan–Yor (CGMY) process [7], where ζ has only recently been identified by Madan and Yor [29], and the normal inverse Gaussian (NIG) process [2], where ζ is an inverse Gaussian process. The popularity of these models is due to their relative simplicity and distributional flexibility.

However, since χ has independent increments, exponential Lévy processes are unable to capture effects due to volatility clustering. One approach discussed in [8], which is related to the price models in [3], is to further time change χ by a stochastic volatility process of the form $T(t) = \int_0^t v(s)\,ds$, where $v(s)$ represents the instantaneous volatility either following a mean reverting Cox–Ingersoll–Ross (CIR) process or a non-Gaussian Ornstein–Uhlenbeck (BNS–OU) model of Barndorff-Nielsen and Shephard [3], specified by the dynamics

$$\mathrm{d}v(t) = -\lambda v(t)\,dt + \vartheta(\lambda\,dt), \tag{1.2}$$

where ϑ is a subordinator we shall call an OU–BDLP. The BNS–OU model $v(t)$, possesses jumps, and has a stationary distribution with laws equating to the class of laws of self-decomposable random variables that remarkably one can choose based on a prescribed choice of ϑ. This latter fact is important for our exposition. In contrast, the integrated volatility $T(t) = \int_0^t v(s)\,ds$ is continuous and has nontypical marginals laws (obviously depending on t). In the case of the CIR process, $v(t)$ is a diffusion having a transition density following a noncentral chi-squared distribution. We will not consider models of this type.

The authors [4] demonstrate that their approach, involving convoluted subordinators, can be viewed as viable variations of the idea in [8]. Furthermore, their work essentially contains the popular model of [3]. However, as noted by the authors, there are practical issues arising for instance in the pricing of options that relate to the marginal distributions of $T(\tau)$ at maturity times τ. While $T(\tau)$ are infinitely divisible, their marginal distributions and also characteristic functions depend on $k(\tau, s)$ and τ, in a nontrivial way. Hence, leading in general to nonfamiliar distributions for $T(\tau)$. Related to this point, is a classical problem where in general it is not clear how to *exactly* sample infinitely divisible random variables, even in the case of a subordinators $\zeta(\tau)$ for each fixed τ. Some notable exceptions for ζ are gamma, positive stable and inverse Gaussian processes whose marginals for each fixed time point are gamma, positive stable and inverse Gaussian random variables, and hence are easily sampled. More generally, one can resort to sampling methods based on truncation of infinite series representations, but these do not yield exact samples and it is not always clear how to control the level of accuracy. Reference [4] do point out that if $T(\tau)$ has an analytically tractable characteristic function then one can apply the popular fast Fourier transform (FFT) techniques in Carr and Madan [10] to obtain explicit option prices. They also provide supporting results for some choices of k and L. However, due to the generality of $k(t, s)$, it appears difficult to apply the (FFT) for option price formula depending on general $T(\tau)$. These points do not reflect a deficiency in the approach of [4] but rather that the class of convoluted subordinators is quite general. The task then becomes how to choose kernels k and subordinators L that are convenient in terms of implementation as well as having general modeling flexibility.

1.1. *Contributions and outline.* In Section 3, we show that quantile clocks, which arise by setting $k(t, s) = Q_R((1 - s/t)_+)$, can be chosen to have continuous trajectories and in general have marginal distributions that, for each fixed time t, equate to a single subordinator ζ. That is $T_R(t) \stackrel{d}{=} \zeta(t)$ for fixed t. We also highlight a very tractable example related to [20, 22]. Of course, in general, the law ζ, that is, the marginal laws of T_R, depends on (R, L), and hence the deterministic quantile function Q_R. However, in Section 4, we show that there are many quantile clocks whose marginal distributions can be chosen such that they do not obviously depend on Q_R. In fact for these given Q_R, we show that one can choose L, and a random variable Y, such that the marginals of T_R have specific laws in the Jurek's [17, 23–25] $\mathcal{U}_\delta$ class of *generalized* s-selfdecomposbale laws, for $\delta > 0$. These classes contain the important class $\mathcal{L}$ of self-decomposable distributions on $\mathbb{R}_+$. See [9] for the relevance of self-decomposable Lévy processes in financial modeling.

This ability to choose specific (familiar) marginal laws for price processes, while allowing for quite varied path properties induced by different quantiles Q_R, gives modelers a great deal of flexibility. It is also reminiscent of how one might choose a BNS–OU model v to have a specific stationary distribution, that is, $v(0) \stackrel{d}{=} v(t)$ for all fixed t, in $\mathcal{L}$ based on the OU–BDLP ϑ appearing in (1.2). However, recall that v has jumps and the law of T_R obviously must depend on t. The precise methods we use to establish these results, and identify Q_R, Y and L, are given in Sections 4, 5 and 6, and should also be of general interest to experts in Lévy processes. In Section 7, we exploit the fact that $T_R(t) \stackrel{d}{=} \zeta(t)$ for each t, and we show that compositions (or time changes) involving quantiles clocks behave marginally like subordinators. In Section 8, we show that as consequences of our results, that we are able to identify price processes whose marginal behavior coincides with those of exponential Lévy price processes. In particular, we identify explicitly many processes whose marginal distributions are equivalent to VG, CGMY and NIG price processes, but whose trajectories are continuous and otherwise quite varied, and additionally exhibit volatility clustering. We also identify models possessing jumps that otherwise have the properties mentioned above. In Section 9, we show how one can use our results for quantile clocks to specify laws for the convoluted subordinator referred as a short memory kernel in [4].

While quantile clocks are our main focus, in Section 2, we also describe results that apply to the practical implementation of log price models $W(T(t))$, considered in [4], where $T(t)$ is based on a general kernel $k(t,s)$. In particular, if L is chosen to have laws in the class of generalized gamma convolutions with finite Thorin measure, see [6, 19, 22], call this class $\mathcal{G}_+$, then the random variable $T(\tau)$ can be exactly sampled in many instances. This is based on a very recent work of Devroye and James [15] where a double coupling from the past (Double CFTP) perfect sampling routine is devised, and also results described in James [19]. Furthermore, for this choice of L, by using a deterministic time-change we obtain a simplified version of the option price formulae given in [4].

1.2. *Preliminaries.* We now present some concepts and notation we shall use throughout. First, for fixed positive numbers (a,b), let γ_a denote a gamma(a) random variable with shape parameter a and scale 1, let $\beta(a,b)$ denote a beta (a,b) random variable. Furthermore, U will always denote a Uniform[0, 1] variable, and recall that for any $\delta > 0$, $U^{1/\delta} \stackrel{d}{=} \beta_{\delta,1}$. ξ_p is a Bernoulli random variable with success probability p. In addition for a generic random variable Y, Y' will denote a variable equivalent in distribution but otherwise independent. $(N(s) : s > 0)$ will denote a homogeneous Poisson process with intensity $\mathbb{E}[N(s)] = s$. For a (nonrandom) function $g(x)$, $g'(x)$ and $g''(x)$ with denote its first and second derivatives.

Formally, recall that a subordinator $\zeta = (\zeta(t); t > 0)$, is an increasing process with right continuous paths and stationary independent increments, whose law is specified by its Laplace transform for some $\omega > 0$

$$\mathbb{E}\left[e^{-\omega\zeta(t)}\right] = e^{-t\psi_\zeta(\omega)}, \tag{1.3}$$

where for some $c \geq 0$, $\omega > 0$

$$\psi_\zeta(\omega) = c\omega + \int_0^\infty (1 - e^{-\omega s})\Lambda_\zeta(ds)$$

is finite and is called the *Laplace exponent* of ζ, $\Lambda_\zeta(ds)$ is its Lévy measure, $\rho_\zeta(s) = \Lambda_\zeta(ds)/ds$ is the Lévy density. We will work with the case where $c = 0$.

It follows that the laws of ζ can be specified by any of these quantities. ζ is said to be of *infinite activity* if $\Lambda_\zeta(\infty) = \infty$ and otherwise of *finite activity*. In the latter case, ζ corresponds to compound Poisson process whose jumps have a common probability density/mass function proportional to ρ_ζ. Throughout we shall reserve the notation ζ, L, Z for generic subordinators, and corresponding random variables, and the notation ϑ for the OU–BDLP. As is well known, for each fixed t, $\zeta(t)$ is a random variable in the class $\mathcal{J}$ of infinitely divisible random variables (taking values in $\mathbb{R}_+$). We now describe the characteristics of some important subclasses of $\mathcal{J}$, say $\mathcal{L}, \mathcal{B}, \mathcal{G}$ and $\mathcal{G}_+$, satisfying $\mathcal{G}_+ \subset \mathcal{G} \subset \mathcal{L} \subset \mathcal{J}$ and $\mathcal{G}_+ \subset \mathcal{G} \subset \mathcal{B} \subset \mathcal{J}$.

We say that a random variable $\zeta(1)$ is in the class $\mathcal{L}$ of self-decomposable variables if $\rho_\zeta(s) = s^{-1}h(s)$, with h decreasing. We also note that from Jurek and Vervaat [26] that, with respect to the OU process in (1.2), there is the relationship

$$\zeta(1) \stackrel{d}{=} v(0) \stackrel{d}{=} v(t) \stackrel{d}{=} \int_0^\infty e^{-s}\vartheta(ds).$$

Note that we will say that ζ is a subordinator in $\mathcal{L}$ to mean that it is a subordinator whose Lévy density corresponds to that of a variable in $\mathcal{L}$, of course $\zeta(t)$ is in $\mathcal{L}$ for each fixed t. Similar statements will apply for other classes. We say that $\zeta(1)$ is a variable in Bondesson's [6], Section 9, $\mathcal{B}$ class, or the class of generalized convolutions of mixtures of exponential distributions (GCMED), if the Lévy density is completely monotone, that is, $\rho_\zeta(s) = \int_0^\infty e^{-sy}\mu(dy)$, for some nonnegative measure μ.

We now describe the classes $\mathcal{G}$ and $\mathcal{G}_+$. $\zeta(1)$ is a variable that is a generalized gamma convolutions (GGC), see [6], if it is in the class $\mathcal{G}$, characterized by

$$\rho_\zeta(s) = s^{-1}\int_0^\infty e^{-sy}\nu(dy) \quad \text{and} \quad \psi_\zeta(\omega) = \int_0^\infty \log(1 + \omega/y)\nu(dy)$$

for some sigma-finite measure ν, formally known as a Thorin measure. We say that ζ is a GGC(ν) subordinator.

A $\zeta(1)$ variable is in the class $\mathcal{G}_+$, if it satisfies

$$\rho_\zeta(s) = \theta s^{-1}\mathbb{E}[e^{-s/R}] \quad \text{and} \quad \psi_\zeta(\omega) = \theta\mathbb{E}[\log(1 + \omega R)] \tag{1.4}$$

for some $\theta > 0$ and some random variable R satisfying $\mathbb{E}[\log(1 + \omega R)] < \infty$. In this case, we say $\zeta(1)$ is a GGC(θ, R) variable. Moreover, $\zeta(t)$ is a GGC$(\theta t, R)$ variable for each fixed t, and ζ is referred to as a GGC(θ, R) subordinator.

We now highlight some important properties of GGC(θ, R) random variables, and subordinators that for instance allow them to be exactly sampled by the methods in [15]. These facts can found in [19] as well as [20, 22], and depend heavily on the results for Dirichlet means in [13]. Letting Z_θ denote a GGC(θ, R) subordinator, it follows that $Z_\theta(t) \stackrel{d}{=} Z_{\theta t}(1)$. Importantly, there is the representation, for any $\kappa \geq \theta > 0$, $Z_\theta(1) \stackrel{d}{=} \gamma_\theta M_\theta = \gamma_\kappa \tilde{M}_\kappa$ where

$$M_\theta \stackrel{d}{=} \beta_{\theta,1} M_\theta + (1 - \beta_{\theta,1}) R \tag{1.5}$$

and

$$\tilde{M}_\kappa \stackrel{d}{=} \beta_{\kappa,1} \tilde{M}_\kappa + (1 - \beta_{\kappa,1}) R \xi_p \tag{1.6}$$

for $p = \theta/\kappa$. That is, a GGC(θ, R) random variable variable is a GGC$(\kappa, R\xi_p)$ variable. In particular, if $0 < \theta t \leq 1$, then $Z_{\theta t}(1) = Z_\theta(t) \stackrel{d}{=} \gamma_1 \tilde{M}_1$, where

$$\tilde{M}_1 \stackrel{d}{=} U \tilde{M}_1 + (1 - U) R \xi_p \tag{1.7}$$

for $p = \theta t$, and it follows from the work of Cifarelli and Regazzini [13] that $\tilde{M}_1$ has density of the form

$$\frac{x^{p-1}}{\pi} \sin(\pi F_{R\xi_p}(x)) \mathrm{e}^{-p\Psi_R(x)} \qquad \text{for } x > 0$$

with

$$\Psi_R(x) = \mathbb{E}\big[\log|x - R| \mathbb{I}_{(R \neq x)}\big].$$

Thus, as pointed out in [19], if one one can evaluate $\Psi_R(x)$ in a suitable fashion, then one can exactly sample any variable $Z_\theta(t)$ for every fixed $0 < t \leq 1/\theta$, by, for instance, rejection sampling. Since any number $s > 0$, can be set to $s = nt$, for some integer n and $0 < t \leq 1/\theta$, it follows that $Z_\theta(s) \stackrel{d}{=} Z_\theta(tn)$ can be exactly sampled by at most exactly sampling n copies of the random variable $Z_\theta(t) \stackrel{d}{=} \gamma_1 \tilde{M}_1$. We note that in general M_θ for $\theta > 0$, does not have a simple expression for its density. So the exact sampling method suggested above relies solely on the ability to sample the variable in $\tilde{M}_1$ in (1.7), for each p. This is possible provided that $\Psi_R(x)$ is analytically tractable. However, since R can be quite arbitrary this will not always be true. Fortunately, there is the recent Double CFTP perfect sampling method by [15] that can be used to exactly sample any of the variables satisfying (1.5), (1.6) or (1.7). This procedure applies provided that R is a bounded variable and one has a method to sample R, but otherwise does not require any potentially complicated calculations. Hence, any GGC(θ, R) variable, with R bounded, can be exactly sampled by drawing an independent gamma variable and applying the Double CFTP. Details may be found in [15], however we shall sketch out the details for a subclass of the variables $T(\tau)$ in the next section.

REMARK 1.1. Letting $Q_R(u)$ denote a quantile function of R, variables M_θ, satisfying (1.5), are called Dirichlet means since they can always be represented as

$$M_\theta \stackrel{d}{=} \int_0^1 Q_R(u) D_{0,\theta}(du|F_U) \stackrel{d}{=} \int_0^\infty y D_\theta(dy|F_R),$$

where

$$D_\theta(y|F_R) \stackrel{d}{=} \sum_{k=1}^\infty P_k \mathbb{I}_{(R_k \le y)} = \sum_{k=1}^\infty V_k \prod_{j=1}^{k-1} (1 - V_j) \mathbb{I}_{(R_k \le y)}$$

is a Dirichlet process with (P_k) a sequence of probabilities having a Poisson Dirichlet law with parameter θ, see [16, 33]. That is, for each k, $\gamma_\theta P_k \stackrel{d}{=} J_k$, where (J_k) are the ranked jumps of a gamma(θ) subordinator. (V_k) are i.i.d. Beta$(1, \theta)$ random variables, and (R_k) are i.i.d. F_R. See [22, 27] for more details.

REMARK 1.2. For $0 < \alpha < 1$, positive stable subordinators $S_\alpha(t)$, where $S_\alpha(1) := S_\alpha$, with $\psi_{S_\alpha}(\omega) = \omega^\alpha$, and corresponding processes $\widehat{S}_\alpha(t)$, with $\psi_{\widehat{S}_\alpha}(\omega) = (1+\omega)^\alpha - 1$, as well as their scaled variations, are in $\mathcal{G}$ but not $\mathcal{G}_+$. Naturally a gamma(θ) subordinator, say $(\gamma_\theta(t); t \ge 0)$, is in $\mathcal{G}_+$. However, S_α, $\widehat{S}_\alpha$ and γ_θ, constitute a family of (generalized gamma) subordinators with Lévy density

$$Cs^{-\alpha-1}e^{-bs}$$

for $0 \le \alpha < 1$ and $b \ge 0$, see [33], Proposition 21. Additionally, heavy tailed variables such as Linnik variables of the form $S_\alpha(\gamma_{\theta t}) \stackrel{d}{=} \gamma_{\theta t}^{1/\alpha} S_\alpha$ are in $\mathcal{G}_+$. As well as their exponentially tilted counterparts $\widehat{S}_\alpha(\gamma_{\theta t} p)$, for some $0 < p < 1$. See [20].

2. Convoluted subordinators. We now give the formal specifications for convoluted subordinators as defined in Bender and Marquardt [4]. Throughout the rest of the paper, let L denote an infinite activity subdordinator. That is the Lévy measure, $\Lambda_L(\infty) = \infty$. In order that the convoluted subordinator

$$T(t) = \int_0^t k(t,s)\, dL(s) \tag{2.1}$$

has strictly continuous and increasing trajectories, k is chosen to satisfy the following regularity conditions:

(a) for fixed $t \in [0, \infty)$, the mapping $s \mapsto k(t,s)$ is integrable,

(b) for fixed $s \in [0, \infty)$, the mapping $t \mapsto k(t,s)$ is continuous and increasing and there is an $\varepsilon > 0$ such that $t \mapsto k(t,s)$ is strictly increasing on $[s, s+\varepsilon]$,

(c) $k(t,s) = 0$ whenever $s > t \ge 0$.

The authors also derive a weighted Black–Scholes pricing formula for European style options as follows. Let

$$\widehat{W}_{\mu}(t) = W(t) + \mu t \tag{2.2}$$

denote a standard Brownian motion with drift parameter μ, that is $W(t)$ is a standard Brownian motion. Recall that for geometric Brownian motion the price process under the risk neutral measure is given by

$$S(t) = S(0)\exp\{rt + \widehat{W}_{-1/2}(\sigma^2 t)\}, \tag{2.3}$$

where $\widehat{W}_{-1/2}$ is defined by (2.2) with $\mu = -1/2$.

Setting $S_\tau = S(\tau)$, the quantity $(S_\tau - K)_+$ is the the payoff function of a European call option with strike $K > 0$ and maturity τ, and $r > 0$ is the risk-free interest rate. Then the Black–Scholes formula for the price at time 0, say $\mathrm{B}(\sigma, K, \tau)$, is given by

$$\mathrm{B}(\sigma, K, \tau) = \mathrm{e}^{-r\tau}\mathbb{E}[(S_\tau - K)_+] = S_0\Phi(d_1(\sigma)) - K\mathrm{e}^{-r\tau}\Phi(d_2(\sigma)), \tag{2.4}$$

where $\Phi(x)$ is the standard normal distribution function

$$d_1(\sigma) = \frac{\log(S_0/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad d_2(\sigma) = d_1(\sigma) - \sigma\sqrt{\tau}.$$

See Schoutens [35] for this notation. In [4], a price model under risk neutral dynamics is specified as

$$\tilde{S}(t) = S_0 \exp\{rt + \widehat{W}_{-1/2}(\sigma^2 T(t))\}, \tag{2.5}$$

where now $\tilde{S}$ is the asset price, and $T(t)$ is a convoluted subordinator. They obtain the following pricing formula.

THEOREM 2.1 (Bender and Marquardt [4], Theorem 4). *For the price model* (2.5), *with* $\tilde{S}_\tau = \tilde{S}(\tau)$. *Let* $(\tilde{S}_\tau - K)_+$ *be the payoff function of a European call option with strike* $K \in \mathbb{R}_+$ *and maturity* τ. *Then the initial fair price of* $(\tilde{S}_\tau - K)_+$, *is given by*

$$\mathrm{e}^{-r\tau}\mathbb{E}[(\tilde{S}_\tau - K)_+] = \mathbb{E}\big[\mathrm{B}\big(\sigma\sqrt{T(\tau)/\tau}, K, \tau\big)\big], \tag{2.6}$$

where for positive y, $\mathrm{B}(y, K, \tau)$ *is the Black–Scholes price given in* (2.4) *with* y *in place of* σ. *Furthermore*, S_0 *is considered fixed*.

As noted in [4], and discussed in the Introduction, the problem with the above result is that it is in general difficult to handle the exact law of $T(\tau)$. However, the authors do point out that if $T(\tau)$ possesses an analytically tractable characteristic function then it is possible to use fast Fourier transform (FFT) methods. They give some special examples where this might be possible, but in general this is not straightforward. This is clear since the Laplace exponent of $T(\tau)$ can be expressed as

$$\psi_{T(\tau)}(\omega) = \tau\mathbb{E}[\psi_L(\omega k(\tau, U\tau))], \tag{2.7}$$

where k and L are quite general and the expression otherwise depends on τ in a nontrivial way.

We believe Theorem 2.1 does have quite a bit of utility provided that one can have more control over the choice of marginal laws exhibited by $T(\tau)$, for each fixed τ. Next, we show that by choosing L to be in $\mathcal{G}_+$ one can (in a practical sense) use Theorem 2.1 for many kernels k. Even those that do not admit nice characteristic functions.

REMARK 2.1. We note that the martingale argument used in [4] is different than that used for standard time changed models. The filtration used by [4] preserves the martingale property for a larger class of models including, of course, time changes by a simple subordinator. However, the usual filtrations used for simple subordinators may not preserve the martingale property for all convoluted subordinators.

2.1. *A general result for L in $\mathcal{G}_+$.* As we just mentioned, we now look at the choice where L is a $\mathrm{GGC}(\theta, Y)$ subordinator where Y is some random variable. In terms of modeling for general $T(t)$, we can view θ as a time parameter that can be manipulated for practical convenience. This is partly because the variable Y can have unknown parameters that can be used for calibration.

THEOREM 2.2. *Let $\widehat{W}_\mu(t)$ denote a Brownian motion with drift as defined in* (2.2). *Let*

$$T(t) = \int_0^t k(t, y) L_\theta(dy) \tag{2.8}$$

denote a convoluted subordinator where L_θ is a $\mathrm{GGC}(\theta, Y)$ *subordinator. For each fixed t, define a random variable $R_t \stackrel{d}{=} k(t, Ut)$. Then the process $(\widehat{W}_\mu(T(t)) : t \ge 0)$ is almost surely continuous and has the following distributional properties*:

(i) *for each fixed t, $T(t)$ is a* $\mathrm{GGC}(\theta t, R_t Y)$ *random variable satisfying $T(t) \stackrel{d}{=} \gamma_{\theta t} M_{\theta t}$, where*

$$M_{\theta t} = \beta_{\theta t,1} M_{\theta t} + (1 - \beta_{\theta t,1}) R_t Y, \tag{2.9}$$

(ii) *if $0 < \theta t = p \le 1$, then $T(t) \stackrel{d}{=} \gamma_1 M_{1,t}$, where*

$$M_{1,t} \stackrel{d}{=} U M_{1,t} + (1 - U) R_t Y \xi_p, \tag{2.10}$$

(iii) *the density of the* $M_{1,t}$ *is given by*

$$\frac{x^{p-1}}{\pi}\sin(\pi\tilde{F}_t(x))\mathrm{e}^{-p\tilde{\Psi}_t(x)} \qquad \textit{for } x>0,$$

where $\tilde{F}_t = F_{R_tY\xi_p}$ *is the c.d.f. of the variable* $R_tY\xi_p$ *and*

$$\tilde{\Psi}_t(x) = \mathbb{E}\big[\log|x - R_tY|\mathbb{I}_{(R_tY\neq x)}\big].$$

PROOF. Following (2.7) and (1.4), it is easy to see that the Laplace exponent of $T(t)$ is given by

$$\psi_{T(t)}(\omega) = t\mathbb{E}[\psi_{L_\theta}(\omega k(t, Ut))] = \theta t\mathbb{E}[\log(1+\omega R_tY)].$$

The results (i), (ii) and (iii) then follow from the material we discussed at the end of Section 1.2. □

We now state a result for European style options, which is immediate from Theorems 2.1 and 2.2.

PROPOSITION 2.1. *For the price model* (2.5), *let* $T(t)$ *be the process specified by* (2.8) *and otherwise consider the setup in Theorem* 2.1. *Let* $(\tilde{S}_\tau - K)_+$ *be the payoff function of a European call option with strike* $K\in\mathbb{R}_+$ *and maturity* τ. *Then the initial fair price of* $(\tilde{S}_\tau - K)_+$, *is now given by*

$$\mathrm{e}^{-r\tau}\mathbb{E}[(\tilde{S}_\tau - K)_+] = \mathbb{E}\big[\mathrm{B}\big(\sigma\sqrt{\gamma_{\theta\tau}M_{\theta,\tau}/\tau}, K, \tau\big)\big]. \tag{2.11}$$

The pricing formula in (2.11) can be expressed in terms of a (VG) process with random scale $M_{\theta\tau}$ specified by (2.9). If for $R_\tau = k(\tau, U\tau)$, $R_\tau Y$ is bounded then one can obtain perfect samples of the distribution of $T(\tau)$, via [15]. For certain k, that are not necessarily bounded, one can use the density formula in (iii) of Theorem 2.2.

The next result introduces a nonrandom time change that leads to a significant reduction in complexity. First, define for $m>0$,

$$\phi_m(\mu) = \sqrt{2+\mu^2m^2}/m + \mu \quad \text{and} \quad b_m(\mu) = \frac{1}{m\sqrt{2+\mu^2m^2}}$$

and $c_m(\mu) = b_m(\mu)/\phi_m(\mu)$.

THEOREM 2.3. *For the convoluted subordinator in Theorem* 2.2, *the time changed process* $(\tilde{X}_\theta(s): s\ge 0) := (\widehat{W}_\mu(T((1-\mathrm{e}^{-s})/\theta)) : s\ge 0)$ *satisfies for each fixed* $s>0$,

$$\tilde{X}_\theta(s) \stackrel{d}{=} \widehat{W}_\mu(\gamma_1 M_{1,s^*}),$$

where M_{1,s^*} *satisfies* (2.10) *for* $t = s^* = (1-\mathrm{e}^{-s})/\theta$ *and* $p = 1-\mathrm{e}^{-s}$.

(i) *Furthermore, for each fixed* s, $\tilde{X}_\theta(s)$ *given* $M_{1,s*} = m^2$, *for* $m > 0$, *follows a double exponential distribution, with density*

$$f_{\tilde{X}_\theta(s)}(z|m) = \begin{cases} b_m(\mu)\mathrm{e}^{z\phi_m(\mu)}, & z \le 0, \\ b_m(\mu)\mathrm{e}^{-z\phi_m(-\mu)}, & z > 0, \end{cases}$$

and distribution function

$$F_\mu(z|m) = \begin{cases} c_m(\mu)\mathrm{e}^{z[\phi_m(\mu)]}, & z \le 0, \\ c_m(\mu) + c_m(-\mu)\bigl(1 - \mathrm{e}^{-z[\phi_m(-\mu)]}\bigr), & z > 0. \end{cases}$$

(ii) *Hence, if the price process in* (2.11) *is based on substituting* $T(t)$ *with the time time changed clock*, $T((1-\mathrm{e}^{-s})/\theta)$ *for* $s > 0$, *then the fair price is given by*

$$\mathrm{e}^{-r\tau}\mathbb{E}[(\tilde{S}_\tau - K)_+] = \mathbb{E}[\mathrm{DE}(\sigma^2 M_{1,\tau^*}, K, \tau)], \tag{2.12}$$

where, $\tau^* = (1-\mathrm{e}^{-\tau})/\theta$, *and for* $z = \log(S_0/K) + r\tau$,

$$\mathrm{DE}(y^2, K, \tau) = S_0 F_{-1/2}(z|y) - \mathrm{e}^{-r\tau} K F_{1/2}(z|y).$$

PROOF. The result follows from the fact that, again,

$$\psi_{T(t)}(\omega) = t\mathbb{E}[\psi_{L_\theta}(\omega k(t, Ut))] = \theta t \mathbb{E}[\log(1 + \omega R_t Y)].$$

Substituting $t = (1-\mathrm{e}^{-s})/\theta = s^*$, $p = \theta s^*$, yields a $\mathrm{GGC}(p, R_{s*}Y)$ variable, which is also a $\mathrm{GGC}(1, R_{s*}Y\xi_p)$. Statement (i) is straightforward. Statement (ii) is also not difficult to verify. □

In order to evaluate the price in (2.12), it remains to evaluate M_{1,τ^*}. We sketch out the details to do this via the Double CFTP perfect sampler in [15]. The deterministic time change allows us to exploit generally the most efficient case, $\theta = 1$, of the Double CFTP.

First, note again that, $R_{\tau^*} \stackrel{d}{=} k(\tau^*, \tilde{U}\tau^*)$ where $k(t, y)$ is a known function, and $\tilde{U}$ is a Uniform[0, 1] random variable. Hence, in order to sample R_{τ^*} we simply need to draw $\tilde{U}$. Note that we write $\tilde{U}$ to distinguish it from the uniform variables we introduce below denoted as U_i. Assuming

$$D \stackrel{d}{=} R_{\tau^*} Y \xi_p \stackrel{d}{=} k(\tau^*, \tilde{U}\tau^*) Y \xi_p$$

is bounded by a positive constant c, the Double CFTP exact sampler in [15] is based on the following steps:

Backward phase. For $i = -1, -2, \ldots$: keep generating (U_i, D_i, D'_i) and storing (D_i, D'_i) until $U_{\mathbb{T}} \le |D_{\mathbb{T}} - D'_{\mathbb{T}}|/(2c)$. Keep $\mathbb{T}$.

Set starting point. Set $M_{1,\tau^*} = D_{\mathbb{T}} \wedge D'_{\mathbb{T}} + 2cU_{\mathbb{T}}$.

Forward phase. For $i = \mathbb{T}+1, \mathbb{T}+2, \ldots, -1$: given $(D_i, D_i', M_{1,\tau^*})$ previously stored, do the following step: generate U' uniform $[0,1]$, $\xi_{1/2}$, and generate U uniform $[0,1]$, and construct $X = (1-U)M_{1,\tau^*} + UD_i\xi_{1/2} + UD_i'(1-\xi_{1/2})$. Repeat this step until:

$$U'\left[\mathbb{I}_{[0,1]}\left(\frac{X - M_{1,\tau^*}}{D_i - M_{1,\tau^*}}\right)\frac{1}{|D_i - M_{1,\tau^*}|} + \mathbb{I}_{[0,1]}\left(\frac{X - M_{1,\tau^*}}{D_i' - M_{1,\tau^*}}\right)\frac{1}{|D_i' - M_{1,\tau^*}|}\right] > 1/c$$

or $X < D_i \wedge D_i'$ or $X > D_i \vee D_i'$. Then set $M_{1,\tau^*} = X$.

Output. Return M_{1,τ^*}.

See [15] for more details.

REMARK 2.2. These results, which are considerably simplified by using the deterministic time change, apply to a wide choice of kernels. It would also be nice to find models for $T(\tau)$ whose marginal distributions were not strongly dependent on the form of the kernel. Even better, would be the ability to specify laws in a manner similar to how one selects the BDLP of an OU to induce general self-decomposable laws for the instantaneous volatility $v(t)$. In the next few sections, we will show that this can be done for convoluted subordinators we refer to as quantile clocks.

3. Quantile clocks.

As in the Introduction, let R denote a positive random variable with continuous strictly increasing cumulative distribution function F_R. Let Q_R denote its corresponding quantile function, that is, the continuous inverse of the cumulative distribution function. Furthermore, assume that $\mathbb{E}[R] < \infty$. Then for a subordinator L, we say that the process $T_R = (T_R(t) : t \geq 0)$, defined as

$$T_R(t) := \int_0^t Q_R\left(\left(1 - \frac{s}{t}\right)_+\right)L(ds) \qquad \text{for } t \geq 0 \tag{3.1}$$

is a *quantile clock* with parameters (R, L). Note here that R does not depend on t. Furthermore, Q_R can be evaluated numerically in many cases, even though it may not have a closed form.

PROPOSITION 3.1. *A quantile clock $T_R = (T_R(t) : t \geq 0)$ with parameters (R, L) has the following properties*:

(i) *If the support of the density of R, say f_R, is of the form $[0, b)$, $b > 0$, that is, $Q_R(0) = 0$, then T_R are random processes with samples paths that are almost sure strictly continuous and strictly increasing.*

(ii) *Suppose the density of R has support starting at $a > 0$, that is, $Q_R(0) = a$, then there is a positive random variable $\tilde{R}$ with $Q_{\tilde{R}}(0) = 0$, such that $R \stackrel{d}{=} \tilde{R} + a$ and $Q_R(u) = Q_{\tilde{R}}(u) + a$ for $u \in [0,1]$. Hence, it follows that the clock can be represented as*

$$T_R(t) = T_{\tilde{R}}(t) + aL(t), \qquad t \geq 0, \tag{3.2}$$

where $T_{\tilde{R}}$ satisfies (i). *Note $T_{\tilde{R}}$ is an $(\tilde{R}, L)$ quantile clock and is obviously not independent of L.*

(iii) *For each fixed t, the marginal distribution*

$$T_R(t) \stackrel{d}{=} \zeta(t),$$

where ζ is a subordinator such that $\zeta(1)$ is a random variable with Laplace exponent

$$\psi_\zeta(\omega) = \mathbb{E}[\psi_L(\omega R)] = \psi_{T_R(1)}(\omega).$$

(iv) *That is, the Lévy density of ζ has the form*

$$\rho_\zeta(s) = \int_0^\infty \rho_L(s/r) r^{-1} F_R(dr).$$

Note furthermore that for a constant c, $T_{cR}(t) \stackrel{d}{=} c\zeta(t)$.

PROOF. Setting $k(t,s) = Q_R((1-\frac{s}{t})_+))$ it follows from [4], Proposition 1, that in order to verify statement (i) we only need to check whether $k(t,s)$ satisfies conditions (a), (b), (c). Conditions (b) and (c) are obvious and it remains to check the integrability condition, which follows from

$$\int_0^t Q_R\left(\left(1-\frac{s}{t}\right)_+\right) ds = t\int_0^1 Q_R(u)\, du = t\mathbb{E}[R] < \infty,$$

since $Q_R(U) \stackrel{d}{=} R$. Now using this, and standard results for linear functionals of Lévy processes, we see that for each fixed t, the Laplace exponent of $T_R(t)$ is given by

$$\int_0^t \psi_L\left(\omega Q_R\left(\left(1-\frac{s}{t}\right)_+\right)\right) ds = t\int_0^1 \psi_L(\omega Q_R(u))\, du = t\mathbb{E}[\psi_L(\omega R)]$$

verifying (iii). Statements (ii) and (iv) follows easily from (i) and (iii). Note that the quantile function of R in statement (ii) violates condition (b). $\square$

We now give an interesting example that has explicit laws.

EXAMPLE 3.1 (Arcsine/Bessel occupation time quantile clocks driven by $L = \gamma_\theta$). First, recall the exponentially tilted stable subordinator $\widehat{S}_\alpha$ discussed in Remark 1.2. Suppose that one specifies $R \stackrel{d}{=} \beta_{1/2,1/2}$ and $L = \gamma_\theta$, a gamma(θ) subordinator. Then

$$Q_{(\beta_{1/2,1/2})}(u) = \sin^2\left(\frac{\pi}{2}u\right), \qquad 0 < u < 1,$$

and the quantile clock is defined as

$$T_{\beta_{1/2,1/2}}(t) = \int_0^t \sin^2\left(\frac{\pi(t-s)_+}{2t}\right)\gamma_\theta(ds)$$

for $t \geq 0$. It follows (see [12, 20, 22]) that for each fixed t,

$$T_{\beta_{1/2,1/2}}(t) \stackrel{d}{=} \widehat{S}_{1/2}(\gamma_{2\theta t}/2) \stackrel{d}{=} \gamma_{\theta t}\beta_{\theta t+1/2,\theta t+1/2}.$$

More generally, for each fixed $0 < \alpha < 1$, let $\mathbb{O}_\alpha(s) = \int_0^s \mathbb{I}_{(B_u>0)}\,du$ denote the time spent positive up till time s of a symmetrized Bessel process $(B_u, u \geq 0)$ of dimension $2 - 2\alpha$, see [1]. Then setting $R \stackrel{d}{=} \mathbb{O}_\alpha(1) := \mathbb{O}_\alpha$, the quantile of $\mathbb{O}_\alpha$ is

$$Q_{\mathbb{O}_\alpha}(u) = \frac{Q_{X_\alpha}(u)}{Q_{X_\alpha}(u) + 1} \qquad \text{where } Q_{X_\alpha}(u) = \left[\frac{\sin(\pi\alpha u)}{\sin(\pi\alpha(1-u))}\right]^{1/\alpha}$$

is the quantile function of the ratio of i.i.d. positive stable random variables $X_\alpha = S_\alpha/S'_\alpha$. Then, from James [20] (see Section 7), the clock $T_{\mathbb{O}_\alpha}$ with parameters $(\mathbb{O}_\alpha, \gamma_\theta)$, satisfies for each fixed t,

$$T_{\mathbb{O}_\alpha}(t) \stackrel{d}{=} \widehat{S}_\alpha(\gamma_{\theta t/\alpha}/2) \stackrel{d}{=} \gamma_{\theta t}\mathbb{O}_{\alpha,\theta t},$$

where

$$\begin{aligned}\mathbb{O}_{\alpha,\theta t} &\stackrel{d}{=} \beta_{\theta t,1}\mathbb{O}_{\alpha,\theta t} + (1-\beta_{\theta t,1})\mathbb{O}_\alpha \\ &\stackrel{d}{=} \beta_{\alpha+\theta t,1-\alpha}\mathbb{O}_{\alpha,\alpha+\theta t} + (1-\beta_{\alpha+\theta t,1-\alpha})\xi_{1/2}\end{aligned}$$

are random variables corresponding to the time spent positive of generalized Bessel bridges as explained in [20], Section 5. These variables can be exactly sampled in various ways as explained in [15]. Furthermore, from [20], Proposition 5.3, it follows that for $0 < p = \theta t \leq 1$,

$$T_{\mathbb{O}_\alpha}(p/\theta) \stackrel{d}{=} \gamma_1\tilde{\mathbb{O}}_{\alpha,p},$$

where $\tilde{\mathbb{O}}_{\alpha,p} \stackrel{d}{=} \beta_{p,1-p}\mathbb{O}_{\alpha,p}$ is $\mathrm{GGC}(1, \mathbb{O}_\alpha\xi_p)$ with density

$$\begin{aligned}f_{\tilde{\mathbb{O}}_{\alpha,p}}(y) = {}& \frac{2^{p/\alpha}}{\pi}y^{p-1}\sin\left(\frac{p}{\alpha}\arctan\left(\frac{(1-y)^\alpha\sin(\pi\alpha)}{(1-y)^\alpha\cos(\pi\alpha)+y^\alpha}\right)\right) \\ & \times [y^{2\alpha} + 2y^\alpha(1-y)^\alpha\cos(\alpha\pi) + (1-y)^{2\alpha}]^{-p/(2\alpha)},\end{aligned} \tag{3.3}$$

$0 < y < 1$. In general, the process $(\widehat{W}_\mu(T_{\mathbb{O}_\alpha}(t)), t \geq 0)$ has almost surely continuous sample paths and satisfies, for each fixed t,

$$\mathbb{E}\big[\mathrm{e}^{i\omega\widehat{W}_\mu(T_{\mathbb{O}_\alpha}(t))}\big] = 2^{\theta t/\alpha}\big(1 + \big(1 + (\omega^2/2 - i\mu\omega)\big)^\alpha\big)^{-\theta t/\alpha}.$$

REMARK 3.1. The last example shows that the class of quantile clocks where L is a $\mathrm{GGC}(\theta, Y)$ subordinator is equivalent in a marginal sense to the representation of GGC variables in terms of Wiener–Gamma integrals as defined and presented in [22]. That manuscript, along with the works of [5, 19–21], yield many examples of quantile functions which can be used to construct quantile clocks with explicit laws, of which quite a few are constructed from Q_{X_α}. We shall encounter some more examples in Section 6, although in a slightly different context.

4. Choosing quantile clocks to have specific laws in $\mathcal{U}_\delta$. The results in the previous section suggest that the the marginal distributions of the quantile clocks, while equating nicely to the marginals of a subordinator ζ, are strongly dependent on a random variable R, induced by Q_R. Noting that $Q_R(u)$ is in fact a deterministic function one would like to be able to choose explicit laws for T_R, regardless of the function Q_R. That is to say, how does one choose L so that $T_R(t)$ has a marginal distribution not obviously depending on R? For example, for each Q_R how does one choose L so that $T_R(t) \stackrel{d}{=} \widehat{S}_\alpha(t)$? Or how does one choose $T_R(t)$ so that a log price process $\widehat{W}_\mu(T_R(t))$ has marginal distributions that are equivalent to a CGMY process? Finally, for different quantile functions Q_{R_1}, Q_{R_2}, R_1 not equivalent in distribution to R_2, how to choose the driving Lévy processes, say L_1 and L_2, such that marginally for each fixed t, $T_{R_1}(t) \stackrel{d}{=} T_{R_2}(t)$?

We saw that this was difficult in the case of general convoluted subordinators as their laws depend strongly on t through the kernel or variable $k(t, Ut)$. However, Proposition 3.1 shows that one can represent

$$T_R(t) \stackrel{d}{=} \int_0^1 Q_R(y) L(t\,dy), \tag{4.1}$$

and there is a clear separation of the effects of t and Q_R. This is similar to the case of the OU models $v(0)$, see [2, 3, 26], where every positive self-decomposable random variable can be represented as

$$v(0) \stackrel{d}{=} v(t) \stackrel{d}{=} \int_{-\infty}^{y} \mathrm{e}^{-\lambda(y-s)} \vartheta(\lambda\,ds) \stackrel{d}{=} \int_0^\infty \mathrm{e}^{-s} \vartheta(ds), \tag{4.2}$$

where ϑ, is a subordinator referred to as a OU–BDLP. More strikingly, there is a simple way of obtaining any desired self-decomposable law for $v(0)$ by choosing the BDLP according to either of the equations

$$\psi_\vartheta(\omega) = \omega \psi'_{v(0)}(\omega) \quad \text{and} \quad \rho_\vartheta(x) = -\rho_{v(0)}(x) - x\rho'_{v(0)}(x). \tag{4.3}$$

We noticed from (4.1) that if $R \stackrel{d}{=} U^{1/\delta} \stackrel{d}{=} \beta_{\delta,1}$ for $\delta > 0$, then

$$T_{\beta_{\delta,1}}(t) \stackrel{d}{=} \int_0^1 u^{1/\delta} Z(t\,du) \stackrel{d}{=} \zeta^{(\delta)}(t),$$

where, we substitute Z for L, and $\zeta^{(\delta)}$ are subordinators having laws in Jurek's [17, 23–25] $\mathcal{U}_\delta$ class of *generalized* s-selfdecomposbale laws, where $\mathcal{U}_\delta \subset \mathcal{J}$. The case of $\delta = 1$, corresponds to Jurek's $\mathcal{U} = \mathcal{U}_1$ class of s-selfdecomposable class. Using Jurek [24, 25], one sees that for $0 < \delta_1 < 1 < \delta_2 < \infty$,

$$\mathcal{G}_+ \subset \mathcal{G} \subset \mathcal{L} \subset \mathcal{U}_{\delta_1} \subset \mathcal{U} \subset \mathcal{U}_{\delta_2} \subset \mathcal{J}.$$

It follows that for each $\zeta^{(\delta)} \in \mathcal{U}_\delta$ there is a, $\mathcal{U}_\delta$-BDLP, Z such that

$$\psi_{\zeta^{(\delta)}}(\omega) = \psi_{T_{\beta_{\delta,1}}(1)}(\omega) = \int_0^1 \psi_Z(\omega u^{1/\delta})\,du = \omega^{-\delta} \int_0^\omega \psi_Z(u) \delta u^{\delta-1}\,du$$

and hence from [14], Lemma 1, which can be verified directly by taking derivatives with respect to ω of

$$\omega^{\delta}\psi_{\zeta^{(\delta)}}(\omega)=\int_0^{\omega}\psi_Z(u)\delta u^{\delta-1}\,du, \tag{4.4}$$

one sees that the $\mathcal{U}_\delta$-BDLP Z is related to $\zeta^{(\delta)}$, and hence $T_{\beta_{\delta,1}}$, by the equation

$$\psi_Z(\omega)=\psi_{\zeta^{(\delta)}}(\omega)+\frac{1}{\delta}\omega\psi'_{\zeta^{(\delta)}}(\omega). \tag{4.5}$$

This is analogous to the relationships between $v(0)$ and its OU–BDLP ϑ, given in (4.2) and (4.3). We shall show that this relationship becomes more explicit as one restricts their choices of laws for $\zeta^{(\delta)}$ to $\mathcal{L}$, $\mathcal{G}$ and $\mathcal{G}_+$.

REMARK 4.1. The specifications in (4.5) and its refinements now allow us to specify any law in $\mathcal{U}_\delta$ for quantile clocks based on $Q_{U^{1/\delta}}$, analogous to the case of the BNS–OU $v(t)$. This, as far as we know, is the first instance where such a property has been noticed for convoluted subordinators. However, in terms of choices of Q_R this is still restrictive. The next results show how, for a large class of quantile functions Q_R, to choose L such that for each fixed t, $T_R(t)\overset{d}{=}\zeta^{(\delta)}(t)\in\mathcal{U}_\delta$.

THEOREM 4.1. *Consider the specifications for a quantile clock*

$$T_R(t)=\int_0^t Q_R\biggl(\biggl(1-\frac{s}{t}\biggr)_+\biggr)L(ds)$$

with parameters (R,L). *Now select* R *so that its density has bounded support and let* Y *denote a positive bounded random variable such that*

$$RY\overset{d}{=}U^{1/\delta}\overset{d}{=}\beta_{\delta,1} \tag{4.6}$$

for a fixed $\delta>0$; *if* $R\overset{d}{=}cU^{1/\delta}$, *then* $Y=1/c$. *Suppose that one wishes to choose* L *such that*

$$T_R(t)\overset{d}{=}\zeta^{(\delta)}(t),$$

where $\zeta^{(\delta)}$ *is a subordinator with*

$$\psi_{\zeta^{(\delta)}}(\omega)=\mathbb{E}[\psi_Z(\omega U^{1/\delta})],$$

where Z *is a* $\mathcal{U}_\delta$*-BDLP, satisfying* (4.5), *and hence* $\zeta^{(\delta)}$ *is in* $\mathcal{U}_\delta$. *Then for this* Z, L *is chosen such that*

$$\psi_L(\omega)=\mathbb{E}[\psi_Z(\omega Y)]\quad\textit{equivalently}\quad\rho_L(x)=\mathbb{E}[\rho_Z(x/Y)Y^{-1}]. \tag{4.7}$$

That is,

$$\psi_L(\omega)=\mathbb{E}\bigl[\psi_{\zeta^{(\delta)}}(\omega Y)\bigr]+\frac{1}{\delta}\omega\mathbb{E}\bigl[Y\psi'_{\zeta^{(\delta)}}(\omega Y)\bigr]. \tag{4.8}$$

Note that Y *is chosen independent of* R *and* Z.

PROOF. The difficulty of this result is envisioning its construction. The proof itself is otherwise straightforward, since (4.7) and (4.5) implies that

$$\psi_{\zeta^{(\delta)}}(\omega) = \mathbb{E}[\psi_L(\omega R)] = \mathbb{E}[\psi_Z(\omega U^{1/\delta})]. \qquad \square$$

We now specialize this result to self-decomposable laws.

THEOREM 4.2. *Consider quantile clocks T_R with parameters (R, L) satisfying (4.6) and (4.7), (4.8). The next result describes further specifications in order for $T_R, (\zeta^{(\delta)})$ to have laws in $\mathcal{L}, \mathcal{G}$ and $\mathcal{G}_+$, respectively.*

I. *$T_R \in \mathcal{L}$: If T_R is selected such that its marginal laws are self-decomposable, then it is known that there exists a subordinator ϑ, such that*

$$T_R(1) \stackrel{d}{=} \zeta^{(\delta)}(1) \stackrel{d}{=} \int_{-\infty}^{y} \mathrm{e}^{-\lambda(y-s)} \vartheta(\lambda\, ds) \stackrel{d}{=} \int_0^{\infty} \mathrm{e}^{-s} \vartheta(ds) \stackrel{d}{=} \upsilon(0).$$

Furthermore, adapting (4.3), one has

$$\psi_\vartheta(\omega) = \omega \psi'_{\zeta^{(\delta)}}(\omega) \quad \textit{and} \quad \rho_\vartheta(x) = -\rho_{\zeta^{(\delta)}}(x) - x\rho'_{\zeta^{(\delta)}}(x). \tag{4.9}$$

(i) *Hence, the BDLP L has to be chosen such that the Lévy density of Z is*

$$\begin{aligned} \rho_Z(x) &= \left(1 - \frac{1}{\delta}\right)\rho_{\zeta^{(\delta)}}(x) - \frac{x}{\delta}\rho'_{\zeta^{(\delta)}}(x) \\ &= \rho_{\zeta^{(\delta)}}(x) + \frac{1}{\delta}\rho_\vartheta(x). \end{aligned}$$

That is,

$$\rho_L(x) = \mathbb{E}\left[\left(\rho_{\zeta^{(\delta)}}(x/Y) + \frac{1}{\delta}\rho_\vartheta(x/Y)\right)Y^{-1}\right].$$

(ii) *Statement* (i) *implies that the subordinators are related as follows*:

$$Z(t) \stackrel{d}{=} \zeta^{(\delta)}(t) + \vartheta(t/\delta), \qquad t \ge 0.$$

II. *$T_R \in \mathcal{G}$: If $T_R, (\zeta^{(\delta)})$ is selected such that its marginal laws are* GGC(ν), *it follows that*

$$\rho_{\zeta^{(\delta)}}(x) = x^{-1}\int_0^{\infty} \mathrm{e}^{-xy}\nu(dy) \quad \textit{and} \quad \rho_\vartheta(x) = \int_0^{\infty} \mathrm{e}^{-xy} y\nu(dy).$$

(i) *Hence, the Lévy density of Z, say ρ_Z, satisfies*

$$\rho_Z(x) = \int_0^{\infty} \mathrm{e}^{-xy}[x^{-1} + y/\delta]\nu(dy).$$

(ii) *Equivalently*,

$$\psi_Z(\omega) = \int_0^\infty \left[\log(1+\omega/y) + \frac{1}{\delta}\frac{\omega}{y+\omega}\right]\nu(dy).$$

III. $T_R \in \mathcal{G}_+$: *If* T_R, $(\zeta^{(\delta)})$ *is selected such that its marginal laws are* $\mathrm{GGC}(\theta, V)$,

(i) *then* L *must be selected such that it is equivalent in distribution to the subordinator*

$$L(s) \stackrel{d}{=} \zeta_{\delta,Y}(s) + \vartheta_Y(s\theta/\delta), \qquad s \geq 0,$$

where $\zeta_{\delta,Y}$ *is a* $\mathrm{GGC}(\theta, VY)$ *subordinator and*

$$\vartheta_Y(s) \stackrel{d}{=} \sum_{k=1}^{N(s)} \gamma_1^{(k)} V_k Y_k, \qquad s \geq 0,$$

where $(\gamma_1^{(k)})$ *are independent* exponential(1) *variables*, (Y_k) *are i.i.d. variables with distribution* F_Y, (V_k) *are i.i.d.* F_V, *and* $N(s)$ *denotes a homogeneous Poisson process with* $\mathbb{E}[N(s)] = s$.

(ii) *As special cases* $T_R(t) \stackrel{d}{=} \gamma_\theta(t)$ *is obtained by setting* $V = 1$.

The proof of this result is fairly immediate from the definitions of the various classes, details are omitted.

REMARK 4.2. Theorem 4.2 shows that in order to specify T_R to have laws in $\mathcal{L}$, one only needs to identify the OU–BDLP ϑ that leads to a corresponding stationary law for $v(t) \stackrel{d}{=} T_R(1) \stackrel{d}{=} \zeta^{(\delta)}(1)$, and use it appropriately to define L. One may consult for instance [3] for many explicit examples ϑ, and the laws they induce.

5. Choosing R and Y such that $RY \stackrel{d}{=} U^{1/\delta}$. The results in the previous section show that for a deterministic quantile function Q_R one can choose quite arbitrary marginal laws for T_R, analogous to the case of $v(0)$, provided that one identifies a variable Y such that $RY \stackrel{d}{=} U^{1/\delta} = \beta_{\delta,1}$ for a fixed $\delta > 0$. Notice that

$$Q_{R^{1/\delta}}(u) = [Q_R(u)]^{1/\delta}. \tag{5.1}$$

The easiest case is to choose $R = U$ and $Y = 1$, which as seen from (5.1) leads to quantile clocks corresponding to the *Holmgren–Liouville* convoluted subordinators discussed in [4]. The equation (5.1) suggests that one may always work with the pair satisfying the solution $RY = U$ and then obviously $R^{1/\delta}Y^{1/\delta} = U^{1/\delta}$. However, the case of $\delta = 1$ may not always be the most obvious.

EXAMPLE 5.1 (Beta variables including the arcsine distribution). Consider the case of products of independent beta variables

$$\beta_{\delta,\kappa-\delta}\beta_{\kappa,1+\delta-\kappa} \stackrel{d}{=} \beta_{\delta,1} \stackrel{d}{=} U^{1/\delta} \tag{5.2}$$

for $\delta \le \kappa \le 1+\delta$. Hence, for each δ, one can choose many (R, Y), ranging over $\delta \le \kappa \le 1+\delta$, such that

$$\begin{aligned} &(R^{1/\delta}, Y^{1/\delta}) \stackrel{d}{=} (\beta_{\delta,\kappa-\delta}, \beta_{\kappa,1+\delta-\kappa}) \quad \text{or} \\ &(R^{1/\delta}, Y^{1/\delta}) = (\beta_{\kappa,1+\delta-\kappa}, \beta_{\delta,\kappa-\delta}). \end{aligned} \tag{5.3}$$

Furthermore, for some $b > 0$, and each fixed δ the variables in (5.3) lead to variables $R^{1/b}$ and $Y^{1/b}$, not having beta distributions, that satisfy $(RY)^{1/b} \stackrel{d}{=} U^{1/b}$.

Lets look at a special case of this in more detail.

EXAMPLE 5.2 (Kumaraswamy and generalized arcsine clocks). Setting $\kappa = 1$ and $\delta = \alpha$, (5.3) leads to the choice of the pair

$$(\beta_{\alpha,1-\alpha}, 1 - U^{1/\alpha}) \stackrel{d}{=} (\beta_{\alpha,1-\alpha}, \beta_{1,\alpha}), \tag{5.4}$$

such that $R^{1/\alpha}Y^{1/\alpha} \stackrel{d}{=} U^{1/\alpha} \stackrel{d}{=} \beta_{\alpha,1}$, where the first component in (5.4) has the *generalized arcsine law* which arises in many studies of random processes. Setting $R^{1/b} \stackrel{d}{=} [1 - U^{1/\alpha}]^{1/b} = K_{\alpha,b}$ leads to the quantile function

$$Q_{K_{\alpha,b}}(u) = [1 - (1-u)^{1/\alpha}]^{1/b}$$

of a Kumaraswamy distribution. Hence, the law of a Kumaraswamy quantile clock $T_{K_{\alpha,b}}$, that is, with parameters $(K_{\alpha,b}, L)$, can be specified such that its marginals satisfy

$$T_{K_{\alpha,b}}(t) \stackrel{d}{=} \zeta^{(\alpha b)}(t),$$

where $\zeta^{(\alpha b)}$ is a subordinator having any law in $\mathcal{U}_{\alpha b}$ and hence in $\mathcal{L}$. Specifically, this is done by the choice of

$$\psi_L(\omega) = \mathbb{E}[\psi_{\zeta^{(\alpha b)}}(\omega(\beta_{\alpha,1-\alpha})^{1/b})] + \frac{1}{\alpha b}\omega\mathbb{E}[\psi'_{\zeta^{(\alpha b)}}(\omega(\beta_{\alpha,1-\alpha})^{1/b})(\beta_{\alpha,1-\alpha})^{1/b}].$$

Note that if we instead choose $R = \beta_{1/2,1/2}$ and hence $R^{1/b} \stackrel{d}{=} (\beta_{1/2,1/2})^{1/b}$ we obtain clocks based on the arcsine law with quantiles

$$Q_{R^{1/b}}(u) := [Q_{(\beta_{1/2,1/2})}(u)]^{1/b} = \sin^{2/b}\left(\frac{\pi}{2}u\right). \tag{5.5}$$

This case can be compared with Example 3.1.

5.1. *Selections based on decompositions of an* exponential(1) *variable.* In general, by rescaling to [0, 1], we see that choosing an R and Y to satisfy (4.6) for some $\delta > 0$ is equivalent to choosing variables ℓ_R and ℓ_Y such that

$$\ell_R + \ell_Y \stackrel{d}{=} \gamma_1/\delta. \tag{5.6}$$

Recall also the relationship between quantiles of a positive variable X and e^{-X},

$$Q_{\mathrm{e}^{-X}}(u) = \mathrm{e}^{-Q_X(1-u)}, \qquad 0 \le u \le 1.$$

There are obviously many pairs satisfying (5.6). We next look at two different types of examples based on suggestions made to us by Prof. Marc Yor.

EXAMPLE 5.3 [Fractional and integer parts of an exponential(1)]. We first note that one of the reasons the following example is interesting is that it identifies a concrete example of a quantile clock that is of the form

$$T_R(t) = T_{\tilde{R}} + aL(t)$$

as specified in statement (ii) of Proposition 3.1, but where we can apply Theorem 4.1. Now, following Chaumont and Yor ([11], page 42, Exercise 2.18), let $[\gamma_1]$ and $\{\gamma_1\}$ denote the fractional part and integer part of an exponential(1) variable γ_1, then (remarkably) these variables are independent and obviously satisfy

$$[\gamma_1] + \{\gamma_1\} = \gamma_1.$$

In this case, $RY \stackrel{d}{=} U$, for

$$R \stackrel{d}{=} \mathrm{e}^{-[\gamma_1]} \stackrel{d}{=} U(1 - e^{-1}) + e^{-1} \quad \text{and} \quad Y \stackrel{d}{=} \mathrm{e}^{-\{\gamma_1\}}, \tag{5.7}$$

where $\{\gamma_1\}$ is a geometric random variable with success probability $1 - e^{-1}$ and values in $\{0, 1, 2, \ldots\}$. We say such a variable is geometric $(1 - e^{-1})$. We can extend this case as follows.

PROPOSITION 5.1. *For* $0 < p \le 1$, *let* $\tilde{U}_p \stackrel{d}{=} Up + (1 - p)$ *and let* X_p *be a* geometric(p) *variable. Then*

$$\tilde{U}_p \mathrm{e}^{-X_p[-\log(1-p)]} \stackrel{d}{=} U.$$

PROOF. It is easy to verify that the Laplace transforms of $-\log(Up + (1 - p))$ and $X_p[-\log(1 - p)]$ are given, respectively, by

$$\frac{1 - (1 - p)^{(1+\omega)}}{p(1 + \omega)} \quad \text{and} \quad \frac{p}{1 - (1 - p)^{(1+\omega)}}.$$

Hence, their product is $1/(1 + \omega)$, which is the desired result. $\square$

That is, there are variables $\tilde{U}_p Y_p \stackrel{d}{=} U$ for

$$R = \tilde{U}_p \stackrel{d}{=} Up + (1-p) \quad \text{and} \quad Y_p = \mathrm{e}^{-X_p[-\log(1-p)]}$$

for X_p a geometric(p) variable for each $0 < p \le 1$. Naturally $\tilde{U}_1 \stackrel{d}{=} U$. Hence, for each fixed p,

$$T_{\tilde{U}_p}(t) = L(t)(1-p) + p\int_0^t \left(1 - \frac{s}{t}\right)_+ L(ds) \qquad \text{for } t \ge 0.$$

We now state a result which now follows obviously from Theorem 4.1 and applies for quantiles clocks based on the variable $\tilde{U}_p^{1/\delta}$.

PROPOSITION 5.2. *For each $\delta > 0$, and $0 < p \le 1$, set $\lambda = -\log(1-p)$, then the quantile clock*

$$T_{\tilde{U}_p^{1/\delta}}(t) = \int_0^t \left[(1-p) + p\left(1 - \frac{s}{t}\right)_+\right]^{1/\delta} L(ds)$$

can be specified such that for each t, $T_{\tilde{U}_p^{1/\delta}}(t) \stackrel{d}{=} \zeta^{(\delta)}(t) \in \mathcal{U}_\delta$ if L is chosen such that

$$\psi_L(\omega) = \mathbb{E}[\psi_{\zeta^{(\delta)}}(\omega \mathrm{e}^{-\lambda X_p/\delta})] + \frac{1}{\delta}\omega \mathbb{E}[\mathrm{e}^{-\lambda X_p/\delta}\psi'_{\zeta^{(\delta)}}(\omega \mathrm{e}^{-\lambda X_p/\delta})],$$

where X_p is a geometric(p) *random variable. When $p = 1$, the quantile clock is continuous, otherwise it has jumps.*

EXAMPLE 5.4 (Splitting the Laplace exponent of γ_1, part I). Next, consider the Laplace exponent of γ_1/δ,

$$\log(1 + \omega/\delta) = \int_0^\infty (1 - \mathrm{e}^{-s\omega/\delta}) s^{-1} \mathrm{e}^{-s}\, ds,$$

then choose ℓ_R and ℓ_Y according to the decomposition of the Lévy density

$$s^{-1}\mathrm{e}^{-s} = \pi_1(s) + \pi_2(s).$$

In particular, ℓ_R and ℓ_Y are infinitely divisible based on the Lévy densities π_1 and π_2, respectively. The simplest case is where

$$s^{-1}\mathrm{e}^{-s} = (1-\alpha)s^{-1}\mathrm{e}^{-s} + \alpha s^{-1}\mathrm{e}^{-s}$$

leading to

$$(RY)^{1/\delta} \stackrel{d}{=} [\mathrm{e}^{-\gamma_\alpha}\mathrm{e}^{-\gamma_{1-\alpha}}]^{1/\delta} \stackrel{d}{=} U^{1/\delta}.$$

So, for instance, the quantile clock with parameters $(\mathrm{e}^{-\gamma_\alpha}, L)$, that is,

$$T_{\mathrm{e}^{-\gamma_\alpha}}(t) = \int_0^t \mathrm{e}^{-Q_{\gamma_\alpha}(1-(1-s/t)_+)} L(ds)$$

is based on a nontrivial quantile which can however be evaluated by various computational packages. Furthermore, our results show that despite the complexity of this clock we can choose quite general marginal laws for $T_{e^{-\gamma_\alpha}}$, not directly depending on the quantile function, by working with a BDLP satisfying

$$\psi_L(\omega) = \mathbb{E}[\psi_Z(\omega e^{-\gamma_{1-\alpha}})] = \int_0^1 \psi_Z(\omega y) \frac{[-\log(y)]^{-\alpha}}{\Gamma(1-\alpha)} \, dy.$$

6. GGC decompositions of a γ subordinator and resulting clocks. We now identify a very large class of variables satisfying (5.6) based on Example 5.4, using variables in $\mathcal{B}$ and in particular, the GGC class $\mathcal{G}_+$.

THEOREM 6.1. *Let Ω_δ be a subordinator in $\mathcal{B}$ with Lévy density $\rho_{\Omega_\delta}(s) = \int_\delta^\infty e^{-sx} q(x)\,dx$, where $q(x)$ is a nonnegative measure such that $q(x) \le 1$ for $x \ge \delta$. Then, there is another subordinator in $\mathcal{B}$, say $\widehat{\Omega}_\delta$, with Lévy density $\rho_{\widehat{\Omega}_\delta}(s) = \int_\delta^\infty e^{-sx}[1 - q(x)]\,dx$, such that the Lévy density of a* gamma(1) *subordinator with scale $1/\delta$, has the decomposition*

$$s^{-1} e^{-s\delta} = \rho_{\widehat{\Omega}_\delta}(s) + \rho_{\Omega_\delta}(s). \tag{6.1}$$

Hence, the gamma subordinator can be expressed as a sum of the subordinators Ω_δ and $\widehat{\Omega}_\delta$, which implies for each fixed θ

$$\gamma_\theta/\delta \stackrel{d}{=} \widehat{\Omega}_\delta(\theta) + \Omega_\delta(\theta).$$

PROOF. The Lévy density of the subordinator $(\gamma_1(t)/\delta : t \ge 0)$ can be expressed as

$$s^{-1} e^{-s\delta} = \int_\delta^\infty e^{-sx}\,dx,$$

leading easily to (6.1). □

We next describe an interesting special case involving variables in $\mathcal{G}_+$.

THEOREM 6.2. *Assume that Ω_δ in Theorem 6.1 is a* GGC$(1, V/\delta)$ *subordinator for V a random variable in* $[0, 1]$, *and let $X \stackrel{d}{=} (1-V)/V$. Let $(\Sigma_t(V) : t \ge 0)$ denote a subordinator with Lévy density denoted as $\rho_{\Sigma_1(V)}$. Note that $\Sigma_1(V)$ is not random in V.*

(i) *The Lévy density of a* gamma(1) *subordinator with scale $1/\delta$ has the decomposition*

$$s^{-1} e^{-s\delta} = \delta \rho_{\Sigma_1(V)}(s\delta) + s^{-1}\mathbb{E}[e^{-s\delta/V}],$$

where

$$\rho_{\Sigma_1(V)}(s) = \int_1^\infty e^{-sx}[1 - F_{1/V}(x)]\,dx = \frac{1}{s}e^{-s}(1 - \mathbb{E}[e^{-sX}]). \tag{6.2}$$

(ii) *If* $0 < \varrho = \mathbb{E}[-\log(V)] < \infty$, *then* $\rho_{\Sigma_1(V)}(s) = \varrho f_{\Delta_{\mathbf{e}}(X)}(s)$, *where* $f_{\Delta_{\mathbf{e}}(X)}(s)$ *is a density of a random variable denoted as* $\Delta_{\mathbf{e}}(X)$, *determined by* (6.2). *In this case* $\Sigma_t(V)$ *is a compound Poisson process representable as*

$$\Sigma_t(V) = \sum_{k=1}^{N(\varrho t)} \Delta_k, \qquad t \geq 0.$$

(Δ_k) *are i.i.d. random variables equal in distribution to* $\Delta_{\mathbf{e}}(X)$.

(iii) *In general, for each fixed* θ,

$$\gamma_\theta \stackrel{d}{=} \Sigma_\theta(V) + \gamma_\theta M_\theta, \tag{6.3}$$

where $M_\theta \stackrel{d}{=} \beta_{\theta,1} M_\theta + (1 - \beta_{\theta,1})V$.

PROOF. From [6], Section 9, we know that Ω_1 corresponds to a GGC$(1, V)$ subordinator if $q(x) = F_{1/V}(x)$. Hence, scaling by δ, and using known properties of variables in $\mathcal{G}_+$ concludes the result. □

Note from, for instance, [6], Example 9.2.3, it follows that for each $\delta > 0$, one can choose $\Omega_\delta(1) \stackrel{d}{=} -\log(\beta_{\delta,\kappa-\delta})$ or $\Omega_\delta(1) \stackrel{d}{=} -\log(\beta_{\kappa,1+\delta-\kappa})$ for the beta variables in Example 5.1 satisfying (5.2). Furthermore, among these, the only choice corresponding to a GGC variable is $\Omega_\delta(1) \stackrel{d}{=} -\log(\beta_{\delta,1})$. However, Theorem 6.2 allows us to construct many quantile clocks based on variables in $\mathcal{G}_+$ whose distributional properties are explicit. We next describe an interesting property of the variable $\Sigma_1(p)$.

PROPOSITION 6.1. *Suppose that* T_R *is a quantile clock with parameters* (R, L) *such that, for an independent variable* Y,

$$RY \stackrel{d}{=} U^{1/\delta} \stackrel{d}{=} \beta_{\delta,1} \stackrel{d}{=} U^p$$

for $\delta = 1/p > 1$. *If* L *is chosen such that*

$$\psi_L(\omega) = \mathbb{E}\big[\psi_Z\big(\omega Y e^{-\Sigma_1(p)}\big)\big], \tag{6.4}$$

where Z *is a* $\mathcal{U}_1$*-BDLP, satisfying* (4.5) *for a subordinator* $\zeta \in \mathcal{U}_1$, *then* $T_R(t) \stackrel{d}{=} \zeta(t)$ *for each fixed* t.

PROOF. Setting $V = p$, it follows from (6.3), with $\theta = 1$, that

$$\gamma_1 \stackrel{d}{=} p\gamma_1 + \Sigma_1(p),$$

which gives the identity

$$U^p \mathrm{e}^{-\Sigma_1(p)} \stackrel{d}{=} U.$$

Hence, (6.4) leads to $\mathbb{E}[\psi_L(\omega R)] = \mathbb{E}[\psi_Z(\omega U)]$. □

6.1. *Interpreting $\Sigma_t(V)$ via diffusions straddling an exponential time.* Provided that $0 < \mathbb{E}[-\log(V)] < \infty$, the random variable $\Delta_{\mathbf{e}}(X)$, with density defined by (6.2) has an interesting interpretation that we now discuss. This will also give us an opportunity to describe some more explicit examples of Q_R. Let $\mathbf{e}/\tilde{X}$ denote an independent exponential(1) time $\mathbf{e}$ divided by an independent variable $\tilde{X}$ with distribution characterized, for bounded measurable functions H, by

$$\mathbb{E}[H(\tilde{X})] = \mathbb{E}[H(X)\log(1+X)]/\varrho.$$

Now let $\{\mathcal{R}_s^{(0,1)}, s \ge 0\}$ denote a recurrent linear diffusion starting at 0 whose inverse local time, in this case, is a gamma(1) subordinator. Define for any $t > 0$,

$$g_t := \sup\{s \le t; \mathcal{R}_s^{(0,1)} = 0\}, \qquad d_t := \inf\{s \ge t, \mathcal{R}_s^{(0,1)} = 0\}.$$

Then given $\tilde{X} = \lambda$, it follows from (6.2) that for an independent exponential(λ) variable $\mathbf{e}/\lambda$, the random variable

$$\Delta_{\mathbf{e}}(\lambda) \stackrel{d}{=} d_{\mathbf{e}/\lambda} - g_{\mathbf{e}/\lambda}$$

corresponds to the length of excursion of $\mathcal{R}^{(0,1)}$ above 0 straddling an exponential(λ) time. See, for instance, [34], Section 4, for this description for more general $\mathcal{R}$ as well as [5, 32, 36]. In addition, see [5, 19, 22] for $\Delta_{\mathbf{e}}(\lambda)$ representation as a variable in $\mathcal{G}_+$. Hence, $\Delta_{\mathbf{e}}(X)$ interprets as $\Delta_{\mathbf{e}}(\lambda)$ but now for a random time $\mathbf{e}/\tilde{X}$, with c.d.f. $F_{\mathbf{e}/\tilde{X}}$ satisfying

$$1 - F_{\mathbf{e}/\tilde{X}}(y) = \mathbb{E}[\mathrm{e}^{-Xy}\log(1+X)]/\varrho.$$

It follows that for $\lambda = (1-p)/p$ and $\varrho = -\log(p)$, that

$$\Sigma_t(p) \stackrel{d}{=} \sum_{k=1}^{N(\varrho t)} d_{\mathbf{e}/\lambda}^{(k)} - g_{\mathbf{e}/\lambda}^{(k)}, \qquad t > 0,$$

where $(d_{\mathbf{e}/\lambda}^{(k)}, g_{\mathbf{e}/\lambda}^{(k)})$ are i.i.d. copies of $(d_{\mathbf{e}/\lambda}, g_{\mathbf{e}/\lambda})$.

6.2. *Some related examples*. From the results in [5] (see also [19, 22]), we can consider more generally $\mathcal{R}^{(\alpha,1)}$, in place of $\mathcal{R}^{(0,1)}$, which, for $0 \le \alpha < 1$ is now a process whose inverse local time is distributed as a generalized gamma subordinator with Lévy density specified by $s^{-\alpha-1}\mathrm{e}^{-s}/\Gamma(1-\alpha)$ for $s > 0$. Furthermore, for $\lambda = 1$, $\Delta_{\mathbf{e}}^{(\alpha,1)}$, is the generalization of $\Delta_{\mathbf{e}}(1) \stackrel{d}{=} \Delta_{\mathbf{e}}^{(0,1)}$, with density

$$\Delta_{\mathbf{e}}^{(\alpha,1)} \stackrel{d}{=} \frac{\alpha x^{-\alpha-1}\mathrm{e}^{-x}(1-\mathrm{e}^{-x})}{[2^{\alpha}-1]\Gamma(1-\alpha)} \qquad \text{for } x > 0. \tag{6.5}$$

Note that the variable $U_{\alpha,\mathbf{e}} \stackrel{d}{=} \mathrm{e}^{-\Delta_{\mathbf{e}}^{(\alpha,1)}}$ has density

$$f_{U_{\alpha,\mathbf{e}}}(u) \stackrel{d}{=} \frac{\alpha[-\log(u)]^{-\alpha-1}(1-u)}{[2^{\alpha}-1]\Gamma(1-\alpha)} \qquad \text{for } 0 < u \le 1. \tag{6.6}$$

In addition, [5] show that $\Delta_{\mathbf{e}}^{(\alpha,1)}$ is $\mathrm{GGC}(1-\alpha, \mathbb{D}_\alpha)$, where $\mathbb{D}_\alpha$ satisfies

$$\mathbb{G}_\alpha \stackrel{d}{=} \frac{1}{\mathbb{D}_\alpha} - 1$$

with

$$\log(X_{1-\alpha}) = \log(S_{1-\alpha}/S'_{1-\alpha}) = \frac{\alpha}{1-\alpha}\log\bigl(\mathbb{G}_\alpha/(1-\mathbb{G}_\alpha)\bigr).$$

Furthermore, $\mathbb{G}_{1/2} \stackrel{d}{=} \beta_{1/2,1/2}$, $\mathbb{G}_1 \stackrel{d}{=} U$ and $1/\mathbb{G}_0 \stackrel{d}{=} 1 + \mathrm{e}^{\pi\eta}$ for η a standard Cauchy variable. Furthermore $\gamma_{1-\alpha}U \stackrel{d}{=} \gamma_1\beta_{1-\alpha,1+\alpha}$ is $\mathrm{GGC}(1-\alpha, \mathbb{G}_\alpha)$. We now look at some special case of Theorem 6.2.

PROPOSITION 6.2. *Let* $0 \le \alpha < 1$.

(i) *Then for* $V = \mathbb{D}_\alpha$ *and* $X = \mathbb{G}_\alpha$,

$$\begin{aligned} \gamma_1 &\stackrel{d}{=} \Delta_{\mathbf{e}}^{(\alpha,1)} + \Sigma_{1-\alpha}(\mathbb{D}_\alpha) + \gamma_\alpha \\ &\stackrel{d}{=} \Delta_{\mathbf{e}}^{(\alpha,1)} + \gamma_\alpha M_\alpha + \Sigma_1(\mathbb{D}_\alpha) \\ &\stackrel{d}{=} \Delta_{\mathbf{e}}^{(\alpha,1)} + \Sigma_1(\xi_{1-\alpha}\mathbb{D}_\alpha), \end{aligned} \tag{6.7}$$

where $\Delta_{\mathbf{e}}^{(\alpha,1)}$ *has density* (6.5). $M_\alpha \stackrel{d}{=} \beta_{\alpha,1}M_\alpha + (1-\beta_{\alpha,1})\mathbb{D}_\alpha$. *When* $\alpha = 1/2$, $\gamma_{1/2}M_{1/2} \stackrel{d}{=} \Delta_{\mathbf{e}}^{(1/2,1)}$, *otherwise*, $\gamma_\alpha M_\alpha$ *has an explicit density given in* [19], *Theorem* 4.2.

(ii) *For* $V = \mathbb{G}_\alpha$ *and* $X = (X_{1-\alpha})^{(1-\alpha)/\alpha}$,

$$\begin{aligned} \gamma_1 &\stackrel{d}{=} \gamma_{1-\alpha}U + \Sigma_{1-\alpha}(\mathbb{G}_\alpha) + \gamma_\alpha \\ &\stackrel{d}{=} \gamma_{1-\alpha}U + \gamma_\alpha\tilde{M}_\alpha + \Sigma_1(\mathbb{G}_\alpha) \\ &\stackrel{d}{=} \gamma_{1-\alpha}U + \Sigma_1(\xi_{1-\alpha}\mathbb{G}_\alpha), \end{aligned} \tag{6.8}$$

where $\tilde{M}_\alpha \stackrel{d}{=} \beta_{\alpha,1}\tilde{M}_\alpha + (1-\beta_{\alpha,1})\mathbb{G}_\alpha$.

PROOF. Note that $\Delta_{\mathbf{e}}^{(\alpha,1)} + \gamma_\alpha M_\alpha$ is $\mathrm{GGC}(1, \mathbb{D}_\alpha)$ and the second equality above is a direct consequence of Theorem 6.1. Additionally, it follows that since $\Delta_{\mathbf{e}}^{(\alpha,1)}$ is $\mathrm{GGC}(1-\alpha, \mathbb{D}_\alpha)$, $\gamma_{1-\alpha} \stackrel{d}{=} \Delta_{\mathbf{e}}^{(\alpha,1)} + \Sigma_{1-\alpha}(\mathbb{D}_\alpha)$. The last equality follows from $\Delta_{\mathbf{e}}^{(\alpha,1)}$ is $\mathrm{GGC}(1, \xi_{1-\alpha}\mathbb{D}_\alpha)$. $\square$

Recall the occupation time variables in Example 3.1, where the variable $\tilde{\mathbb{O}}_{\alpha,p}$ has density (3.3), and additionally $\mathbb{O}_0 = 1/2$ and $\mathbb{O}_1 \stackrel{d}{=} \xi_{1/2}$. Then using Theorems 6.1 or 6.2, we obtain the following result.

PROPOSITION 6.3. *Let* $0 \le \alpha \le 1$ *and* $0 < p \le 1$.

(i) *Then for* $V = \mathbb{O}_\alpha$ *and* $X = X_\alpha$,

$$\gamma_1 \stackrel{d}{=} \gamma_1 \tilde{\mathbb{O}}_{\alpha,p} + \Sigma_p(\mathbb{O}_\alpha) + \gamma_{1-p}. \tag{6.9}$$

(ii) *When* $\alpha = 1/2$, $V \stackrel{d}{=} \beta_{1/2,1/2}$ *and* $X \stackrel{d}{=} \gamma_{1/2}/\gamma'_{1/2}$,

$$\gamma_1 \stackrel{d}{=} \gamma_p \beta_{p+1/2,p+1/2} + \Sigma_p(\beta_{1/2,1/2}) + \gamma'_{1-p}. \tag{6.10}$$

REMARK 6.1. The diffusions above belong to a more general family, $\mathcal{R}^{(\alpha,b)}$ for $0 \le \alpha < 1, b \ge 0$ with inverse local time corresponding to a generalized gamma subordinator with Lévy density $Cs^{-\alpha-1}\mathrm{e}^{-bs}$. Hence, the density of variables $\Delta_{\mathbf{e}}^{(\alpha,b)}$ is proportional to $s^{-\alpha-1}\mathrm{e}^{-bs}(1-e^{-s})$. In particular, for $b = 0$, the variable $\Delta_{\mathbf{e}}^{(\alpha,0)} \stackrel{d}{=} \gamma_{1-\alpha}/U^{1/\alpha}$ is a $\mathrm{GGC}(1-\alpha, 1/\mathbb{G}_\alpha)$ variable. See [5, 19, 22] for more details.

REMARK 6.2. Reference [5], Theorem 1.4, yields the following decomposition of γ_1, for $0 \le \alpha \le 1$:

$$\gamma_1 \stackrel{d}{=} \gamma_1 \mathbb{G}_\alpha + \gamma'_1 \mathbb{G}_{1-\alpha}.$$

As a special case, with $\alpha = 0$ or 1,

$$\gamma_1 \stackrel{d}{=} \gamma_1 U + \gamma'_1/(1+\mathrm{e}^{\pi\eta}).$$

Combining this fact with with (6.3), with $V \stackrel{d}{=} \mathbb{G}_0 \stackrel{d}{=} 1/(1+\mathrm{e}^{\pi\eta})$, leads to the interesting identity

$$\Sigma_1(\mathbb{G}_0) \stackrel{d}{=} \gamma_1 \mathbb{G}_0$$

with Lévy density given by (6.2). This follows since $\gamma_1 U$ is a $\mathrm{GGC}(1, \mathbb{G}_0)$ variable.

7. Composition of quantile clocks. We now highlight an important property of the general class of quantile clocks. Recall from Proposition 3.1 that for each fixed t, the marginal distribution of a quantile clock T_R, with parameters (R, L), satisfies

$$T_R(t) \stackrel{d}{=} \zeta(t),$$

where ζ is a subordinator such that $\zeta(1)$ has Laplace exponent

$$\psi_\zeta(\omega) = \mathbb{E}[\psi_L(\omega R)] = \psi_{T_R(1)}(\omega).$$

An important operation for Lévy processes is the composition of Lévy processes, in financial applications this is associated with time changed processes. The fact that quantile clocks behave marginally like a subordinator allows us to obtain the following results.

First, we can discuss the composition of two independent quantile clocks T_{R_1}, T_{R_2}, with parameters (R_1, L_1) and (R_2, L_2), respectively, which can be written as

$$T_{R_1}(T_{R_2}(t)) = \int_0^{T_{R_2}(t)} Q_{R_1}\left(\left(1 - \frac{s}{T_{R_2}(t)}\right)_+\right) L_1(ds) \tag{7.1}$$

for $t \geq 0$. The apparently complicated random process appearing in (7.1) is no longer a quantile clock. However, as the next result shows, its marginals are easy to describe. We use the notation $\circ$ to denote the composition of functions, so, for instance, $T_{R_1} \circ T_{R_2}$ means the operation in (7.1).

PROPOSITION 7.1. *Let T_{R_i}, $i = 1, \dots, k$, denote independent quantile clocks such that pointwise $T_{R_i}(t) \stackrel{d}{=} \zeta_i(t)$ for independent subordinators with corresponding Laplace exponents $\psi_{\zeta_i}(\omega)$ for $i = 1, \dots, k$. Then the composition $\widehat{T}_k := (\widehat{T}_k(t) = T_{R_1} \circ \cdots \circ T_{R_k}(t) : t \geq 0)$ is an increasing process such that for each fixed t,*

$$\widehat{T}_k(t) \stackrel{d}{=} \widehat{\zeta}_k(t) = \zeta_1 \circ \cdots \circ \zeta_k(t),$$

where $\widehat{\zeta}_k$ is a subordinator with Laplace exponent

$$\psi_{\zeta_k} \circ \cdots \circ \psi_{\zeta_1}(\omega).$$

If each T_{R_i} is a continuous process, then $\widehat{T}_k$ is a continuous process.

PROOF. It suffices to show this for $k = 2$. But this is immediate from Proposition 3.1, since given T_{R_2},

$$\psi_{T_{R_1}(T_{R_2}(t))}(\omega) = T_{R_2}(t)\psi_{\zeta_1}(\omega). \qquad \square$$

Of course one can also compose these clocks with subordinators as follows. The next result is immediate.

PROPOSITION 7.2. *Let T_R denote a quantile clock that satisfies $T_R(t) \stackrel{d}{=} \zeta_1(t)$ for some subordinator ζ_1. Furthermore, let ζ_2 denote a subordinator independent of T_R and ζ_1. Then for each fixed t*

$$T_R(\zeta_2(t)) \stackrel{d}{=} \zeta_1(\zeta_2(t)) \quad \textit{and} \quad \zeta_2(T_R(t)) \stackrel{d}{=} \zeta_2(\zeta_1(t)).$$

We now illustrate an important special case.

PROPOSITION 7.3. *For $0 < \alpha < 1$ and $0 < \beta < 1$, one can use the specifications in Theorem 4.2 to construct independent clocks T_{R_1} and T_{R_2} such that marginally $T_{R_1}(t) \stackrel{d}{=} \widehat{S}_\alpha(t)$ and $T_{R_2}(t) \stackrel{d}{=} \widehat{S}_\beta(t)$, where $\widehat{S}_\alpha$ and $\widehat{S}_\beta$ are independent with Laplace exponents $[(1+\omega)^\alpha - 1]$ and $[(1+\omega)^\beta - 1]$. Then for each fixed t*

$$T_{R_1}(T_{R_2}(t)) \stackrel{d}{=} T_{R_2}(T_{R_1}(t)) \stackrel{d}{=} T_{R_1}(\widehat{S}_\beta(t)) \stackrel{d}{=} \widehat{S}_\alpha(T_{R_2}(t)) \stackrel{d}{=} \widehat{S}_{\alpha\beta}(t),$$

that is, for each fixed t the Laplace exponent is $t[(1+\omega)^{\alpha\beta} - 1]$. Additionally, the first two compositions can be specified such that the resulting processes are continuous, but the latter compositions always correspond to processes with jumps.

8. Continuous VG, CGMY, NIG and other price processes. Summarizing, we have demonstrated that quantile clocks T_R can either be chosen to have strictly continuous and increasing paths or can be expressed as $T_{\tilde{R}}(t) + aL(t)$, where $T_{\tilde{R}}$ is a continuous increasing quantile clock. In general, quantile clocks have marginals that are equivalent to those of a subordinator, ζ, for each t, that is, $T_R(t) \stackrel{d}{=} \zeta(t)$. Moreover, we have shown that for a large class of quantiles Q_R we can choose T_R to have any desired marginal law in $\mathcal{U}_\delta$, by choosing a random variable Y and the subordinator L in a clearly prescribed fashion. Furthermore, our results in the last section show that composition operations involving quantile clocks or quantile clocks with subordinators are marginally equivalent in distribution to compositions of subordinators. All these properties make them highly desirable components in pricing models based on time changes. For example, the processes

$$\big(\widehat{W}_\mu(T_R(t)) : t \geq 0\big) \quad \text{and} \quad \big(\widehat{W}_\mu(T_{R_1}(T_{R_2}(t))) : t \geq 0\big)$$

can be chosen such that they are processes with continuous trajectories, but have simple and familiar marginal laws. In addition, for a subordinator $\tilde{\zeta}$, the processes

$$\big(\widehat{W}_\mu(\tilde{\zeta}(T_R(t))) : t \geq 0\big) \quad \text{and} \quad \big(\widehat{W}_\mu(T_R(\tilde{\zeta}(t))) : t \geq 0\big)$$

have jumps, exhibit volatility clustering, and otherwise may be chosen to have familiar marginal distributions, in fact the same marginal, for many choices of Q_R. We illustrate these points through some examples that equate these processes marginally with some of the most popular Lévy processes.

EXAMPLE 8.1 (Continuous variance gamma processes). As a first example, it follows from III of Theorem 4.2, that if $RY \stackrel{d}{=} U$, then for each $\delta > 0$, a quantile clock $T_{R^{1/\delta}}$ with parameters $(R^{1/\delta}, L_\delta)$, can be chosen such that for each fixed t,

$$\widehat{W}_\mu(T_{R^{1/\delta}}(t)) \stackrel{d}{=} \widehat{W}_\mu(\gamma_\theta(t)),$$

that is, it has marginal distributions equivalent to the log price of a variance gamma (VG) process [28], not depending on δ, if for each $\delta > 0$, the subordinator (depending on δ)

$$L_\delta(s) \stackrel{d}{=} \zeta_{\delta, Y^{1/\delta}}(s) + \sum_{k=1}^{N(\theta s/\delta)} \gamma_1^{(k)} Y_k^{1/\delta}, \qquad s \geq 0,$$

where $\zeta_{\delta, Y^{1/\delta}}$ is a GGC$(\theta, Y^{1/\delta})$ subordinator.

We next show how to obtain price processes whose marginal laws are equivalent to a Carr–Geman–Madan–Yor (CGMY) process [7] but otherwise possesses continuous sample paths.

EXAMPLE 8.2 (Continuous CGMY processes). For this example, we follow the exposition in [29]. Let

$$A = \frac{G-M}{2} \quad \text{and} \quad B = \frac{G+M}{2},$$

then the Lévy density of the log prices of a CGMY process, say χ_{CGMY}, is given by

$$\rho_{\chi_{\mathrm{CGMY}}(1)}(x) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \mathrm{e}^{Ax - B|x|}|x|^{-d-1} \qquad \text{for } -\infty < x < \infty \tag{8.1}$$

and $0 < d = 2\alpha < 2$. Madan and Yor [29], show that the log price of a CGMY process has an explicit representation in terms of a time changed brownian motion, $\chi_{\mathrm{CGMY}}(t) := \widehat{W}_A(\zeta(t))$, where ζ is a subordinator with Lévy density

$$\begin{aligned} \rho_\zeta(s) &= \frac{2^\alpha \Gamma(\alpha)}{\Gamma(2\alpha)} \mathrm{e}^{(A^2 - B^2)s/2} s^{-\alpha-1} \mathbb{E}[\mathrm{e}^{-s(B^2/2)(\gamma_\alpha/\gamma_{1/2})}] \\ &= \frac{2^\alpha \Gamma(\alpha)}{\Gamma(2\alpha)} s^{-\alpha-1} \mathbb{E}[\mathrm{e}^{-sV}] \end{aligned} \tag{8.2}$$

for

$$V \stackrel{d}{=} \left(4MG + B^2 \frac{\gamma_\alpha}{\gamma_{1/2}}\right) \Big/ 2. \tag{8.3}$$

It is evident from (8.2) that $\zeta \in \mathcal{G}$. We now give the specifications for a quantile clock to have marginals with Lévy density (8.2) hence inducing price processes that have the marginal distribution of a CGMY process. Notice that

$$
\begin{aligned}
-s\rho_\zeta'(s) &= \frac{2^\alpha \Gamma(\alpha)}{\Gamma(2\alpha)} s^{-\alpha-1}\mathbb{E}\bigl[\bigl((1+\alpha)+sV\bigr)\mathrm{e}^{-sV}\bigr] \\
&= (1+\alpha)\rho_\zeta(s) + \frac{2^\alpha \Gamma(\alpha)}{\Gamma(2\alpha)} s^{-\alpha}\mathbb{E}[V\mathrm{e}^{-sV}].
\end{aligned} \tag{8.4}
$$

Hence, $\psi_\zeta(\omega) = \mathbb{E}[\psi_Z(\omega U^{1/\delta})]$, for

$$
\rho_Z(s) = (1+\alpha/\delta)\rho_\zeta(s) + \frac{2^\alpha \Gamma(\alpha)}{\delta\Gamma(2\alpha)} s^{-\alpha}\mathbb{E}[V\mathrm{e}^{-sV}]. \tag{8.5}
$$

PROPOSITION 8.1. *Suppose that T_R is a quantile clock with parameters (R, L), such that there exists a variable Y satisfying $RY \stackrel{d}{=} U^{1/\delta}$ for some $\delta > 0$. Then for each fixed t,*

$$
\widehat{W}_A(T_R(t)) \stackrel{d}{=} \chi_{\mathrm{CGMY}}(t)
$$

specified by (8.1) if the subordinator L is chosen such that

$$
\begin{aligned}
\rho_L(s) = (1+\alpha/\delta)&\frac{c_\alpha 2^\alpha \Gamma(\alpha)}{\Gamma(2\alpha)} s^{-\alpha-1}\mathbb{E}[\mathrm{e}^{-sV/Y_\alpha}] \\
&+ \frac{c_\alpha 2^\alpha \Gamma(\alpha)}{\delta\Gamma(2\alpha)} s^{-\alpha}\mathbb{E}[(V/Y_\alpha)\mathrm{e}^{-sV/Y_\alpha}],
\end{aligned} \tag{8.6}
$$

where V is defined by (8.3), $c_\alpha = \mathbb{E}[Y^\alpha]$ and Y_α is the random variable whose distribution is proportional to $y^\alpha F_Y(dy)$. When $Y = 1$, $L := Z$ satisfying (8.5). Note also that $\mathbb{E}[V^\alpha]$ is finite only if $\alpha < 1/2$. Hence,

$$
s^{-\alpha}\mathbb{E}[(V/Y_\alpha)\mathrm{e}^{-sV/Y_\alpha}]
$$

is the Lévy density of a compound Poisson process only in the case where $\alpha < 1/2$.

PROOF. The result is a special case of Theorem 4.2 and the specifications we derived above. In particular, (8.5). $\square$

As a specific example with continuous paths, consider again the Kumaraswamy quantile clock with

$$
T_{K_{p,b}}(t) = \int_0^t \bigl[1 - \bigl(1-(1-s/t)_+\bigr)^{1/p}\bigr]^{1/b} L(ds)
$$

for $R \stackrel{d}{=} K_{p,b} \stackrel{d}{=} (1-U^{1/p})^{1/b} \stackrel{d}{=} \beta_{1,p}^{1/b}$. Then for each fixed t,

$$
\widehat{W}_A(T_{K_{p,b}}(t)) \stackrel{d}{=} \chi_{\mathrm{CGMY}}(t),
$$

if L is selected according to (8.6) with $Y \stackrel{d}{=} \beta_{p,1-p}^{1/b}$ and $\delta = bp$.

If we consider the arcsine clock using (5.5) with

$$T_{\beta_{1/2,1/2}^{1/b}}(t) := \int_0^t \sin^{2/b}\left(\frac{\pi}{2}\left(1-\frac{s}{t}\right)_+\right) L(ds)$$

then for each fixed t,

$$\widehat{W}_A(T_{\beta_{1/2,1/2}^{1/b}}(t)) \stackrel{d}{=} \chi_{\mathrm{CGMY}}(t),$$

if L is selected according to (8.6) with $Y \stackrel{d}{=} (1-U^2)^{1/b}$ and $\delta = b/2$.

The next two cases are from Section 6.2 involving quantile functions that can be evaluated numerically. If we consider the clock $T_{U_{\alpha,\mathbf{e}}^{1/\delta}}$ based on the variable $U_{\alpha,\mathbf{e}}$ with density (6.6), it follows from Proposition 6.2 that

$$\widehat{W}_A(T_{U_{\alpha,\mathbf{e}}^{1/\delta}}(t)) \stackrel{d}{=} \chi_{\mathrm{CGMY}}(t),$$

if L is selected according to (8.6) with

$$Y \stackrel{d}{=} \mathrm{e}^{-[\Sigma_{1-\alpha}(\mathbb{D}_\alpha)+\gamma_\alpha]/\delta}.$$

If we consider the variables in Remark 6.2 then this leads to a quantile clock based on the variable $\gamma_1 \mathbb{G}_\alpha$. Hence,

$$\widehat{W}_A(T_{\mathrm{e}^{-\gamma_1 \mathbb{G}_\alpha/\delta}}(t)) \stackrel{d}{=} \chi_{\mathrm{CGMY}}(t),$$

if L is selected according to (8.6) with

$$Y \stackrel{d}{=} \mathrm{e}^{-\gamma_1 \mathbb{G}_{1-\alpha}/\delta}.$$

Other examples using (6.9) and (6.10) are based on the pairs

$$\left(\mathrm{e}^{-\gamma_1 \tilde{\mathbb{O}}_{\alpha,p}}, \mathrm{e}^{-[\Sigma_p(\mathbb{O}_\alpha)+\gamma_{1-p}]}\right) \quad \text{and} \quad \left(\mathrm{e}^{-\gamma_p \beta_{p+1/2,p+1/2}}, \mathrm{e}^{-[\Sigma_p(\beta_{1/2,1/2})+\gamma'_{1-p}]}\right).$$

Finally, if instead one uses the clock

$$T_{\tilde{U}_p^{1/\delta}}(t) := \int_0^t \left[(1-p) + p\left(1-\frac{s}{t}\right)_+\right]^{1/\delta} L(ds),$$

then for each fixed t,

$$\widehat{W}_A(T_{\tilde{U}_p^{1/\delta}}(t)) \stackrel{d}{=} \chi_{\mathrm{CGMY}}(t),$$

if L is selected according to (8.6) with $Y = \mathrm{e}^{-X_p[-\log(1-p)]/\delta}$ where again X_p is geometric(p). Hence, for each $0 < p < 1$, the resulting process has CGMY marginals, exhibits volatility clustering, but also has jumps. If $p = 1$, then the process is continuous and the quantile clock coincides with the Holmgren–Liouville clock discussed in [4].

It is evident that the specifications (8.6) for L appearing in Proposition 8.1 can be modified such that $T_R(t)$ has marginals equivalent to a subordinator with Lévy density

$$\rho_\zeta(s) = Cs^{-\alpha-1}\mathbb{E}[\mathrm{e}^{-sV}]$$

for some positive constant C, where V is a much more general random variable. That is, L is specified by

$$\begin{aligned}\rho_L(s) = (1+\alpha/\delta)c_\alpha Cs^{-\alpha-1}\mathbb{E}[\mathrm{e}^{-sV/Y_\alpha}] \\ + (c_\alpha/\delta)Cs^{-\alpha}\mathbb{E}[(V/Y_\alpha)\mathrm{e}^{-sV/Y_\alpha}].\end{aligned} \tag{8.7}$$

As a specific example, we next look at the case corresponding to NIG and related processes.

EXAMPLE 8.3 (Processes with NIG and related marginals). For this example, let $\widehat{S}_\alpha(t)$ denote any subordinator with Lévy density

$$\rho_\alpha(s) = \frac{\alpha}{\Gamma(1-\alpha)}s^{-\alpha-1}\mathrm{e}^{-s}$$

and define the Lévy process on $\mathbb{R}$ by $\chi_\alpha(t) := \widehat{W}_\mu(\widehat{S}_\alpha(t))$. It follows that

$$\chi_{1/2}(t) := \chi_{\mathrm{NIG}}(t)$$

is a normal inverse Gaussian (NIG) process [2]. If $T_R^{(\alpha)}$ denotes a quantile clock such that $RY = U^{1/\delta}$, and L is specified according to (8.7), specifically using the Lévy density ρ_α, with $V = 1$, then $\widehat{W}_\mu(T_R^{(\alpha)}(t)) \stackrel{d}{=} \chi_\alpha(t)$. In particular setting $\alpha = 1/2$, it follows that for each t,

$$\widehat{W}_\mu\big(T_R^{(1/2)}(t)\big) \stackrel{d}{=} \chi_{\mathrm{NIG}}(t).$$

Thus, yielding processes with continuous trajectories but NIG marginals. In addition, choosing α and $0 < \beta \le 1$ such that $\alpha\beta = 1/2$, it follows from Proposition 7.3 that

$$\chi_\beta\big(T^{(\alpha)}(t)\big) \stackrel{d}{=} \widehat{W}_\mu\big(T^{(\alpha)}(\widehat{S}_\beta(t))\big) \stackrel{d}{=} \chi_{\mathrm{NIG}}(t),$$

corresponding to processes with jumps, dependent increments and NIG marginal distributions. Note that when $Y = 1$, corresponding to the quantile clock $T_{U^{1/\delta}}$, then similar to (8.5),

$$L(t) = \widehat{S}_\alpha\big((1+\alpha/\delta)t\big) + \sum_{k=1}^{N(\alpha s/\delta)} \gamma_{1-\alpha}^{(k)} \qquad \text{for } t > 0. \tag{8.8}$$

9. Choosing laws for the short memory kernel. We now apply our results for quantile clocks to a convoluted subordinator that [4] refer to as a short memory kernel. We note that this convoluted subordinator is not a quantile clock.

THEOREM 9.1. *Let ζ denote a subordinator with self-decomposable laws such that the quantile clock with parameters (U, Z), that is, T_U has marginals $T_U(t) \stackrel{d}{=} \zeta(t) \in \mathcal{L}$. This is achieved by setting the Lévy density of Z to be*

$$\rho_Z(x) = -x\rho'_\zeta(x) = \rho_\zeta(x) + \rho_\vartheta(x), \tag{9.1}$$

where ϑ is the OU–BDLP of $v(0) \stackrel{d}{=} T_U(1) \stackrel{d}{=} \zeta(1)$. Then for Z satisfying (9.1), the short memory convoluted subordinator constructed as

$$\tilde{T}_\varepsilon(t) = \int_0^t \min\left(1, \frac{(t-s)_+}{\varepsilon}\right) Z(ds)$$

has the following distributional properties:

(i) *For each fixed t, the Laplace exponent of the r.v. $\tilde{T}_\varepsilon(t)$, is given by*

$$\psi_{\tilde{T}_\varepsilon(t)}(\omega) = \begin{cases} t\psi_\zeta\left(\omega \dfrac{t}{\varepsilon}\right), & t \le \varepsilon, \\ t\psi_\zeta(\omega) + (t-\varepsilon)\omega\psi'_\zeta(\omega), & t > \varepsilon. \end{cases}$$

(ii) *For each fixed t, the marginal distribution of $\tilde{T}_\varepsilon(t)$ is given by*

$$\tilde{T}_\varepsilon(t) \stackrel{d}{=} \begin{cases} \dfrac{t}{\varepsilon}\zeta(t), & t \le \varepsilon, \\ \zeta(t) + \vartheta(t-\varepsilon), & t > \varepsilon. \end{cases}$$

PROOF. First, notice that in general, for each fixed t, the Lévy exponent of the random variable $\tilde{T}_\varepsilon(t)$ is given by

$$\psi_{\tilde{T}_\varepsilon(t)}(\omega) = \int_0^t \psi_Z\left(\omega \min\left(1, \frac{(t-s)_+}{\varepsilon}\right)\right) ds. \tag{9.2}$$

So for $t \le \varepsilon$, (9.2) can be expressed as

$$\int_0^t \psi_Z\left(\omega \min\left(1, \frac{(t-s)_+}{\varepsilon}\right)\right) ds = t\mathbb{E}\left[\psi_Z\left(\omega \frac{t}{\varepsilon} U\right)\right] = t\psi_\zeta\left(\omega \frac{t}{\varepsilon}\right),$$

where the last equality follows from (9.1). For $t > \varepsilon$, split the interval $[0, t]$ into $[0, t-\varepsilon]$ and $(t-\varepsilon, t]$ then (9.2) becomes

$$\varepsilon\mathbb{E}[\psi_Z(\omega U)] + (t-\varepsilon)\psi_Z(\omega). \tag{9.3}$$

Now use (9.1) to show that (9.3) is equal to

$$\varepsilon\psi_\zeta(\omega) + (t-\varepsilon)[\psi_\zeta(\omega) + \psi_\vartheta(\omega)]$$

yielding the result. $\square$

Our result now allows one to choose more convenient laws for $\tilde{T}_\epsilon$ which allows one to easily apply the option pricing formula of [4], as displayed in Theorem 2.1, either by exact simulation or FFT methods. We illustrate this in the next example.

EXAMPLE 9.1 (Short memory convoluted subordinator with NIG related marginals). First, it is interesting to recall from [3] that the OU–BDLP, ϑ, leading to $v(0) \stackrel{d}{=} \widehat{S}_\alpha(1)$, as specified in Example 8.3, has Lévy density

$$\rho_\vartheta(s) = \frac{\alpha}{\Gamma(1-\alpha)} s^{-\alpha-1}[\alpha + s]\mathrm{e}^{-s}.$$

Hence,

$$\vartheta(s) \stackrel{d}{=} \widehat{S}_\alpha(\alpha s) + \sum_{k=1}^{N(\alpha s)} \gamma_{1-\alpha}^{(k)} \qquad \text{for } s > 0.$$

Now using (8.8) with $\delta = 1$, setting the subordinator

$$Z(s) = \widehat{S}_\alpha\big((1+\alpha)s\big) + \sum_{k=1}^{N(\alpha s)} \gamma_{1-\alpha}^{(k)} \qquad \text{for } s > 0$$

leads to the following marginal behavior of the corresponding short-memory model, for each fixed t,

$$\tilde{T}_\varepsilon(t) \stackrel{d}{=} \begin{cases} \dfrac{t}{\varepsilon}\widehat{S}_\alpha(t), & t \le \varepsilon, \\ \widehat{S}_\alpha\big(t + \alpha(t-\epsilon)\big) + \displaystyle\sum_{k=1}^{N(\alpha(t-\varepsilon))} \gamma_{1-\alpha}^{(k)}, & t > \varepsilon. \end{cases}$$

Note however that $\tilde{T}_\varepsilon$ is a continuous process since Z is an infinite activity process that satisfies the conditions in [4]. This leads to price processes (2.5) with continuous trajectories that have the following marginal behavior. $\widehat{W}_{-1/2}(\sigma^2\tilde{T}_\varepsilon(t))$ is equivalent in distribution to

$$\begin{cases} \widehat{W}_{-1/2}\left(\sigma^2 \dfrac{t}{\varepsilon}\widehat{S}_\alpha(t)\right), & t \le \varepsilon, \\ \widehat{W}_{-1/2}\big(\sigma^2\widehat{S}_\alpha\big(t+\alpha(t-\epsilon)\big)\big) + \widehat{W}'_{-1/2}\left(\sigma^2 \displaystyle\sum_{k=1}^{N(\alpha(t-\varepsilon))} \gamma_{1-\alpha}^{(k)}\right), & t > \varepsilon. \end{cases}$$

If one sets $\alpha = 1/2$, then this reduces to

$$\begin{cases} \widehat{W}_{-1/2}\left(\sigma^2 \dfrac{t}{\varepsilon}\widehat{S}_{1/2}(t)\right), & t \le \varepsilon, \\ \widehat{W}_{-1/2}\big(\sigma^2\widehat{S}_{1/2}\big((3t-\varepsilon)/2\big)\big) + \widehat{W}'_{-1/2}\left(\sigma^2 \displaystyle\sum_{k=1}^{N((t-\varepsilon)/2)} \gamma_{1/2}^{(k)}\right), & t > \varepsilon. \end{cases}$$

Hence, in this case, for each fixed $t \le \varepsilon$, the marginal distribution of the price process (2.5) follows an NIG distribution, with scale parameters depending on t.

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