The Annals of Applied Probability
2011, Vol. 21, No. 4, 1282–1321
DOI: 10.1214/10-AAP717
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SMOOTHNESS AND ASYMPTOTIC ESTIMATES OF DENSITIES FOR SDES WITH LOCALLY SMOOTH COEFFICIENTS AND APPLICATIONS TO SQUARE ROOT-TYPE DIFFUSIONS

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We study smoothness of densities for the solutions of SDEs whose coefficients are smooth and nondegenerate only on an open domain D. We prove that a smooth density exists on D and give upper bounds for this density. Under some additional conditions (mainly dealing with the growth of the coefficients and their derivatives), we formulate upper bounds that are suitable to obtain asymptotic estimates of the density for large values of the state variable ("tail" estimates). These results specify and extend some results by Kusuoka and Stroock [*J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **32** (1985) 1–76], but our approach is substantially different and based on a technique to estimate the Fourier transform inspired from Fournier [*Electron. J. Probab.* **13** (2008) 135–156] and Bally [Integration by parts formula for locally smooth laws and applications to equations with jumps I (2007) The Royal Swedish Academy of Sciences]. This study is motivated by existing models for financial securities which rely on SDEs with non-Lipschitz coefficients. Indeed, we apply our results to a square root-type diffusion (CIR or CEV) with coefficients depending on the state variable, that is, a situation where standard techniques for density estimation based on Malliavin calculus do not apply. We establish the existence of a smooth density, for which we give exponential estimates and study the behavior at the origin (the singular point).

1. Introduction.

It is well known that Malliavin calculus is a tool which allows, among other, to prove that the law of a diffusion process admits a smooth density. More precisely, if one assumes that the coefficients of an SDE are bounded C^∞ functions with bounded derivatives of any order and that, on the other hand, the Hormandër condition holds, then the solution of the equation is a smooth functional in Malliavin's sense, and it is nondegenerate at any fixed positive time. Then the general criterion given by Malliavin [15] allows one to say that the law of such a random variable is absolutely continuous with respect to the Lebesgue measure, and its density is a smooth function (see [16] for a general presentation of this topic).

Received April 2009; revised May 2010.

[1]Supported by the European Science Foundation (ESF) within the framework of the ESF activity entitled "Advanced Mathematical Methods for Finance" for the first semester of 2009.

MSC2010 subject classifications. Primary 60J35; secondary 60H10, 60H07, 60J60, 60J70.

Key words and phrases. Smoothness of densities, stochastic differential equations, locally smooth coefficients, tail estimates, Malliavin calculus, square-root process.

The aim of this paper is to relax the aforementioned conditions on the coefficients: roughly speaking, we assume that the coefficients are smooth only on an open domain D and have bounded partial derivatives therein. Moreover, we assume that the nondegeneracy condition on the diffusion coefficient holds true on D only. Under these assumptions, we prove that the law of a strong solution to the equation admits a smooth density on D (Theorem 2.1). Furthermore, when D is the complementary of a compact ball and the coefficients satisfy some additional assumptions on D (mainly dealing with their growth and the one of derivatives), we give upper bounds for the density for large values of the state variable (Theorem 2.2). We will occasionally refer to these aymptotic estimates of the density as "tail estimates" or estimates on the density's "tails."

Local results have already been obtained by Kusuoka and Stroock in [13], Section 4. Here the authors work under local regularity and nondegeneracy hypotheses too, but the bounds they provide on the density are mostly significant on the diagonal (i.e., close to starting point) and in the small time limit, while they are not appropriate for tail estimates. Moreover, the constants appearing in the estimates are not explicit (cf. (4.7)–(4.9) in Theorem 4.5 and the corresponding estimates in Corollary 4.10, [13]). In the present paper, we provide upper bounds that are suitable for tail estimates, and we find out the explicit dependence of the bounding constants with respect to the coefficients of the SDE and their derivatives. Our bounds turn out to be applicable to the case of diffusions with tails stronger than gaussian. This is the case for square-root diffusions, which are our major example of interest (see Section 3). Also, our approach is substantially different from the one in [13]. In particular, we rely on a Fourier transform argument, employing a technique to estimate the Fourier transform of the process inspired from the work of Fournier in [8] and of Bally in [2] and relying on specifically-designed Malliavin calculus techniques. We estimate the density $p_t(y)$ of the diffusion at a point $y \in D$ performing an integration by parts that involves the contribution of the Brownian noise only on an arbitrarily small time interval $[t-\delta, t]$. This allows us to gain a free parameter δ that we can eventually optimize, and the appropriate choice of δ proves to be a key point in our argument. We do not study here the regularity with respect to initial condition (which may be the subject of future work).

Our study is motivated by applications to Finance, in particular by the study of models for financial securities which rely on SDEs with non-Lipschitz coefficients. As it is well known, a celebrated process with square-root diffusion coefficient was proposed by Cox, Ingersoll and Ross in [5] as a model for short interest rates and was later employed by Heston in [10] to model the stochastic volatilities of assets. The stochastic-$\alpha\beta\rho$ or SABR model in [9] is based on the following mixing of local and stochastic volatility dynamics:

$$\begin{cases} dX_t = \sigma_t X_t^{\beta}\, dW_t^1, \\ d\sigma_t = \nu\sigma_t\, dB_t, \qquad \sigma_0 = \alpha, \end{cases}$$

where $0 < \beta < 1$, $B_t = \rho W_t^1 + \sqrt{1-\rho^2} W_t^2$, (W^1, W^2) is a standard Brownian motion, and $\rho \in [-1, 1]$ is the correlation parameter. In this paper, we apply our results to one-dimensional SDEs of the form

$$X_t = x + \int_0^t \big(a(X_s) - b(X_s)X_s\big)\,ds + \int_0^t \gamma(X_s)X_s^\alpha\,dW_t, \tag{1.1}$$

where $\alpha \in [1/2, 1)$ and a, b and γ are C_b^∞ functions. When the coefficients a, b, σ are constant, the solutions to this class of equations include the classical CIR process ($\alpha = 1/2$) and a subclass of the CEV local volatility diffusions (when $a = 0$ and $b = -r$). As pointed out by Bossy and Diop in [4], SDEs with square-root terms and coefficients depending on the level of the state variable arise as well in the modeling of turbulent flows in fluid mechanics. It is well known that for a CIR process, the density of X_t is known explicitly. The main contribution of our results lies in the fact that they apply to the more general framework of SDEs whose coefficients are functions of the state variable, thus when explicit computations are no longer possible. Theorem 2.2 directly applies to (1.1) and allows one to show that X_t admits a smooth density on $(0, +\infty)$: under some additional conditions on the coefficients (mainly dealing with their asymptotic behavior at ∞ and zero), we give exponential-type upper bounds for the density at infinity (Proposition 3.3) and study the explosive behavior of the density at zero (Proposition 3.4). The explicit expression of the density for the classical CIR process shows that our estimates are in the good range.

The paper is organized follows: in Section 2 we present our main results on SDEs with locally smooth coefficients (Section 2.1), and we collect all the technical elements we need to give their proofs. In particular, in Section 2.2 we recall the basic tools of Malliavin calculus on the Wiener space, which will be used in Section 2.3 to obtain some explicit estimates of the L^2-norms of the weights involved in the integration by parts formula. This is done following some standard techniques of estimation of Sobolev norms and inverse moments of the determinant of the Malliavin matrix (as in [16] and [6], Section 4), but in our computations we explicitly pop out the dependence with respect to the coefficients of the SDE and their derivatives. This further allows us to obtain the explicit asymptotic estimates on the density. Section 2.4 is devoted to the proof of the theorems stated in Section 2.1. We employ the Fourier transform argument and the optimized integration by parts we have discussed above. Finally, in Section 3 we apply our results to the solutions of (1.1).

2. Smoothness and tail estimates of densities for SDEs with locally smooth coefficients.

2.1. *Main results.* In what follows, b and σ_j are measurable functions from $\mathbb{R}^m$ into $\mathbb{R}^m$, $j = 1, \dots, d$. For $y_0 \in \mathbb{R}^m$ and $R > 0$, we denote by $B_R(y_0)$ [resp., $\overline{B}_R(y_0)$] the open (resp., closed) ball $B_R(y_0) = \{y \in \mathbb{R}^m : |y - y_0| < R\}$ [resp.,

$\overline{B}_R(y_0) = \{y \in \mathbb{R}^m : |y - y_0| \le R\}]$, where $|\cdot|$ stands for the Euclidean norm. We follow the usual notation denoting $C_b^\infty(A)$ the class of infinitely differentiable functions on the open set $A \subseteq \mathbb{R}^m$ which are bounded together with their partial derivatives of any order. For a multi-index $\alpha \in \{1, \dots, m\}^k$, $k \ge 1$, ∂_α denotes the partial derivative $\frac{\partial^k}{\partial_{x_{\alpha_1}} \cdots \partial_{x_{\alpha_k}}}$.

Let $0 < R \le 1$ and $y_0 \in \mathbb{R}^m$ be given. We consider the SDE

$$X_t^i = x^i + \int_0^t b^i(X_s)\, ds + \sum_{j=1}^d \int_0^t \sigma_j^i(X_s)\, dW_s^j, \tag{2.1}$$
$$t \in [0, T], i = 1, \dots, m,$$

for a finite $T > 0$ and $x \in \mathbb{R}^m$, and assume that the following hold:

(H1) (*local smoothness*) $b, \sigma_j \in C_b^\infty(B_{5R}(y_0); \mathbb{R}^m)$;
(H2) (*local ellipticity*) $\sigma\sigma^*(y) \ge c_{y_0,R} I_m$ for every $y \in B_{3R}(y_0)$, for some $0 < c_{y_0,R} < 1$;
(H3) existence of strong solutions holds for the couple (b, σ).

Let then $(X_t; t \in [0, T])$ denote a strong solution of (2.1). Our first main result follows.

THEOREM 2.1. *Assume* (H1), (H2) *and* (H3). *Then for any initial condition $x \in \mathbb{R}^m$ and any $0 < t \le T$, the random vector X_t admits an infinitely differentiable density p_{t,y_0} on $B_R(y_0)$. Furthermore, for any integer $k \ge 3$ there exists a positive constant Λ_k depending also on y_0, R, T, m, d and on the coefficients of* (2.1) *such that, setting*

$$P_t(y) = \mathbb{P}\big(\inf\{|X_s - y| : s \in [(t-1) \vee t/2, t]\} \le 3R\big),$$

then one has

$$p_{t,y_0}(y) \le P_t(y_0)\left(1 + \frac{1}{t^{m3/2}}\right)\Lambda_3 \tag{2.2}$$

for any $y \in B_R(y_0)$. Analogously, for every $\alpha \in \{1, \dots, m\}^k$, $k \ge 1$,

$$|\partial_\alpha p_{t,y_0}(y)| \le P_t(y_0)\left(1 + \frac{1}{t^{m(2k+3)/2}}\right)\Lambda_{2k+3} \tag{2.3}$$

for every $y \in B_R(y_0)$.

The functional dependence of Λ_k with respect to y_0, R, T and to the bounds on the coefficients b and σ is known explicitly. We provide the expression of Λ_k in Section 2.4 in a more detailed version of Theorem 2.1 (Theorem 2.4) which we do not give here for the simplicity of notation.

When the coefficients of (2.1) are smooth outside a compact ball and have polynomial growth together with their derivatives therein, according to Theorem 2.1 a smooth density exists outside the same compact set, and one can deduce some more easily-read bounds on the tails. More precisely, we consider the following assumptions:

(H1′) There exist $\eta \geq 0$ such that b, σ_j are of class C^∞ on $\mathbb{R}^m \setminus \overline{B}_\eta(0)$, and (H2) holds for any $R > 0$ and y_0 such that $\overline{B}_{3R}(y_0) \subset \mathbb{R}^m \setminus \overline{B}_\eta(0)$;

(H4) there exist $q, \overline{q} > 0$, and positive constants $0 < C_0 < 1$ and C_k, $k \geq 1$, such that for any $\alpha \in \{1, \dots, m\}^k$

$$|\partial_\alpha b^i(y)| + |\partial_\alpha \sigma^i_j(y)| \leq C_k(1 + |y|^q) \tag{2.4}$$

and

$$\sigma\sigma^*(y) \geq C_0 |y|^{-\overline{q}} I_m \tag{2.5}$$

hold for $|y| > \eta$.

THEOREM 2.2. *Assume* (H1′) *and* (H3).

(a) *For any initial condition $x \in \mathbb{R}^m$ and for any $0 < t \leq T$, X_t admits a smooth density on $\mathbb{R}^m \setminus \overline{B}_\eta(0)$.*

(b) *Assume* (H4) *as well. Then estimates* (2.2) *and* (2.3) *hold with $R = 1$ and*

$$\Lambda_k = \Lambda_k(y_0) := C_{k,T}\big(1 + |y_0|^{q'_k(q)}\big), \tag{2.6}$$

for every $|y_0| > \eta + 5$. The value of the exponent $q'_k(q)$ is explicitly known (*and provided in Theorem* 2.5).

(c) *If moreover $\sup_{0 \leq s \leq t} |X_s|$ has finite moments of all orders, then for every $p > 0$ and $k \geq 1$ there exist positive constants $C_{k,p,T}$ such that*

$$\begin{aligned} |p_t(y)| &\leq C_{3,p,T}\left(1 + \frac{1}{t^{m3/2}}\right)|y|^{-p}, \\ |\partial_\alpha p_t(y)| &\leq C_{k,p,T}\left(1 + \frac{1}{t^{m(2k+3)/2}}\right)|y|^{-p}, \qquad \alpha \in \{1, \dots, m\}^k, \end{aligned} \tag{2.7}$$

for every $0 < t \leq T$ and every $|y| > \eta + 5$.

In the above, the $C_{k,p,T}$ are positive constants depending on k, p, T and also on m, d and on the bounds (2.4) *and* (2.5) *on the coefficients.*

The proofs of these results will be given in Section 2.4.

Notation. Through the rest of the paper, $\langle \cdot, \cdot \rangle$ will denote the Euclidean scalar product in $\mathbb{R}^m$, while the notation $|\cdot|$ will be used both for the absolute value of real numbers and for the Euclidean norm in $\mathbb{R}^m$. Furthermore, when $\Theta = \theta_1, \dots, \theta_\nu$ is a family of parameters, unless differently specified by C_Θ, we denote a constant depending on the θ_i's but not on any of the other existing variables. All constants of such a type may vary from line to line, but always depend only on the θ_i's. For functions of one variable, the kth derivative will be denoted by $\frac{d^k f}{dx^k}$ or $f^{(k)}$. We will follow the convention of summation over repeated indexes, wherever present.

2.2. *Elements of Malliavin calculus.* We recall hereafter some elements of Malliavin calculus on the Wiener space, following [16].

Let $W = (W_t^1, \dots, W_t^d; t \geq 0)$ be a d-dimensional Brownian motion defined on the canonical space $(\Omega, \mathcal{F}, \mathbb{P})$. For fixed $T > 0$, let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = L^2([0, T]; \mathbb{R}^d)$. For any $h \in \mathcal{H}$ we set $W(h) = \sum_{j=1}^d \int_0^T h^j(s)\, dW_s^j$, and consider the family $\mathcal{S} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ of smooth random variables defined by

$$\mathcal{S} = \{F : F = f(W(h_1), \dots, W(h_n)); h_1, \dots, h_n \in \mathcal{H}; f \in C_{\mathrm{pol}}^\infty(\mathbb{R}^n); n \geq 1\},$$

where C_{pol}^∞ denotes the class of C^∞ functions which have polynomial growth together with their derivatives of any order.

The Malliavin derivative of $F \in \mathcal{S}$ is the d-dimensional stochastic process $DF = (D_r^1 F, \dots, D_r^d F; r \in [0, T])$ defined by

$$D_r^j F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n)) h_i^j(r), \qquad j = 1, \dots, d.$$

For any positive integer k, the kth order derivative of F is obtained by iterating the derivative operator: for any multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, d\}^k$ and $(r_1, \dots, r_k) \in [0, T]^k$, we set $D_{r_1, \dots, r_k}^{\alpha_1, \dots, \alpha_k} F := D_{r_1}^{\alpha_1} \cdots D_{r_k}^{\alpha_k} F$. Given $p \geq 1$ and positive integer k, for every $F \in \mathcal{S}$ we define the seminorm

$$\|F\|_{k,p} = \left(\mathbb{E}[|F|^p] + \sum_{h=1}^k \mathbb{E}[\|D^{(h)} F\|_{\mathcal{H}^{\otimes h}}^p]\right)^{1/p},$$

where

$$\|D^{(k)} F\|_{\mathcal{H}^{\otimes k}} = \left(\sum_{|\alpha|=k} \int_{[0,T]^k} |D_{r_1, \dots, r_k}^{\alpha_1, \dots, \alpha_k} F|^2\, dr_1 \cdots dr_k\right)^{1/2},$$

and the sum is taken over all the multi-indexes $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, d\}^k$. We denote with $\mathbb{D}^{k,p}$ the completion of $\mathcal{S}$ with respect to the seminorms $\|\cdot\|_{k,p}$, and we set $\mathbb{D}^\infty = \bigcap_{p \geq 1} \bigcap_{k \geq 1} \mathbb{D}^{k,p}$. We may occasionally refer to $\|F\|_{k,p}$ as the stochastic Sobolev norm of F.

In a similar way, for any separable Hilbert space V we can define the analogous spaces $\mathbb{D}^{k,p}(V)$ and $\mathbb{D}^\infty(V)$ of V-valued random variables with the corresponding $\|\cdot\|_{k,p,V}$ semi-norms (the smooth functionals being now of the form

$F = \sum_{j=1}^{n} F_j v_j$, where $F_j \in \mathcal{S}$ and $v_j \in V$). In particular, for any $\mathbb{R}^d$-valued process $(u_s; s \le t)$ such that $u_s \in \mathbb{D}^{k,p}$ for all $s \in [0,t]$ and

$$\|u\|_{\mathcal{H}} + \sum_{h=1}^{k} \|D^{(h)}u\|_{\mathcal{H}^{\otimes h+1}} < \infty, \qquad \mathbb{P}\text{-a.s.},$$

we have

$$\|u\|_{k,p,\mathcal{H}} = \left(\mathbb{E}[\|u\|_{\mathcal{H}}^p] + \sum_{h=1}^{k} \mathbb{E}[\|D^{(h)}u\|_{\mathcal{H}^{\otimes h+1}}^p]\right)^{1/p}.$$

Finally, we denote by δ the adjoint operator of D.

One of the main applications of Malliavin calculus consists of showing that the law of a nondegenerate random vector $F = (F^1, \dots, F^m) \in (\mathbb{D}^\infty)^m$ admits an infinitely differentiable density. The property of nondegeneracy, understood in the sense of the Malliavin covariance matrix, is introduced in the following:

DEFINITION 2.1. A random vector $F = (F^1, \dots, F^m) \in (\mathbb{D}^\infty)^m$, $m \ge 1$, is said to be nondegenerate if its Malliavin covariance matrix σ_F, defined by

$$(\sigma_F)_{i,j} = \langle DF^i, DF^j \rangle_{\mathcal{H}}, \qquad i,j = 1, \dots, m,$$

is invertible a.s. and moreover

$$\mathbb{E}[\det(\sigma_F)^{-p}] < \infty$$

for all $p \ge 1$.

The key tool to prove smoothness of the density for a nondegenerate random vector is the following integration by parts formula (cf. [16]).

PROPOSITION 2.1. *Let $F = (F^1, \dots, F^m) \in (\mathbb{D}^\infty)^m$, $m \ge 1$, be a nondegenerate random vector. Let $G \in \mathbb{D}^\infty$ and $\phi \in C_{\text{pol}}^\infty(\mathbb{R}^m)$. Then for any $k \ge 1$ and any multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, m\}^k$ there exists a random variable $H_\alpha(F,G) \in \mathbb{D}^\infty$ such that*

$$\mathbb{E}[\partial_\alpha \phi(F) G] = \mathbb{E}[\phi(F) H_\alpha(F,G)], \tag{2.8}$$

where the $H_\alpha(F,G)$ are recursively defined by

$$H_\alpha(F,G) = H_{(\alpha_k)}\big(F, H_{(\alpha_1,\dots,\alpha_{k-1})}(F,G)\big),$$

$$H_{(i)}(F,G) = \sum_{j=1}^{m} \delta\big(G(\sigma_F^{-1})_{i,j} DF^j\big).$$

2.3. *Explicit bounds on integration by parts formula for diffusion processes.* The notation of this section is somehow cumbersome, as we try to keep our bounds as general and as accurate as possible. The framework will nevertheless considerably simplify in Section 2.4, when we will give the proofs of the results stated in Section 2.1.

Throughout this section, $X = (X_t; t \geq 0)$ will denote the unique strong solution of the SDE

$$(2.9)\qquad X_t^i = x^i + \int_0^t B^i(X_s)\,ds + \sum_{j=1}^d \int_0^t A_j^i(X_s)\,dW_s^j, \qquad t \geq 0, i = 1, \dots, m,$$

where $x \in \mathbb{R}^m$ and $B^i, A_j^i \in \mathcal{C}_b^\infty(\mathbb{R}^m)$ for all $i = 1, \dots, m$ and $j = 1, \dots, d$. We assume that the diffusion matrix A satisfies the following ellipticity condition at starting point x:

(E) $A(x)A(x)^* \geq c_* I_m$, for some $c_* > 0$, where $\cdot^*$ stays for matrix transposition. Without loss of generality, we will suppose $c_* < 1$.

We recall that the first-variation process of X is the matrix-valued process

$$(Y_t)_{i,j} = \frac{\partial X_t^i}{\partial x_j}, \qquad i, j = 1, \dots, m,$$

which satisfies the following equation, written in matrix form:

$$dY_t = I_m + \int_0^t \partial B(X_s) Y_s\,ds + \sum_{l=1}^d \int_0^t \partial A_l(X_s) Y_s\,dW_s^l,$$

where ∂B and ∂A_l are, respectively, the $m \times m$ matrices of components $(\partial B)_{i,j} = \partial_j B^i$ and $(\partial A_l)_{i,j} = \partial_j A_l^i$. By means of Itô's formula, one shows that Y_t is invertible and that the inverse $Z_t := Y_t^{-1}$ satisfies the equation

$$(2.10)\qquad \begin{aligned} Z_t = I_m &- \int_0^t Z_s \left\{ \partial B(X_s) - \sum_{l=1}^d (\partial A_l(X_s))^2 \right\} ds \\ &- \sum_{l=1}^d \int_0^t Z_s\,\partial A_l(X_s)\,dW_s^l. \end{aligned}$$

Additional notation. For $k \geq 0$, we define

$$(2.11)\qquad \begin{aligned} |B|_k &= 1 + \sum_{i=1}^m \sum_{0 \leq |\alpha| \leq k} \sup_{x \in \mathbb{R}^m} |\partial_\alpha B^i(x)|, \\ |A|_k &= 1 + \sum_{i,j} \sum_{0 \leq |\alpha| \leq k} \sup_{x \in \mathbb{R}^m} |\partial_\alpha A_j^i(x)|, \end{aligned}$$

where $|\alpha|$ is the length of the multi-index α. Then, for $p \geq 1$ and $t \geq 0$ we set

$$e_p(t) := e^{t^{p/2}(t^{1/2}|B|_1+|A|_1)^p} \tag{2.12}$$

and

$$e_p^Z(t) := e^{t^{p/2}(t^{1/2}(|B|_1+|A|_1^2)+|A|_1)^p}. \tag{2.13}$$

The constants in (2.12) and (2.13) naturally arise when estimating the moments of the random variables X_t, Y_t and Z_t. Indeed, the results given in the following proposition can be easily obtained from (2.9) and (2.10) applying Burkholder's inequality and Gronwall's lemma.

PROPOSITION 2.2. *For every $p > 1$ there exists a positive constant $C_{p,m}$ depending on p and m but not on the bounds on B and A and their derivatives such that, for every $0 \leq s \leq t \leq T$,*

$$\text{(i)} \quad \mathbb{E}\Big[\sup_{s \leq r \leq t} |X_r^i - X_s^i|^p\Big] \leq C_{p,m}(t-s)^{p/2}\big((t-s)^{1/2}|B|_0 + |A|_0\big)^p, \tag{2.14}$$

$$\text{(ii)} \quad \sup_{s \leq t} \mathbb{E}[|(Z_s)_{i,j}|^p] \leq C_{p,m} e_p^Z(t)^{C_{p,m}} \tag{2.15}$$

for all $i, j = 1, \ldots, m$.

For any $t > 0$, the iterated Malliavin derivative of X_t is the solution of a linear SDE. The coefficients of this equation are bounded, and hence it is once again a straightforward application of Gronwall's lemma to show that the random variables $D_{r_1,\ldots,r_k}^{\alpha_1,\ldots,\alpha_k} X_t$ have moments of any order which are finite and uniformly bounded in $r_1, \ldots, r_k$. This is indeed the content of [16], Theorem 2.2.2. The following lemma highlights the explicit constants appearing in the estimates of the L^p-norms of the iterated derivative, expressing them in terms of the bounds (2.11) on A and B.

LEMMA 2.1. *For every $k \geq 1$ and every $p > 1$ there exist a positive integer $\gamma_{k,p}$ and a positive constant $C_{k,p}$ depending on k, p but not on the bounds on B and A and their derivatives such that, for any $t > 0$,*

$$\begin{aligned} &\sup_{r_1,\ldots,r_k \leq t} \mathbb{E}[|D_{r_1,\ldots,r_k}^{j_1,\ldots,j_k} X_t^i|^p] \\ &\quad \leq C_{k,p}|A|_{k-1}^{kp}(t^{1/2}|B|_k + |A|_k)^{(k+1)^2 p} e_p(t)^{\gamma_{k,p}}, \end{aligned} \tag{2.16}$$

for all $i = 1, \ldots, m$ and $(j_1, \ldots, j_k) \in \{1, \ldots, d\}^k$.

The proof of this result is based on some standard but rather cumbersome computations; hence we leave it for Appendix A.1. We rather give hereafter the proof of some estimates which follow easily from Lemma 2.1 and will be useful in the following sections.

COROLLARY 1. *For any $k \geq 1$ and $p > 1$, there exists a positive constant $C_{k,p}$ depending only on k and p such that, for any $t > 0$,*

$$\text{(i)}\quad \mathbb{E}[\|D^{(k)}X_t^i\|_{\mathcal{H}^{\otimes k}}^p]^{1/p} \leq C_{k,p}t^{k/2}|A|_{k-1}^k(t^{1/2}|B|_k + |A|_k)^{(k+1)^2} \times e_p(t)^{\gamma_{k,p}}; \tag{2.17}$$

$$\text{(ii)}\quad \|\phi(X_t)\|_{k,p} \leq C_{k,p}|\phi|_k(1 + (t \vee t^k)^{1/2}) \times |A|_{k-1}^k(t^{1/2}|B|_k + |A|_k)^{(k+2)^2} e_p(t)^{k\gamma_{k,p}}, \tag{2.18}$$

where (i) *holds for $i = 1, \ldots, m$ and* (ii) *for any $\phi \in C^\infty(\mathbb{R}^m)$.*

PROOF. (i) Employing the definition of $\|\cdot\|_{\mathcal{H}^{\otimes k}}$ and Lemma 2.1, a simple computation holds.

$$\begin{aligned}
&\mathbb{E}[\|D^{(k)}X_t^i\|_{\mathcal{H}^{\otimes k}}^p]^{1/p} \\
&\quad \leq C_{k,p}\left\{t^{k(p/2-1)}\int_{[0,t]^k}\mathbb{E}\Big[\sup_{|\alpha|=k}|D_{r_1,\ldots,r_k}^{\alpha_1,\ldots,\alpha_k}X_t^i|^p\Big]dr_1\cdots dr_k\right\}^{1/p} \\
&\quad \leq C_{k,p}t^{k/2}|A|_{k-1}^k(t^{1/2}|B|_k + |A|_k)^{(k+1)^2}e_p(t)^{\gamma_{k,p}/p},
\end{aligned}$$

hence we get bound (2.17).

(ii) We start from the definition of $\|\cdot\|_{k,p}$ and write

$$\begin{aligned}
\|\phi(X_t)\|_{k,p} &= \left(\mathbb{E}[|\phi(X_t)|^p] + \sum_{h=1}^k \mathbb{E}[\|D^{(h)}\phi(X_t)\|_{\mathcal{H}^{\otimes h}}^p]\right)^{1/p} \\
&\leq \|\phi\|_0 + \sum_{h=1}^k \mathbb{E}[\|D^{(h)}\phi(X_t)\|_{\mathcal{H}^{\otimes h}}^p]^{1/p}.
\end{aligned} \tag{2.19}$$

Using the notation introduced in the proof Lemma 2.1, we have

$$D^{(h)}\phi(X_t) = \sum_{I_1,\ldots,I_\nu=\{1,\ldots,h\}} \partial_{k_1}\cdots\partial_{k_\nu}\phi(X_t)\prod_{l=1}^\nu D^{(\mathrm{card}(I_l))}X_t^{k_l},$$

where, with a slight abuse of notation, we have now written $D^{(h)}$ for the generic derivative of order h. Repeatedly applying Hölder's inequality for Sobolev norms and using bound (2.17), we get

$$\begin{aligned}
&\mathbb{E}[\|D^{(h)}\phi(X_t)\|_{\mathcal{H}^{\otimes h}}^p] \\
&\quad \leq c_{h,p}\sum_{\substack{h_1,\ldots,h_\nu=1,\ldots,h \\ h_1+\cdots+h_\nu=h}} \mathbb{E}\left[\left\|\partial_{k_1}\cdots\partial_{k_\nu}\phi(X_t)\prod_{l=1}^\nu D^{(h_l)}X_t^{k_l}\right\|_{\mathcal{H}^{\otimes h}}^p\right]
\end{aligned}$$

$$\leq c_{h,p}\|\phi\|_h^p \sum_{\substack{h_1,\dots,h_\nu=1,\dots,h \\ h_1+\cdots+h_\nu=h}} \sup_{i=1,\dots,m} \prod_{l=1}^{\nu} \mathbb{E}[\|D^{(h_l)}X_t^i\|_{\mathcal{H}^{\otimes h_l}}^{2^l p}]^{1/2^l}$$

$$\leq c_{h,p}\|\phi\|_h^p \{t^{h/2}|A|_{h-1}^h (t^{1/2}|B|_h + |A|_h)^{(h+2)^2}\}^p e_p(t)^{h\gamma_{h,p}}.$$

By means of this bound, from (2.19) we get the desired estimate when setting $C_{k,p} \geq \max\{c_{h,p} : h \leq k\}$. $\square$

We need a last preliminary result on the inverse moments of the determinant of the Malliavin covariance matrix of X_t. This result is again achieved with some standard arguments, but, as in Lemma 2.1, the next lemma finds out the explicit constants appearing in the estimate of the L^p-norms of $\det(\sigma_{X_t})^{-1}$.

LEMMA 2.2. *For every $p > 1$ and $t > 0$,*

$$\mathbb{E}[|\det\sigma_{X_t}|^{-p}]^{1/p} \leq C_{p,m,d}\, e^Z_{4(mp+1)}(t)^{C_{p,m,d}} K_m(t, c_*), \tag{2.20}$$

where

$$K_m(t,c_*) = 1 + \left(\frac{4}{tc_*}+1\right)^m + \frac{1}{c_*^{2(m+1)}}(t^{1/2}\|B\|_0\|A\|_2^3 + \|A\|_1^2)^{2(m+1)}$$

for some positive constant $C_{p,m,d}$ depending on p, m and d but not on the bounds on B and A and their derivatives.

The proof is once again postponed to Appendix A.1.

We now come to the main result of this section. We give an estimate of the L^2-norm of the random variables H_α involved in the integration by parts formula (2.8), when $F = X_t$. The proof follows the arguments of [6], proof of Lemma 4.11, but is given in the general setting of an integration by parts of order $k \in \mathbb{N}$, and moreover it takes advantage of the explicit bounds which have been obtained in Corollary 1 and Lemma 2.2.

We give this result employing some slightly more compact notation, defining

$$P_k(t) = t^{1/2}|B|_k + |A|_k,$$
$$P_k^A(t) = |A|_k P_{k+1}(t).$$

THEOREM 2.3. *For every $k \geq 1$ there exists a positive constant $C_k = C_{k,m,d}$ such that, for any multi-index $\alpha \in \{1,\dots,m\}^k$, any $G \in \mathbb{D}^\infty$ and $t > 0$,*

$$\|H_\alpha(X_t,G)\|_{0,2} \leq C_k\|G\|_{k,2^{k+1}}(t^{-k/2} \vee t^{k(k-1)/2})(t^m K_m(t,c_*))^{k(k+3)/2} \times (P_k^A)^{\phi_k}(e_8(t) \vee e_{2^{k+2}}(t))^{C_k}(e^Z_{32m+4}(t) \vee e^Z_{2^{k+4}m+4}(t))^{C_k}, \tag{2.21}$$

where $K_m(t,c_*)$ has been defined in Lemma 2.2, and

$$\phi_k = 3m(k+4)^2.$$

REMARK 2.1. Estimate (2.21) is rather involved. For our purposes, the most important elements are the dependence with respect to time of the factor $t^{-k/2} \vee t^{k(k-1)/2}$ and the coefficient P_k^A containing the bounds on the derivatives of the coefficients. We remark that the factor $t^m K_m(t, c_*)$ is bounded for t close to zero. Moreover, when $t < 1$, the factor $t^{-k/2} \vee t^{k(k-1)/2}$ reduces to $t^{-k/2}$.

PROOF OF THEOREM 2.3. We write $\sigma_t = \sigma_{X_t}$ for simplicity of notation. We first use the continuity of δ (see [6], Proposition 4.5) and Hölder's inequalities for Sobolev norms to obtain

$$\begin{aligned} &\|H_{(\alpha_1,\dots,\alpha_k)}(X_t, G)\|_{0,2} \\ &\qquad = \|H_{(\alpha_k)}(X_t, H_{(\alpha_1,\dots,\alpha_{k-1})}(X_t, G))\|_{0,2} \\ &\qquad = \left\| \sum_{j=1}^m \delta\big(H_{(\alpha_1,\dots,\alpha_{k-1})}(X_t, G)(\sigma_t^{-1})_{\alpha_k, j} DX_t^j\big) \right\|_{0,2} \\ &\qquad \le C_m \|H_{(\alpha_1,\dots,\alpha_{k-1})}(X_t, G)\|_{1,4} \sum_{j=1}^m \|(\sigma_t^{-1})_{\alpha_k, j}\|_{1,8} \|DX_t^j\|_{1,8,\mathcal{H}}. \end{aligned} \tag{2.22}$$

To estimate the last factor we can directly use the definition of $\|\cdot\|_{k,p,\mathcal{H}}$ and apply Corollary 1. The major part of the efforts in the rest of the proof will be targeted on the estimation of $\|(\sigma_t^{-1})_{i,j}\|_{k,p}$.

We claim that for any $k \ge 1$, $p > 1$ and for all $i, j = 1, \dots, m$,

$$\begin{aligned} \|(\sigma_t^{-1})_{i,j}\|_{k,p} &\le c_{k,p}(t^{-1} \vee t^{k/2-1})(t^m K_m(t, c_*))^{1+k} \\ &\qquad \times P_k^A(t)^{\phi_k' + 2(k+4)^2} e_p(t)^{c_{k,p}} e^Z_{4(mp+1)}(t)^{c_{k,p}}, \end{aligned} \tag{2.23}$$

where

$$\phi_k' = 2(k+1)(m-1),$$

and $c_{k,p}$ is a positive constant depending also on m, d but not on t and on the bounds on B and A and their derivatives. Iterating process (2.22) and repeatedly using estimates (2.23) and (2.17), one easily obtains the desired estimate

$$\begin{aligned} &\|H_{(\alpha_1,\dots,\alpha_k)}(X_t, G)\|_{0,2} \\ &\qquad \le C_{k,m,d} \|G\|_{k,2^{k+1}} (t^m K_m(t, c_*))^k \\ &\qquad\quad \times \prod_{h=1}^k (t^{-1} \vee t^{h/2-1})(t \vee t^h)^{1/2} (t^m K_m(t, c_*))^h \\ &\qquad\qquad \times P_k^A(t)^{k(\phi_k' + 2(k+4)^2 + (k+1)^2)} \\ &\qquad\quad \times \prod_{h=1}^k e_{2^{h+2}}(t)^{c_{h,m,d}} e^Z_{4(2^{h+2}m+1)}(t)^{c_{h,m,d}} \end{aligned}$$

$$\leq C_{k,m,d}\|G\|_{k,2^{k+1}}(t^m K_m(t,c_*))^{k(k+3)/2} \times \big(t^{-k/2}\vee t^{k(k-1)/2}\big)P_k^A(t)^{\phi_k} \times \big(e_8(t)\vee e_{2^{k+2}}(t)\big)^{C_{k,m,d}}\big(e^Z_{32m+4}(t)\vee e^Z_{2^{k+4}m+4}(t)\big)^{C_{k,m,d}}.$$

PROOF OF (2.23). We follow [6], proof of Lemma 4.11. We start from the definition of $\|\cdot\|_{k,p}$ and write

$$\|(\sigma_t^{-1})_{i,j}\|_{k,p}=\left(\mathbb{E}[|(\sigma_t^{-1})_{i,j}|^p]+\sum_{h=1}^k\mathbb{E}[\|D^{(h)}(\sigma_t^{-1})_{i,j}\|^p_{\mathcal{H}^{\otimes h}}]\right)^{1/p}. \tag{2.24}$$

For the first term, we simply use Cramer's formula for matrix inversion,

$$|(\sigma_t^{-1})_{i,j}|=(\det\sigma_t)^{-1}\sigma_t^{(i,j)},$$

where $\sigma_t^{(i,j)}$ denotes the (i,j) minor of σ_t. We then apply Hölder's inequality and bounds (2.17) and (2.20) and get

$$\begin{aligned}\mathbb{E}[|(\sigma_t^{-1})_{i,j}|^p] &\leq c^{(1)}_{p,m}\{\mathbb{E}[\det(\sigma_t)^{-2p}]\mathbb{E}[|\sigma_t^{(i,j)}|^{-2p}]\}^{1/2}\\ &\leq c^{(1)}_{p,m}\Big\{\mathbb{E}[\det(\sigma_t)^{-2p}]\mathbb{E}\Big[\sup_i\|DX^i_t\|_{\mathcal{H}}^{4(m-1)p}\Big]\Big\}^{1/2}\\ &\leq c^{(1)}_{p,m,d}t^{-p}(t^mK_m(t,c_*))^p\\ &\quad\times\{|A|_0(t^{1/2}|B|_1+|A|_1)^4\}^{2(m-1)p}\\ &\quad\times e_p(t)^{c^{(1)}_{p,m,d}}e^Z_{4(mp+1)}(t)^{c^{(1)}_{p,m,d}},\end{aligned} \tag{2.25}$$

where $K_m(t,c_*)$ is the constant defined in Lemma 2.2. To estimate the second term, as done in [6], proof of Lemma 4.11, we iterate the chain rule for D

$$D(\sigma_t^{-1})_{i,j}=-\sum_{a,b=1}^m(\sigma_t^{-1})_{i,a}D(\sigma_t)_{a,b}(\sigma_t^{-1})_{b,j}.$$

We take advantage of the notation introduced in the proof of Lemma 2.1 and for $(\beta_1,\dots,\beta_k)\in\{1,\dots,m\}^k$, $k\geq 1$, we write

$$\begin{aligned}&|D^{\beta_1,\dots,\beta_k}_{r_1,\dots,r_k}(\sigma_t^{-1})_{i,j}|\\ &\quad\leq\sum_{I_1\cup\cdots\cup I_\nu=\{1,\dots,k\}}\sum_{\substack{a_1,\dots,a_\nu=1\\ b_1,\dots,b_\nu=1}}^m|(\sigma_t^{-1})_{i,a_1}(\sigma_t^{-1})_{b1,a_2}\cdots\\ &\qquad\times(\sigma_t^{-1})_{b_{\nu-1},a_\nu}(\sigma_t^{-1})_{b_\nu,j}|\\ &\qquad\times|D^{\beta(I_1)}_{r(I_1)}(\sigma_t)_{a_1,b_1}\cdots D^{\beta(I_\nu)}_{r(I_\nu)}(\sigma_t)_{a_\nu,b_\nu}|.\end{aligned} \tag{2.26}$$

We repeatedly apply Hölder's inequality for Sobolev norms to (2.26) and get

$$\begin{aligned}
&\mathbb{E}[\|D^{(k)}(\sigma_t^{-1})_{i,j}\|^p_{\mathcal{H}^{\otimes k}}]\\
&\quad\le c^{(2)}_{k,p,m}\sum_{\substack{k_1,\dots,k_\nu=1,\dots,k\\ k_1+\cdots+k_\nu=k}}\Big\{\sup_{\substack{a,a_1,\dots,a_\nu=1,\dots,m\\ b,b_1,\dots,b_\nu=1,\dots,m}}\mathbb{E}[\|(\sigma_t^{-1})^{\nu+1}_{a,b}\\
&\qquad\qquad\times D^{(k_1)}(\sigma_t)_{a_1,b_1}\cdots\\
&\qquad\qquad\times D^{(k_\nu)}(\sigma_t)_{a_\nu,b_\nu}\|^p_{\mathcal{H}^{\otimes k}}]\Big\}\qquad (2.27)\\
&\quad\le c^{(2)}_{k,p,m}\\
&\qquad\times\sup_{\substack{k_1,\dots,k_\nu=1,\dots,k\\ k_1+\cdots+k_\nu=k}}\Big\{\sup_{a,b=1,\dots,m}\mathbb{E}[|(\sigma_t^{-1})_{a,b}|^{(\nu+1)p}]\\
&\qquad\qquad\times\prod_{l=1}^{\nu}\sup_{a_l,b_l=1,\dots,m}\mathbb{E}[\|D^{(k_l)}(\sigma_t)_{a_l,b_l}\|^{2^lp}_{\mathcal{H}^{\otimes k_l}}]^{1/2^l}\Big\},
\end{aligned}$$

where, as in the proof of Corollary 1, we have written $D^{(k_l)}$ for the generic derivative of order k_l. To estimate $D^{(k_l)}(\sigma_t)_{a_l,b_l}$ we use bound (2.17) and get

$$\begin{aligned}
&\mathbb{E}[\|D^{(k)}(\sigma_t)_{i,j}\|^p_{\mathcal{H}^{\otimes k}}]\\
&\quad\le\mathbb{E}\left[\left\|\sum_{h=0}^{k}\binom{k}{h}\int_0^t D^{(h)}D_sX_t^i\cdot D^{(k-h)}D_sX_t^j\right\|^p_{\mathcal{H}^{\otimes k}}\right]\\
&\quad\le c^{(3)}_{k,p}\sum_{h=0}^{k}\mathbb{E}[\|D^{(h)}DX_t^i\|^{2p}_{\mathcal{H}^{\otimes h+1}}]^{1/2}\qquad (2.28)\\
&\qquad\qquad\times\mathbb{E}[\|D^{(k-h)}DX_t^j\|^{2p}_{\mathcal{H}^{\otimes k-h+1}}]^{1/2}\\
&\quad\le c^{(3)}_{k,p}t^{(k/2+1)p}|A|_k^{(k+2)p}(t^{1/2}|B|_{k+1}+|A|_{k+1})^{2(k+2)^2p}e_p(t)^{2\gamma_{k+1,p}},
\end{aligned}$$

where we have once again applied Hölder's inequality for Sobolev norms in the second step.

Using (2.28) together with (2.27), bound (2.25) and (2.24) and observing that $t^mK_m(t)$ is greater than one for all the values of t, we finally obtain

$$\begin{aligned}
\|(\sigma_t^{-1})_{i,j}\|_{k,p}&\le c_{k,p}(t^{-1}\vee t^{k/2-1})(t^mK_m(t,c_*))^{1+k}\\
&\quad\times|A|_k^{\phi'_k+k(k+2)}(t^{1/2}|B|_{k+1}+|A|_{k+1})^{\phi'_k+2(k+4)^2}\\
&\quad\times e_p(t)^{c_{k,p}}e^Z_{4(mp+1)}(t)^{c_{k,p}},
\end{aligned}$$

for a positive constant $c_{k,p}$ depending also on m,d. Estimate (2.23) follows. □

2.4. *Proofs of Theorems* 2.1 *and* 2.2. We now come to the proof of the results stated in Section 2.1. We recall that an $\mathbb{R}^m$-valued random vector X is said to admit a density on an open set $A \in \mathbb{R}^m$ if $\mathcal{L}_X|_A$ possesses a density, $\mathcal{L}_X$ being the law of X. It is equivalent to say that

$$\mathbb{E}[f(X)] = \int_{\mathbb{R}} f(x)p(x)\,dx \tag{2.29}$$

holds for all $f \in C_b(\mathbb{R})$ such that $\mathrm{supp}(f) \subset A$, for some positive $p \in L^1(A)$.

We refer to the setting of Section 2.1. We recall that $X = (X_t; t \in [0, T])$ denotes a strong solution of

$$X_t^i = x^i + \int_0^t b^i(X_s)\,ds + \sum_{j=1}^d \int_0^t \sigma_j^i(X_s)\,dW_s^j, \qquad t \in [0, T], i = 1, \dots, m, \tag{2.30}$$

where b and σ satisfy the assumptions (H1)–(H3). For $k \geq 1$ and $f \in C^k(\mathbb{R}^m)$, we denote

$$|f|_{k,B_R(y_0)} = 1 + \sum_{|\alpha| \leq k} \sup_{x \in B_R(y_0)} |\partial_\alpha f(x)|, \tag{2.31}$$

where the sum is taken over all the multi-index $\alpha \in \{1, \dots, m\}^k$. Let us define the following "local" version of the constants appearing in the estimates of the previous section:

$$\begin{aligned}
P_k(t, y_0) &= t^{1/2}|b|_{k,B_{5R}(y_0)} + |\sigma|_{k,B_{5R}(y_0)}, \\
P_k^\sigma(t, y_0) &= |\sigma|_{k,B_{5R}(y_0)} P_{k+1}(t, y_0), \\
P_1^Z(t, y_0) &= t^{1/2}\big(|b|_{1,B_{5R}(y_0)} + |\sigma|_{1,B_{5R}(y_0)}^2\big) + |\sigma|_{1,B_{5R}(y_0)}, \\
P_m^C(t, y_0) &= \big(t^{1/2}|b|_{0,B_{5R}(y_0)}|\sigma|_{2,B_{5R}(y_0)}^3 + |\sigma|_{1,B_{5R}(y_0)}^2\big)^{2(m+1)}, \\
C_m(t, y_0) &= t^m + \frac{4^m}{c_{y_0}^{2(m+1)}}\big(1 + P_m^C(t, y_0)\big), \\
e_p(t, y_0) &= \exp(t^{p/2}P_1(t, y_0)^p), \\
e_p^Z(t, y_0) &= \exp(t^{p/2}P_1^Z(t, y_0)^p).
\end{aligned}$$

In order to prove Theorem 2.1, we simplify this rather heavy notation introducing a constant that contains the factors appearing in estimate (2.21) in Theorem 2.3 (recall the constant ϕ_k defined there)

$$\begin{aligned}
\Theta_k(t, y_0, \gamma) = C_m(t, y_0)^{mk(mk+3)/2} & P_{mk}^\sigma(t, y_0)^{\phi_{mk}+(mk+2)^2} \\
&\times \big(e_8(t, y_0) \vee e_{2^{mk+2}}(t, y_0)\big)^\gamma \\
&\times \big(e_{32m+4}^Z(t, y_0) \vee e_{2^{mk+4}m+4}^Z(t, y_0)\big)^\gamma.
\end{aligned}$$

As addressed in Section 2.1, the following theorem is a more detailed version of Theorem 2.1. In particular, it provides the explicit expression of the constant Λ_k appearing in estimates (2.2) and (2.3).

THEOREM 2.4. *Assume* (H1), (H2) *and* (H3). *Then, for any initial condition* $x \in R^m$ *and any* $0 < t \le T$, *the random vector* X_t *admits an infinitely differentiable density* p_{t,y_0} *on* $B_R(y_0)$. *Furthermore, for every* $k \ge 1$ *there exists a positive constant* $C_k = C_{k,m,d}$ *such that, setting*

$$\Lambda_k(t, y_0) = C_k R^{-mk}\big(P_0(t, y_0)^{mk} + \Theta_k(t, y_0, C_k)\big) \tag{2.32}$$

and

$$P_t(y) = \mathbb{P}\big(\inf\{|X_s - y| : s \in [(t-1) \vee t/2, t]\} \le 3R\big),$$

then one has

$$p_{t,y_0}(y) \le P_t(y_0)\left(1 + \frac{1}{t^{m3/2}}\right)\Lambda_3(t \wedge 1, y_0) \tag{2.33}$$

for every $y \in B_R(y_0)$. *Analogously, for any* $\alpha \in \{1, \dots, m\}^k$, $k \ge 1$,

$$|\partial_\alpha p_{t,y_0}(y)| \le P_t(y_0)\left(1 + \frac{1}{t^{m3/2}}\right)\Lambda_{2k+3}(t \wedge 1, y_0) \tag{2.34}$$

for every $y \in B_R(y_0)$.

To prove this result we rely on the following classical criterion for smoothness of laws based on a Fourier transform argument (cf. [16], Lemma 2.1.5).

PROPOSITION 2.3. *Let* μ *be a probability law on* $\mathbb{R}^m$, *and* $\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{i\langle \xi, y\rangle} \times \mu(dy)$ *its characteristic function. If* $\widehat{\mu}$ *is integrable, then* μ *is absolutely continuous w.r.t. the Lebesgue measure, and*

$$p(y) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{-i\langle \xi, y\rangle} \widehat{\mu}(\xi)\, d\xi \tag{2.35}$$

is a continuous version of its density. If moreover

$$\int_{\mathbb{R}^m} |\xi|^k |\widehat{\mu}(\xi)|\, d\xi < \infty \tag{2.36}$$

holds for any $k \in \mathbb{N}$, *then* p *is of class* C^∞ *and for any multi-index* $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, m\}^k$,

$$\partial_\alpha p(y) = (-i)^k \int_{\mathbb{R}} \left(\prod_{j=1}^k \xi^{\alpha_j}\right) e^{-i\langle \xi, y\rangle} \widehat{\mu}(\xi)\, d\xi.$$

PROOF OF THEOREM 2.4. *Step* 1 ("*localized*" *characteristic function*). Fix a t in $(0,T]$. Let $\phi_R \in C_b^\infty(\mathbb{R}^m)$ be such that $1_{B_R(0)} \le \phi_R \le 1_{B_{2R}(0)}$ and $|\phi_R|_k \le 2^k R^{-k}$. We first observe that if $m_0 = \mathbb{E}[\phi_R(X_t - y_0)]$ is zero, then it just follows that $p \equiv 0$ is a density for X_t on $B_R(y_0)$. Otherwise, we consider $\mathcal{L}_{t,y_0}$ the law on $\mathbb{R}^m$ such that

$$\int_{\mathbb{R}^m} f(y)\mathcal{L}_{t,y_0}(dy) = \frac{1}{m_0}\mathbb{E}[f(X_t)\phi_R(X_t - y_0)], \tag{2.37}$$

for all $f \in C_b(\mathbb{R}^m)$. If $\mathcal{L}_{t,y_0}$ possesses a density, say p'_{t,y_0}, it follows that $p_{t,y_0}(y) := m_0 p'_{t,y_0}$ is a density for X_t on $B_R(y_0)$. Indeed, for any $f \in C_b$ such that $\operatorname{supp}(f) \subset B_R(y_0)$, (2.37) implies

$$\begin{aligned}\int_{\mathbb{R}^m} f(y)p_{t,y_0}(y)\,dy &= \int_{\mathbb{R}^m} f(y)m_0 p'_{t,y_0}(y)\,dy\\ &= m_0\int_{\mathbb{R}^m} f(y)\mathcal{L}_{t,y_0}(dy)\\ &= \mathbb{E}[f(X_t)].\end{aligned}$$

If the characteristic function of $\mathcal{L}_{t,y_0}$

$$\widehat{p}_{t,y_0}(\xi) = \int_{\mathbb{R}^m} e^{i\langle\xi,y\rangle}\mathcal{L}_{t,y_0}(dy) = \frac{1}{m_0}\mathbb{E}\big[e^{i\langle\xi,X_t\rangle}\phi_R(X_t - y_0)\big]$$

is integrable, then by Proposition 2.3 $\mathcal{L}_{t,y_0}$ admits a density. Hence, we focus on the integrability of $\widehat{p}_{t,y_0}$; in particular, we show that condition (2.36) of Proposition 2.3 holds true for all $k \in \mathbb{N}$.

Moreover, the inversion formula (2.35) yields the representation for p_{t,y_0}

$$\begin{aligned}p_{t,y_0}(y) := m_0 p'_{t,y_0}(y) &= \frac{m_0}{(2\pi)^m}\int_{\mathbb{R}^m} e^{-i\langle\xi,y\rangle}\widehat{p}_{t,y_0}(\xi)\,d\xi\\ &= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{-i\langle\xi,y\rangle}\mathbb{E}\big[e^{i\langle\xi,X_t\rangle}\phi_R(X_t - y_0)\big]\,d\xi.\end{aligned} \tag{2.38}$$

Step 2 (*localization*). We define the coefficients

$$\begin{aligned}\overline{b}^i(y) &= b^i\big(\psi(y-y_0)\big),\\ \overline{\sigma}^i_j(y) &= \sigma^i_j\big(\psi(y-y_0)\big),\end{aligned} \tag{2.39}$$

where $\psi \in C^\infty(\mathbb{R}^m;\mathbb{R}^m)$ (a *truncation* function) is defined by

$$\psi(y) = \begin{cases} y, & \text{if } |y| \le 4R,\\ 5\dfrac{y}{|y|}, & \text{if } |y| \ge 5R,\end{cases}$$

and $\psi(y) \in \overline{B_{5R}}(0)$ for all $y \in \mathbb{R}^m$. ψ can be defined in such a way that, for all $i = 1,\dots,m$, $\|\psi^i\|_1 \le 1$ and $\|\psi^i\|_k \le 2^{k-2}R^{-(k-1)}$ for all $k \ge 2$. As a consequence of (H1), the $\overline{b}$ and $\overline{\sigma}$ defined in this way are C_b^∞-extensions of $b|_{B_{4R}(y_0)}$

and $\sigma|_{B_{4R}(y_0)}$. Furthermore, there exist constants $c = c_{k,m}$ such that

$$
\begin{aligned}
|\overline{b}^i|_k &\le c_{k,m} R^{-(k-1)} |b^i|_{k,B_{5R}(y_0)}, \\
|\overline{\sigma}^i_j|_k &\le c_{k,m} R^{-(k-1)} |\sigma^i_j|_{k,B_{5R}(y_0)}
\end{aligned} \tag{2.40}
$$

and by (H2), for any $y \in B_{3R}(y_0)$ the matrix $\overline{\sigma}(y)$ is elliptic

$$
\overline{\sigma}\,\overline{\sigma}^*(y) \ge c_{y_0,R} I_m, \qquad y \in B_{3R}(y_0). \tag{2.41}
$$

For $y \in \mathbb{R}^m$ we denote by $\overline{X}(y) = (\overline{X}_s(y); 0 \le s \le t)$ the unique strong solution of the equation

$$
\begin{aligned}
\overline{X}^i_s(y) = y^i + \int_0^s \overline{b}^i(\overline{X}_u(y))\,du + \sum_{j=1}^d \int_0^s \overline{\sigma}^i_j(\overline{X}_u(y))\,dW^j_u, \\
0 \le s \le t, i = 1, \dots, m.
\end{aligned} \tag{2.42}
$$

Let now $0 < \delta < t/2 \wedge 1$. We employ an up-down crossing argument to estimate the increments of X in the neighborhood of y_0 by replacing them with the increments of $\overline{X}$. More precisely, let $\nu = \nu_{t,\delta}$ and $\tau = \tau_{t,\delta}$ be the stopping times defined by

$$
\begin{aligned}
\nu_{t,\delta} &= \inf\{s \ge t - \delta : X_s \in B_{3R}(y_0)\}, \\
\tau_{t,\delta} &= \inf\{s \ge \nu_{t,\delta} : X_s \notin B_{4R}(y_0)\}
\end{aligned} \tag{2.43}
$$

and $\inf\{\varnothing\} = \infty$. Suppose that $\phi_R(X_t - y_0) > 0$, so that $X_t \in B_{2R}(y_0)$ and $\nu < t$. On this set, if $\nu > t - \delta$, then $|X_{t\wedge\tau} - X_\nu| \ge R$. This implies $|\overline{X}_{t\wedge\tau-\nu}(X_\nu) - X_\nu| = |X_{t\wedge\tau} - X_\nu| \ge R$. Here we are employing the fact that on the interval $[\nu, \tau]$, X stays in $B_{4R}(y_0)$, hence in the region where the truncated coefficients $\overline{b}, \overline{\sigma}$ coincide with the original ones b, σ. On this interval, both X and $\overline{X}$ satisfy (2.42) for which pathwise uniqueness holds; hence we can replace X by $\overline{X}$ and employ the flow property for $\overline{X}$. Notice that flow property may not hold true for X [due to possible lack of uniqueness for the couple (b, σ)], but it always does for $\overline{X}$.

Analogously, if $\nu = t - \delta$ and $\tau < t$, then $|X_\tau - X_\nu| = |\overline{X}_{\tau-\nu}(X_\nu) - X_\nu| \ge R$. In both cases, $\sup_{0\le s\le\delta}|\overline{X}_s(X_\nu) - X_\nu| \ge R$. Hence, we conclude that

$$
\begin{aligned}
\{\phi_R(X_t - y_0) > 0\} &= \{\phi_R(X_t - y_0) > 0, t - \delta = \nu < t < \tau\} \\
&\quad \cup \Big\{\phi_R(X_t - y_0) > 0, \sup_{0\le s\le\delta} |\overline{X}_s(X_\nu) - X_\nu| \ge R\Big\}
\end{aligned}
$$

and $\widehat{p}_{t,y_0}$ rewrites as

$$
\begin{aligned}
m_0 \widehat{p}_{t,y_0}(\xi) &= \mathbb{E}\big[e^{i\langle\xi,X_t\rangle} \phi_R(X_t - y_0) 1_{\{\phi_R(X_t-y_0)>0, \sup_{0\le s\le\delta} |\overline{X}_s(X_\nu)-X_\nu|\ge R\}}\big] \\
&\quad + \mathbb{E}\big[e^{i\langle\xi,X_t\rangle} \phi_R(X_t - y_0) 1_{\{\phi_R(X_t-y_0)>0, t-\delta=\nu<t<\tau\}}\big].
\end{aligned}
$$

We now claim that for all $q > 0$ the following estimate holds:

$$\mathbb{P}\Big(\phi_R(X_t - y_0) > 0, \sup_{0\le s\le\delta} |\overline{X}_s(X_\nu) - X_\nu| \ge R\Big) \le c_{q,m} R^{-q}\delta^{q/2} P_0(\delta, y_0)^q \mathbb{P}\Big(\inf_{t-\delta\le s\le t} |X_s - y_0| \le 3R\Big), \tag{2.44}$$

for some positive constant $c_{q,m}$. Estimate (2.44) will be proved later on.

On the other hand,

$$\begin{aligned} &\mathbb{E}[e^{i\langle\xi, X_t\rangle}\phi_R(X_t - y_0)1_{\{\phi_R(X_t-y_0)>0, t-\delta=\nu<t<\tau\}}] \\ &\quad = \mathbb{E}[\mathbb{E}[e^{i\langle\xi,\overline{X}_\delta(y)\rangle}\phi_R(\overline{X}_\delta(y) - y_0)|X_{t-\delta} = y]1_{\{t-\delta=\nu<t<\tau\}}] \\ &\quad \le \mathbb{P}(|X_{t-\delta} - y_0| < 3R) \sup_{y\in B_{3R}(y_0)} \big|\mathbb{E}[e^{i\langle\xi,\overline{X}_\delta(y)\rangle}\phi_R(\overline{X}_\delta(y) - y_0)]\big|. \end{aligned} \tag{2.45}$$

Step 3 (*integration by parts*). We apply integration by parts formula (2.8) to estimate the last term in (2.45). By (2.40), (2.41) and Lemma 2.2, $\overline{X}_\delta(y)$ is a smooth and nondegenerate random vector for any $\delta > 0$ and $y \in B_{3R}(y_0)$. Then, for a given $k \ge 1$ we define the multi-index

$$\alpha = (\underbrace{1, \dots, 1}_{k \text{ times}}, \dots, \underbrace{m, \dots, m}_{k \text{ times}}),$$

such that $|\alpha| = km$. Hence, recalling that $\partial_{x_k} e^{i\langle\xi,x\rangle} = i\xi^k e^{i\langle\xi,x\rangle}$,

$$\begin{aligned} &\big|\mathbb{E}[e^{i\langle\xi,\overline{X}_\delta(y)\rangle}\phi_R(\overline{X}_\delta(y) - y_0)]\big| \\ &\quad \le \frac{1}{\prod_{i=1}^m |\xi^i|^k} \big|\mathbb{E}[\partial_\alpha e^{i\langle\xi,\overline{X}_\delta(y)\rangle}\phi_R(\overline{X}_\delta(y) - y_0)]\big| \\ &\quad \le \frac{1}{\prod_{i=1}^m |\xi^i|^k} \mathbb{E}[|H_\alpha(\overline{X}_\delta(y), \phi_R(\overline{X}_\delta(y) - y_0))|], \end{aligned} \tag{2.46}$$

for any $y \in B_{3R}(y_0)$.

We need to separately estimate $\|\phi(\overline{X}_\delta(y) - y_0)\|_{|\alpha|,2^{|\alpha|+1}}$. By Corollary 1, this is given by

$$\begin{aligned} \|\phi(\overline{X}_\delta(y) - y_0)\|_{mk,2^{mk+1}} &\le c_{k,m} R^{-mk}(1 + \delta^{1/2})|\sigma|^{mk}_{mk-1,B_{5R}(y_0)} \\ &\qquad \times P_{mk}(y_0, \delta)^{(mk+2)^2} e_{2^{mk+1}}(\delta)^{c_{k,m}} \\ &\le c_{k,m} R^{-mk} P^\sigma_{mk}(y_0, \delta)^{(mk+2)^2} e_{2^{mk+1}}(\delta)^{c_{k,m}} \end{aligned}$$

for some positive constant $c_{k,m}$. Then, from (2.44), (2.45), (2.46) and Theorem 2.3 it follows that

$$m_0|\widehat{p}_{t,y_0}(\xi)| \le C_{k,q} P_R(\delta, t, y_0) I_{k,q}(\xi, \delta, y_0) \tag{2.47}$$

for some constant $C_{k,q}$ depending also on m and d, with

$$P_R(\delta, t, y_0) = \mathbb{P}\Big(\inf_{t-\delta\le s\le t} |X_s - y_0| \le 3R\Big)$$

and

$$I_{k,q}(\xi, \delta, y_0) = R^{-q}\delta^{q/2} P_0(\delta, y_0)^q + \frac{R^{-mk}}{\prod_{i=1}^m |\xi^i|^k}\delta^{-mk/2}\Theta_k(\delta, y_0, C_{k,q}).$$

Estimate (2.47) holds simultaneously for any $\xi \in \mathbb{R}^m$, $0 < \delta < t/2 \wedge 1$, $q > 0$ and $k \ge 1$. The constant $\Theta_k(\delta, y_0, C_{k,q})$ appears when applying estimate (2.21).

Step 4 (*optimization*). We show that for any ξ and any $l \ge 1$, δ can always be chosen in such a way that there exist q and k such that $I_{k,q}(\xi, \delta, y_0)$ goes to zero at ∞ faster than $(\prod_{i=1}^m |\xi^i|)^{-(l+2)}$.

Denoting $\|\xi\| = \prod_{i=1}^m |\xi^i|$, we set

$$\delta := \delta(\xi) = t/2 \wedge 1 \wedge \|\xi\|^{-a}$$

for some $a > 0$ that is to be identified hereafter. For this choice of δ,

$$P_R(\delta(\xi), t, y_0) \le \mathbb{P}\Big(\inf_{t/2\vee(t-1)\le s\le t} |X_s - y_0| \le 3R\Big) = P_t(y_0)$$

and

$$\begin{aligned} I_{k,q}(\xi, \delta(\xi), y_0) \le{} & R^{-q}\big(\|\xi\|^{-qa/2} \wedge (t\wedge 1)^{q/2}\big) P_0(t\wedge 1, y_0)^q \\ & + R^{-mk}\big(\|\xi\|^{-k(1-ma/2)} \vee \|\xi\|^{-k}(t\wedge 1)^{-mk/2}\big) \\ & \times \Theta_k(t\wedge 1, y_0, C_{k,q}), \end{aligned} \tag{2.48}$$

since $\delta \to P_0(\delta, y_0)$ and $\delta \to \Theta_k(\delta, y_0, C_{k,q})$ are increasing; hence $P_0(\delta(\xi), y_0) \le P_0(t\wedge 1, y_0)$ and the same holds for Θ_k.

We consider the leading terms determining the decay of $I_{k,q}(\xi, \delta(\xi), y_0)$ with respect to ξ and impose

$$\frac{qa}{2} = k\Big(1 - \frac{ma}{2}\Big). \tag{2.49}$$

Setting $a = 1/m$, (2.49) yields $q = mk$, hence $\frac{qa}{2} = k(1 - \frac{ma}{2}) = \frac{k}{2}$. Therefore, we get the bound

$$\begin{aligned} I_{k,q_k^*}(\xi, \delta(\xi), y_0) \le{} & R^{-mk}\big(\|\xi\|^{-k/2} \wedge (t\wedge 1)^{mk/2}\big) P_0(t\wedge 1, y_0)^{mk} \\ & + R^{-mk}\big(\|\xi\|^{-k/2} \vee (t\wedge 1)^{-mk/2}\|\xi\|^{-k}\big) \\ & \times \Theta_k(t\wedge 1, y_0, C_{k,q_k^*}) \end{aligned} \tag{2.50}$$

with $q_k^* = mk$. Estimate (2.50) holds for any $k \ge 1$ and $\xi \ne 0$, and then it proves that the function $p_{t,y_0}(y)$ defined in (2.38) is in fact well defined and infinitely differentiable with respect to y.

Let us come to estimate (2.33). We take (2.38) and cut off the integration over a region I of finite Lebesgue measure on which $\|\xi\| = \prod_{i=1}^m |\xi^i|$ remains smaller than a given constant. That is, we write

$$\begin{aligned} p_{t,y_0}(y) &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{-i\langle \xi, y\rangle} \mathbb{E}\big[e^{i\langle \xi, X_t\rangle} \phi_R(X_t - y_0)\big]\, d\xi \\ &\le \frac{1}{(2\pi)^m} \bigg[\int_I \mathbb{E}[\phi_R(X_t - y_0)]\, d\xi \\ &\qquad + \int_{I^c} e^{-i\langle \xi, y\rangle} \mathbb{E}\big[e^{i\langle \xi, X_t\rangle} \phi_R(X_t - y_0)\big]\, d\xi\bigg] \\ &\le \frac{1}{(2\pi)^m} \bigg[\mathbb{P}(|X_t - y_0| < 2R) \lambda_m(I) \\ &\qquad + C_{k,q_k^*} P_t(y_0) \int_{I^c} I_{k,q_k^*}(\xi, \delta(\xi), y_0)\, d\xi\bigg], \end{aligned}$$

where λ_m denotes the Lebesgue measure on $\mathbb{R}^m$. As we have seen, the last term is such that

$$\begin{aligned} &\int_{I^c} I_{k,q_k^*}(\xi, \delta(\xi), y_0)\, d\xi \\ &\quad \le R^{-mk} P_0(t \wedge 1, y_0)^{mk} \int_{I^c} |\xi|^{-k/2}\, d\xi \\ &\qquad + R^{-mk} \Theta_k(t \wedge 1, y_0, C_{k,q_k^*}) \bigg((t \wedge 1)^{-mk/2} \int_{I^c \cap \{\xi : |\xi| < (t\wedge 1)^{-m}\}} |\xi|^{-k}\, d\xi \\ &\qquad\qquad + \int_{I^c \cap \{\xi : |\xi| \ge (t\wedge 1)^{-m}\}} |\xi|^{-k/2}\, d\xi\bigg). \end{aligned}$$

Now, since

$$\int_{I^c \cap \{\xi : |\xi| \ge (t\wedge 1)^{-m}\}} |\xi|^{-k/2}\, d\xi \le \int_{I^c} |\xi|^{-k/2}\, d\xi = c_k^{(1)} < \infty$$

and

$$(t \wedge 1)^{-mk/2} \int_{I^c \cap \{\xi : |\xi| < (t\wedge 1)^{-m}\}} |\xi|^{-k}\, d\xi \le (t \wedge 1)^{-mk/2} c_k^{(2)} < \infty$$

hold for any $k \ge 3$, we then take $k = 3$ and get the estimate

$$p_{t,y_0}(y) \le C^* P_t(y_0)\big[1 + R^{-3m}\big(P_0(t \wedge 1, y_0)^{3m} + (t \wedge 1)^{-3m/2} \Theta_3(t \wedge 1, y_0, C_{m,d})\big)\big]$$

for every $y \in B_R(y_0)$, for a positive constant C^*, estimate (2.33) then follows. For estimate (2.34) on the derivatives we proceed in the same way, observing that for $\alpha \in \{1, \dots, m\}^l$, $|\xi|^{-k/2} \times \prod_{j=1}^l |\xi^{\alpha_j}|$ is integrable at ∞ as soon as $k \ge 2l + 3$.

PROOF OF (2.44). We remark that $\{\phi_R(X_t - y_0)\} \subseteq \{t-\delta \le \nu \le t\} \subseteq \{t-\delta \le \nu \le t, X_\nu \in \overline{B}_{3R}(y_0)\}$, hence

$$\begin{aligned}
&\mathbb{P}\Big(\phi_R(X_T - y_0) > 0, \sup_{0\le s\le\delta} |\overline{X}_s(X_\nu) - X_\nu| \ge R\Big) \\
&\quad \le \mathbb{P}\Big(t-\delta \le \nu \le t, X_\nu \in \overline{B}_{3R}(y_0), \sup_{0\le s\le\delta} |\overline{X}_s(X_\nu) - X_\nu| \ge R\Big) \\
&\quad \le R^{-q}\mathbb{P}(t-\delta \le \nu \le t) \sup_{y\in\overline{B}_{3R}(y_0)} \mathbb{E}\Big[\sup_{0\le s\le\delta} |\overline{X}_s(y) - y|^q\Big].
\end{aligned}$$

Using boundedness of coefficients of (2.42), it is easy to show that

$$\mathbb{E}\Big[\sup_{0\le s\le\delta} |\overline{X}_s(y) - y|^q\Big] \le c_{q,m}\delta^{q/2}P_0(\delta, y_0)^q$$

for some positive constant $c_{q,m}$. □

As addressed in Section 2.1, the constants appearing in the definition of Λ_k (2.32) can be considerably simplified under assumptions (H1′) and (H4), resulting in some polynomial-type bounds. The following result corresponds to Theorem 2.2; in the presents statement, we explicitly give the expression of the exponent $q'_k(q)$ appearing in bound (2.6).

THEOREM 2.5. *Assume* (H1′) *and* (H3).

(a) *For any initial condition $x \in \mathbb{R}^m$ and any $0 < t \le T$, X_t admits a smooth density on $\mathbb{R}^m \setminus \overline{B}_\eta(0)$.*

(b) *Assume* (H4) *as well. Then the constant Λ_k defined in Theorem 2.4 is such that*

$$\Lambda_k(t, y_0) \le C_{k,T}\big(1 + |y_0|^{q'_k(q)}\big), \tag{2.51}$$

for every $0 < t \le T$ and every $|y| > \eta + 5$. The exponent $q'_k(q)$ is worth

$$q'_k(q) = mk(\overline{q}+4)(m+1)(mk+3) + 2qm\big(\phi_{mk} + (mk+2)^2\big).$$

(c) *If moreover $\sup_{0\le s\le t} |X_s|$ has finite moments of all orders, then for every $p > 0$ and every $k \ge 1$ there exist positive constants $C_{k,p,T}$ such that*

$$\begin{aligned}
|p_t(y)| &\le C_{3,p,T}\left(1 + \frac{1}{t^{m3/2}}\right)|y|^{-p}, \\
|\partial_\alpha p_t(y)| &\le C_{k,p,T}\left(1 + \frac{1}{t^{m(2k+3)/2}}\right)|y|^{-p}, \qquad \alpha \in \{1, \dots, m\}^k,
\end{aligned} \tag{2.52}$$

for every $0 < t \le T$ and every $|y| > \eta + 5$.

The $C_{k,p,T}$ are positive constants depending also on m, d and on bounds (2.4) *and* (2.5) *on the coefficients.*

PROOF. (a) We no longer need to distinguish between y_0 and the (close) point y where the density is evaluated; hence we just set $y = y_0$ and consider suitable radii. For $|y| > \eta$, we set $R_y = \frac{1}{10}\,\mathrm{dist}(y, \overline{B}_\eta(0)) \wedge 1$. By (H1′), b and σ are of class C_b^∞ on $B_{5R_y}(y)$ and satisfy (H2) on $B_{3R_y}(y)$. From Theorem 2.4 it follows that X_t admits a smooth density on $B_{R_y}(y)$. This holds true for every ball $B_{R_y}(y)$ with center y in $\mathbb{R}^m \setminus \overline{B}_\eta(0)$; hence statement (a) follows.

(b) Without loss of generality, we take $R = 1$. As a consequence of (2.4), the constants introduced before Theorem 2.4 can be bounded as follows, for $0 \le t \le T$ and $|y| > \eta + 5$:

$$
\begin{aligned}
P_k(t, y) &\le c_k^{(1)}\bigl(1 + (|y| + 5)^q\bigr) \le c_k^{(1)}|y|^q, \\
P_k^\sigma(t, y) \vee P_1^Z(t, y) &\le c_k^{(1)}|y|^{2q}, \\
P_m^C(t, y) &\le c^{(1)}(|y|^4 + |y|^2)^{2(m+1)} \le c^{(1)}|y|^{8(m+1)}, \\
C_m(t, y) &\le \frac{c^{(1)}}{C_0^{2(m+1)}}|y|^{2\overline{q}(m+1)}|y|^{8(m+1)} \\
&\le c^{(1)}|y|^{2(\overline{q}+4)(m+1)},
\end{aligned}
\tag{2.53}
$$

for some constants $c^{(1)}$ and $c_k^{(1)}$ depending also on m, q and on the bounds on b, σ and their derivatives in (2.4) and (2.5).

The exponential factors e and e^Z must be treated on a specific basis. Indeed, $e_\cdot(t, y)$ and $e_\cdot^Z(t, y)$ may explode when $|y| \to +\infty$. Nevertheless, explosion can be avoided stepping further into the optimization procedure set up in the proof of Theorem 2.4. More precisely, we restart from step 4 and force the state variable y to appear in the choice of δ, setting

$$\delta(\xi, y) = t/2 \wedge 1 \wedge |\xi|^{-a} \wedge |y|^{-4q}.$$

Now, whatever the value of p is, e_p and e_p^Z are reduced to

$$
\begin{aligned}
e_p(\delta(\xi, y)) &\le e_p(1 \wedge |y|^{-4q}, y) \\
&\le \exp\bigl(c_1^{(1)}(1 \wedge |y|^{-2qp})(1 + 1 \wedge |y|^{-2q})^p|y|^{qp}\bigr) \\
&\le \exp\bigl(c_p(1 \wedge |y|^{-2qp})|y|^{qp}\bigr) \\
&\le \exp(c_p)
\end{aligned}
$$

and

$$
\begin{aligned}
e_p^Z(\delta(\xi, y)) &\le e_p^Z(1 \wedge |y|^{-4}, y) \\
&= \exp\bigl(c_1^{(1)}(1 \wedge |y|^{-2qp})(1 + 1 \wedge |y|^{-2q})^p|y|^{2qp}\bigr) \\
&\le \exp(c_p).
\end{aligned}
$$

We then perform the integration over ξ as done in the last step of the proof of Theorem 2.4, and employing (2.53) we obtain estimate (2.51) for Λ_k, for $|y| > \eta + 5$. The value of $q_k'(q)$ is obtained from the definition of Θ_k and (2.53).

(c) From boundedness of moments of $\sup_{s\le t}|X_s|$, for any interval $I_t \subseteq [0,t]$ one can easily deduce the estimate

$$\begin{aligned}
&\mathbb{P}(\inf\{|X_s - y| : s \in I_t\} \le 3)\\
&\quad\le \mathbb{P}(\sup\{|X_s| : s \in I_t\} \ge |y| - 3)\\
&\quad\le \frac{1}{(|y|-3)^r}\mathbb{E}\Big[\sup_{s\le t}|X_s|^r\Big] \le c_r^{(2)}\frac{1}{|y|^r},
\end{aligned}\tag{2.54}$$

for any $r > 0$, $0 \le t \le T$ and $|y| > 3$. It is then easy to obtain the desired estimate on p_t with Theorem 2.4: for a given $p > 0$, we employ (2.33) with $y_0 = y$ and (2.54) with $r > p + q_3'(q)$. Similarly, to obtain the estimate on derivatives, one employs (2.34) and (2.54) with $r > p + q_{2k+3}'(q)$. □

3. A square root-like (CIR/CEV) process with local coefficients. We apply our results to the solution of (1.1). We will be able to refine the polynomial estimate on the density at $+\infty$ giving exponential-type upper bounds. Under some additional assumptions on the coefficients, we also study the asymptotic behavior of the density at zero, that is, the point where the diffusion coefficient is singular.

We first collect some basic facts concerning the solution of (1.1). Let us recall the SDE

$$\begin{cases} dX_t = \big(a(X_t) - b(X_t)X_t\big)\,dt + \gamma(X_t)X_t^\alpha\,dW_t, & t \ge 0, \alpha \in [1/2, 1),\\ X_0 = x \ge 0. \end{cases}\tag{3.1}$$

When $\alpha = 1/2$ and a, b and γ are constant, the solution to (3.1) is the celebrated Cox–Ingersoll–Ross process (see [5]), appearing in finance as a model for short interest rates. It is well known that, in spite of the lack of globally Lipschitz-continuous coefficients, existence and uniqueness of strong solutions hold for the equation of a CIR process. If $a \ge 0$, the solution stays a.s. in $\mathbb{R}_+ = [0, +\infty)$; furthermore, a solution starting at $x > 0$ stays a.s. in $\mathbb{R}_> = (0, +\infty)$ if the Feller condition $2a \ge \gamma^2$ is achieved (cf. [14] for details). The following proposition gives the (straightforward) generalization of the previous statements to the case of coefficients a, b, γ that are functions of the underlying process. The proof is left to Appendix A.2.

PROPOSITION 3.1. *Assume*:

(s0) $\alpha \in [1/2, 1)$; a, b *and* $\gamma \in C_b^1$ *with* $a(0) \ge 0$ *and* $\gamma(x)^2 > 0$ *for every* $x > 0$.

Then, for any initial condition $x \ge 0$ *there exists a unique strong solution to* (3.1) *which is such that* $\mathbb{P}(X_t \ge 0; t \ge 0) = 1$. *Let then* $x > 0$ *and* $\tau_0 = \inf\{t \ge 0 : X_t = 0\}$, *with* $\inf\{\varnothing\} = \infty$.

- *If* $\alpha > 1/2$ *and*

  (s1)′ $a(0) > 0$ *and* $z \mapsto \frac{1}{\gamma^2(z)z^{2\alpha-1}}$ *is integrable at* 0^+, *then*

$$\mathbb{P}(\tau_0 = \infty) = 1. \tag{3.2}$$

- *If* $\alpha = 1/2$ *and*

  (s1) $\frac{1}{\gamma^2}$ *is integrable at zero*,

  (s2) *there exists* $\overline{x} > 0$ *such that* $\frac{2a(x)}{\gamma(x)^2} \geq 1$ *for* $0 < x < \overline{x}$,

  then the same conclusion on τ_0 *holds*.

When X is a CIR process, the moment-generating function of X_t can be computed explicitly, leading to the knowledge of the density. Setting $L_t = (1 - e^{-bt})\gamma^2/4b$, then X_t/L_t follows a noncentral chi-square law with $\delta = 4a/\gamma^2$ degrees of freedom and parameter $\zeta_t = 4xb/(\gamma^2(e^{bt} - 1))$ (recall that x is here the initial condition). The density of X_t is then given by (cf. [14])

$$p_t(y) = \frac{e^{-\zeta_t/2}}{2^{\delta/2} L_t} e^{-y/(2L_t)} \left(\frac{y}{L_t}\right)^{\delta/2-1} \sum_{n=0}^{\infty} \frac{(y/(4L_t))^n}{n!\Gamma(\delta/2+n)} \zeta_t^n, \qquad y > 0.$$

We incidentally remark that p_t is in general unbounded, since $y^{\delta/2-1}$ diverges at zero when $\delta/2 - 1 = 2a/\gamma^2 - 1$ is negative (in fact, fixed a value of $\delta/2 - 1$, there exists a $n \geq 0$ such that $\frac{d^n}{dy^n} p_t$ is unbounded).

The standard techniques of Malliavin calculus cannot be directly applied to study the existence of a smooth density for the solution of (3.1), as the diffusion coefficient in general is not (depending on γ) globally Lipschitz continuous. Actually, Alos and Ewald [1] have shown that if X is CIR process, then X_t, $t > 0$, belongs to $\mathbb{D}^{1,2}$ when the Feller condition $2a \geq \gamma^2$ is achieved. Higher order of differentiability (in the Malliavin sense) can be proven, requiring a stronger condition on a and γ, and the authors apply these results to option pricing within the Heston model. If we are interested in density estimation, the results of the previous sections allow us to overcome the problems related to the singular behavior of the diffusion coefficient and to directly establish the existence of a smooth density, independently from any Feller-type condition [provided that (s0) is satisfied]. More precisely, we can give the following preliminary result:

PROPOSITION 3.2 (Preliminary result). *Assume* (s0) *and let* a, b, γ *be of class* C_b^∞. *Let* $X = (X_t; t \geq 0)$ *be the strong solution of* (3.1) *starting at* $x \geq 0$. *For any* $t > 0$, X_t *admits a smooth density* p_t *on* $(0, +\infty)$. p_t *is such that* $\lim_{y\to\infty} p_t(y) y^p = 0$ *for any* $p > 0$.

PROOF. It is easy so see that, under the current assumptions, the drift and diffusions coefficients of (3.1) satisfy (H1′) with $\eta = 0$ and (H4) with $q = 1$. (H3) holds as well, by Proposition 3.1. As the coefficients have sub-linear growth, for any $t > 0$, $\sup_{s\leq t} X_s$ has finite moments of any order. The conclusion follows from Theorem 2.5(c). □

3.1. *Exponential decay at* ∞. In order to further develop our study of the density, we could take advantage of some of the generalized-chaining tools settled by Viens and Vizcarra in [17]. In particular notice that, in order to estimate the density by means of Theorem 2.5, we need to deal with the probability term $P_t(y)$ appearing therein. For our present purposes, we can rely on alternative strategies involving time-change arguments and the existence of quadratic exponential moments for suprema of Brownian motions (Fernique's theorem) in the current section, and a detailed analysis of negative moments of the process X in Section 3.2.

From now on, condition (s0) is assumed, the coefficients a, b, γ are of class C_b^∞ and $(X_t; t \le T)$ denotes the unique strong of (3.1) on $[0, T]$, $T > 0$. We make explicit the dependence with respect to the initial condition denoting $p_t(x, \cdot)$ the density at time t of X starting at $x \ge 0$. The following result improves Proposition 3.2 in the estimate of the density for $y \to \infty$.

PROPOSITION 3.3. *Assume that*

$$\varliminf_{x\to\infty} b(x)x^{1-\alpha} > -\infty. \tag{3.3}$$

Then there exist positive constants γ_0 *and* $C_k(T)$, $k \ge 3$, *such that*

$$p_t(x, y) \le C_3(T)\left(1 + \frac{1}{t^{3/2}}\right)\exp\left(-\gamma_0 \frac{(y-x)^{2(1-\alpha)}}{2Ct}\right) \tag{3.4}$$

and

$$p_t^{(k)}(x, y) \le C_k(T)\left(1 + \frac{1}{t^{(2k+3)/2}}\right)\exp\left(-\gamma_0 \frac{(y-x)^{2(1-\alpha)}}{2Ct}\right) \tag{3.5}$$

for every $y > x + 1$, *with* $C = 2^{3-2\alpha} + 2|\gamma|_0^2(1-\alpha)^2$. *The* $C_k(T)$ *also depend on* α *and on the coefficients* a, b *and* γ.

REMARK 3.1. In the case of constant coefficients and $a = b = 0$, the bound (3.4) can be compared to the density of the CEV process as provided, for example, in [7], Theorem 1.6 (see also the references therein). The comparison shows that our estimate is in the good range on the log-scale.

PROOF OF PROPOSITION 3.3. In the spirit of Lamperti's change-of-scale argument (cf. [12], page 294), let $\varphi \in C^2((0, \infty))$ be defined by

$$\varphi(x) = \int_0^x \frac{1}{|\gamma|_0 y^\alpha}\, dy = \frac{1}{|\gamma|_0(1-\alpha)} x^{1-\alpha},$$

so that $\varphi'(x) = \frac{1}{|\gamma|_0 x^\alpha}$. Let moreover $\theta \in C_b^\infty(\mathbb{R})$ be such that $1_{[2,\infty)} \le \theta \le 1_{[1,\infty)}$ and $\theta' \le 1$. We set

$$\rho(x) = \begin{cases} \theta(x)\varphi(x), & x > 0, \\ 0, & x \le 0, \end{cases}$$

so that ρ is of class $C^2(\mathbb{R})$. We define the auxiliary process $Y_t = X_t - x$, $t \geq 0$, which is such that $\mathbb{P}(Y_t \geq -x) = 1$, $t \geq 0$. An application of Itô's formula yields

$$\rho(Y_t) = \int_0^t f(Y_s)\,ds + M_t,$$

where

$$f(y) = \rho'(y)\big(a(y) - b(y)y\big) + \tfrac{1}{2}\rho''(y)\gamma(y)^2 y^{2\alpha}$$

and

$$M_t = \int_0^t \rho'(Y_s)\gamma(Y_s)Y_s^\alpha\,dW_s.$$

The key point is the fact that f is bounded from above on $(-x, \infty)$ and, on the other hand, M is a martingale with bounded quadratic variation. Indeed, f is continuous, it is zero for $y \leq 1$ and for $y > 2$ one has

$$f(y) = \frac{a(y)}{|\gamma|_0 y^\alpha} - \frac{b(y)}{|\gamma|_0} y^{1-\alpha} - \frac{\alpha}{2}\frac{\gamma(y)^2}{|\gamma|_0} y^{\alpha-1},$$

and hence, recalling that a is bounded, $\overline{\lim}_{y\to\infty} f(y) < \infty$ is ensured by condition (3.3). Then we set $C_1 = \sup_{y\geq 0} f(y)$. For M, one has

$$\begin{aligned}\langle M\rangle_t &= \int_0^t \rho'(Y_s)^2\gamma(Y_s)^2 Y_s^{2\alpha}\,ds \\ &\leq \int_0^t \big(\varphi(2) + \varphi'(Y_s)\big)^2\gamma(Y_s)^2 Y_s^{2\alpha}\,ds \\ &\leq 2\big(\varphi(2)^2 + 1\big)t,\end{aligned} \tag{3.6}$$

and hence we set $C_2 = 2(\varphi(2)^2 + 1) = 2(\frac{2^{2(1-\alpha)}}{|\gamma|_0^2(1-\alpha)^2} + 1)$. Now, since ρ is strictly increasing, $\{Y_t > y\} = \{\rho(Y_t) > \rho(y)\}$ for any $y > 0$. Moreover, $\{\rho(Y_t) > \rho(y)\} \subseteq \{M_t + C_1 t > \rho(y)\} \subseteq \{2M_t^2 + 2C_1^2t^2 > \rho(y)^2\}$. We set $I_t = [(t-1)\vee t/2, t]$ and $\tau = \inf\{s \geq 0 : Y_s \geq 3/2\}$. The quadratic variation of M is strictly increasing after τ, since $(\rho(Y_{\tau\vee t})\gamma(Y_{\tau\vee t})Y_{\tau\vee t}^\alpha)^2 > 0$: hence, by Dubins and Schwarz's theorem (cf. Theorem 3.4.6 in [12]) there exists a one-dimensional Brownian motion $(b_t; t \geq 0)$ such that $M_{\tau\vee t} = b_{\langle M\rangle_{\tau\vee t}}$. Clearly one has $\{Y_t > 2\} \subseteq \{\tau < t\}$, so that for $y > 2$

$$\begin{aligned}\overline{P}_t(y) &= \mathbb{P}(\exists s \in I_t : Y_s > y) \leq \mathbb{P}(\exists s \in I_t : Y_s > y, \tau < s) \\ &\leq \mathbb{P}\big(\exists s \in I_t : 2M_s^2 + 2C_1^2 s^2 > \rho(y)^2, \tau < s\big) \\ &\leq \mathbb{P}\Big(\sup_{\tau < s \leq t}(2M_s^2 + 2C_1^2 s^2) > \rho(y)^2\Big) \\ &\leq \mathbb{P}\Big(\sup_{0 < s \leq t} b_{\langle M\rangle_s}^2 + C_1^2 t^2 > \frac{1}{2}\rho(y)^2\Big) \\ &\leq \mathbb{P}\Big(\sup_{s \leq C_2 t} b_s^2 + C_1^2 t^2 > \frac{1}{2}\rho(y)^2\Big).\end{aligned}$$

We now employ the scaling property for the Brownian motion $(b_s; s \geq 0) \sim (\sqrt{a} b_{s/a}; s \geq 0)$, $a > 0$, and Fernique's Theorem (cf. [11], page 402). The latter tells that there exists a positive constant γ_0 such that $\exp(\gamma_0 \sup_{s \leq 1} b_s^2)$ is integrable, hence

$$\begin{aligned}\overline{P}_t(y) &\leq \mathbb{P}\left(\gamma_0 \sup_{s \leq 1} b_s^2 + \gamma_0 \frac{C_1^2}{C_2} t > \frac{\gamma_0}{2C_2 t} \rho(y)^2\right) \\ &\leq \exp\left(-\gamma_0 \frac{\rho(y)^2}{2C_2 t}\right) \mathbb{E}[e^{\gamma_0 C_1^2/C_2 t + \gamma_0 \sup_{s \leq 1} b_s^2}] \\ &\leq C_0 \exp\left(-\gamma_0 \frac{\rho(y)^2}{2C_2 t} + \gamma_0 \frac{C_1^2}{C_2} t\right),\end{aligned}$$

where $C_0 = \mathbb{E}[\exp(\gamma_0 \sup_{s \leq 1} b_s^2)]$ is a universal constant. The estimates on the density of X_t and its derivatives now follow from Theorem 2.4 [estimates (2.33) and (2.34)] and Theorem 2.5(b), using $X_t - x = Y_t$, the value of the constant C_2 and taking, for example, $R = 1/6$. $\square$

3.2. *Asymptotics at* 0. We have established conditions under which the solution of (3.1) admits a smooth density p_t on $(0, +\infty)$. According to Proposition 3.1, the process remains almost surely in $\mathbb{R}_+$: this trivially means that for any $t > 0$, X_t has an identically zero density on $(-\infty, 0)$, which can be extended to 0 when $\tau_0 = \infty$ a.s. We are now wondering what are sufficient conditions for p_t to converge to zero at the origin, hence providing the existence of a continuous (eventually differentiable, eventually C^∞) density on the whole real line.

What we have in mind is the application of Theorem 2.5 to the inversed process $Y_t = \frac{1}{X_t}$ (considered on the event $\{\tau_0 = \infty\}$). An application of Itô's formula yields

$$dY_t = J_\alpha(Y_t)\, dt - \widehat{\gamma}(Y_t) Y_t^{2-\alpha}\, dW_t, \tag{3.7}$$

where

$$J_\alpha(Y_t) = -\widehat{a}(Y_t) Y_t^2 + \widehat{b}(Y_t) Y_t + \widehat{\gamma}(Y_t)^2 Y_t^{3-2\alpha}$$

with the notation $\widehat{f}(y) = f(1/y)$, $y > 0$, for $f = a, b, \sigma$. Equation (3.7) has superlinear coefficients, in particular condition (2.4) of Theorem 2.2 holds with $q = 2$. Willing to apply Theorem 2.5(c), we first need some preliminary results on the moments of Y. The proof of the next statement is based on the techniques employed in [3], proof of Lemma 2.1, that we adapt to our framework.

LEMMA 3.1.

(1) *If* $\alpha > 1/2$, *assume* (s1)$'$. *Then for any initial condition* $x > 0$, *for any* $t > 0$ *and* $p > 0$

$$\mathbb{E}\left[\sup_{s \leq t} \frac{1}{X_s^p}\right] \leq C, \tag{3.8}$$

for some positive constant C depending on x, p, α, t and on the coefficients of (3.1).

(2) *If* $\alpha = 1/2$, *then assume* (s1) *and* (s2) *and let*

$$l^* = \varliminf_{x \to 0} \frac{2a(x)}{\gamma(x)^2} > 1. \tag{3.9}$$

Then (3.8) *holds for* $p > 0$ *such that*

$$p + 1 < l^*. \tag{3.10}$$

PROOF. Let τ_n be the stopping time defined by $\tau_n = \inf\{t \geq 0 : X_t \leq 1/n\}$. The application of Itô's formula to $X_{t\wedge\tau_n}^p$, $p > 0$, yields

$$\mathbb{E}\left[\frac{1}{X_{t\wedge\tau_n}^p}\right] = \frac{1}{x^p} + \mathbb{E}\int_0^{t\wedge\tau_n} \varphi(X_s)\,ds, \tag{3.11}$$

where

$$\varphi(x) = p\frac{b(x)}{x^p} + \frac{p}{x^{p+1}} \underbrace{\left(\frac{p+1}{2}\gamma(x)^2 x^{2\alpha-1} - a(x)\right)}_{g(x)}, \qquad x > 0.$$

It is easy to see that, if

$$\varlimsup_{x\to 0} g(x) < 0, \tag{3.12}$$

there exists a positive constant C such that $\varphi(x) < p\frac{|b|_0}{x^p} + C$, for every $x > 0$. If (3.12) holds, from (3.11) we get

$$\mathbb{E}\left[\sup_{s\leq t} \frac{1}{X_{s\wedge\tau_n}^p}\right] \leq \frac{1}{x^p} + Ct + p|b|_0 \int_0^t \mathbb{E}\left[\sup_{u\leq s} \frac{1}{X_{u\wedge\tau_n}^p}\right] ds,$$

and hence, by Gronwall's lemma,

$$\mathbb{E}\left[\sup_{s\leq t} \frac{1}{X_{s\wedge\tau_n}^p}\right] \leq \left(\frac{1}{x^p} + Ct\right) e^{p|b|_0 t}. \tag{3.13}$$

We verify (3.12), distinguishing the two cases.

Case $\alpha > 1/2$. We simply observe that $\lim_{x\to 0} g(x) = -a(0) < 0$. Estimate (3.8) then follows by taking the limit $n \to \infty$ in (3.13) and using Proposition 3.1 under assumption (s1)$'$.

Case $\alpha = 1/2$. We have $g(x) = \frac{p+1}{2}\gamma^2(x) - a(x)$. If p satisfies (3.10), then (3.9) ensures that $\varlimsup_{x\to 0} g(x) < 0$. We conclude again taking the limit $n \to \infty$ and using Proposition 3.1 under assumptions (s1) and (s2). □

We are now provided with the tools to prove the following:

PROPOSITION 3.4.

(1) *If* $\alpha > 1/2$, *assume* (s1)$'$. *Then for every* $t > 0$, *every* $p \geq 0$ *and every* $k > 0$ *the density* p_t *of* X_t *on* $(0, +\infty)$ *is such that*

$$\lim_{y \to 0^+} y^{-p} |p_t(y)| = 0, \qquad \lim_{y \to 0^+} y^{-p} \bigl|p_t^{(k)}(y)\bigr| = 0. \tag{3.14}$$

(2) *If* $\alpha = 1/2$, *then assume* (s1) *and* (s2) *and define* l^* *as in Lemma* 3.1. *If*

$$l^* > 3 + q_3'(2) \tag{3.15}$$

[*where* $q_\cdot'(\cdot)$ *has been defined in Theorem* 2.5], *then*

$$\lim_{y \to 0^+} y^{-p} p_t(y) = 0 \tag{3.16}$$

for every $0 \leq p < l^* - (3 + q_3'(2))$. *Moreover, if*

$$l^* > 2k + 3 + q_{2k+3}'(2), \tag{3.17}$$

then

$$\lim_{y \to 0^+} y^{-p} \bigl|p_t^{(k)}(y)\bigr| = 0 \tag{3.18}$$

for every $0 \leq p < l^* - (2k + 3 + q_{2k+3}'(2))$.

PROOF. We apply Theorem 2.2 to $Y = 1/X$. For simplicity of notation, we write p for p_t and p_Y for the density of Y_t. As Y satisfies (3.7), from Theorem 2.2(b) it follows that the bound (2.51) on Λ_k holds with $q_k'(q) = q_k'(2)$. Hence, from Theorem 2.2(c) it follows that

$$\lim_{y' \to +\infty} p_Y(y') |y'|^{p'} = 0 \tag{3.19}$$

and, for a given $k \geq 0$,

$$\lim_{y' \to +\infty} p_Y^{(k)}(y') |y'|^{p'} = 0, \tag{3.20}$$

if $\sup_{s \leq t} Y_s$ has a finite moment of order $r > p' + q_{2k+3}'(2)$.

Now, it is easy to see that

$$p(y) = \frac{1}{y^2} p_Y\left(\frac{1}{y}\right), \tag{3.21}$$

and hence, after some rather straightforward computations,

$$\bigl|p^{(k)}(y)\bigr| \leq C_k \left(\frac{1}{y}\right)^{2(k+1)} \sum_{j=0}^{k} \sum_{\nu=1}^{j} \frac{d^\nu}{dy^\nu} p_Y\left(\frac{1}{y}\right), \qquad 0 < y < 1. \tag{3.22}$$

Once again, we distinguish the two cases.

Case $\alpha > 1/2$. If $1/2 < \alpha < 1$, by Lemma 3.1, (3.19) and (3.20) hold for any $p' > 0$. Then (3.14) easily follows from (3.21) and (3.22).

Case $\alpha = 1/2$. By Lemma 3.1, (3.15) is the condition for $\sup_{s \le t} Y_s$ to have finite moment of order strictly greater than $2 + q_3'(2) + p$, with $p < l^* - (3 + q_3'(2))$. By (3.21), in this case (3.19) holds true with $k = 0$ and $p' = 2 + p$, and hence (3.16) holds. Similarly, by (3.22) estimate (3.18) holds if (3.19) holds with $p' = p + 2(k+1)$. The latter condition is achieved if $\sup_{s \le t} Y_s$ has finite moment of order strictly greater than $2(k+1) + q'_{2(k+3)}(2) + p$, which is in turn ensured by (3.17). □

REMARK 3.2. Proposition 3.3 states that p_t decays exponentially at infinity for any value of α, as far as condition (3.3) holds true. When $\alpha > 1/2$, Proposition 3.4 states that p_t and all its derivatives tend to zero at the origin, while the price to pay for the same conclusion to hold is higher when $\alpha = 1/2$ [cf. conditions (3.15) and (3.17), which become rapidly strong for growing values of k]. With regard to this behavior at zero, we recall that Proposition 3.4 only provides sufficient conditions for estimates (3.16) and (3.18) to hold. We do not give any conclusion on the behavior of the density at zero when condition (3.15) [or (3.17) for the derivatives] fail to hold.

APPENDIX

We collect here the proofs of some of the more technical results.

A.1. Proofs of Lemmas 2.1 and 2.2.

PROOF OF LEMMA 2.1. We refer to the notation introduced in the proof of [16], Theorem 2.2.2, allowing us to write the equation satisfied by the kth Malliavin derivative in a compact form. This is stated as follows: for any subset $K = \{h_1, \dots, h_\eta\}$ of $\{1, \dots, k\}$, one sets $j(K) = j_{h_1}, \dots, j_{h_\eta}$ and $r(K) = r_{h_1}, \dots, r_{h_\eta}$. Then, one defines

$$\begin{aligned}\alpha^i_{l, j_1, \dots, j_k}(s, r_1, \dots, r_k) &:= D^{j_1, \dots, j_k}_{r_1, \dots, r_k} A^i_l(X_s) \\ &= \sum \partial_{k_1} \cdots \partial_{k_\nu} A^i_l(X_s) \\ &\quad \times D^{j(I_1)}_{r(I_1)} X^{k_1}_s \cdots D^{j(I_\nu)}_{r(I_\nu)} X^{k_\nu}_s\end{aligned}$$

and

$$\begin{aligned}\beta^i_{j_1, \dots, j_k}(s, r_1, \dots, r_k) &:= D^{j_1, \dots, j_k}_{r_1, \dots, r_k} B^i(X_s) \\ &= \sum \partial_{k_1} \cdots \partial_{k_\nu} B^i(X_s) \\ &\quad \times D^{j(I_1)}_{r(I_1)} X^{k_1}_s \cdots D^{j(I_\nu)}_{r(I_\nu)} X^{k_\nu}_s,\end{aligned}$$

where in both cases the sum is extended to the set of all partitions of $\{1,\dots,k\} = I_1 \cup \cdots \cup I_\nu$. Finally, one sets $\alpha^i_j(s) = A^i_j(X_s)$. Making use of this notation, it is shown that the equation satisfied by the kth derivative reads as

$$
\begin{aligned}
D^{j_1,\dots,j_k}_{r_1,\dots,r_k} X^i_t = {} & \sum_{\varepsilon=1}^{k} \alpha^i_{j_\varepsilon, j_1,\dots,j_{\varepsilon-1}, j_{\varepsilon+1},\dots,j_k}(r_\varepsilon, r_1,\dots,r_{\varepsilon-1}, r_{\varepsilon+1},\dots,r_k) \\
& + \int_{r_1\vee\cdots\vee r_k}^{t} (\beta^i_{j_1,\dots,j_k}(s, r_1,\dots,r_k)\,ds \\
& \qquad + \alpha^i_{l,j_1,\dots,j_k}(s, r_1,\dots,r_k)\,dW^l_s),
\end{aligned} \tag{A.1}
$$

if $t \ge r_1 \vee \cdots \vee r_k$, and $D^{j_1,\dots,j_k}_{r_1,\dots,r_k} X^i_t = 0$ otherwise. We prove (2.16) by induction. The estimate is true for $k=1$, with $\gamma_{1,p} = 2C_{1,p}$: this simply follows with an application of Burkholder's inequality and Gronwall's lemma to (A.1) taken for $k=1$. Let us suppose that (2.16) is true up to $k-1$. As done for $k=1$, we apply Burkholder's inequality to (A.1), and, setting $r = r_1 \vee \cdots \vee r_k$, we get

$$
\begin{aligned}
& \mathbb{E}[|D^{j_1,\dots,j_k}_{r_1,\dots,r_k} X^i_t|^p] \\
& \quad \le C_{k,p} \Biggl\{ \sum_{\varepsilon=1}^{k} \mathbb{E}[|\alpha^i_{j_\varepsilon, j_1,\dots,j_{\varepsilon-1}, j_{\varepsilon+1},\dots,j_k}(r_\varepsilon, r_1,\dots,r_{\varepsilon-1}, r_{\varepsilon+1},\dots,r_k)|^p] \\
& \qquad + (t-r)^{p/2-1} \\
& \qquad \times \sum_{\substack{I_1\cup\cdots\cup I_\nu \\ \mathrm{card}(I)\le k-1}} \int_r^t \mathbb{E}\Biggl[\Biggl((t-r)^{1/2} |\partial_{k_1}\cdots\partial_{k_\nu} B^i(X_s)| \\
& \qquad\qquad + \sum_{l=1}^{d} |\partial_{k_1}\cdots\partial_{k_\nu} A^i_l(X_s)|\Biggr)^p \\
& \qquad\qquad \times |D^{j(I_1)}_{r(I_1)} X^{k_1}_s \cdots D^{j(I_\nu)}_{r(I_\nu)} X^{k_\nu}_s|^p\Biggr]\,ds \\
& \qquad + (t-r)^{p/2-1} \int_r^t \mathbb{E}\Biggl[\Biggl((t-r)^{1/2} |\partial_k B^i(X_s)| \\
& \qquad\qquad + \sum_{l=1}^{d} |\partial_k A^i_l(X_s)|\Biggr)^p \\
& \qquad\qquad \times |D^{j_1,\dots,j_k}_{r_1,\dots,r_k} X^k_s|^p\Biggr]\,ds\Biggr\},
\end{aligned} \tag{A.2}
$$

where, in the last line, we have isolated the term depending on $D^{j_1,\dots,j_k}_{r_1,\dots,r_k} X$.

To estimate the second term in (A.2) we notice that, for any partition $I_1 \cup \cdots \cup I_\nu$ of $\{1, \ldots, k\}$ such that $\mathrm{card}(I) \le k-1$, using (2.16) up to order $k-1$ we have

$$\begin{aligned}
&\mathbb{E}\Biggl[\Biggl((t-r)^{1/2}|\partial_{k_1}\cdots\partial_{k_\nu}B^i(X_s)| \\
&\qquad + \sum_{l=1}^{d}|\partial_{k_1}\cdots\partial_{k_\nu}A_l^i(X_s)|\Biggr)^p |D_{r(I_1)}^{j(I_1)}X_s^{k_1}\cdots D_{r(I_\nu)}^{j(I_\nu)}X_s^{k_\nu}|^p\Biggr] \\
&\quad \le C\{|A|_{k-2}^k(t^{1/2}|B|_k+|A|_k)^{\chi_k}\}^p e_p(t)^{\lambda_{k,p}^{(1)}},
\end{aligned} \tag{A.3}$$

where we have defined

$$\lambda_{k,p}^{(1)} := \sup_{\substack{I_1\cup\cdots\cup I_\nu=\{1,\ldots,k\}\\ \mathrm{card}(I)\le k-1}} \{\gamma_{\mathrm{card}(I_1),p}+\cdots+\gamma_{\mathrm{card}(I_\nu),p}\}$$

and

$$\chi_k = 1+\sum_{l=1}^{\nu}(\mathrm{card}(I_l)+1)^2.$$

It is easy to see that

$$\chi_k \le (k+1)^2,$$

since

$$\sum_{l=1}^{\nu}\mathrm{card}(I_l)^2 = (k-1)^2\sum_{l=1}^{\nu}\frac{\mathrm{card}(I_l)^2}{(k-1)^2} \le (k-1)^2\sum_{l=1}^{\nu}\frac{\mathrm{card}(I_l)}{(k-1)} = (k-1)k,$$

so that

$$\begin{aligned}
\chi_k &= 1+\sum_{l=1}^{\nu}\mathrm{card}(I_l)^2+2\sum_{l=1}^{\nu}\mathrm{card}(I_l)+\nu \\
&\le 1+(k-1)k+2k+\nu \\
&\le 1+k^2-k+3k=(k+1)^2.
\end{aligned}$$

To estimate the first term in (A.2), notice that we have as well

$$\begin{aligned}
&\mathbb{E}[|\alpha^i_{j_\varepsilon,j_1,\ldots,j_{\varepsilon-1},j_{\varepsilon+1},\ldots,j_k}(r_\varepsilon,r_1,\ldots,r_{\varepsilon-1},r_{\varepsilon+1},\ldots,r_k)|^p] \\
&\quad \le C\{|A|_{k-1}^k(r_\varepsilon^{1/2}|B|_{k-1}+|A|_{k-1})^{k^2}\}^p e_p(t)^{\lambda_{k,p}^{(2)}},
\end{aligned} \tag{A.4}$$

with

$$\lambda_{k,p}^{(2)} := \sup_{I_1\cup\cdots\cup I_\nu=\{1,\ldots,k-1\}}\{\gamma_{\mathrm{card}(I_1),p}+\cdots+\gamma_{\mathrm{card}(I_\nu),p}\}.$$

We remark that $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$ are defined by means of the γ's up to order $k-1$.

Collecting (A.2), (A.3) and (A.4), we get

$$\begin{aligned}
&\mathbb{E}[|D^{j_1,\dots,j_k}_{r_1,\dots,r_k} X^i_t|^p] \\
&\quad \le C_{k,p}\Bigg\{ |A|^{kp}_{k-1}(t^{1/2}|B|_{k-1}+|A|_{k-1})^{k^2 p} e_p(t)^{\lambda^{(2)}_{k,p}} \\
&\qquad + (t-r)^{p/2}|A|^k_{k-2}(t^{1/2}|B|_k+|A|_k)^{(k+1)^2 p} e_p(t)^{\lambda^{(1)}_{k,p}} \\
&\qquad + (t-r)^{p/2-1}(t^{1/2}|B|_1+|A|_1)^p \sum_{k=1}^m \int_r^t \mathbb{E}|D^{j_1,\dots,j_k}_{r_1,\dots,r_k} X^i_s|^p\,ds \Bigg\} \\
&\quad \le C_{k,p}|A|^{kp}_{k-1}(t^{1/2}|B|_k+|A|_k)^{(k+1)^2 p} e_p(t)^{\lambda^{(1)}_{k,p}\vee\lambda^{(2)}_{k,p}} \\
&\qquad \times \big(1 + C_{k,p} t^{p/2}(t^{1/2}|B|_1+|A|_1)^p e_p(t)^{C_{k,p}}\big) \\
&\quad \le C_{k,p}|A|^{kp}_{k-1}(t^{1/2}|B|_k+|A|_k)^{(k+1)^2 p} e_p(t)^{\lambda^{(1)}_{k,p}\vee\lambda^{(2)}_{k,p}+2C_{k,p}},
\end{aligned}$$

where we have applied Gronwall's lemma to get the second inequality. The constant $C_{k,p}$ may vary from line to line, but never depends on t nor on the bounds on B and A. We recursively define $\gamma_{k,p}$ by setting $\gamma_{k,p} := \lambda^{(1)}_{k,p} \vee \lambda^{(2)}_{k,p} + 2C_{k,p}$, and we finally obtain (2.16). $\square$

PROOF OF LEMMA 2.2. *Step* 1. We first use the decomposition $D_s X_t = Y_t Z_s A(X_s)$ (see, e.g., [16]) and write

$$\begin{aligned}
\sigma_{X_t} &= Y_t \int_0^t Z_s A(X_s) A(X_s)^* Z_s^*\,ds\, Y_t^* \\
&= Y_t U_t Y_t^*,
\end{aligned} \tag{A.5}$$

where we have set $U_t = \int_0^t Z_s A(X_s) A(X_s)^* Z_s^*\,ds$. Notice that U_t is a positive operator, and that for any $\xi \in \mathbb{R}^m$ we have

$$\langle \xi, U_t \xi\rangle = \int_0^t \langle A(X_s)^* Z_s^* \xi, A(X_s)^* Z_s^* \xi\rangle\,ds = \sum_{j=1}^d \int_0^t \langle Z_s A_j(X_s), \xi\rangle^2.$$

From identity (A.5) it follows that $\det \sigma_{X_t} = (\det Y_t)^2 \det U_t = (\det Z_t)^{-2} \det U_t$. Hence, applying Hölder's inequality,

$$\begin{aligned}
\mathbb{E}[|\det \sigma_{X_t}|^{-p}] &\le (\mathbb{E}[|\det Z_t|^{4p}]\mathbb{E}[(\det U_t)^{-2p}])^{1/2} \\
&\le C_{p,m}(e^Z_{4p}(t)^m \mathbb{E}[(\det U_t)^{-2p}])^{1/2},
\end{aligned} \tag{A.6}$$

where in the last step we have used bound (2.15) on the entries of Z_t.

Step 2. Let $\lambda_t = \inf_{|\xi|=1}\langle \xi, U_t\xi\rangle$ be the smallest eigenvalue of U_t, so that $\mathbb{E}[(\det U_t)^{-2p}] \le \mathbb{E}[\lambda_t^{-2mp}]$. We evaluate $\mathbb{P}(\lambda_t \le \varepsilon)$.

For any ξ such that $|\xi| = 1$, using the elementary inequality $(a+b)^2 \ge a^2/2 - b^2$ we get

$$\begin{aligned}\sum_{j=1}^d \langle Z_s A_j(X_s), \xi\rangle^2 &\ge \frac{1}{2}\sum_{j=1}^d \langle A_j(x), \xi\rangle^2 - \sum_{j=1}^d \langle Z_s A_j(X_s) - A_j(x), \xi\rangle^2\\ &\ge \frac{1}{2}c_* - \sum_{j=1}^d |Z_s A_j(X_s) - A_j(x)|^2,\end{aligned}$$

where in the last step we have used the ellipticity assumption (E). For any $\varepsilon > 0$ and $a > 0$ such that $a\varepsilon < t$, the previous inequality gives

$$\mathbb{P}(\lambda_t \le \varepsilon) \le \mathbb{P}\left(\frac{1}{2}ac_*\varepsilon - \sup_{s\le a\varepsilon}\left\{a\varepsilon \sum_{j=1}^d |Z_s A_j(X_s) - A_j(x)|^2\right\} \le \varepsilon\right),$$

and thus, if we take $a = 4/c_*$ in order to have $ac_*/2 = 2$ and apply Markov's inequality, we obtain

$$\begin{aligned}\mathbb{P}(\lambda_t \le \varepsilon) &\le \mathbb{P}\left(\sup_{s\le a\varepsilon}\left\{\sum_{j=1}^d |Z_s A_j(X_s) - A_j(x)|^2\right\} \ge \frac{c_*}{4}\right)\\ &\le d^{q-1}\frac{4^q}{c_*^q}\sum_{j=1}^d \mathbb{E}\Big[\sup_{s\le a\varepsilon}|Z_s A_j(X_s) - A_j(x)|^{2q}\Big],\end{aligned} \tag{A.7}$$

where the last holds for all $q > 1$. Now, to estimate the last term we claim that, for all $j = 1, \dots, d$,

$$\mathbb{E}\Big[\sup_{s\le t}|Z_s A_j(X_s) - A_j(x)|^{2q}\Big] \le Ct^q(t^{1/2}|B|_0|A|_2^3 + |A|_1^2)^{2q} e_{2q}^Z(t)^C, \tag{A.8}$$

for a constant C depending on q, m, d but not on the bounds on B and A. From (A.7) and this last estimate, it follows that

$$\mathbb{P}(\lambda_t \le \varepsilon) \le C_{q,m,d}\frac{\varepsilon^q}{c_*^{2q}}(t^{1/2}|B|_0|A|_2^3 + |A|_1^2)^{2q} e_{2q}^Z(t)^{C_{q,m,d}},$$

for any ε such that $4\varepsilon/c_* < 1 \wedge t$.

Step 3. We finally estimate $\mathbb{E}[\lambda_t^{-2mp}]$. We write

$$\begin{aligned}\mathbb{E}[\lambda_t^{-2mp}] &= \mathbb{E}\big[\lambda_t^{-2mp}1_{\{\lambda_t>1\}}\big] + \sum_{k=1}^\infty \mathbb{E}\big[\lambda_t^{-2mp}1_{\{1/(k+1)<\lambda_t\le 1/k\}}\big]\\ &\le 1 + \sum_{k=1}^\infty (k+1)^{2mp}\mathbb{P}\big(1/(k+1) < \lambda_t \le 1/k\big),\end{aligned}$$

and separate the contribution of the sum over $k > \frac{4}{tc_*}$ to obtain

$$\begin{aligned}
\mathbb{E}[\lambda_t^{-2mp}] &\le 1 + \sum_{1\le k\le 4/(tc_*)} (k+1)^{2mp}\mathbb{P}\bigl(1/(k+1) < \lambda_t \le 1/k\bigr) \\
&\quad + \sum_{k>4/(tc_*)} (k+1)^{2mp}\mathbb{P}(\lambda_t \le 1/k) \\
&\le 1 + \left(\frac{4}{tc_*}+1\right)^{2mp}\mathbb{P}(\lambda_t \le 1) \\
&\quad + C_{q,m,d}e_{2q}^Z(t)^{C_{q,m,d}}\frac{1}{c_*^{2q}}(t^{1/2}|B|_0|A|_2^3 + |A|_1^2)^{2q} \\
&\quad \times \sum_{k>4/(tc_*)} (k+1)^{2mp}\frac{1}{k^q}.
\end{aligned}$$

We finally take $q = 2mp + 2$ in order to get convergent series. This last estimate, together with (A.6), gives the desired result.

PROOF OF (A.8). We apply Itô's formula to the product $Z_t A_j(X_t)$ and get

$$\begin{aligned}
d(Z_tA_j(X_t)) &= Z_t\left\{(\partial A_jB - \partial BA) + \sum_{l=1}^d \partial A_l(\partial A_lA_j - \partial A_jA_l)\right\}(X_t)\,dt \\
&\quad + Z_t\left(\frac{1}{2}\partial_{k_1}\partial_{k_2}A_jA_l^{k_1}A_l^{k_2}\right)(X_t)\,dt \qquad \text{(A.9)} \\
&\quad + Z_t\sum_{l=1}^d(\partial A_jA_l - \partial A_lA_j)(X_t)\,dW^l(t).
\end{aligned}$$

Hence, by Burkholder's inequality,

$$\begin{aligned}
&\sup_{i=1,\dots,m}\mathbb{E}\Bigl[\sup_{s\le t}\bigl|\bigl(Z_sA_j(X_s) - A_j(x)\bigr)^i\bigr|^{2q}\Bigr] \\
&\quad \le C\{t^{2q-1}(|B|_0|A|_1 + |B|_1|A|_0 + |A|_1^2|A|_0 + |A|_2|A|_0^2)^{2q} \\
&\qquad\qquad + t^{q-1}(|A|_1|A|_0)^{2q}\} \\
&\qquad \times \int_0^t \mathbb{E}\Bigl[\sup_{h,k=1,\dots,m}|(Z_u)_{h,k}|^{2q}\Bigr]\,du \\
&\quad \le Ct^q(t^{1/2}|B|_0|A|_2^3 + |A|_1^2)^{2q}e_{2q}^Z(t)^C,
\end{aligned}$$

where the constant C depends on q, m and d, but not on t and on the bounds on B and A and their derivatives. In the last step, we have once again used bound (2.15) on the entries of Z. □

A.2. Proof of Proposition 3.1. We first collect the basic facts we need to give the proof of Proposition 3.1. We will start by proving existence and uniqueness of strong solutions for the following equation:

$$X_t = x + \int_0^t \big(a(X_s) - b(X_s)X_s\big)\,ds + \int_0^t \gamma(X_s)|X_s|^\alpha\,dW_s, \qquad t \ge 0, \alpha \in [1/2, 1), \tag{A.10}$$

whose coefficients are defined on the whole real line [a, b and γ are the functions appearing in (3.1)]. Once we have established that the unique strong solution of (A.10) is a.s. positive, then (A.10) will coincide with the original equation (3.1).

The proof of Proposition 3.1 is split in the following two short lemmas.

LEMMA A.1. *Assume condition* (s0) *of Proposition* 3.1. *Then existence and uniqueness of strong solutions hold for* (A.10). *Moreover, for any initial condition* $x \ge 0$ *the solution is a.s. positive*, $\mathbb{P}(X_t \ge 0; t \ge 0) = 1$.

PROOF. Existence of nonexplosive weak solutions for (A.10) follows from continuity and sub-linear growth of drift and diffusion coefficients. The existence of weak solutions together with pathwise uniqueness imply the existence of strong solutions (cf. [12], Proposition 5.3.20 and Corollary 5.3.23). Pathwise uniqueness follows in its turn from a well-known theorem of uniqueness of Yamada and Watanabe (cf. [12], Proposition 5.2.13). Indeed, as $a, b, \gamma \in C_b^1$ the diffusion coefficient of (A.10) is locally Hölder-continuous of exponent $\alpha \ge 1/2$ and the drift coefficient is locally Lipschitz-continuous. We apply the standard localization argument for locally Lipschitz coefficients and the Yamada–Watanabe theorem to establish that solutions are pathwise unique up to their exit time from a compact ball, and hence pathwise uniqueness holds for (A.10). □

Lemma A.2 deals with the second part of Proposition 3.1, that is, the behavior at zero. The proof is based on Feller's test for explosions of solutions of one-dimensional SDEs (cf. [12], Theorem 5.5.29). Letting τ denote the exit time from $(0, \infty)$, that is, $\tau = \inf\{t \ge 0 : X_t \notin (0, \infty)\}$ with $\inf \varnothing = \infty$, we have to verify that

$$\lim_{x \to 0} p_c(x) = -\infty \tag{A.11}$$

with p_c defined by

$$p_c(x) := \int_c^x \exp\left(-2\int_c^y \frac{a(z) - b(z)z}{\gamma(z)^2 z^{2\alpha}}\,dz\right)dy, \qquad x > 0, \tag{A.12}$$

for a fixed $c > 0$. Property (A.11) implies that $\mathbb{P}(\tau = \infty) = 1$, then $\tau \equiv \tau_0$ with τ_0 as defined in Proposition 3.1, because the solution of (A.10) does not explode at

∞ (cf. Lemma A.1). The inner integral in (A.12) is well defined and finite for any $y > 0$ because $\gamma(z)^2 > 0$ for any $z > 0$ and γ is continuous.

REMARK A.1. The conclusion does not depend on the choice of $c \in (0, \infty)$.

LEMMA A.2. *Assume* (s0), *and let* $X = (X_t; t \geq 0)$ *denote the unique strong solution of* (A.10) *for initial condition* $x > 0$. *Then the statements of Proposition* 3.1 *on the stopping time* τ_0 *hold true.*

PROOF. We prove (A.11), for $c = 1$. We assume without restriction that $x < 1$ and distinguish the two cases.

Case $\alpha > 1/2$. We have $a(z) \geq a(0) - |a|_1 z$, $z > 0$. Then

$$\frac{a(z) - b(z)z}{\gamma(z)^2 z^{2\alpha}} \geq \frac{a(0) - (|a|_1 + b(z))z}{\gamma(z)^2 z^{2\alpha}}$$
$$\geq \frac{a(0)}{|\gamma|_0^2 z^{2\alpha}} - \frac{|a|_1 + |b|_0}{\gamma(z)^2 z^{2\alpha-1}},$$

$\frac{1}{\gamma(z)^2 z^{2\alpha-1}}$ is integrable at zero by (s1)$'$, and then there exists a positive constant K such that

$$-2\int_1^y \frac{a(z) - b(z)z}{\gamma(z)^2 z^{2\alpha}} \geq \frac{2a(0)}{|\gamma|_0^2}\int_y^1 \frac{dz}{z^{2\alpha}} + K$$
$$= \frac{2a(0)}{(2\alpha - 1)|\gamma|_0^2}\left(\frac{1}{y^{2\alpha-1}} - 1\right) + K$$

hence

$$p_1(x) \leq -C\int_x^1 \exp\left(\frac{2a(0)}{(2\alpha - 1)|\gamma|_0^2}\frac{1}{y^{2\alpha-1}}\right)dy$$
$$= -C\int_1^{1/x}\frac{1}{t^2}\exp\left(\frac{2a(0)}{(2\alpha - 1)|\gamma|_0^2}t^{2\alpha-1}\right)dt \underset{x\to 0^+}{\to} -\infty.$$

Case $\alpha = 1/2$. By (s2),

$$\frac{2a(z)}{\gamma(z)^2 z} \geq \frac{1}{z},$$

for $z < \overline{x}$. Hence

$$2\frac{a(z) - b(z)z}{\gamma(z)^2 z} \geq \frac{1}{z} - 2|b|_0\frac{1}{\gamma(z)^2},$$

and thus, $\frac{1}{\gamma^2}$ being integrable at zero, for $x < \overline{x}$ we have

$$p_1(x) \le -C \int_x^1 \exp\biggl(\int_y^1 \frac{1}{z}\,dz\biggr)\,dy$$
$$= -C \int_x^1 \frac{1}{y}\,dy \underset{x\to 0^+}{\to} -\infty. \qquad \square$$

Acknowledgments. I am grateful to Vlad Bally of University Paris-Est Marne-la-Vallée for introducing me to this subject and for many stimulating discussions and to Alexander Yu. Veretennikov of University of Leeds for useful insights. I sincerely thank an anonymous referee for reading a previous version of this paper and for providing helpful comments to improve the presentation.

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