# UPPER LARGE DEVIATIONS FOR THE MAXIMAL FLOW THROUGH A DOMAIN OF $\mathbb{R}^d$ IN FIRST PASSAGE PERCOLATION

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We consider the standard first passage percolation model in the rescaled graph $\mathbb{Z}^d/n$ for $d \geq 2$ and a domain $\Omega$ of boundary $\Gamma$ in $\mathbb{R}^d$. Let $\Gamma^1$ and $\Gamma^2$ be two disjoint open subsets of $\Gamma$ representing the parts of $\Gamma$ through which some water can enter and escape from $\Omega$. We investigate the asymptotic behavior of the flow $\phi_n$ through a discrete version $\Omega_n$ of $\Omega$ between the corresponding discrete sets $\Gamma_n^1$ and $\Gamma_n^2$. We prove that under some conditions on the regularity of the domain and on the law of the capacity of the edges, the upper large deviations of $\phi_n/n^{d-1}$ above a certain constant are of volume order, that is, decays exponentially fast with $n^d$. This article is part of a larger project in which the authors prove that this constant is the a.s. limit of $\phi_n/n^{d-1}$.

**1. First definitions and main result.** We use notation introduced in [8] and [9]. Let $d \geq 2$. We consider the graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$ having for vertices $\mathbb{Z}_n^d = \mathbb{Z}^d/n$ and for edges $\mathbb{E}_n^d$, the set of pairs of nearest neighbors for the standard $L^1$ norm. With each edge $e$ in $\mathbb{E}_n^d$ we associate a random variable $t(e)$ with values in $\mathbb{R}^+$. We suppose that the family $(t(e), e \in \mathbb{E}_n^d)$ is independent and identically distributed with a common law $\Lambda$; this is the standard model of first passage percolation on the graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$. We interpret $t(e)$ as the capacity of the edge $e$; it means that $t(e)$ is the maximal amount of fluid that can go through the edge $e$ per unit of time.
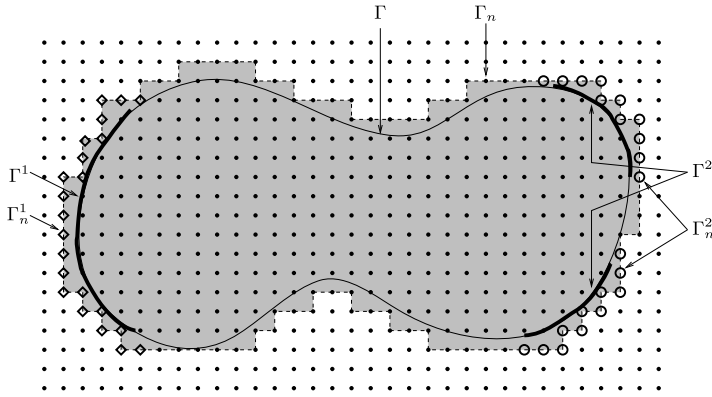
We consider an open bounded connected subset $\Omega$ of $\mathbb{R}^d$ such that the boundary $\Gamma = \partial\Omega$ of $\Omega$ is piecewise of class $\mathcal{C}^1$ [in particular $\Gamma$ has finite area, $\mathcal{H}^{d-1}(\Gamma) < \infty$]. It means that $\Gamma$ is included in the union of a finite number of hypersurfaces of class $\mathcal{C}^1$, that is, in the union of a finite number of $C^1$ submanifolds of $\mathbb{R}^d$ of codimension 1. Let $\Gamma^1$, $\Gamma^2$ be two disjoint subsets of $\Gamma$ that are open in $\Gamma$. We want to define the maximal flow from $\Gamma^1$ to $\Gamma^2$ through $\Omega$ for the capacities $(t(e), e \in \mathbb{E}_n^d)$. We consider a discrete version $(\Omega_n, \Gamma_n, \Gamma_n^1, \Gamma_n^2)$ of $(\Omega, \Gamma, \Gamma^1, \Gamma^2)$ defined by

$$
\begin{cases}
\Omega_n = \{x \in \mathbb{Z}_n^d \mid d_\infty(x, \Omega) < 1/n\}, \\
\Gamma_n = \{x \in \Omega_n \mid \exists y \notin \Omega_n, \langle x, y \rangle \in \mathbb{E}_n^d\}, \\
\Gamma_n^i = \{x \in \Gamma_n \mid d_\infty(x, \Gamma^i) < 1/n, d_\infty(x, \Gamma^{3-i}) \geq 1/n\}, \quad \text{for } i = 1, 2,
\end{cases}
$$

FIG. 1. *Domain Ω.*

where $d_\infty$ is the $L^\infty$-distance and the notation $\langle x, y \rangle$ corresponds to the edge of endpoints $x$ and $y$ (see Figure 1).

We shall study the maximal flow from $\Gamma_n^1$ to $\Gamma_n^2$ in $\Omega_n$. Let us properly define the maximal flow $\phi(F_1 \to F_2 \text{ in } C)$ [also denoted by $\phi(F_1 \to F_2 \text{ in } C \cap \mathbb{Z}^d/n)$] from $F_1$ to $F_2$ in $C$ for $C \subset \mathbb{R}^d$ (or by commodity the corresponding graph $C \cap \mathbb{Z}^d/n$). We will say that an edge $e = \langle x, y \rangle$ belongs to a subset $A$ of $\mathbb{R}^d$, which we denote by $e \in A$ if the interior of the segment joining $x$ to $y$ is included in $A$. We define $\widetilde{\mathbb{E}}_n^d$ as the set of all the oriented edges, that is, an element $\widetilde{e}$ in $\widetilde{\mathbb{E}}_n^d$ is an ordered pair of vertices which are nearest neighbors. We denote an element $\widetilde{e} \in \widetilde{\mathbb{E}}_n^d$ by $\langle\!\langle x, y \rangle\!\rangle$, where $x, y \in \mathbb{Z}_n^d$ are the endpoints of $\widetilde{e}$ and the edge is oriented from $x$ toward $y$. We consider the set $\mathcal{S}$ of all pairs of functions $(g, o)$, with $g : \mathbb{E}_n^d \to \mathbb{R}^+$ and $o : \mathbb{E}_n^d \to \widetilde{\mathbb{E}}_n^d$ such that $o(\langle x, y \rangle) \in \{\langle\!\langle x, y \rangle\!\rangle, \langle\!\langle y, x \rangle\!\rangle\}$, satisfying the following:

(1) for each edge $e$ in $C$ we have

$$0 \le g(e) \le t(e),$$

(2) for each vertex $v$ in $C \setminus (F_1 \cup F_2)$ we have

$$\sum_{e \in C \,:\, o(e) = \langle\!\langle v, \cdot \rangle\!\rangle} g(e) = \sum_{e \in C \,:\, o(e) = \langle\!\langle \cdot, v \rangle\!\rangle} g(e),$$

where the notation $o(e) = \langle\!\langle v, \cdot \rangle\!\rangle$ [resp., $o(e) = \langle\!\langle \cdot, v \rangle\!\rangle$] means that there exists $y \in \mathbb{Z}_n^d$ such that $e = \langle v, y \rangle$ and $o(e) = \langle\!\langle v, y \rangle\!\rangle$ [resp., $o(e) = \langle\!\langle y, v \rangle\!\rangle$]. A couple $(g, o) \in \mathcal{S}$ is a possible stream in $C$ from $F_1$ to $F_2$; $g(e)$ is the amount of fluid that goes through the edge $e$ and $o(e)$ gives the direction in which the fluid goes through $e$. The two conditions on $(g, o)$ express only the fact that the amount of fluid that can go through an edge is bounded by its capacity and that there is no loss of fluid in the graph. With each possible stream we associate the corresponding

flow

$$\text{flow}(g, o) = \sum_{u \in F_2, v \notin C \,:\, \langle u,v \rangle \in \mathbb{E}_n^d} g(\langle u, v \rangle) \mathbb{1}_{o(\langle u,v \rangle) = \langle\!\langle u,v \rangle\!\rangle} - g(\langle u, v \rangle) \mathbb{1}_{o(\langle u,v \rangle) = \langle\!\langle v,u \rangle\!\rangle}$$

$$= \sum_{u \in F_1, v \notin C \,:\, \langle u,v \rangle \in \mathbb{E}_n^d} g(\langle u, v \rangle) \mathbb{1}_{o(\langle u,v \rangle) = \langle\!\langle v,u \rangle\!\rangle} - g(\langle u, v \rangle) \mathbb{1}_{o(\langle u,v \rangle) = \langle\!\langle u,v \rangle\!\rangle}.$$

This is the amount of fluid that crosses $C$ from $F_1$ to $F_2$ if the fluid respects the stream $(g, o)$. The two definitions are equivalent since the stream satisfies the node law at each vertex of $C \setminus (F_1 \cup F_2)$. The maximal flow through $C$ from $F_1$ to $F_2$ is the supremum of this quantity over all possible choices of streams,

$$\phi(F_1 \to F_2 \text{ in } C) = \sup\{\text{flow}(g, o) \mid (g, o) \in \mathcal{S}\}.$$

We recall that we consider an open bounded connected subset $\Omega$ of $\mathbb{R}^d$ whose boundary $\Gamma$ is piecewise of class $\mathcal{C}^1$ and two disjoint open subsets $\Gamma^1$ and $\Gamma^2$ of $\Gamma$. We denote by

$$\phi_n = \phi(\Gamma_n^1 \to \Gamma_n^2 \text{ in } \Omega_n)$$

the maximal flow from $\Gamma_n^1$ to $\Gamma_n^2$ in $\Omega_n$. We will investigate the asymptotic behavior of $\phi_n / n^{d-1}$ when $n$ goes to infinity. More precisely, we will show that the upper large deviations of $\phi_n$ above a certain constant $\widetilde{\phi}_\Omega$ are of volume order. Here we state the precise theorem.

THEOREM 1. *We suppose that $d(\Gamma^1, \Gamma^2) > 0$, where $d$ is the Euclidean distance between these two subsets of $\mathbb{R}^d$. If the law $\Lambda$ of the capacity of an edge admits an exponential moment,*

$$\exists \theta > 0 \qquad \int_{\mathbb{R}^+} e^{\theta x} \, d\Lambda(x) < +\infty,$$

*then there exists a finite constant $\widetilde{\phi}_\Omega$ such that for all $\lambda > \widetilde{\phi}_\Omega$,*

$$\limsup_{n \to \infty} \frac{1}{n^d} \log \mathbb{P}[\phi_n \geq \lambda n^{d-1}] < 0.$$

The description of $\widetilde{\phi}_\Omega$ will be given in Section 3. As we will explain in Section 2.3, this constant is relevant in the sense that we prove in the companion papers [3] and [4] that under added geometric assumptions, $\widetilde{\phi}_\Omega$ is the almost sure limit of $\phi_n / n^{d-1}$. Theorem 1 is needed to prove this a.s. convergence.

REMARK 1. In Theorem 1 we need to impose that $d(\Gamma^1, \Gamma^2) > 0$ because otherwise we cannot be sure that $\widetilde{\phi}_\Omega < \infty$, as we will see in Section 5. Moreover, if $d(\Gamma^1, \Gamma^2) = 0$, the upper large deviations of $\phi_n / n^{d-1}$ may not be of volume order (see Theorem 10 in Section 2.2 below).

REMARK 2.    The large deviations we obtain are of the relevant order. Indeed, if all the edges in $\Omega_n$ have a capacity which is abnormally big, then the maximal flow $\phi_n$ will be abnormally big too. The probability for these edges to have an abnormally large capacity is of order $\exp -Cn^d$ for a constant $C$ because the number of edges in $\Omega_n$ is $C'n^d$ for a constant $C'$.

The rest of the article is structured as follows. The background is presented in Section 2; we first give some added definitions in Section 2.1, then we present the existing results concerning maximal flows in first passage percolation in Section 2.2 and finally, we explain the role of this article in the comprehension of maximal flows problems in Section 2.3. The constant $\widetilde{\phi}_\Omega$ is computed in Section 3. Section 4 gives a detailed sketch of the proof of Theorem 1. The rest of the article is devoted to the proof itself.

## 2. Background.

### 2.1. *Some definitions*.

2.1.1. *Geometric notation.*   We start with some geometric definitions. For a subset $X$ of $\mathbb{R}^d$, we denote by $\mathcal{H}^s(X)$ the $s$-dimensional Hausdorff measure of $X$ (we will use $s = d - 1$ and $s = d - 2$). The $r$-neighborhood $\mathcal{V}_i(X, r)$ of $X$ for the distance $d_i$ that can be the Euclidean distance if $i = 2$ or the $L^\infty$-distance if $i = \infty$, is defined by

$$\mathcal{V}_i(X, r) = \{y \in \mathbb{R}^d \mid d_i(y, X) < r\}.$$

If $X$ is a subset of $\mathbb{R}^d$ included in a hyperplane of $\mathbb{R}^d$ and of codimension 1 (e.g., a nondegenerate hyperrectangle), we denote by $\mathrm{hyp}(X)$ the hyperplane spanned by $X$ and we denote by $\mathrm{cyl}(X, h)$ the cylinder of basis $X$ and of height $2h$ defined by

$$\mathrm{cyl}(X, h) = \{x + tv \mid x \in X, t \in [-h, h]\},$$

where $v$ is one of the two unit vectors orthogonal to $\mathrm{hyp}(X)$ (see Figure 2). For $x \in \mathbb{R}^d$, $r \geq 0$ and a unit vector $v$, we denote by $B(x, r)$ the Euclidean closed ball centered at $x$ of radius $r$.

2.1.2. *Flow in a cylinder.*   Here are some particular definitions of flows through a box. It is important to know them because all our work consists of comparing the maximal flow $\phi_n$ in $\Omega_n$ with the maximal flows in small cylinders. Let $A$ be a nondegenerate hyperrectangle, that is, a box of dimension $d - 1$ in $\mathbb{R}^d$. All hyperrectangles will be supposed to be closed in $\mathbb{R}^d$. We denote by $v$ one of the two unit vectors orthogonal to $\mathrm{hyp}(A)$. For $h$ a positive real number, we consider the cylinder $\mathrm{cyl}(A, h)$. The set $\mathrm{cyl}(A, h) \setminus \mathrm{hyp}(A)$ has two connected components
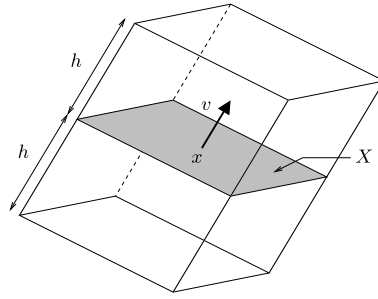
FIG. 2.  *Cylinder* cyl$(X, h)$.

which we denote by $\mathcal{C}_1(A, h)$ and $\mathcal{C}_2(A, h)$. For $i = 1, 2$, let $A_i^h$ be the set of the points in $\mathcal{C}_i(A, h) \cap \mathbb{Z}_n^d$ which have a nearest neighbor in $\mathbb{Z}_n^d \setminus \mathrm{cyl}(A, h)$,

$$A_i^h = \{x \in \mathcal{C}_i(A, h) \cap \mathbb{Z}_n^d \mid \exists y \in \mathbb{Z}_n^d \setminus \mathrm{cyl}(A, h), \langle x, y \rangle \in \mathbb{E}_n^d\}.$$

Let $T(A, h)$ [resp., $B(A, h)$] be the top (resp., the bottom) of cyl$(A, h)$, that is,

$$T(A, h) = \{x \in \mathrm{cyl}(A, h) \mid \exists y \notin \mathrm{cyl}(A, h),$$
$$\langle x, y \rangle \in \mathbb{E}_n^d \text{ and } \langle x, y \rangle \text{ intersects } A + hv\}$$

and

$$B(A, h) = \{x \in \mathrm{cyl}(A, h) \mid \exists y \notin \mathrm{cyl}(A, h),$$
$$\langle x, y \rangle \in \mathbb{E}_n^d \text{ and } \langle x, y \rangle \text{ intersects } A - hv\}.$$

For a given realization $(t(e), e \in \mathbb{E}_n^d)$, we define the variable $\tau(A, h) = \tau(\mathrm{cyl}(A, h), v)$ by

$$\tau(A, h) = \tau(\mathrm{cyl}(A, h), v) = \phi\big(A_1^h \to A_2^h \text{ in } \mathrm{cyl}(A, h)\big)$$

and the variable $\phi(A, h) = \phi(\mathrm{cyl}(A, h), v)$ by

$$\phi(A, h) = \phi(\mathrm{cyl}(A, h), v) = \phi\big(B(A, h) \to T(A, h) \text{ in } \mathrm{cyl}(A, h)\big),$$

where $\phi(F_1 \to F_2 \text{ in } C)$ is the maximal flow from $F_1$ to $F_2$ in $C$, for $C \subset \mathbb{R}^d$ (or by commodity the corresponding graph $C \cap \mathbb{Z}^d/n$) defined previously. The dependence in $n$ is implicit here, in fact we can also write $\tau_n(A, h)$ and $\phi_n(A, h)$ if we want to emphasize this dependence on the mesh of the graph.

2.1.3. *Max-flow min-cut theorem.*  The maximal flow $\phi(F_1 \to F_2 \text{ in } C)$ can be expressed differently thanks to the max-flow min-cut theorem [1]. We need some definitions to state this result. A path on the graph $\mathbb{Z}_n^d$ from $v_0$ to $v_m$ is a sequence $(v_0, e_1, v_1, \ldots, e_m, v_m)$ of vertices $v_0, \ldots, v_m$ alternating with edges $e_1, \ldots, e_m$ such that $v_{i-1}$ and $v_i$ are neighbors in the graph joined by the edge $e_i$,

for $i$ in $\{1, \ldots, m\}$. A set $E$ of edges in $C$ is said to cut $F_1$ from $F_2$ in $C$ if there is no path from $F_1$ to $F_2$ in $C \setminus E$. We call $E$ an $(F_1, F_2)$-cut if $E$ cuts $F_1$ from $F_2$ in $C$ and if no proper subset of $E$ does. With each set $E$ of edges we associate its capacity which is the variable

$$V(E) = \sum_{e \in E} t(e).$$

The max-flow min-cut theorem states that

$$\phi(F_1 \rightarrow F_2 \text{ in } C) = \min\{V(E) \mid E \text{ is a } (F_1, F_2)\text{-cut}\}.$$

### 2.2. *State of the art.*

2.2.1. *Existing laws of large numbers.* In this section and the next one we consider the standard first passage percolation model on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ instead of the rescaled graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$.

In dimension two, classical problems of distance in first passage percolation and problems of flows are closely related. By the max-flow min-cut theorem, we know that the maximal flow from the top to the bottom of a cylinder is equal to the minimal capacity of a set of edges that cuts the top from the bottom of the cylinder and we can notice that in dimension two the dual of such a cutset—for the standard duality of planar graphs—is a path from the left-hand side to the right-hand side of the dual cylinder; thus, if we give the same capacity at an edge and its dual but we interpret it as the time needed to cross the dual edge in the dual graph, the minimal capacity of a cutset in the original cylinder is equal to the minimal time needed to go from left to right in the dual cylinder. Thus, results concerning maximal flows in two-dimensional first passage percolation can be obtained thanks to the known results concerning problems of distance in first passage percolation (see [7] and [8]). However, such a correspondence does not exist in dimension three or more; the "dual" of an edge in dimension three is a small plaquette (as defined by Kesten [9]) orthogonal to this edge that cuts it in its middle of side-length one and whose sides are parallel to the axis of coordinates. Thus, the dual of a cutset is a "surface" of plaquettes and there exists no result concerning this kind of object in the literature of first passage percolation in terms of problems of distance.

The existing results concerning maximal flows in first passage percolation in dimension three or more follow. The maximal flow has been studied almost exclusively through cylinders, since cylinders have good properties of symmetry and stacking. The most natural flow to study in a cylinder $\text{cyl}(nA, h(n))$ (where the height function $h$ satisfies $\lim_{n \rightarrow \infty} h(n) = +\infty$) is $\phi(nA, h(n))$ but $\tau(nA, h(n))$ has better properties so it is easer to deal with; it is almost subadditive.

Using a subadditive argument and concentration inequalities, Rossignol and Théret have proved in [12] that $\tau(nA, h(n))$ satisfies a law of large numbers.

THEOREM 2 (Rossignol and Théret). *We suppose that*

$$\int_{[0,\infty[} x \, d\Lambda(x) < \infty.$$

*For every unit vector $v$, for every nondegenerate hyperrectangle $A$ orthogonal to $v$, for every height function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, we have*

$$\lim_{n\to\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu(v) \qquad \text{in } L^1.$$

*Moreover, if the origin of the graph belongs to $A$ or if*

$$\int_{[0,\infty[} x^{1+1/(d-1)} \, d\Lambda(x) < \infty,$$

*then*

$$\lim_{n\to\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu(v) \qquad a.s.$$

Indeed, thanks to the max-flow min-cut theorem, we know that $\tau(nA, h(n))$ is equal to the minimal capacity of a $((nA)_1^{h(n)}, (nA)_2^{h(n)})$-cutset. Roughly speaking, such a cutset has its boundary fixed along $\partial(nA)$ which lies between $(nA)_1^{h(n)}$ and $(nA)_2^{h(n)}$. This property implies that $\tau(nA, h(n))$ is almost subadditive, thus, its convergence is not surprising. In the case where $h(n)$ is negligible compared to $n$, then $\phi(nA, h(n))$ satisfies the same law of large numbers as $\tau(nA, h(n))$ since the cylinder $\text{cyl}(nA, h(n))$ is asymptotically very flat, thus, a $(B(nA, h(n)), T(nA, h(n)))$-cutset has also its boundary very close to $\partial(nA)$.

We recall some geometric properties of the map $\nu : v \in S^{d-1} \mapsto \nu(v)$ under the only condition on $\Lambda$ that $\mathbb{E}(t(e)) < \infty$. They have been stated in [12], Section 4.4. There exists a unit vector $v_0$ such that $\nu(v_0) = 0$ if and only if for all unit vector $v$, $\nu(v) = 0$ and it happens if and only if $\Lambda(0) \geq 1 - p_c(d)$, where $p_c(d)$ is the critical parameter of the bond percolation on $\mathbb{Z}^d$. This property has been proved by Zhang [15]. Moreover, $\nu$ satisfies the weak triangle inequality, that is, if $(ABC)$ is a nondegenerate triangle in $\mathbb{R}^d$ and $v_A$, $v_B$ and $v_C$ are the exterior normal unit vectors to the sides $[BC]$, $[AC]$, $[AB]$ in the plane spanned by $A$, $B$, $C$, then

$$\mathcal{H}^1([AB])\nu(v_C) \leq \mathcal{H}^1([AC])\nu(v_B) + \mathcal{H}^1([BC])\nu(v_A).$$

This implies that the homogeneous extension $\nu_0$ of $\nu$ to $\mathbb{R}^d$, defined by $\nu_0(0) = 0$ and for all $w$ in $\mathbb{R}^d$,

$$\nu_0(w) = |w|_2 \nu(w/|w|_2)$$

is a convex function; in particular, since $\nu_0$ is finite, it is continuous on $\mathbb{R}^d$. We denote by $\nu_{\min}$ (resp., $\nu_{\max}$) the infimum (resp., supremum) of $\nu$ on $S^{d-1}$.

Kesten, Zhang, Rossignol and Théret have studied the maximal flow between the top and the bottom of straight cylinders. Let us denote by $D(\mathbf{k}, m)$ the cylinder

$$D(\mathbf{k}, m) = \prod_{i=1}^{d-1} [0, k_i] \times [0, m],$$

where $\mathbf{k} = (k_1, \ldots, k_{d-1}) \in \mathbb{R}^{d-1}$. We denote by $\phi(\mathbf{k}, m)$ the maximal flow in $D(\mathbf{k}, m)$ from its top $\prod_{i=1}^{d-1} [0, k_i] \times \{m\}$ to its bottom $\prod_{i=1}^{d-1} [0, k_i] \times \{0\}$. Kesten [9] proved the following result.

THEOREM 3 (Kesten).    *Let $d = 3$. We suppose that $\Lambda(0) < p_0$ for some fixed $p_0 \geq 1/27$ and that*

$$\exists \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} d\Lambda(x) < \infty.$$

*If $m = m(\mathbf{k})$ goes to infinity with $k_1 \geq k_2$ in such a way that*

$$\exists \delta > 0 \qquad \lim_{k_1 \geq k_2 \to \infty} k^{-1+\delta} \log m(\mathbf{k}) = 0,$$

*then*

$$\lim_{k_1 \geq k_2 \to \infty} \frac{\phi(\mathbf{k}, m)}{k_1 k_2} = \nu((0, 0, 1)) \qquad \textit{a.s. and in } L^1.$$

*Moreover, if $\Lambda(0) > 1 - p_c(d)$, where $p_c(d)$ is the critical parameter for the standard bond percolation model on $\mathbb{Z}^d$ and if*

$$\int_{[0,+\infty[} x^6 d\Lambda(x) < \infty,$$

*there exists a constant $C = C(F) < \infty$ such that for all $m = m(\mathbf{k})$ that goes to infinity with $k_1 \geq k_2$ and satisfies*

$$\liminf_{k_1 \geq k_2 \to \infty} \frac{m(\mathbf{k})}{k_1 k_2} > C$$

*for all $k_1 \geq k_2$ sufficiently large, we have*

$$\phi(\mathbf{k}, m) = 0 \qquad \textit{a.s.}$$

Zhang [16] improved this result.

THEOREM 4 (Zhang).    *Let $d \geq 2$. We suppose that*

$$\exists \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} d\Lambda(x) < \infty.$$

*Then for all $m = m(\mathbf{k})$ that goes to infinity when all the $k_i$, $i = 1, \ldots, d - 1$ go to infinity in such a way that*

$$\exists \delta \in\ ]0, 1] \qquad \log m(\mathbf{k}) \leq \max_{i=1,\ldots,d-1} k_i^{1-\delta},$$

*we have*

$$\lim_{k_1,\ldots,k_{d-1}\to\infty} \frac{\phi(\mathbf{k}, m)}{\prod_{i=1}^{d-1} k_i} = \nu((0,\ldots,0,1)) \qquad a.s.\ and\ in\ L^1.$$

*Moreover, this limit is positive if and only if* $\Lambda(0) < 1 - p_c(d)$.

To show this theorem, Zhang first obtains an important control on the number of edges in a minimal cutset. Finally, Rossignol and Théret [12] improved Zhang's result in the particular case where the dimensions of the basis of the straight cylinder go to infinity all at the same speed. They obtain the following result.

THEOREM 5 (Rossignol and Théret). *We suppose that*

$$\int_{[0,\infty[} x\,d\Lambda(x) < \infty.$$

*For every straight hyperrectangle* $A = \prod_{i=1}^{d-1}[0, a_i] \times \{0\}$ *with* $a_i > 0$ *for all* $i$, *for every height function* $h : \mathbb{N} \to \mathbb{R}^+$ *satisfying* $\lim_{n\to\infty} h(n) = +\infty$ *and* $\lim_{n\to\infty} \log h(n)/n^{d-1} = 0$, *we have*

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu((0,\ldots,0,1)) \qquad a.s.\ and\ in\ L^1.$$

In dimension two more results are known, as explained previously. We present here the two results that do not come from the literature of problems of distance in first passage percolation. Rossignol and Théret [11] studied the maximal flow from the top to the bottom of a tilted cylinder in dimension two and proved the following theorem ([11], Corollary 2.10).

THEOREM 6 (Rossignol and Théret). *Let* $v$ *be a unit vector, let* $A$ *be a nondegenerate line-segment orthogonal to* $v$ *and* $h : \mathbb{N} \to \mathbb{R}^+$ *a height function satisfying* $\lim_{n\to\infty} h(n) = +\infty$ *and* $\lim_{n\to\infty} \log h(n)/n = 0$. *We suppose that there exists* $\alpha \in [0, \pi/2]$ *such that*

$$\lim_{n\to\infty} \frac{2h(n)}{\mathcal{H}^1(nA)} = \tan\alpha.$$

*Then, if*

$$\int_{[0,\infty[} x\,d\Lambda(x) < \infty,$$

*we have*

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{\mathcal{H}^1(nA)} = \inf\left\{ \frac{\nu(v')}{v \cdot v'} \,\Big|\, v' \text{ satisfies } v \cdot v' \geq \cos\alpha \right\} \qquad in\ L^1.$$

*Moreover, if the origin of the graph is the middle of A, or if*

$$\int_{[0,\infty[} x^2 \, d\Lambda(x) < \infty,$$

*then we have*

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{\mathcal{H}^1(nA)} = \inf\left\{\frac{\nu(v')}{v \cdot v'} \,\Big|\, v' \text{ satisfies } v \cdot v' \geq \cos\alpha\right\} \qquad a.s.$$

Garet [6] studied the maximal flow $\sigma(A)$ between a convex bounded set $A$ and infinity in the case $d = 2$. By an extension of the max-flow min-cut theorem to nonfinite graphs, Garet [6] proves that this maximal flow is equal to the minimal capacity of a set of edges that cuts all paths from $A$ to infinity. Let $\partial A$ be the boundary of $A$ and $\partial^* A$ the set of the points $x \in \partial A$ at which $A$ admits a unique exterior normal unit vector $v_A(x)$ in a measure theoretic sense (see [2], Section 13, for a precise definition). If $A$ is a convex set, the set $\partial^* A$ is also equal to the set of the points $x \in \partial A$ at which $A$ admits a unique exterior normal vector in the classical sense and this vector is $v_A(x)$. Garet proved the following theorem.

THEOREM 7 (Garet). *Let $d = 2$. We suppose that $\Lambda(0) < 1 - p_c(2) = 1/2$ and that*

$$\exists \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} \, d\Lambda(x) < \infty.$$

*Then for all convex bounded set $A$ containing $0$ in its interior, we have*

$$\lim_{n\to\infty} \frac{\sigma(nA)}{n} = \int_{\partial^* A} \nu(v_A(x)) \, d\mathcal{H}^1(x) = \mathcal{I}(A) > 0 \qquad a.s.$$

*Moreover, for all $\varepsilon > 0$ there exist constants $C_1, C_2 > 0$ depending on $\varepsilon$ and $\Lambda$ such that*

$$\forall n \geq 0 \qquad \mathbb{P}\left[\frac{\sigma(nA)}{n\mathcal{I}(A)} \notin \,]1 - \varepsilon, 1 + \varepsilon[\,\right] \leq C_1 \exp(-C_2 n).$$

Nevertheless, a law of large numbers for the maximal flow from the top to the bottom of a tilted cylinder for $d \geq 3$ was not proved yet. In fact, the lack of symmetry of the graph induced by the slope of the box is a major issue to extend the existing results concerning straight cylinders to tilted cylinders. The theorem of Garet was not extended to dimension $d \geq 3$ either.

2.2.2. *Large deviations results.* The upper and lower large deviations of the maximal flows $\phi(nA, h(n))$ and $\tau(nA, h(n))$ have been studied in the cases where the laws of large numbers are known. Let us start with the existing results concerning the upper large deviations of these maximal flows. They are studied in [14] and [13]. Theorem 4 in [14] deals with the upper large deviations of the variable $\phi(nA, h(n))$ above $\nu(v)$.

THEOREM 8. *We suppose that*

$$\exists \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} \, d\Lambda(x) < \infty.$$

*Then for every unit vector $v$ and every nondegenerate hyperrectangle $A$ orthogonal to $v$, for every height function $h : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{n\to\infty} h(n) = +\infty$ and for every $\lambda > \nu(v)$ we have*

$$\liminf_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)h(n)} \log \mathbb{P}\left[ \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq \lambda \right] > 0.$$

We shall rely on this result for proving Theorem 1. Moreover, Theorem 1 is a generalization of Theorem 8 where we work in the domain $\Omega$ instead of a parallelepiped. We stress the fact that $\nu(v)$ is not in general the a.s. limit on $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$. The corresponding large deviation principle is proved only in the case of straight cylinders (see Theorems 2 and 3 in [13] *that are gathered here*).

THEOREM 9. *We consider the maximal flow $\phi^{[h]}(n)$ through the straight cylinder $[0,n]^{d-1} \times [0, h(n)]$ from its top to its bottom. We suppose that the height function $h : \mathbb{N} \to \mathbb{R}^+$ satisfies*

$$\lim_{n\to\infty} \frac{h(n)}{\log n} = +\infty.$$

*Then for every $\lambda$ in $\mathbb{R}^+$, the limit*

$$\psi(\lambda) = \lim_{n\to\infty} \frac{-1}{n^{d-1}h(n)} \log \mathbb{P}[\phi^{[h]}(n) \geq \lambda n^{d-1}]$$

*exists and is independent of $h$. Moreover, $\psi$ is convex on $\mathbb{R}^+$, finite and continuous on the set $\{\lambda \mid \Lambda([\lambda, +\infty[) > 0\}$. If*

$$\int_{[0,+\infty[} x \, d\Lambda(x) < \infty,$$

*then $\psi$ vanishes on $[0, \nu((0, \ldots, 0, 1))]$. If*

$$\exists \gamma > 0 \qquad \int_{[0,+\infty[} e^{\gamma x} \, d\Lambda(x) < \infty,$$

*then $\psi$ is strictly positive on $]\nu((0, \ldots, 0, 1)), +\infty[$, and the sequence*

$$\left( \frac{\phi^{[h]}(n)}{n^{d-1}} \right)_{n\in\mathbb{N}}$$

*satisfies a large deviation principle with speed $n^{d-1}h(n)$ and governed by the good rate function $\psi$.*

The lack of symmetry makes it difficult to extend this large deviation principle to the case of tilted cylinders. The upper large deviations for $\tau$ depend a lot on the moments of $\Lambda$, as proved in Theorem 3 of [14].

THEOREM 10. *Let $A$ be a nondegenerate hyperrectangle and $\vec{v}$ one of the two unit vectors normal to $A$. Let $h : \mathbb{N} \to \mathbb{R}^+$ be a height function satisfying $\lim_{n\to\infty} h(n) = +\infty$. The upper large deviations of $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ depend on the tail of the distribution of the capacities. Indeed, we obtain that*:

(i) *If the law of the capacity of the edges has bounded support, then for every $\lambda > \nu(\vec{v})$ we have*

$$\liminf_{n\to\infty} \frac{-1}{\mathcal{H}^{d-1}(nA)\min(h(n), n)} \log \mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq \lambda\right] > 0;$$

*the upper large deviations are then of volume order for height functions $h$ such that $h(n)/n$ is bounded and of order $n^d$ if $\lim_{n\to\infty} h(n)/n = +\infty$.*

(ii) *If the capacity of the edges follows the exponential law of parameter $1$, then there exists $n_0(d, A, h)$ and for every $\lambda > \nu(\vec{v})$ there exists a positive constant $D$ depending only on $d$ and $\lambda$ such that for all $n \geq n_0$ we have*

$$\mathbb{P}[\tau(nA, h(n)) \geq \lambda \mathcal{H}^{d-1}(nA)] \geq \exp(-D\mathcal{H}^{d-1}(nA)).$$

(iii) *If the law of the capacity of the edges satisfies*

$$\forall \theta > 0 \qquad \int_{[0,+\infty[} e^{\theta x}\, d\Lambda(x) < \infty,$$

*then for all $\lambda > \nu(\vec{v})$ we have*

$$\lim_{n\to\infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq \lambda\right] = -\infty.$$

This dependence on the moment conditions on $\Lambda$ comes from the fact that the distance between the part of the boundary of the cylinder through which the water can enter and the part through which it can escape is null.

We just say a few words about lower large deviations, since it is not the purpose of this article. Rossignol and Théret [12] proved that under some moment conditions, the lower large deviations of $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$ [resp., for $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ when $h(n)$ is negligible compared to $n$ or when cyl$(nA, h(n))$ is straight] are of surface order, that is, the probability that these rescaled flows are abnormally small decays exponentially fast with $\mathcal{H}^{d-1}(nA)$. They also prove the corresponding large deviation principles.

From now on, we work in the rescaled graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$ again.

2.3. *Global project.*   This article is part of a global project in which we prove that $\phi_n / n^{d-1}$ converges a.s. toward $\widetilde{\phi}_\Omega$ and we find the right order of the upper and lower large deviations. Indeed, we prove in [4] the following result.

THEOREM 11.   *If the law $\Lambda$ of the capacity of an edge admits an exponential moment,*

$$\exists \theta > 0 \qquad \int_{\mathbb{R}^+} e^{\theta x} \, d\Lambda(x) < +\infty$$

*and if $\Lambda(0) < 1 - p_c(d)$, then there exists a finite constant $\phi_\Omega$ such that for all $\lambda < \phi_\Omega$,*

$$\limsup_{n \to \infty} \frac{1}{n^{d-1}} \log \mathbb{P}[\phi_n \le \lambda n^{d-1}] < 0.$$

The definition of the constant $\phi_\Omega$ is given in [4]; it is of the same kind as the one of $\widetilde{\phi}_\Omega$ we will give in Section 3 but slightly different. Finally we prove in [3] through a completely geometrical study that the constants $\phi_\Omega$ and $\widetilde{\phi}_\Omega$ are equal and we investigate when they are strictly positive. Thus, by a simple Borel–Cantelli's lemma, we obtain in [3] the following result.

THEOREM 12.    *We suppose that $\Omega$ is a Lipschitz domain and that $\Gamma$ is included in the union of a finite number of oriented hypersurfaces $\mathcal{S}_1, \ldots, \mathcal{S}_r$ of class $\mathcal{C}^1$ which are transverse to each other. We also suppose that $\Gamma^1$ and $\Gamma^2$ are open in $\Gamma$, that their relative boundaries $\partial_\Gamma \Gamma^1$ and $\partial_\Gamma \Gamma^2$ in $\Gamma$ have null $\mathcal{H}^{d-1}$ measure and that $d(\Gamma^1, \Gamma^2) > 0$. We suppose that the law $\Lambda$ of the capacity of an edge admits an exponential moment*

$$\exists \theta > 0 \qquad \int_{\mathbb{R}^+} e^{\theta x} \, d\Lambda(x) < +\infty.$$

*Then there exists a finite constant $\phi_\Omega \ge 0$ such that*

$$\lim_{n \to \infty} \frac{\phi_n}{n^{d-1}} = \phi_\Omega \qquad a.s.$$

*Moreover, this equivalence holds*:

$$\phi_\Omega > 0 \quad \iff \quad \Lambda(0) < 1 - p_c(d).$$

Combining Theorems 12, 11 and 1, we prove that the rescaled maximal flow $\phi_n / n^{d-1}$ converges a.s. toward a constant $\phi_\Omega$, that its upper large deviations are of volume order and that its lower large deviations are of surface order. These theorems apply to the maximal flow from the top to the bottom of a tilted cylinder. Thus, they generalize the existing results concerning the variable $\phi(A, h)$ in straight cylinders, in the particular case where all the dimensions of the cylinder go to infinity at the same speed (or, equivalently, the cylinder is fixed and the mesh of the graph go to zero isotropically). The large deviation principles from above and below still remain to be proved.

**3. Computation of $\widetilde{\phi}_\Omega$.** We give here a definition of $\widetilde{\phi}_\Omega$ in terms of the map $\nu$. When a hypersurface $\mathcal{S}$ is piecewise of class $\mathcal{C}^1$, we say that $\mathcal{S}$ is transverse to $\Gamma$ if for all $x \in \mathcal{S} \cap \Gamma$, the normal unit vectors to $\mathcal{S}$ and $\Gamma$ at $x$ are not collinear; if the normal vector to $\mathcal{S}$ (resp., to $\Gamma$) at $x$ is not well defined, this property must be satisfied by all the vectors which are limits of normal unit vectors to $\mathcal{S}$ (resp., $\Gamma$) at $y \in \mathcal{S}$ (resp., $y \in \Gamma$) when we send $y$ to $x$—there is at most a finite number of such limits. We say that a subset $P$ of $\mathbb{R}^d$ is polyhedral if its boundary $\partial P$ is included in the union of a finite number of hyperplanes. For each point $x$ of such a set $P$ which is on the interior of one face of $\partial P$, we denote by $\nu_P(x)$ the exterior unit vector orthogonal to $P$ at $x$. For $A \subset \mathbb{R}^d$, we denote by $\mathring{A}$ the interior of $A$. We define $\widetilde{\phi}_\Omega$ by

$$\widetilde{\phi}_\Omega = \inf\{\mathcal{I}_\Omega(P) \mid P \subset \mathbb{R}^d, \overline{\Gamma}^1 \subset \mathring{P}, \overline{\Gamma}^2 \subset \overbrace{\mathbb{R}^d \setminus P}^{\circ}, P \text{ is polyhedral},$$

$$\partial P \text{ is transverse to } \Gamma\},$$

where

$$\mathcal{I}_\Omega(P) = \int_{\partial P \cap \Omega} \nu(\nu_P(x)) \, d\mathcal{H}^{d-1}(x).$$

See Figure 3 for an example of such a polyhedral set $P$.

The definition of the constant $\widetilde{\phi}_\Omega$ is not very intuitive. We propose to define the notion of a continuous cutset to have a better understanding of this constant. We say that $\mathcal{S} \subset \mathbb{R}^d$ cuts $\Gamma^1$ from $\Gamma^2$ in $\overline{\Omega}$ if every continuous path from $\Gamma^1$ to $\Gamma^2$ in $\overline{\Omega}$ intersects $\mathcal{S}$. In fact, if $P$ is a polyhedral set of $\mathbb{R}^d$ such that

$$\overline{\Gamma}^1 \subset \mathring{P} \quad \text{and} \quad \overline{\Gamma}^2 \subset \overbrace{\mathbb{R}^d \setminus P}^{\circ},$$
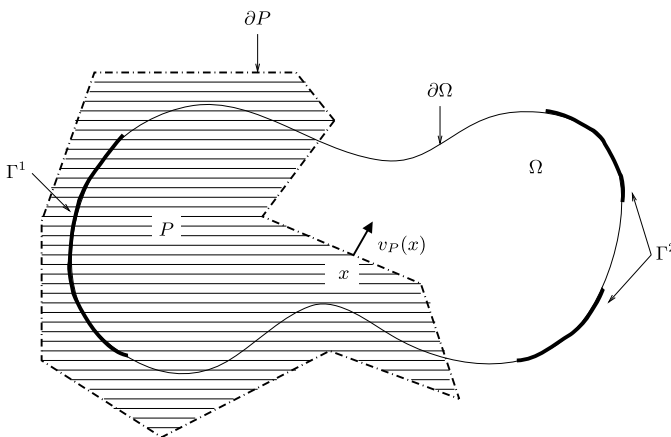


FIG. 3.   *A polyhedral set $P$ as in the definition of $\widetilde{\phi}_\Omega$.*

then $\partial P \cap \overline{\Omega}$ is a continuous cutset from $\Gamma^1$ to $\Gamma^2$ in $\overline{\Omega}$. Since $\nu(v)$ is the average amount of fluid that can cross a hypersurface of area one in the direction $v$ per unit of time, it can be interpreted as the capacity of a unitary hypersurface orthogonal to $v$. Thus, $\mathcal{I}_\Omega(P)$ can be interpreted as the capacity of the continuous cutset $\partial P \cap \overline{\Omega}$ defined by $P$. The constant $\widetilde{\phi}_\Omega$ is the solution of a min-cut problem because it is equal to the infimum of the capacity of a continuous cutset that satisfies some specific properties.

We remark that the capacity $\mathcal{I}_\Omega$ of a continuous cutset is exactly the same as the one defined by Garet in [6] in dimension two (see Theorem 7), except that we consider a maximal flow through a bounded domain so our capacity is adapted to the problems of boundaries that arise. Moreover, $\widetilde{\phi}_\Omega$ has the same form as the limit observed in dimension two in Theorem 6.

**4. Sketch of the proof.**   To prove Theorem 1, we have to study the probability

(1) $$\mathbb{P}[\phi_n \geq (\widetilde{\phi}_\Omega + \varepsilon)n^{d-1}]$$

for a positive $\varepsilon$.

*Step* 1. We first prove that $\widetilde{\phi}_\Omega$ is finite, that is, that there exists a polyhedral set $P \subset \mathbb{R}^d$ such that $\partial P$ is transverse to $\Gamma$ and

$$\overline{\Gamma}^1 \subset \mathring{P}, \qquad \overline{\Gamma}^2 \subset \overset{\circ}{\overbrace{\mathbb{R}^d \setminus P}}.$$

For that purpose, at each point $x$ of $\overline{\Gamma}^1$ we associate a cube of center $x$, of strictly positive side-length, which is transverse to $\Gamma^1$ and at positive distance of $\Gamma^2$ [this is possible thanks to the regularity of $\Gamma$ and the fact that $d(\Gamma^1, \Gamma^2) > 0$]. From this family of cubes we extract by compactness a finite family of cubes that covers $\overline{\Gamma}^1$. The set $P$ which is defined as the union of the cubes satisfies the desired properties.

*Step* 2. We consider a polyhedral set $P$ as in the definition of $\widetilde{\phi}_\Omega$ such that $\mathcal{I}_\Omega(P)$ is very close to this constant. We want to construct sets of edges near $\partial P \cap \Omega$ that cut $\Gamma_n^1$ from $\Gamma_n^2$ in $\Omega_n$. Because we took a discrete approximation of $\Omega$ from the outside, we need to enlarge a little $\Omega$, because some flow might go from $\Gamma_n^1$ to $\Gamma_n^2$ using paths that lies partly in $\Omega_n \setminus \Omega$. Thus, we construct a set $\Omega'$ which contains a small neighborhood of $\Omega$ (hence, also $\Omega_n$ for all $n$ large enough) which is transverse to $\partial P$ and which is small enough to ensure that $\mathcal{I}_{\Omega'}(P)$ is still very close to $\phi_\Omega$. To construct this set, we cover $\partial\Omega$ with small cubes, by compactness we extract a finite subcover of $\partial\Omega$ and finally we add the cubes of the subcover to $\Omega$ to obtain $\Omega'$. We construct these cubes so that their boundaries are transverse to $\partial P$ and their diameters are uniformly smaller than a small constant, so that $\Omega'$ is included in a neighborhood of $\Omega$ as small as we need. Since $\partial P$ is transverse to $\Gamma$, if we take this constant small enough, we can control $\mathcal{H}^{d-1}(\partial P \cap (\Omega' \setminus \Omega))$ and thus the difference between $\mathcal{I}_{\Omega'}(P)$ and $\mathcal{I}_\Omega(P)$.

*Step* 3. Then we construct a family of $Cn$ (where $C > 0$) disjoint sets of edges that cut $\Gamma_n^1$ from $\Gamma_n^2$ in $\Omega_n$ and that lie near $\partial P$. We consider the neighborhood $P'$

of $P$ inside $\Omega'$ at distance smaller than a tiny constant $h$ and we partition $P' \setminus P$ into slabs $\mathcal{M}'(k)$ of width of order $1/n$, so we have $Cn$ such slabs which look like translations of $\partial P \cap \Omega'$ that are slightly deformed and thickened. We prove that each path from $\Gamma_n^1$ to $\Gamma_n^2$ in $\Omega_n$ must contain at least one edge that lies in the set $\mathcal{M}'(k)$ for each $k$, that is, each set $\mathcal{M}'(k)$ contains a cutset. Thus, we have found a family of $Cn$ disjoint cutsets.

*Step* 4. We almost cover $\partial P \cap \Omega'$ by a finite family of disjoint cylinders $B_{i,j}$, whose bases are hyperrectangles of side length $l$, that are orthogonal to $\partial P$, of height $h$ and such that the part of $\partial P$ which is missing in this covering is very small. Thus, we obtain that

$$(2) \qquad \mathcal{I}_{\Omega'}(P) \text{ is close to } \sum \nu(v_{i,j}) l^{d-1},$$

where $v_{i,j}$ gives the direction toward which the cylinder $B_{i,j}$ is tilted (it is the unit vector which is orthogonal to the face of $\partial P$ that cuts $B_{i,j}$).

We want to compare $\phi_n$ with the sum of the maximal flows $\phi(B_{i,j}, v_{i,j})$. For each $(i, j)$, let $E_{i,j}$ be a set of edges that cuts the top from the bottom of $B_{i,j}$. The set $\bigcup E_{i,j}$ does not cut $\Gamma_n^1$ from $\Gamma_n^2$ in $\Omega_n$ in general; to create such a cutset we must add two sets of edges.

(i) A set of edges that covers the part of $\partial P \cap \Omega'$ that is missing in the covering by the cylinders $B_{i,j}$.

(ii) A set of edges that glues together all the previous sets of edges [the sets $E_{i,j}$ and the set described in (i)].

In fact, we have already constructed $Cn$ possible sets of edges as in (i): the edges that lie in $\mathcal{M}'(k) \setminus (\bigcup B_{i,j})$ for $k = 1, \ldots, Cn$. We denote these sets by $M(k)$. We can also find $C'n$ ($C' > 0$) disjoint sets of edges that can be the glue described in (ii); we denote these sets by $W(l)$ for $l = 1, \ldots, C'n$. Indeed, we can choose different sets because we provide the glue more or less in the interior of the cylinders $B_{i,j}$. Thus, we obtain that

$$\forall k \in \{1, \ldots, Cn\} \ \forall l \in \{1, \ldots, C'n\} \qquad \bigcup E_{i,j} \cup M(k) \cup W(l)$$

$$\text{cuts } \Gamma_n^1 \text{ from } \Gamma_n^2 \text{ in } \Omega_n.$$

Then

$$(3) \qquad \phi_n \leq \sum \phi(B_{i,j}, v_{i,j}) + \min_{k=1,\ldots,Cn} V(M(k)) + \min_{l=1,\ldots,C'n} V(W(l)).$$

Combining (2) and (3) we see that if $\phi_n \geq (\widetilde{\phi}_\Omega + \varepsilon) n^{d-1}$, one of the following events must happen:

(a) $\exists j \in J \ \phi(B_j, v_j) \geq (\nu(v_j) + \varepsilon/2) l^{d-1} n^{d-1}$,
(b) $\forall k \in \{1, \ldots, Cn\} \ V(M(k)) \geq \eta n^{d-1}$,
(c) $\forall l \in \{1, \ldots, C'n\} \ V(W(l)) \geq \eta n^{d-1}$,

where $\eta$ is a very small constant (depending on $\varepsilon$ and $\phi_\Omega$).

*Step* 5. It consists in taking care of the probability that the events (a), (b) or (c) happen. The probability of (a) has already been studied in [14]; cf. Theorem 8 above; the upper large deviations of the variable $\phi$ in a cylinder above $\nu$ are of volume order. The events (b) and (c) are of the same type and their probability is of the form

$$(4) \qquad \mathbb{P}\left[\sum_{m=1}^{\lceil \alpha n^{d-1} \rceil} t_m \geq \eta n^{d-1}\right]^{Dn},$$

where $(t_m)_{m \in \mathbb{N}}$ is a family of i.i.d. variables of distribution function $\Lambda$, $D$ is a constant, $\eta$ is a very small constant and $\alpha n^{d-1}$ is the cardinality of the family of variables we consider. If $\alpha < \eta \mathbb{E}[t_1]^{-1}$ and if the law $\Lambda$ admits one exponential moment, the Cramér theorem in $\mathbb{R}$ states that the probability (4) decays exponentially fast with $n^d$. Note the role of the optimization over $Dn$ different probabilities to obtain the correct speed of decay. The proof would have been slightly simpler if we would have proven only that the decay of the probability (1) is at least exponential in $n^{d-1}$.

*Step* 6. To complete the proof, it is enough to control the cardinality of the sets $M(k)$ and $W(l)$ for each $k$, $l$, to ensure that we can use the Cramér theorem as explained in Step 5. This can be done using the geometrical properties of $\partial P$ (it is polyhedral and transverse to $\partial \Omega'$).

## 5. The constant $\widetilde{\phi}_\Omega$ is finite.

To prove that $\widetilde{\phi}_\Omega < \infty$, it is sufficient to exhibit a set $P$ satisfying all the conditions given in the definition of $\widetilde{\phi}_\Omega$. Indeed, if such a set $P$ exists, then

$$\widetilde{\phi}_\Omega \leq \nu_{\max} \mathcal{H}^{d-1}(\partial P \cap \Omega) < \infty$$

since a polyhedral set has finite perimeter in $\Omega$. We will construct such a set $P$. The idea of the proof is the following. We will cover $\overline{\Gamma}^1$ with small hypercubes which are transverse to $\Gamma^1$ and at positive distance of $\overline{\Gamma}^2$. Then, by compactness, we will extract a finite covering. We will denote by $P$ the union of the hypercubes of this finite covering. Then $P$ satisfies the desired properties.

We prove a geometric lemma.

LEMMA 1. *Let $\Gamma$ be an hypersurface (i.e., a $C^1$ submanifold of $\mathbb{R}^d$ of codimension 1) and let $K$ be a compact subset of $\Gamma$. There exists a positive $M = M(\Gamma, K)$ such that*

$$\forall \varepsilon > 0 \; \exists r > 0 \; \forall x, y \in K \quad |x - y|_2 \leq r \quad \Rightarrow \quad d_2(y, \tan(\Gamma, x)) \leq M\varepsilon|x - y|_2$$

*[$\tan(\Gamma, x)$ is the tangent hyperplane of $\Gamma$ at $x$].*

Proof. By a standard compactness argument, it is enough to prove the following local property:

$$\forall x \in \Gamma \ \exists M(x) > 0 \ \forall \varepsilon > 0 \ \exists r(x, \varepsilon) > 0 \ \forall y, z \in \Gamma \cap B(x, r(x, \varepsilon))$$

$$d_2(y, \tan(\Gamma, z)) \leq M(x)\varepsilon |y - z|_2.$$

Indeed, if this property holds, we cover $K$ by the open balls $\overset{\circ}{B}(x, r(x, \varepsilon)/2)$, $x \in K$, we extract a finite subcover $\overset{\circ}{B}(x_i, r(x_i, \varepsilon)/2)$, $1 \leq i \leq k$, and we set

$$M = \max\{M(x_i): 1 \leq i \leq k\}, \qquad r = \min\{r(x_i, \varepsilon)/2: 1 \leq i \leq k\}.$$

Now let $y, z$ belong to $K$ with $|y - z|_2 \leq r$. Let $i$ be such that $y$ belongs to $B(x_i, r(x_i, \varepsilon)/2)$. Since $r \leq r(x_i, \varepsilon)/2$, then both $y, z$ belong to the ball $B(x_i, r(x_i, \varepsilon))$ and it follows that

$$d_2(y, \tan(\Gamma, z)) \leq M(x_i)\varepsilon |y - z|_2 \leq M\varepsilon |y - z|_2.$$

We turn now to the proof of the above local property. Since $\Gamma$ is an hypersurface, for any $x$ in $\Gamma$ there exists a neighborhood $V$ of $x$ in $\mathbb{R}^d$, a diffeomorphism $f : V \mapsto \mathbb{R}^d$ of class $C^1$ and a $(d-1)$-dimensional vector space $Z$ of $\mathbb{R}^d$ such that $Z \cap f(V) = f(\Gamma \cap V)$ (see, e.g., [5], 3.1.19). Let $A$ be a compact neighborhood of $x$ included in $V$. Since $f$ is a diffeomorphism, the maps $y \in A \mapsto df(y) \in \mathrm{End}(\mathbb{R}^d)$, $u \in f(A) \mapsto df^{-1}(u) \in \mathrm{End}(\mathbb{R}^d)$ are continuous. Therefore, they are bounded

$$\exists M > 0 \ \forall y \in A \qquad \|df(y)\| \leq M, \qquad \forall u \in f(A) \qquad \|df^{-1}(u)\| \leq M$$

[here $\|df(x)\| = \sup\{|df(x)(y)|_2 : |y|_2 \leq 1\}$ is the standard operator norm in $\mathrm{End}(\mathbb{R}^d)$]. Since $f(A)$ is compact, the differential map $df^{-1}$ is uniformly continuous on $f(A)$,

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall u, v \in f(A) \qquad |u - v|_2 \leq \delta \quad \Rightarrow \quad \|df^{-1}(u) - df^{-1}(v)\| \leq \varepsilon.$$

Let $\varepsilon$ be positive and let $\delta$ be associated to $\varepsilon$ as above. Let $\rho$ be positive and small enough so that $\rho < \delta/2$ and $B(f(x), \rho) \subset f(A)$ [since $f$ is a $C^1$ diffeomorphism, $f(A)$ is a neighborhood of $f(x)$]. Let $r$ be such that $0 < r < \rho/M$ and $B(x, r) \subset A$. We claim that $M$ associated to $x$ and $r$ associated to $\varepsilon, x$ answer the problem. Let $y, z$ belong to $\Gamma \cap B(x, r)$. Since $[y, z] \subset B(x, r) \subset A$, and $\|df(\zeta)\| \leq M$ on $A$, then

$$|f(y) - f(x)|_2 \leq M|y - x|_2 \leq Mr < \rho, \qquad |f(z) - f(x)|_2 < \rho,$$

$$|f(y) - f(z)|_2 < \delta, \qquad |f(y) - f(z)|_2 < M|y - z|_2.$$

We next apply a classical lemma of differential calculus (see [10], I, 4, Corollary 2) to the map $f^{-1}$ and the interval $[f(z), f(y)]$ [which is included in $\overset{\circ}{B}(f(x), \rho) \subset f(A)$] and the point $f(z)$

$$|y - z - df^{-1}(f(z))(f(y) - f(z))|_2$$

$$\leq |f(y) - f(z)|_2 \sup\{\|df^{-1}(\zeta) - df^{-1}(f(z))\| : \zeta \in [f(z), f(y)]\}.$$

The right-hand member is less than $M|y - z|_2 \varepsilon$. Since $z + df^{-1}(f(z))(f(y) - f(z))$ belongs to $\tan(\Gamma, z)$, the proof is complete. $\square$

We come back to our case. The boundary $\Gamma$ of $\Omega$ is piecewise of class $\mathcal{C}^1$, that is, it is included in a finite union of $\mathcal{C}^1$ hypersurfaces which we denote by $(S_1, \ldots, S_p)$. The hypersurfaces $S_1, \ldots, S_p$ being $\mathcal{C}^1$ and the set $\Gamma$ compact, the maps $x \in \Gamma \mapsto v_{S_k}(x)$, $1 \le k \le p$ [where $v_{S_k}(x)$ is the unit normal vector to $S_k$ at $x$] are uniformly continuous:

$$\forall \delta > 0 \; \exists \eta > 0 \; \forall k \in \{1, \ldots, p\} \; \forall x, y \in S_k \cap \Gamma$$

$$|x - y|_2 \le \eta \quad \Rightarrow \quad |v_{S_k}(x) - v_{S_k}(y)|_2 < \delta.$$

Let $\eta^*$ be associated to $\delta = 1$ by this property. Let $k \in \{1, \ldots, p\}$. The set $S_k \cap \Gamma$ is a compact subset of the hypersurface $S_k$. Applying the previous lemma, we get

$$\exists M_k \; \forall \delta_0 > 0 \; \exists \eta_k > 0 \; \forall x, y \in S_k \cap \Gamma$$

$$|x - y|_2 \le \eta_k \quad \Rightarrow \quad d_2(y, \tan(S_k, x)) \le M_k \delta_0 |x - y|_2.$$

Let $M_0 = \max_{1 \le k \le p} M_k$ and let $\delta_0$ in $]0, 1/2[$ be such that $M_0 \delta_0 < 1/2$. For each $k$ in $\{1, \ldots, p\}$, let $\eta_k$ be associated to $\delta_0$ as in the above property and let

$$\eta_0 = \min\left( \min_{1 \le k \le p} \eta_k, \eta^*, \frac{1}{8d} \operatorname{dist}(\Gamma^1, \Gamma^2) \right).$$

We build a family of cubes $Q(x, r)$, indexed by $x \in \Gamma$ and $r \in ]0, r_\Gamma[$ such that $Q(x, r)$ is a cube centered at $x$ of side length $r$ which is transverse to $\Gamma$. For $x \in \mathbb{R}^d$ and $k \in \{1, \ldots, p\}$, let $p_k(x)$ be a point of $S_k \cap \Gamma$ such that

$$|x - p_k(x)|_2 = \inf\{|x - y|_2 : y \in S_k \cap \Gamma\}.$$

Such a point exists since $S_k \cap \Gamma$ is compact. We define then for $k \in \{1, \ldots, p\}$

$$\forall x \in \mathbb{R}^d \qquad v_k(x) = v_{S_k}(p_k(x)).$$

We also define

$$d_r = \inf_{v_1, \ldots, v_p \in S^{d-1}} \max_{\substack{b \in \mathcal{B}_d \\ e \in b}} \min_{1 \le k \le p} (|e - v_k|_2, |-e - v_k|_2),$$

where $\mathcal{B}_d$ is the collection of the orthonormal basis of $\mathbb{R}^d$ and $S^{d-1}$ is the unit sphere of $\mathbb{R}^d$. Let $\eta$ be associated to $d_r/4$ as in the above continuity property. We set

$$r_\Gamma = \frac{\eta}{2d}.$$

Let $x \in \Gamma$. By the definition of $d_r$, there exists an orthonormal basis $b_x$ of $\mathbb{R}^d$ such that

$$\forall e \in b_x \; \forall k \in \{1, \ldots, p\} \qquad \min(|e - v_k(x)|_2, |-e - v_k(x)|_2) > \frac{d_r}{2}.$$

Let $Q(x, r)$ be the cube centered at $x$ of sidelength $r$ whose sides are parallel to the vectors of $b_x$. We claim that $Q(x, r)$ is transverse to $\Gamma$ for $r < r_\Gamma$. Indeed, let $y \in Q(x, r) \cap \Gamma$. Suppose that $y \in S_k$ for some $k \in \{1, \ldots, p\}$, so that $v_k(y) = v_{S_k}(y)$ and $|x - p_k(x)|_2 < dr_\Gamma$. In particular, we have $|y - p_k(x)|_2 < 2dr_\Gamma < \eta$ and $|v_{S_k}(y) - v_k(x)|_2 < d_r/4$. For $e \in b_x$,

$$\frac{d_r}{2} \le |e - v_k(x)|_2 \le |e - v_{S_k}(y)|_2 + |v_{S_k}(y) - v_k(x)|_2$$

whence,

$$|e - v_{S_k}(y)|_2 \ge \frac{d_r}{2} - \frac{d_r}{4} = \frac{d_r}{4}.$$

This is also true for $-e$, therefore, the faces of the cube $Q(x, r)$ are transverse to $S_k$.

Now we consider the collection

$$(\mathring{Q}(x, r), x \in \overline{\Gamma}^1, r < r_\Gamma).$$

It covers $\overline{\Gamma}^1$. By compactness of $\overline{\Gamma}^1$, we can extract a finite covering $(\mathring{Q}(x_i, r_i), i \in I)$ from this collection. We define

$$P = \bigcup_{i \in I} Q(x_i, r_i).$$

We claim that $P$ satisfies all the hypotheses in the definition of $\widetilde{\phi}_\Omega$. Indeed, $P$ is obviously polyhedral and transverse to $\Gamma$. Moreover, we know that

$$\overline{\Gamma}^1 \subset \mathring{P}$$

and since $d(P, \overline{\Gamma}^2) > 0$, we also obtain that

$$\overline{\Gamma}^2 \subset \overset{\circ}{\overbrace{\mathbb{R}^d \setminus P}}.$$

## 6. Definition of the set $\Omega'$.

Let $\lambda$ be in $]\widetilde{\phi}_\Omega, +\infty[$. We are studying

$$\mathbb{P}[\phi_n \ge \lambda n^{d-1}].$$

Suppose first that $\widetilde{\phi}_\Omega > 0$. There exists a positive $s$ such that $\lambda > \widetilde{\phi}_\Omega(1 + s)^2$. By definition of $\widetilde{\phi}_\Omega$, for every positive $s$, there exists a polyhedral subset $P$ of $\mathbb{R}^d$, such that $\partial P$ is transverse to $\Gamma$,

$$\overline{\Gamma}^1 \subset \mathring{P}, \qquad \overline{\Gamma}^2 \subset \overset{\circ}{\overbrace{\mathbb{R}^d \setminus P}}$$

and

$$\mathcal{I}_\Omega(P) \le \widetilde{\phi}_\Omega(1 + s).$$

Then $\lambda > \mathcal{I}_\Omega(P)(1+s)$ and

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P}[\phi_n \geq \mathcal{I}_\Omega(P)(1+s)n^{d-1}].$$

Since $\partial P$ is transverse to $\Gamma$, we know that there exists $\delta_0 > 0$ (depending on $\lambda$, $P$ and $\Gamma$) such that for all $\delta \leq \delta_0$,

$$\mathcal{H}^{d-1}\big(\partial P \cap (\mathcal{V}_2(\Omega, \delta) \setminus \Omega)\big) \leq \frac{s\mathcal{I}_\Omega(P)}{2\nu_{\max}}.$$

Thus, for any set $\Omega'$ satisfying $\Omega \subset \Omega' \subset \mathcal{V}_2(\Omega, \delta_0)$, we have

$$\int_{\partial P \cap \Omega'} \nu(v_P(x)) \, d\mathcal{H}^{d-1}(x) \leq \mathcal{I}_\Omega(P)(1+s/2),$$

then $\lambda > (1+s/2)(\int_{\partial P \cap \Omega'} \nu(v_P(x)) \, d\mathcal{H}^{d-1}(x))$ and

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P}\bigg[\phi_n \geq \bigg(\int_{\partial P \cap \Omega'} \nu(v_P(x)) \, d\mathcal{H}^{d-1}(x)\bigg)(1+s/2)n^{d-1}\bigg].$$

Suppose now that $\widetilde{\phi}_\Omega = 0$. Then for an arbitrarily fixed $s \in {]0, 1[}$, there exists a polyhedral subset $P$ of $\mathbb{R}^d$, such that $\partial P$ is transverse to $\Gamma$,

$$\overline{\Gamma}^1 \subset \overset{\circ}{P}, \qquad \overline{\Gamma}^2 \subset \overset{\circ}{\widetilde{\mathbb{R}^d \setminus P}}$$

and

$$\mathcal{I}_\Omega(P) \leq \frac{\lambda}{1+s}$$

and, thus, $\lambda > \mathcal{I}_\Omega(P)(1+s)$. If $\mathcal{I}_\Omega(P) > 0$, we can use exactly the same argument as previously. We suppose that $\mathcal{I}_\Omega(P) = 0$. We know as previously that there exists $\delta_0 > 0$ (depending on $\lambda$, $P$ and $\Gamma$) such that for all $\delta \leq \delta_0$,

$$\mathcal{H}^{d-1}\big(\partial P \cap (\mathcal{V}_2(\Omega, \delta) \setminus \Omega)\big) < \frac{\lambda}{\nu_{\max}(1+s/2)}.$$

Thus, in any case, we obtain that there exists $\delta_0 > 0$ such that, for any set $\Omega'$ satisfying $\Omega \subset \Omega' \subset \mathcal{V}_2(\Omega, \delta_0)$, we have

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P}\bigg[\phi_n \geq \bigg(\int_{\partial P \cap \Omega'} \nu(v_P(x)) \, d\mathcal{H}^{d-1}(x)\bigg)(1+s/2)n^{d-1}\bigg].$$

We will construct a particular set $\Omega'$ satisfying $\Omega \subset \Omega' \subset \mathcal{V}_2(\Omega, \delta_0)$. In the previous section we have associated to each couple $(x, r)$ in $\Gamma \times {]0, r_\Gamma[}$ a hypercube $Q(x, r)$ centered at $x$, of sidelength $r$, and which is transverse to $\Gamma$. Using exactly the same method, we can build a family of hypercubes

$$\big(Q'(x, r), x \in \Gamma, r < r_{(\Gamma, P)}\big)$$

such that $Q'(x, r)$ is centered at $x$, of sidelength $r$ and it is transverse to $\Gamma$ and $\partial P$. The family

$$(\mathring{Q}'(x, r), x \in \Gamma, r < \min(r_{(\Gamma, P)}, \delta_0/(2d)))$$

is a covering of the compact set $\Gamma$, thus we can extract a finite covering from this collection; we denote it by $(\mathring{Q}'(x_i, r_i), i \in J)$. We define

$$\Omega' = \Omega \cup \bigcup_{i \in J} \mathring{Q}'(x_i, r_i).$$

Since $r_i \leq \delta_0/(2d)$ for all $i \in J$, we have $\Omega' \subset \mathcal{V}_2(\Omega, \delta_0)$. Moreover, $\partial P$ is transverse to the boundary $\Gamma'$ of $\Omega'$. Finally, if we define

$$\delta_1 = \min_{i \in J} r_i/2,$$

we know that $\mathcal{V}_2(\Omega, \delta_1) \subset \Omega'$ and thus, for all $n \geq 2d/\delta_1$, we have $\Omega_n \subset \Omega'$.

**7. Existence of a family of $(\Gamma_n^1, \Gamma_n^2)$-cuts.** In this section we prove that we can construct a family of disjoint $(\Gamma_n^1, \Gamma_n^2)$-cuts in $\Omega_n$. Let $\zeta$ be a fixed constant larger than $2d$. We consider a parameter $h < h_0 = d(\partial P, \Gamma^1 \cup \Gamma^2)$. For $k \in \{0, \ldots, \lfloor hn/\zeta \rfloor\}$ we define

$$P(k) = \{x \in \mathbb{R}^d \mid d(x, P) \leq k\zeta/n\}$$

and for $k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}$ we define

$$\mathcal{U}(k) = \overbrace{(\mathbb{R}^d \setminus P_{k+1})}^{\circ} \setminus \mathring{P}_k$$

$$= \{x \in \mathbb{R}^d \mid k\zeta/n \leq d(x, P) < (k+1)\zeta/n\}$$

and $\mathcal{M}'(k) = \mathcal{U}(k) \cap \Omega'$ (see Figure 4). We will prove the following lemma.

LEMMA 2. *There exists $N$ large enough such that for all $n \geq N$, every path on the graph $(\mathbb{Z}_n^d, \mathbb{E}_n^d)$ from $\Gamma_n^1$ to $\Gamma_n^2$ in $\Omega_n$ contains at least one edge which is included in the set $\mathcal{M}'(k)$ for $k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}$.*

This lemma states precisely that for all $k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}$, $\mathcal{M}'(k)$ contains a $(\Gamma_n^1, \Gamma_n^2)$-cut in $\Omega_n$.

PROOF OF LEMMA 2. Let $k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}$. Let $\gamma$ be a discrete path from $\Gamma_n^1$ to $\Gamma_n^2$ in $\Omega_n$. In particular, $\gamma$ is continuous, so we can parametrize it: $\gamma = (\gamma_t)_{0 \leq t \leq 1}$. There exists $N$ large enough such that for all $n \geq N$, we have

$$\Omega_n \subset \Omega', \qquad \Gamma_n^1 \subset \mathcal{V}_2(\Gamma^1, 2d/n) \subset \mathring{P}_k$$
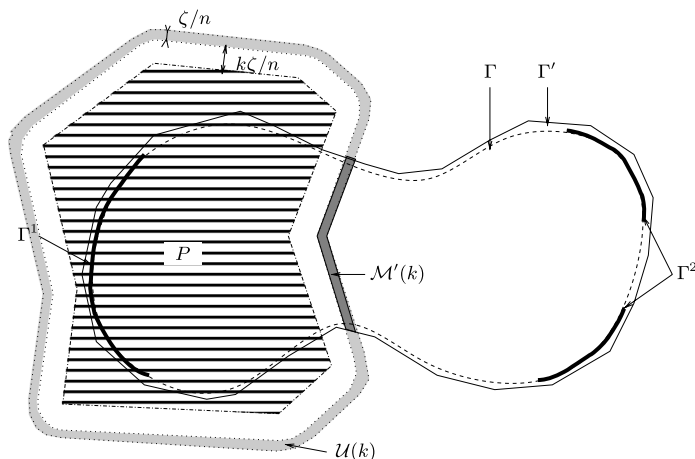
FIG. 4. *The sets $P$, $\mathcal{U}(k)$ and $\mathcal{M}'(k)$.*

and

$$\Gamma_n^2 \subset \mathcal{V}_2(\Gamma^2, 2d/n) \subset \overbrace{\mathbb{R}^d \setminus \overset{\circ}{P_{k+1}}}.$$

Since $\gamma$ is continuous, we know that there exists $t_1, t_2 \in ]0, 1[$ such that

$$t_1 = \sup\{t \in [0, 1] \mid \gamma_t \in \overset{\circ}{P_k}\},$$

$$t_2 = \inf\{t \geq t_1 \mid \gamma_t \in \overbrace{\mathbb{R}^d \setminus \overset{\circ}{P_{k+1}}}\}.$$

Since

$$\overset{\circ}{P_k} \cup \mathcal{U}(k) \cup \overbrace{\mathbb{R}^d \setminus \overset{\circ}{P_{k+1}}}$$

is a partition of $\mathbb{R}^d$, we know that $(\gamma_t)_{t_1 \leq t < t_2}$, which is a continuous path, is included in $\mathcal{U}(k)$. The length of $(\gamma_t)_{t_1 \leq t < t_2}$ is larger than $d(\gamma_{t_1}, \gamma_{t_2})$. The segment $[\gamma_{t_1}, \gamma_{t_2}]$ intersects

$$\{x \in \mathbb{R}^d \mid d(x, P) = (k + 1/2)\zeta/n\}$$

at a point $z$ and we know that

$$\mathcal{V}_2(z, \zeta/(2n)) \subset \overset{\circ}{\widetilde{V(k)}}.$$

Thus, $d(\gamma_{t_1}, \gamma_{t_2}) \geq \zeta/n$ and then the length of $(\gamma_t)_{t_1 \leq t < t_2}$ is larger than $\zeta/n$. Finally, $\gamma$ is composed of edges of length $1/n$ and $\zeta \geq 2d$ so $(\gamma_t)_{t_1 \leq t < t_2}$ and, thus, $\gamma$ contains at least one edge which is included in $\mathcal{U}(k)$. Noticing that for all $n \geq N$,

$$\gamma \subset \Omega_n \subset \Omega',$$

we obtain that this edge belongs to $\mathcal{U}(k) \cap \Omega' = \mathcal{M}'(k)$. $\quad \square$

**8. Covering of $\partial P \cap \Omega'$ by cylinders.** From now on we only consider $n \geq N$. According to Lemma 2, we know that each set $\mathcal{M}'(k)$ for $k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}$ contains a $(\Gamma_n^1, \Gamma_n^2)$-cut in $\Omega_n$, thus, if we denote by $M'(k)$ the set of the edges included in $\mathcal{M}'(k)$, we obtain

$$\phi_n \leq \min\{V(M'(k)), k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}\}.$$

However, we do not have estimates on $V(M'(k))$ that allow us to control $\phi_n$ using only the previous inequality. What we can use are the upper large deviations for the maximal flow from the top to the bottom of a cylinder (Theorem 8). In this section, we will transform our family of cuts $(M'(k))$ by replacing a huge part of the edges in each $\mathcal{M}'(k)$ by the edges of minimal cutsets in cylinders.

We denote by $H_i$, $i = 1, \ldots, \mathcal{N}$, the intersection of the faces of $\partial P$ with $\Omega'$. For each $i = 1, \ldots, \mathcal{N}$, we denote by $v_i$ the exterior normal unit vector to $P$ along $H_i$. We will cover $\partial P \cap \Omega'$ by cylinders except a surface of $\mathcal{H}^{d-1}$ measure controlled by a parameter $\varepsilon$. To explain the construction of a cutset we will do with a huge number of cylinders, we present first the simpler construction of a cutset using one cylinder. Let $R$ be a hyperrectangle that is included in $H_j$ for a $j \in \{1, \ldots, \mathcal{N}\}$ and let $B$ be the cylinder defined by

$$B = \{x + t v_j \mid x \in R, t \in [0, h]\},$$

where $h \leq h_0$ is the same parameter as previously. The cylinder $B$ is built on $\partial P \cap \Omega'$, in $\mathbb{R}^d \setminus \overset{\circ}{P}$. We recall that $h_0 = d(\partial P, \Gamma^1 \cup \Gamma^2) > 0$, so we know that $d(B, \Gamma^1 \cup \Gamma^2) > 0$. We denote by $E_a$ the set of the edges included in

$$\mathcal{E}_a = \{x + t v_j \mid x \in R, d(x, \partial R) < \zeta/n, t \in [0, h]\}.$$

The set $\mathcal{E}_a$ is a neighborhood in $B$ of the "vertical" faces of $B$, that is, the faces of $B$ that are collinear to $v_j$. We denote by $E_b$ a set of edges in $B$ that cuts the top $R + h v_j$ from the bottom $R$ of $B$. Let $M'(k)$ be the set of the edges included in $\mathcal{M}'(k)$, for a $k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}$. Let $B'$ be the thinner cylinder

$$B' = \{x + t v_j \mid x \in R, d(x, \partial R) \geq \zeta/n, t \in [0, h]\}.$$

Thus, for all $k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}$, the set of edges

$$\left(M'(k) \cap (\mathbb{R}^d \setminus B')\right) \cup E_a \cup E_b$$

cuts $\Gamma_n^1$ from $\Gamma_n^2$ in $\Omega_n$. Indeed, the set of edges $M'(k)$ is already a cut between $\Gamma_n^1$ and $\Gamma_n^2$ in $\Omega_n$. We remove from it the edges that are inside $B'$ which is in the interior of $B$ and we add to it a cutset $E_b$ from the top to the bottom of $B$ and the set of edges $E_a$ that glue together $E_b$ and $M'(k) \cap (\mathbb{R}^d \setminus B')$. This property is illustrated in the Figure 5.

REMARK 3. In this figure, we have represented $E_b$ as a surface (so a path in dimension 2) that separates the top from the bottom of the cylinder to illustrate the
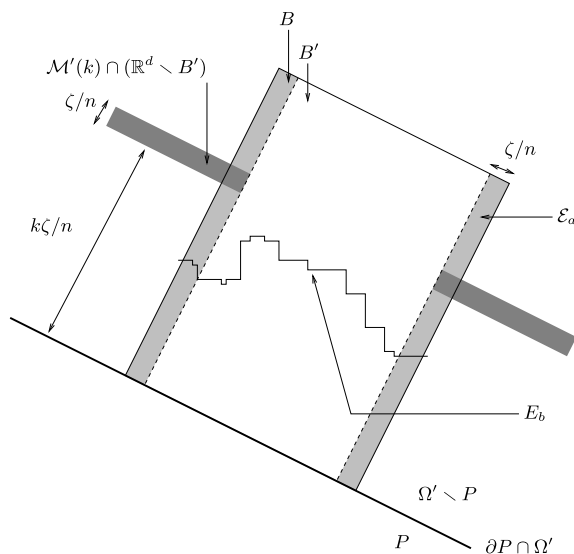
FIG. 5. *Construction of a $(\Gamma_n^1, \Gamma_n^2)$-cut in $\Omega_n$ using a cutset in a cylinder.*

fact that $E_b$ cuts all discrete paths from the bottom to the top of $B$. Actually, we can mention that it is possible to define an object which could be the dual of an edge in dimension $d \geq 2$ (as a generalization of the dual of a planar graph). This object is a plaquette, that is, a hypersquare of sidelength $1/n$ that is orthogonal to the edge and cuts it in its middle and whose sides are parallel to the hyperplanes of the axis. Then the dual of a cutset is a hypersurface of plaquettes, thus, Figure 5 is somehow intuitive.

We use exactly the same construction, but with a large number of cylinders, that will almost cover $\partial P \cap \Omega'$. We consider a fixed $\varepsilon > 0$. There exists a $l$ sufficiently small (depending on $F$, $P$ and $\varepsilon$) such that there exists a finite collection $(R_{i,j}, i = 1, \ldots, \mathcal{N}, j = 1, \ldots, N_i)$ of hypersquares of side $l$ of disjoint interiors satisfying $R_{i,j} \subset H_i$ for all $i \in \{1, \ldots, \mathcal{N}\}$ and $j \in \{1, \ldots, N_i\}$, and for all $i \in \{1, \ldots, \mathcal{N}\}$,

$$\{x \in H_i \mid d(x, \partial H_i) \geq \varepsilon \mathcal{H}^{d-2}(\partial H_i)^{-1} \mathcal{N}^{-1}\}$$

$$\subset \bigcup_{j=1}^{N_i} R_{i,j} \subset \{x \in H_i \mid d(x, \partial H_i) \geq \varepsilon \mathcal{H}^{d-2}(\partial H_i)^{-1} \mathcal{N}^{-1} 2^{-1}\}.$$

We immediately obtain that

$$\mathcal{H}^{d-1}\left((\partial P \cap \Omega') \setminus \bigcup_{i=1}^{\mathcal{N}} \bigcup_{j=1}^{N_i} R_{i,j}\right) \leq \varepsilon.$$

We remark that

$$\int_{\partial P \cap \Omega'} \nu(v_P(x)) \, d\mathcal{H}^{d-1}(x) \geq \sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i),$$

so that

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P}\left[\phi_n \geq (1 + s/2)n^{d-1} \sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i)\right].$$

Let $h < h_0$. For all $i \in \{1, \ldots, \mathcal{N}\}$ and $j \in \{1, \ldots, N_i\}$, we define

$$B_{i,j} = \{x + tv_i \mid x \in R_{i,j}, t \in [0, h]\}.$$

Since all the $B_{i,j}$ are at strictly positive distance of $\partial H_i$, there exists a positive $h_1$ such that for all $h < h_1$, the cylinders $B_{i,j}$ have pairwise disjoint interiors. We thus consider $h < \min(h_0, h_1)$ (see, e.g., Figure 6). At this point, we could define a neighborhood of the vertical faces of each cylinder $B_{i,j}$, and do the same construction as in the previous example with one cylinder. Actually, we need to choose a little bit more carefully the sets of edges we define along the vertical faces of the cylinders. We will not consider only each cylinder $B_{i,j}$, but also thinner versions of these cylinders of the type

$$B_{i,j}(k) = \{x + tv_j \mid x \in R_{i,j}, d(x, \partial R_{i,j}) > k\zeta/n, t \in [0, h]\}$$

for different values of $k$. We will then consider the edges included in a neighborhood of the vertical faces of each $B_{i,j}(k)$ [see the set $\mathcal{W}_{i,j}(k)$ above] and choose
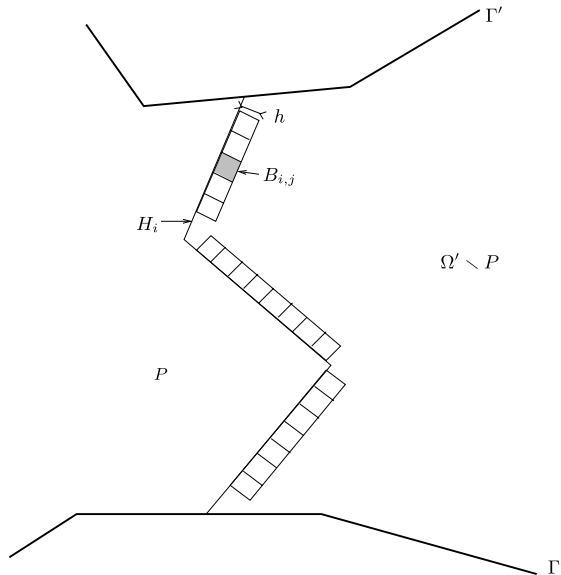


FIG. 6.   *Covering of $\partial P \cap \Omega'$ by cylinders.*

$k$ to minimize the capacity of the union over $i$ and $j$ of these edges. The reason why we need this optimization is also the reason why we built a family $(M'(k))$ of cutsets and not only one cutset from $\Gamma_n^1$ to $\Gamma_n^2$ in $\Omega_n$; we will try to explain it in Remark 4.

Here are the precise definitions of the sets of edges. We still consider the same constants $\zeta$ bigger than $2d$ and $h < \min(h_0, h_1)$. We define another positive constant $\eta$ that we will choose later (depending on $P$, $s$ and $\Omega$). For $i$ in $\{1, \ldots, \mathcal{N}\}$ and $j$ in $\{1, \ldots, N_i\}$ we recall the definition of $B_{i,j}$:

$$B_{i,j} = \{x + tv_i \mid x \in R_{i,j}, t \in [0, h]\}$$

and we define the following subsets of $\mathbb{R}^d$:

$$B'_{i,j} = \{x + tv_i \mid x \in R_{i,j}, d(x, \partial R_{i,j}) > \eta, t \in [0, h]\},$$

$$\forall k \in \{0, \ldots, \lfloor \eta n/\zeta - 1 \rfloor\}$$

$$\mathcal{W}_{i,j}(k) = \{x \in B_{i,j} \mid k\zeta/n \leq d_2(x, \partial R_{i,j} + \mathbb{R}v_i) < (k+1)\zeta/n\},$$

$$\forall k \in \{0, \ldots, \lfloor hn\kappa/\zeta - 1 \rfloor\}$$

$$\mathcal{M}(k) = \mathcal{M}'(k) \setminus \left( \bigcup_{i,j} B'_{i,j} \right)$$

(see Figures 7 and 8). We denote by $W_{i,j}(k)$ the set of the edges included in $\mathcal{W}_{i,j}(k)$ and we define $W(k) = \bigcup_{i,j} W_{i,j}(k)$. We also denote by $M(k)$ the edges included in $\mathcal{M}(k)$. Exactly as in the construction of a cutset with one cylinder, we obtain a cutset that is built with cutsets in each cylinders $B_{i,j}$. Indeed, if we denote by $E_{i,j}$ a set of edges that is a cutset from the top to the bottom of $B_{i,j}$
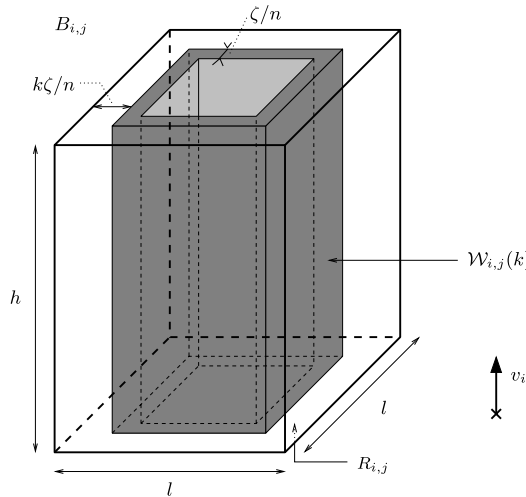
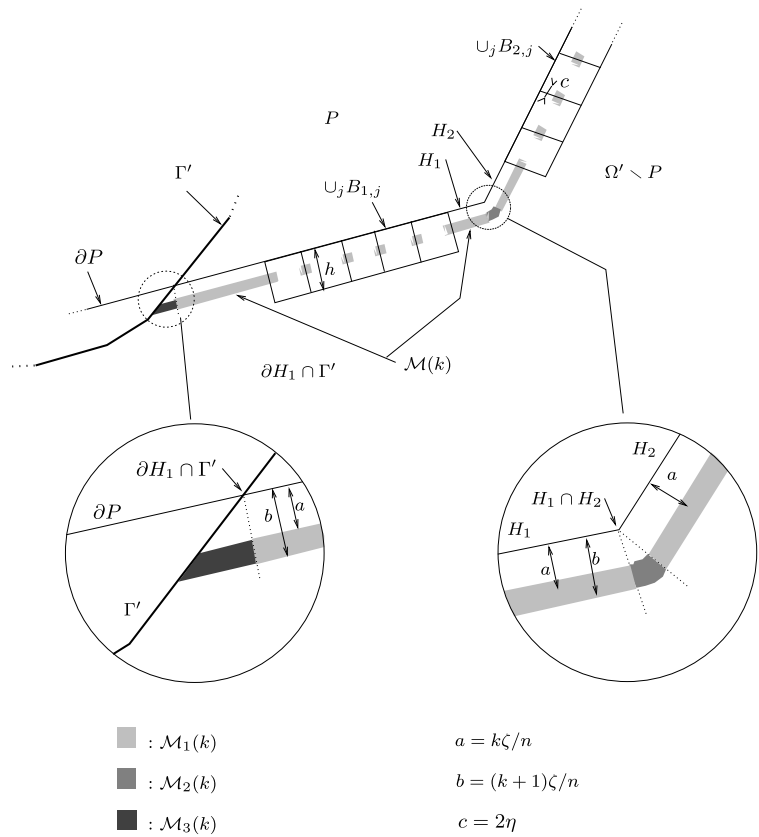

FIG. 7.  *The set $\mathcal{W}_{i,j}(k)$.*

FIG. 8.  *The set $\mathcal{M}(k)$.*

(oriented toward the direction given by $v_i$), then for each $k_1 \in \{0, \dots, \lfloor \eta n/\zeta - 1 \rfloor\}$ and $k_2 \in \{0, \dots, \lfloor hn/\zeta - 1 \rfloor\}$, the set of edges

$$\bigcup_{\substack{i=1,\dots,\mathcal{N} \\ j=1,\dots,N_i}} E_{i,j} \cup W(k_1) \cup M(k_2)$$

contains a cutset from $\Gamma_n^1$ to $\Gamma_n^2$ in $\Omega_n$. We deduce that

$$(5) \qquad \phi_n \leq \sum_{i,j} \phi_{B_{i,j}} + \min_{k_1} V(W(k_1)) + \min_{k_2} V(M(k_2)).$$

**9. Control of the cardinality of the sets of edges $W$ and $M$.**  For the sake of clarity, we do not recall the sets in which the parameters take its values; we always assume that they are the following: $i \in \{1, \dots, \mathcal{N}\}$, $j \in \{1, \dots, N_i\}$, $k_1 \in \{0, \dots, \lfloor \eta n/\zeta - 1 \rfloor\}$ and $k_2 \in \{0, \dots, \lfloor hn/\zeta - 1 \rfloor\}$. We have to evaluate the number

of edges in the sets $W(k_1)$ and $M(k_2)$ to control the terms $\min_{k_1} V(W(k_1))$ and $\min_{k_2} V(M(k_2))$ in (5). There exist constants $c_1(d, \Omega)$, $c_2(P, d, \Omega)$ such that

$$\operatorname{card} W(k_1) \leq c_1 \frac{\mathcal{H}^{d-1}(\partial P \cap \Omega')}{l^{d-1}} \zeta l^{d-2} h n^{d-1} \leq c_2 l^{-1} h n^{d-1}.$$

The cardinality of $M(k_2)$ is a little bit more complicated to control. We will divide $M(k)$ [resp., $\mathcal{M}(k)$] into three parts: $M(k) \subset M_1(k) \cup M_2(k) \cup M_3(k)$ [resp., $\mathcal{M}(k) \subset \mathcal{M}_1(k) \cup \mathcal{M}_2(k) \cup \mathcal{M}_3(k)$], that are represented in Figure 8.

We define $R'_{i,j} = \{x \in R_{i,j} \mid d(x, \partial R_{i,j}) > \eta\}$ which is the basis of $B'_{i,j}$. The set $\mathcal{M}_1(k)$ is a translation of the sets $H_i \setminus (\bigcup_{j=1}^{N_i} R'_{i,j})$ along the direction given by $v_i$ enlarged with a thickness $\zeta/(n\kappa)$,

$$\mathcal{M}_1(k) \subset \bigcup_{i=1}^{\mathcal{N}} \left\{ x + t v_i \mid x \in H_i \setminus \left( \bigcup_{j=1}^{N_i} R'_{i,j} \right), t \in [k\zeta/n, (k+1)\zeta/n[ \right\}.$$

Here we have an inclusion and not an equality because $\mathcal{M}_1(k)$ can be a truncated version of this set (truncated at the junction between the translates of two different faces). Since we know that

$$\mathcal{H}^{d-1} \left( (\partial P \cap \Omega') \setminus \bigcup_{i=1}^{\mathcal{N}} \bigcup_{j=1}^{N_i} R_{i,j} \right) \leq \varepsilon$$

and

$$\mathcal{H}^{d-1} \left( \bigcup_{i=1}^{\mathcal{N}} \bigcup_{j=1}^{N_i} (R_{i,j} \setminus R'_{i,j}) \right) \leq \frac{\mathcal{H}^{d-1}(\partial P \cap \Omega')}{l^{d-1}} l^{d-2} \eta = \mathcal{H}^{d-1}(\partial P \cap \Omega') l^{-1} \eta,$$

we have the following bound on the cardinality of $M_1(k)$:

$$\operatorname{card}(M_1(k)) \leq c_3 (\varepsilon + l^{-1} \eta) n^{d-1}$$

for a constant $c_3(d, P, \Omega, \Omega')$.

The part $M_2(k)$ corresponds to the edges included in the "bends" of the neighborhood of $\partial P$ located around the boundary of the faces of $\partial P$ in $\Omega'$, denoted by $\mathcal{M}_2(k)$, that is,

$$\mathcal{M}_2(k) \subset \bigcup_{i,j} (\mathcal{V}_2(H_i \cap H_j, (k+1)\zeta/n) \setminus \mathcal{V}_2(H_i \cap H_j, k\zeta/n))$$

and there exists a constant $c_4(d, P, \Omega')$ such that

$$\operatorname{card} M_2(k) \leq c_4 |k\zeta/n|^{d-2} n^{d-1} \leq c_4 h^{d-2} n^{d-1}.$$

The last part $\mathcal{M}_3(k)$ corresponds to the part of $\mathcal{M}(k)$ that is near the boundary $\Gamma'$ of $\Omega'$. Indeed, $\Gamma'$ is not orthogonal to $\partial P$, thus, for some $k$, the set $\mathcal{M}(k)$ may

contain edges that are not included in

$$\bigcup_{i=1}^{\mathcal{N}}\left\{x+tv_i \mid x \in H_i \setminus \left(\bigcup_{j=1}^{N_i} R'_{i,j}\right), t \in [k\zeta/n, (k+1)\zeta/n[\right\},$$

nor in

$$\bigcup_{i,j}(\mathcal{V}_2(H_i \cap H_j, (k+1)\zeta/n) \setminus \mathcal{V}_2(H_i \cap H_j, k\zeta/n))$$

(see Figure 8). However, $\mathcal{M}(k) \subset \mathcal{U}(k)$, the problem is to evaluate the difference of cardinality between the different $M(k)$ due to the intersection of $\mathcal{U}(k)$ with $\Omega'$. We have constructed $\Omega'$ such that $\Gamma'$ is transverse to $\partial P$ precisely to obtain this control. The sets $\Gamma'$ and $\partial P$ are polyhedral surfaces which are transverse. We denote by $(\mathcal{H}_i, i \in I)$ [resp., $(\mathcal{H}'_j, j \in J)$] the hyperplanes that contain $\partial P$ (resp., $\Gamma'$) and by $v_i$ (resp., $v'_j$) the exterior normal unit vector to $P$ along $\mathcal{H}_i$ (resp., $\Omega'$ along $\mathcal{H}'_j$). The set $\Gamma' \cap \partial P$ is included in the union of a finite number of intersections $\mathcal{H}_i \cap \mathcal{H}'_j$ of transverse hyperplanes. To each such intersection $\mathcal{H}_i \cap \mathcal{H}'_j$, we can associate the angles between $v_i$ and $v'_j$ and between $v_i$ and $-v'_j$ in the plane of dimension 2 spanned by $v_i$ and $v'_j$. Each such angle is strictly positive because $\mathcal{H}_i$ is transverse to $\mathcal{H}'_j$ and so the minimum $\theta_0$ over the finite number of defined angles is strictly positive. This $\theta_0$ and the measure $\mathcal{H}^{d-2}(\partial P \cap \Gamma')$ give to us a control on the volume of $\mathcal{M}_3(k)$ and, thus, on $\mathrm{card}(M_3(k))$, as soon as these sets belong to a neighborhood of $\partial P \cap \Gamma'$ (see Figure 9). Thus, there exist $h_2(\Omega', P) > 0$ and a constant $c_5(d, P, \Omega, \Omega')$ such that for all $h \le h_2$,

$$\mathrm{card}(M_3)(k) = c_5 h n^{d-1}.$$



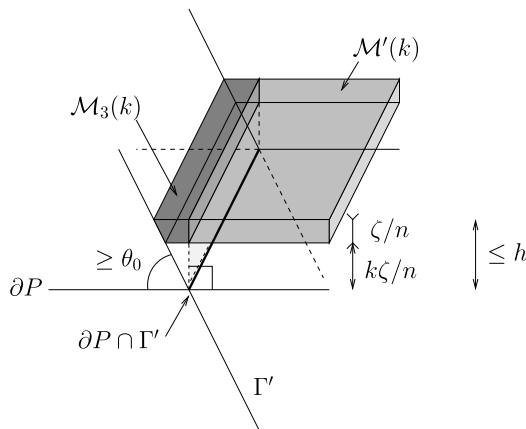FIG. 9.    *The set $\mathcal{M}_3(k)$.*

We conclude that there exists a positive constant $c_6(d, P, \Omega, \Omega')$ such that

$$\operatorname{card} M(k) \leq c_6(\varepsilon + l^{-1}\eta + h^{d-2} + h)n^{d-1}.$$

**10. Calibration of the constants.** We remark that the sets $W(k)$ [resp., the sets $M(k)$] are pairwise disjoint for different $k$. Then we obtain that

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}]$$

$$\leq \mathbb{P}\left[\phi_n \geq (1 + s/2)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i)\right]$$

$$\leq \mathbb{P}\left[\sum_{i=1}^{\mathcal{N}}\sum_{j=1}^{N_i} \phi_{B_{i,j}} \geq (1 + s/4)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i)\right]$$

$$+ \mathbb{P}\left[\min_{k_1} V(W(k_1)) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i)\right]$$

$$+ \mathbb{P}\left[\min_{k_2} V(M(k_2)) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i)\right]$$

$$\leq \sum_{i=1}^{\mathcal{N}}\sum_{j=1}^{N_i}\left(\max_{i,j} \mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1}\nu(v_i)(1 + s/4)n^{d-1}]\right)$$

$$+ \mathbb{P}\left[\sum_{i=1}^{\lceil c_2 l^{-1} h n^{d-1}\rceil} t(e_i) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i)\right]^{\lfloor \eta n/\zeta\rfloor}$$

$$+ \mathbb{P}\left[\sum_{i=1}^{\lceil c_6(\varepsilon + l^{-1}\eta + h^{d-2} + h)n^{d-1}\rceil} t(e_i) \geq (s/8)n^{d-1}\sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i)\right]^{2\lfloor hn/\zeta\rfloor}.$$

The terms

$$\mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1}\nu(v_i)(1 + s/4)n^{d-1}]$$

have already been studied in [14] (we recalled it as Theorem 8 in this paper).

It remains to study two terms of the type

$$\mathcal{P}(n) = \mathbb{P}\left(\sum_{i=1}^{\alpha n^{d-1}} t(e_i) \geq \beta n^{d-1}\right).$$

As soon as $\beta > \alpha\mathbb{E}(t)$ and the law of the capacity of the edges admits an exponential moment, the Cramér theorem in $\mathbb{R}$ allows us to affirm that

$$\limsup_{n\to\infty} \frac{1}{n^{d-1}} \log \mathcal{P}(n) < 0.$$

Moreover, for all

$$\varepsilon \le \varepsilon_0 = \frac{1}{2\nu_{\max}} \int_{\mathcal{P} \cap \Omega'} \nu(v_P(x)) \, d\mathcal{H}^{d-1}(x),$$

we have

$$\sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i) \ge \int_{\partial P \cap \Omega'} \nu(v_P(x)) \, d\mathcal{H}^{d-1}(x) - \varepsilon \nu_{\max}$$

$$\ge \frac{1}{2} \int_{\partial P \cap \Omega'} \nu(v_P(x)) \, d\mathcal{H}^{d-1}(x)$$

$$\ge \frac{\nu_{\min}}{2} \mathcal{H}^{d-1}(\partial P \cap \Omega').$$

Thus, for all $\varepsilon < \varepsilon_0$ and $h < \min(h_0, h_1, h_2)$, if the constants satisfy the two following conditions:

(6) $$c_2 l^{-1} h < \mathcal{H}^{d-1}(\partial P \cap \Omega') \nu_{\min} \mathbb{E}(t(e)) s / 16$$

and

(7) $$c_6(\varepsilon + l^{-1}\eta + h^{d-2} + h) < \mathcal{H}^{d-1}(\partial P \cap \Omega') \nu_{\min} \mathbb{E}(t(e)) s / 16,$$

thanks to Theorem 8 and the Cramér theorem in $\mathbb{R}$, we obtain that

$$\limsup_{n \to \infty} \frac{1}{n^d} \log \mathbb{P}[\phi_n \ge \lambda n^{d-1}] < 0$$

and Theorem 1 is proved. We claim that it is possible to choose the constants such that conditions (6) and (7) are satisfied. Indeed, we first choose $\varepsilon < \varepsilon_0$ such that

$$\varepsilon < \frac{1}{4} \frac{\mathcal{H}^{d-1}(\partial P \cap \Omega) \nu_{\min} \mathbb{E}(t(e)) s}{16 c_6}.$$

To this fixed $\varepsilon$ corresponds a $l$. Knowing $\varepsilon$ and $l$, we choose $h \le \min(h_0, h_1, h_2)$ and $\eta$ such that

$$\max(h, h^{d-2}, l^{-1}h, l^{-1}\eta) < \frac{1}{4} \frac{\mathcal{H}^{d-1}(\partial P \cap \Omega') \nu_{\min} \mathbb{E}(t(e)) s}{16 \max(c_2, c_6)}.$$

The proof of Theorem 1 is complete.

REMARK 4. We try here to explain why we built several sets $W(k_1)$ and $M(k_2)$, and not only one couple of such sets, that would have been sufficient to construct a cutset from $\Gamma_n^1$ to $\Gamma_n^2$ in $\Omega_n$. To use estimates of upper large deviations of maximal flows in cylinder we already know, we want to compare $\phi_n$ with $\sum_{i,j} \phi_{B_{i,j}}$. Heuristically, to construct a $(\Gamma_n^1, \Gamma_n^2)$-cut in $\Omega_n$ from the union of cutsets in each cylinder $B_{i,j}$, we have to add edges to glue together the different cutsets at the common boundary of the small cylinders and to extend these cutsets

to $(\partial P \cap \Omega_n) \setminus \bigcup_{i=1}^{\mathcal{N}} \bigcup_{j=1}^{N_i} R_{i,j}$. Yet we want to prove that the upper large deviations of $\phi_n$ are of volume order. If we only consider one possible set $E$ of edges such that

$$\phi_n \leq \sum_{i,j} \phi_{B_{i,j}} + V(E),$$

we will obtain that

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \sum_{i,j} \mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1} \nu(v_i)(1+s/4)n^{d-1}]$$

$$+ \mathbb{P}\left[V(E) \geq n^{d-1} \sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i)s/4\right].$$

We can choose such a set $E$ so that it contains less than $\delta n^{d-1}$ edges for a small $\delta$ [e.g., $E$ is equal to $W(k_1) \cup M(k_2)$ for a fixed couple $(k_1, k_2)$] but the probability

$$\mathbb{P}\left[\sum_{i=1}^{\delta n^{d-1}} t(e_i) \geq C n^{d-1}\right]$$

does not decay exponentially fast with $n^d$ in general. To obtain this speed of decay, we have to make an optimization over the possible choices of the set $E$, that is, we choose $E$ among a set of $C'n$ possible disjoint sets of edges $E_1, \ldots, E_{C'n}$; in this case, we obtain that

$$\phi_n \leq \sum_{i,j} \phi_{B_{i,j}} + \min_{k=1,\ldots,C'n} V(E_k)$$

and so

$$\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \sum_{i,j} \mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1} \nu(v_i)(1+s/4)n^{d-1}]$$

(8)

$$+ \prod_{k=1}^{C'n} \mathbb{P}\left[V(E_k) \geq n^{d-1} \sum_{i=1}^{\mathcal{N}} N_i l^{d-1} \nu(v_i)s/4\right].$$

It is then sufficient to prove that for all $k$, $\mathbb{P}[V(E_k) \geq C''n^{d-1}]$ decays exponentially fast with $n^{d-1}$ to conclude that the last term in (8) decays exponentially fast with $n^d$. Theorem 8 gives a control on the terms

$$\mathbb{P}[\phi_{B_{i,j}} \geq l^{d-1} \nu(v_i)(1+s/4)n^{d-1}].$$

The conclusion is that to obtain the volume order of the upper large deviations, the optimization over the different possible values of $k_1$ and $k_2$ is really important, even if it is not needed if we only want to prove that $\mathbb{P}(\phi_n \geq \lambda n^{d-1})$ goes to zero when $n$ goes to infinity.

## REFERENCES

[1] BOLLOBÁS, B. (1979). *Graph Theory*: *An Introductory Course*. *Graduate Texts in Mathematics* **63**. Springer, New York. MR0536131

[2] CERF, R. (2006). *The Wulff Crystal in Ising and Percolation Models*. *Lecture Notes in Math*. **1878**. Springer, Berlin. MR2241754

[3] CERF, R. and THÉRET, M. (2011). Law of large numbers for the maximal flow through a domain of $\mathbb{R}^d$ in first passage percolation. *Trans. Amer. Math. Soc.* **363** 3665–3702.

[4] CERF, R. and THÉRET, M. (2011). Lower large deviations for the maximal flow through a domain of $\mathbb{R}^d$ in first passage percolation. *Probab. Theory Related Fields*. To appear. Available at arxiv.org/abs/0907.5501.

[5] FEDERER, H. (1969). *Geometric Measure Theory*. *Die Grundlehren der Mathematischen Wissenschaften* **153**. Springer, New York. MR0257325

[6] GARET, O. (2009). Capacitive flows on a 2D random set. *Ann. Appl. Probab.* **19** 641–660. MR2521883

[7] GRIMMETT, G. and KESTEN, H. (1984). First-passage percolation, network flows and electrical resistances. *Z. Wahrsch. Verw. Gebiete* **66** 335–366. MR0751574

[8] KESTEN, H. (1986). Aspects of first passage percolation. In *École D'été de Probabilités de Saint-Flour, XIV—1984*. *Lecture Notes in Math*. **1180** 125–264. Springer, Berlin. MR0876084

[9] KESTEN, H. (1987). Surfaces with minimal random weights and maximal flows: A higher-dimensional version of first-passage percolation. *Illinois J. Math*. **31** 99–166. MR0869483

[10] LANG, S. (1985). *Differential Manifolds*, 2nd ed. Springer, New York. MR0772023

[11] ROSSIGNOL, R. and THÉRET, M. (2010). Law of large numbers for the maximal flow through tilted cylinders in two-dimensional first passage percolation. *Stochastic Process. Appl*. **120** 873–900. MR2610330

[12] ROSSIGNOL, R. and THÉRET, M. (2010). Lower large deviations and laws of large numbers for maximal flows through a box in first passage percolation. *Ann. Inst. H. Poincaré Probab. Statist*. **46** 1093–1131. MR2744888

[13] THÉRET, M. (2007). Upper large deviations for the maximal flow in first-passage percolation. *Stochastic Process. Appl*. **117** 1208–1233. MR2343936

[14] THÉRET, M. (2009). Upper large deviations for maximal flows through a tilted cylinder. *ESAIM Probab. Stat.* To appear. Available at arxiv.org/abs/0907.0614.

[15] ZHANG, Y. (2000). Critical behavior for maximal flows on the cubic lattice. *J. Stat. Phys*. **98** 799–811. MR1749233

[16] ZHANG, Y. (2007). Limit theorems for maximum flows on a lattice. Available at arxiv.org/abs/0710.4589.

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