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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE INVERSE BOUNDARY CROSSING PROBLEM FOR DIFFUSIONS

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We study the inverse boundary crossing problem for diffusions. Given a diffusion process X_t, and a survival distribution p on $[0, \infty)$, we demonstrate that there exists a boundary $b(t)$ such that $p(t) = \mathbb{P}[\tau > t]$, where τ is the first hitting time of X_t to the boundary $b(t)$. The approach taken is analytic, based on solving a parabolic variational inequality to find b. Existence and uniqueness of the solution to this variational inequality were proven in earlier work. In this paper, we demonstrate that the resulting boundary b does indeed have p as its boundary crossing distribution. Since little is known regarding the regularity of b arising from the variational inequality, this requires a detailed study of the problem of computing the boundary crossing distribution of X_t to a rough boundary. Results regarding the formulation of this problem in terms of weak solutions to the corresponding Kolmogorov forward equation are presented.

1. Introduction. Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion defined on a filtered probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ satisfying the usual conditions. We consider a diffusion process $\{X_t\}_{t\geq 0}$ defined by the stochastic differential equation

$$dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)\,dB_t \qquad \forall t > 0,$$

where $\mu, \sigma : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ are smooth bounded functions, with bounded derivatives[4] and $\inf_{\mathbb{R}\times\mathbb{R}_+} \sigma > 0$. We assume that X_0 is independent of B, and has initial distribution $\mathbb{P}(X_0 \leq x) = p_0(x, 0)$, with density $\rho_0(x, 0)$. For $y \in \mathbb{R}$ and $t \geq s$, we further denote by $F(y, s; \cdot, t)$ and $\rho(y, s; \cdot, t)$ the transition distribution and density, respectively, of X_t given $X_s = y$:

$$F(y, s; x, t) := \mathbb{P}(X_t \leq x | X_s = y), \qquad \rho(y, s; x, t) := \frac{\partial F(y, s; x, t)}{\partial x}.$$

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[4]The main results in the paper are true under less restrictive assumptions on the coefficients, although we have not sought to determine the minimal assumptions under which they hold. The stated assumptions ensure that the applications of various results from the theory of PDEs, a change of variables, and an application of Itô's lemma are all valid.

In the sequel, we denote by $\rho_0(\cdot, t)$ and $p_0(\cdot, t)$ the density[5] and cumulative distribution of X_t:

$$p_0(x,t) := \mathbb{P}(X_t \le x) = \int_{\mathbb{R}} \rho_0(y,0) F(y,0;x,t)\, dy,$$

$$\rho_0(x,t) := \frac{\partial p_0(x,t)}{\partial x} := \int_{\mathbb{R}} \rho_0(y,0) \rho(y,0;x,t)\, dy.$$

For a given function $b : \mathbb{R}_+ \to [-\infty, \infty)$, the first boundary crossing time $\hat{\tau}$ is defined to be

$$\hat{\tau} = \inf\{t > 0 | X_t \le b(t)\}. \tag{1.1}$$

We shall also have occasion to consider the related, but less commonly used, first time that X_t goes strictly below b:

$$\tau = \inf\{t > 0 | X_t < b(t)\}. \tag{1.2}$$

We are interested in the following two problems.

1. *The boundary crossing problem*: for a given function $b : \mathbb{R}_+ \to [-\infty, \infty)$, compute the survival distribution of the first time that X crosses b; that is,

$$p(t) = \mathbb{P}(\hat{\tau} \ge t). \tag{1.3}$$

   In this case, we denote $p = \mathcal{P}[b]$.
2. *The inverse boundary crossing problem*: for a given survival distribution p on $(0, \infty)$ find a function b such that b satisfies (1.1), (1.3). If such a boundary exists and is unique, we denote it by $\mathcal{B}[p]$.

The boundary crossing problem is classical, and the subject of a large literature. The inverse boundary crossing problem has recently been the subject of increased interest by probabilists and researchers in mathematical finance. The main purpose of this paper is to show that the inverse boundary crossing problem is well posed.

According to Zucca and Sacerdote (2009), the problem was originally posed by A. N. Shiryayev in 1976, for the special case where X_t is a Brownian motion and p is the exponential distribution. Dudley and Gutmann (1977) and Anulova (1980) showed that there exists a *stopping time* with the given distribution; however, this stopping time is not realized as the first time the process X crosses a boundary b. Recently, there has been an increase of interest in the problem due to its importance in applications. In mathematical finance, with X_t an indicator of a firm's financial health, and p the distribution of its time to default (estimated from the prices of market instruments), the problem is to find a default barrier that reproduces the given default distribution. Many authors have proposed numerical methods for finding such a boundary, including Hull and White (2001), Iscoe and

[5] At $t = 0$, ρ_0 is interpreted as the distributional derivative of p_0.

Kreinin (2002), Huang and Tian (2006), Avellaneda and Zhu (2001) and Zucca and Sacerdote (2009). A formulation of the problem in terms of nonlinear Volterra integral equations has been given by Peskir (2002) [see also Peskir and Shiryayev (2005) for a more detailed discussion].

The numerical method proposed by Avellaneda and Zhu (2001) is most relevant to our work. They note that for sufficiently smooth boundaries b, the function $U(x,t) = \partial_x \mathbb{P}(\hat{\tau} \geq t, X_t \leq x)$ should be the solution of the free boundary problem

$$\begin{cases} \mathcal{L}_1 U(x,t) = 0, & \text{for } x > b(t),\ t > 0, \\ U(x,t) = 0, & \text{for } x \leq b(t),\ t > 0, \\ U(x,0) = \rho_0(x,0), & \text{for } x \in \mathbb{R}, \end{cases} \tag{1.4}$$

with the free boundary condition

$$\dot{p}(t) = -\tfrac{1}{2}(\sigma^2 U)_x|_{x=b(t)} \qquad \forall t \geq 0, \tag{1.5}$$

where $\mathcal{L}_1$ is the differential operator

$$\mathcal{L}_1 \phi := \frac{\partial \phi}{\partial t} - \frac{1}{2}\frac{\partial^2[\sigma^2 \phi]}{\partial x^2} + \frac{\partial[\mu\phi]}{\partial x}. \tag{1.6}$$

Avellaneda and Zhu (2001) perform a change of variables to "straighten out the boundary," and then solve the resulting transformed PDE numerically using finite differences.

An analytic study of the inverse boundary crossing problem was initiated by Cheng et al. (2006). In that work, we defined

$$w(x,t) = \int_x^\infty U(y,t)\, dy. \tag{1.7}$$

Formally, direct calculation from (1.4), (1.5) shows that w should satisfy the free boundary problem

$$\begin{cases} \mathcal{L} w(x,t) = 0, & \text{for } x > b(t),\ t > 0, \\ w(x,t) = p(t), & \text{for } x \leq b(t),\ t > 0, \\ w_x(x,t) = 0, & \text{for } x \leq b(t),\ t > 0, \\ w(x,0) = 1 - p_0(x,0), & \text{for } x \in \mathbb{R}, \end{cases} \tag{1.8}$$

where $\mathcal{L}$ is the differential operator

$$\mathcal{L}\phi := \frac{\partial \phi}{\partial t} - \frac{1}{2}\frac{\partial}{\partial x}\left(\sigma^2 \frac{\partial \phi}{\partial x}\right) + \mu \frac{\partial \phi}{\partial x}. \tag{1.9}$$

Formally,

$$w(x,t) = \mathbb{P}(\hat{\tau} \geq t, X_t > x), \tag{1.10}$$

providing the connection between the probabilistic problems (1.1), (1.3) and our analytic approach. Based on the free boundary problem (1.8), one can infer that w should satisfy the variational inequality

$$\begin{aligned} &\max\{\mathcal{L} w, w - p\} = 0 \qquad \text{in } L^\infty\big(\mathbb{R} \times (0,\infty)\big), \\ &w(\cdot,0) = 1 - p_0(\cdot,0) \qquad \text{on } \mathbb{R}, \end{aligned} \tag{1.11}$$

and b can be recovered from w by

$$b(t) := \inf\{x | p(t) - w(x,t) > 0\} \qquad \forall t > 0. \tag{1.12}$$

In Cheng et al. (2006), the existence and uniqueness of a viscosity solution to (1.11) was proved. However, no attempt was made to connect the resulting functions w, b to the original probabilistic formulation of the inverse boundary crossing problem. In this paper, we show that b does in fact give a boundary that reproduces the survival distribution p. This is complicated by the fact that it is very difficult to prove the regularity of the boundary b derived from the variational inequality.[6] As a consequence, in order to verify that b has the required hitting distribution, we must first study the problem of computing the boundary crossing probabilities of diffusions to nonsmooth boundaries. To this end, for a given function $b : (0, \infty) \to [-\infty, \infty)$, we define

$$b^*(t) := \max\left\{b(t), \overline{\lim_{s \to t}} b(s)\right\}, \qquad b^*_-(t) := \overline{\lim_{s \nearrow t}} b(s) \qquad \forall t > 0. \tag{1.13}$$

When needed, we also define $b^*(0) := \overline{\lim}_{s \searrow 0} b(s)$. We also employ the notation $Q_b := \{(x,t) | x > b(t), t > 0\}$. It will turn out that the inverse boundary crossing problem is most naturally formulated in the following spaces:

$$B_0 := \left\{b : (0,\infty) \to [-\infty,\infty) | b = b^* = b^*_-, \mathbb{P}\left(\bigcup_{\varepsilon > 0} \bigcap_{s \in (0,\varepsilon)} \{X_s \geq b(s)\}\right) = 1\right\},$$

$$P_0 := \{p \in C([0,\infty)) | p(0) = 1 \geq p(s) \geq p(t) > 0 \ \forall t > s \geq 0\}.$$

The main result of this paper is the following theorem.

THEOREM 1.

1. *For every $p \in P_0$, there exists a unique viscosity solution, w, for the survival distribution of the inverse boundary crossing problem associated with p [i.e., a viscosity solution of problem (1.11)]. In addition the unique solution, w, satisfies*

$$\begin{aligned} 0 \leq 1 - [w(x,t) + p_0(x,t)] \leq 1 - p(t), \\ w(x,t) \leq w(y,t) \qquad \forall t \geq 0, x \in \mathbb{R}, x \geq y. \end{aligned} \tag{1.14}$$

Consequently, the operator $\mathcal{B}$

$$b(t) = \mathcal{B}[p](t) := \inf\{x \in \mathbb{R} | w(x,t) < p(t)\} \qquad \forall t > 0, \tag{1.15}$$

is well defined on P_0.

[6] The problem of the regularity of the boundary has subsequently been investigated by Chen (2011).

2. *For every* $p \in P_0$, $\mathcal{B}[p] \in B_0$ *and* $(\mathcal{P} \circ \mathcal{B})[p] = p$, *where* $(\mathcal{P}[b])(t) = \mathbb{P}(\hat{\tau} \geq t)$ *and* $\hat{\tau}$ *is defined as in* (1.1).
   (This implies that for a given $p \in P_0$, $b := \mathcal{B}[p]$ is a solution of the inverse problem since $\mathcal{P}[b] = (\mathcal{P} \circ \mathcal{B})[p] = p$.)
3. *For every* $b \in B_0$, $\mathcal{P}[b] \in P_0$ *and* $(\mathcal{B} \circ \mathcal{P})[b] = b$.
   (This implies that for a given $p \in P_0$, if $\tilde{b} \in B_0$ is a solution of the inverse problem, i.e., $\mathcal{P}[\tilde{b}] = p$, then $\tilde{b}$ can be identified as $\tilde{b} = (\mathcal{B} \circ \mathcal{P})[\tilde{b}] = \mathcal{B}[p]$, the viscosity solution of the inverse problem.)
4. *If* (w, b) *is a classical* (*i.e.*, $w + p_0 \in C(\mathbb{R} \times [0, \infty))$, $\partial_x w \in C(\mathbb{R} \times (0, \infty)$, $\partial_t w, \partial_{xx} w \in C(Q_b)$) *solution of the free boundary problem* (1.8), *then* b *is the solution of the inverse boundary crossing problem associated with* p, *that is*, $\mathcal{P}[b] = p$. *Similarly, if* (U, b) *is a classical* (*i.e.*, $U - \rho_0 \in C(\mathbb{R} \times [0, \infty))$, $\partial_t U, \partial^2_{xx} U \in C(Q_b)$) *solution of the free boundary problem* (1.4), (1.5), *then* $\mathcal{P}[b] = p$.

The proof of the above theorem proceeds as follows. We begin by studying the direct problem of computing the distribution of $\hat{\tau}$, and the function $w(x, t) = \mathbb{P}(\hat{\tau} \geq t, X_t > x)$ for boundaries $b \in B_0$. By considering a carefully constructed discrete approximation scheme motivated by (1.4) when b is known, we are able to show that w is the unique viscosity solution to (1.8). Elementary calculations verify that the viscosity solution of the variational inequality (1.11) also solves (1.8). Once we have also determined that $\{x > b\} = \{w < p\}$, the verification proceeds by relatively straightforward arguments.

We note that the sequence of stopping times constructed in our discrete time approximation actually converges to the first time that X is strictly below the boundary b, τ as given by (1.2). This definition of the boundary crossing time is slightly different from the standard one (1.1) for $\hat{\tau}$. We have found that for the analytic approach we take here (particularly for rough boundaries), our definition is more convenient. In Section 2 below, we show that for boundaries with minimal regularity properties (including those arising from the solution to the inverse boundary crossing problem, $b \in B_0$), $\mathbb{P}(\tau = \hat{\tau}) = 1$.

The remainder of the paper is structured as follows. The second section proves measurability properties of τ and $\hat{\tau}$, and proves that these times are almost surely equal. In addition, it gives preliminary results that are needed for the study of our approximation scheme. The third section studies the approximation scheme in detail, and proves convergence. The convergence provides a rigorous connection between the probabilistic definition of the survival probability p and the PDE definition of the survival distribution w. The fourth section formulates viscosity solutions for the *direct problem* of computing p for a given $b \in B_0$, and shows that the survival distribution w gives the unique viscosity solution for the direct problem. The fifth section provides the link between the variational inequality studied in Cheng et al. (2006) and the inverse boundary crossing problem. It also provides a sufficient condition under which the resulting boundary b is continuous.

2. Crossing times of upper-semi-continuous boundaries. We calculate boundary crossing distributions for rough boundaries based on discrete time approximations to be studied in the next section. In order to ensure convergence of our approximation scheme, the time points used must be chosen carefully. We refer to the selected points as the "landmark points" of the boundary. In this section, we begin by defining the landmark points and investigating their properties. Using these properties, we study the measurability of τ and $\hat{\tau}$, show that the boundary crossing times of b and b^* are equal and that $\mathbb{P}(\tau = \hat{\tau}) = 1$ for $b \in B_0$.

DEFINITION 1. Let $b : (0, \infty) \to [-\infty, \infty)$, and b^* be its upper-semi-continuous envelope. The set of landmark points of b, denoted by $\mathbb{A}(b)$, is defined as follows:

$$\mathbb{A}(b) := \bigcup_{n \in \mathbb{N}} \mathbb{A}_n(b), \qquad \mathbb{A}_n(b) := \{t_n^i | i \in \mathbb{N}\}, \tag{2.1}$$

$$t_n^i := \inf\left\{ t \in \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right] \middle| b^*(t) \geq \sup_{[2^{-n}i, 2^{-n}(i+1)]} b(\cdot) \right\}. \tag{2.2}$$

The following lemma summarizes some properties of the landmark points that are used in the paper.

LEMMA 2.1. *Let $b : (0, \infty) \to [-\infty, \infty)$, and let its landmark points $\mathbb{A}(b)$ be defined as in* (2.2).

1. *For $i, n \in \mathbb{N}$, $b^*(t_n^i) \geq b^*(s)$ for every $s \in [2^{-n}i, 2^{-n}(i+1))$.*
2. *For $i, n \in \mathbb{N}$, either $t_n^i = t_{n+1}^{2i}$ or $t_n^i = t_{n+1}^{2i+1}$, so $\mathbb{A}_n(b) \subset \mathbb{A}_{n+1}(b)$.*

PROOF. 1. If $s \in (2^{-n}i, 2^{-n}(i+1))$, then by definition $b^*(t_n^i) \geq b(t)$ for $t \in (s - \varepsilon, s + \varepsilon)$ with ε small enough. $b^*(t_n^i) \geq b^*(s)$ follows immediately. If $s = 2^{-n}i$, and $b^*(s) \geq b^*(t_n^i) \geq \sup\{b(t) | t \in [2^{-n}i, 2^{-n}(i+1)]\}$ then $s = t_n^i$ by (2.2).

2. We first claim that if $t_n^i \in [2^{-n}i, 2^{-(n+1)}(2i+1))$ then $t_n^i = t_{n+1}^{2i}$. Clearly, $b^*(t_n^i) \geq \sup\{b(s) | s \in [2^{-n}i, 2^{-n}(i+1)]\} \geq \sup\{b(s) | s \in [2^{-n}i, 2^{-(n+1)}(2i+1)]\}$, so by definition $t_{n+1}^{2i} \leq t_n^i$. If the inequality is strict, there is a $\delta > 0$ small enough so that $(t_n^i - \delta, t_n^i + \delta) \subseteq (2^{-n}i, 2^{-(n+1)}(2i+1))$, and since $b^*(t_{n+1}^{2i}) \geq b$ on this interval, we obtain $b^*(t_{n+1}^{2i}) \geq b^*(t_n^i)$, contradicting the definition of t_n^i. A similar proof shows that if $t_n^i \in [2^{-(n+1)}(2i+1), 2^{-n}(i+1))$, then $t_n^i = t_{n+1}^{2i+1}$. Finally, it is easy to see that if $t_n^i = 2^{-n}(i+1)$, then $\sup\{b(s) | s \in [2^{-n}i, 2^{-n}(i+1)]\} = \sup\{b(s) | s \in [2^{-(n+1)}(2i+1), 2^{-n}(i+1)]\} \geq \sup\{b(s) | s \in [2^{-n}i, 2^{-(n+1)}(2i+1)]\}$, after which repeating the same argument by contradiction ensures that $t_n^i = t_{n+1}^{2i+1}$. □

The following lemma collects some properties of upper-semi-continuous functions that are used throughout the paper. The proofs are elementary, and are omitted.

LEMMA 2.2. *Let $b:(0,\infty)\to[-\infty,\infty)$ be upper-semi-continuous.*

1. *If $x:[0,\infty)\to(-\infty,\infty)$ is continuous, then for all $t>0$,*

$$\inf\{s>0|x(s)\le b(s)\}>t \quad\Longleftrightarrow\quad x(s)>b(s) \qquad \forall s\in(0,t]$$

2. *The set $Q_b:=\{(x,t)|x>b(t),t>0\}$ is open.*

The next proposition addresses two main issues. First, it considers the measurability of τ and $\hat{\tau}$, to ensure that the various functions considered in the remainder of the paper are well defined. Second, it shows that for the purposes of computing the distribution of τ, it is enough to consider the upper-semi-continuous envelope, b^*, of the boundary b. We observe that the result for τ holds with minimal assumptions on the function b (we have not even assumed measurability). Furthermore, we note that some of the results on measurability could be derived by applying more general theorems [e.g., $\hat{\tau}$ is the first hitting time of the two-dimensional process (X_t,t) to the set $\{(x,s)|x\le b(s)\}$, which is closed when b is upper-semi-continuous]. However, we have decided to present elementary proofs of these assertions to make the paper more self-contained.

PROPOSITION 1. *Let $b:(0,\infty)\to[-\infty,\infty)$.*

1. *Let b^* be as in (1.13) and $\mathbb{A}(b)$ be as in (2.1) and (2.2). Then for every $t>0$,*

$$\bigcap_{s\in(0,t)}\{X_s\ge b(s)\}=\bigcap_{s\in(0,t)\cap\mathbb{A}(b)}\{X_s\ge b^*(s)\}. \tag{2.3}$$

Consequently, we can define the first boundary crossing time $\tau:\Omega\to[0,\infty]$, *the* survival probability $p:[0,\infty)\to[0,1]$, *and the* survival distribution $w:\mathbb{R}\times[0,\infty)\to[0,1]$ *by*

$$\tau(\omega):=\inf\{s>0|X_s(\omega)<b(s)\} \qquad \forall\omega\in\Omega, \tag{2.4}$$

$$p(t):=\mathbb{P}\big(X_s\ge b(s)\ \forall s\in(0,t)\big) \qquad \forall t\ge 0, \tag{2.5}$$

$$w(x,t):=\mathbb{P}\big(X_s\ge b(s)\ \forall s\in(0,t),X_t>x\big) \qquad \forall x\in\mathbb{R},t\ge 0. \tag{2.6}$$

In addition, τ is an optional time with respect to the filtration generated by the process X, $\{\tau\ge t\}\in\mathcal{F}_t^X$, $\forall t\ge 0$. Also

$$p(t)=\mathbb{P}(\tau\ge t), \qquad w(x,t)=\mathbb{P}(\tau\ge t,X_t>x).$$

2. *Let the* (conventional) first crossing time $\hat{\tau}:\Omega\to[0,\infty]$ *be defined by*

$$\hat{\tau}(\omega):=\inf\{s>0|X_s(\omega)\le b(s)\} \qquad \forall\omega\in\Omega.$$

If b is upper-semi-continuous, i.e., $b=b^*$, then $\hat{\tau}$ is a stopping time with respect to the filtration generated by the process X, $\{\omega\in\Omega|\hat{\tau}(\omega)>t\}\in\mathcal{F}_t^X$ $\forall t\ge 0$, so that we can define $\hat{p}(t):=\mathbb{P}(\hat{\tau}(\omega)>t)$ $\forall t\ge 0$.

PROOF. 1. When $t = 0$, we have $\{\tau \geq 0\} = \Omega = \{\omega \in \Omega | X_s(\omega) \geq b(s), \text{Con } \forall s \in (0,t)\}$ and $p(0) = \mathbb{P}(\Omega) = 1$. Now we assume that $t > 0$. It is easy to verify that

$$\bigcap_{s\in(0,t)} \{X_s \geq b(s)\} = \{X_s \geq b(s)\ \forall s \in (0,t)\} = \{\tau \geq t\}.$$

Hence, to complete the proof of the first assertion, it suffices to verify (2.3). Suppose $\omega \in \{X_s \geq b(s)\ \forall s \in (0,t)\}$. Then $X_s(\omega) \geq b(s)$ for every $s \in (0,t)$. For every $\hat{s} \in (0,t)$, by the continuity of $X_\bullet(\omega)$,

$$X_{\hat{s}}(\omega) = \lim_{s\to\hat{s}} X_s(\omega) \geq \max\Big\{b(\hat{s}), \overline{\lim_{s\to\hat{s}}}\, b(s)\Big\} = b^*(\hat{s}).$$

As $\hat{s} \in (0,t)$ is arbitrary, we have $\omega \in \{X_s \geq b^*\ \forall s \in (0,t)\}$. Thus, $\{X_s \geq b(s)\ \forall s \in (0,t)\} \subset \{X_s \geq b^*(s)\ \forall s \in (0,t)\} \subset \{X_s \geq b^*(s)\ \forall s \in (0,t) \cap \mathbb{A}(b)\}$.

Next, suppose $\omega \in \{X_s(\omega) \geq b^*(s)\ \forall s \in (0,t) \cap \mathbb{A}(b)\}$. Let $\hat{s} \in (0,t)$ be arbitrary. We want to show that $X_{\hat{s}} \geq b(\hat{s})$. For each integer n satisfying $2^{-n} \leq \hat{s}$, let i_n be the integer such that $\hat{s} \in [i_n 2^{-n}, (i_n+1)2^{-n})$. Then $t_n^{i_n} \in \mathbb{A}(b)$ and $X_{t_n^{i_n}}(\omega) \geq b^*(t_n^{i_n}) \geq b^*(\hat{s})$ by Lemma 2.1. Hence,

$$X_{\hat{s}}(\omega) = \lim_{n\to\infty} X_{t_n^{i_n}}(\omega) \geq \overline{\lim_{n\to\infty}}\, b^*(t_n^{i_n}) \geq b^*(\hat{s}) \geq b(\hat{s}).$$

Since $\hat{s}$ is arbitrary, we see that $\omega \in \{X_s \geq b(s)\ \forall s \in (0,t)\}$. Consequently,

$$\begin{aligned}\{\tau \geq t\} &= \bigcap_{s\in(0,t)} \{X_s \geq b(s)\} = \bigcap_{s\in(0,t)} \{X_s \geq b^*(s)\} \\ &= \bigcap_{s\in(0,t)\cap\mathbb{A}(b)} \{X_s \geq b^*(s)\} \in \mathcal{F}_t^X.\end{aligned}$$

This proves (2.3) and also the first assertion.

2. Assume that b is usc, that is, $b = b^*$. Then by the continuity of the sample paths of X, if $\hat{s} > 0$ and $X_{\hat{s}}(\omega) > b(\hat{s})$ then there exists $\delta > 0$ such that $X_s(\omega) > b(s) + \delta$ for all $s \in [\hat{s}-\delta, \hat{s}+\delta]$. By the Heine–Borel theorem, if $X_s > b(s)$ for every $s \in [a,c] \subset (0,\infty)$, then there exists a large integer i such that $X_s > b(s) + 2^{-i}$ for every $s \in [a,c]$. Hence, for every $t > 0$,

$$\begin{aligned}\{\hat{\tau} > t\} &= \bigcap_{s\in(0,t]} \{X_s > b(s)\} = \bigcap_{n\in\mathbb{N}} \bigcap_{s\in[2^{-n}t,t]} \{X_s > b(s)\} \\ &= \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \bigcap_{s\in[2^{-n}t,t]} \{X_s \geq b(s) + 2^{-i}\} \qquad (2.7) \\ &= \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \bigcap_{s\in\{2^{-n}t,t\}\cup(\mathbb{A}(b)\cap[2^{-n}t,t])} \{X_s \geq b(s) + 2^{-i}\} \in \mathcal{F}_t^X.\end{aligned}$$

This completes the proof of the second assertion. □

The following proposition justifies our choice to work with τ, the first time the process is strictly below the boundary, rather than $\hat{\tau}$, the first time the process hits the boundary. In particular, the second assertion implies that these times are almost surely equal, and hence they have the same distributions (so solving the inverse boundary crossing problem for τ is the same as solving it for $\hat{\tau}$). We will see in the next section that it is easier to work with τ in defining approximations for the boundary crossing problem.

PROPOSITION 2. *Let* $b:(0,\infty)\to[-\infty,\infty)$.

1. *Let* b^* *and* b^*_- *be as in* (1.13). *Then*

$$\{\omega\in\Omega|\tau(\omega)=t \text{ or } \hat{\tau}(\omega)=t\}\subset\{\omega\in\Omega|X_t(\omega)\in[b^*_-(t),b^*(t)]\}\qquad\forall t>0.$$

2. *The set* $\{\tau\neq\hat{\tau}\}$ *has probability zero, so that*

$$p(t)=p(t-)=\hat{p}(t-)=\mathbb{P}(\tau\geq t)\qquad\forall t>0,$$
$$\hat{p}(t)=\hat{p}(t+)=p(t+)=\mathbb{P}(\tau>t)\qquad\forall t\geq 0.$$

Consequently, if $b^*=b^*_-$, *then* $p\in C((0,\infty))$, $\hat{p}=p$ *on* $(0,\infty)$, *and* $\hat{p}\in C([0,\infty))$.

PROOF. To prove the first assertion, let $t>0$ and $\omega\in\{\tau=t\}\cup\{\hat{\tau}=t\}$. Then $X_s(\omega)\geq b(s)$ for all $s\in(0,t)$ so $X_t(\omega)\geq\overline{\lim}_{s\nearrow t}b(s)=b^*_-(t)$. Also, $\tau(\omega)=t$ or $\hat{\tau}(\omega)=t$ implies that there exists a sequence $\{s_i\}$ of positive numbers such that $\lim_{i\to\infty}s_i=t$ and $X_{s_i}(\omega)\leq b(s_i)$ for all i. This implies that $X_t(\omega)=\lim_{i\to\infty}X_{s_i}(\omega)\leq\overline{\lim}_{i\to\infty}b(s_i)\leq b^*(t)$. Hence, $X_t(\omega)\in[b^*_-(t),b^*(t)]$. Also, note that if $b^*_-(t)=b^*(t)$, then $\mathbb{P}(\{\tau=t\}\cup\{\hat{\tau}=t\})\leq\mathbb{P}(\{X_t=b^*(t)\})=0$.

Since the family $\{\tau>t+\varepsilon\}_{\varepsilon\geq -t}$ of sets is monotonic in ε, we see that

$$\begin{aligned}p(t+)&=\lim_{\varepsilon\searrow 0}p(t+\varepsilon)=\lim_{\varepsilon\searrow 0}\mathbb{P}(\tau\geq t+\varepsilon)\\&=\mathbb{P}\Big(\bigcup_{\varepsilon>0}\{\tau\geq t+\varepsilon\}\Big)=\mathbb{P}(\tau>t)\qquad\forall t\geq 0,\\p(t-)&=\lim_{\varepsilon\searrow 0}p(t-\varepsilon)=\lim_{\varepsilon\searrow 0}\mathbb{P}(\tau\geq t-\varepsilon)\\&=\mathbb{P}\Big(\bigcap_{\varepsilon>0}\{\tau\geq t-\varepsilon\}\Big)=\mathbb{P}(\tau\geq t)=p(t)\qquad\forall t>0.\end{aligned}$$

Clearly, if $b^*_-(t)=b^*(t)$, then $p(t-)-p(t+)=\mathbb{P}(\tau=t)=0$ so p is continuous at t. Similarly, when $b^*=b$ so $\hat{p}$ is well defined, we have $\hat{p}(t)=\mathbb{P}(\hat{\tau}>t)=\hat{p}(t+)$ and $\hat{p}(t-)=\mathbb{P}(\hat{\tau}\geq t)$.

To complete the proof, it remains to show that the set $\{\tau\neq\hat{\tau}\}$ has probability zero. For every $\omega\in\Omega$,

$$\tau(\omega)=\inf\{s>0|X_s(\omega)<b(s)\}\geq\inf\{s>0|X_s(\omega)\leq b(s)\}=\hat{\tau}(\omega).$$

Now, suppose $\tau(\omega) \neq \hat{\tau}(\omega)$. Then we must have $\hat{\tau}(\omega) < \tau(\omega)$. Set $t = \tau(\omega)$. Then $X_s \geq b(s)$ $\forall s \in (0,t)$. By continuity, we also have $X_s \geq b^*(s)$ $\forall s \in (0,t)$. Set $\hat{t} = \hat{\tau}(\omega)$. If $\hat{t} = 0$, then by definition, there exists $r \in (0,t)$ such that $X_r(\omega) \leq b(r)$. If $\hat{t} > 0$, then by definition, there exists $r \in [\hat{t}, t)$ such that $X_r(\omega) \leq b(r)$. As $X_s(\omega) \geq b^*(s)$ for all $s \in (0,t)$, in either case, we have $r \in (0,t)$ and $X_r(\omega) = b^*(r)$. Taking $r_1 \in \mathbb{A}(b) \cap (0,r)$ and $r_2 \in \mathbb{A}(b) \cap (r,t)$ we obtain $\min_{s\in[r_1,r_2]}\{X_s(\omega) - b^*(s)\} = 0$. Hence,

$$\{\tau \neq \hat{\tau}\} = \{\hat{\tau} < \tau\} \subset \bigcup_{r_1 \in \mathbb{A}(b)} \bigcup_{r_2 \in \mathbb{A}(b)\cap(r_1,\infty)} \mathbf{B}(r_1, r_2),$$

where for every $0 < a < c < \infty$,

$$\mathbf{B}(a,c) = \Big\{\omega \in \Omega \Big| \min_{s\in[a,c]}\{X_s(\omega) - b^*(s)\} = 0\Big\}.$$

Note that for each $c > a > 0$, $\mathbf{B}(a,c)$ is $\mathcal{F}_c$ measurable since

$$\mathbf{B}(a,c) = \Big\{\min_{s\in[a,c]}\{X_s - b_s^*\} \geq 0\Big\} \setminus \bigcup_{n=1}^{\infty}\Big\{\min_{s\in[a,c]}\{X_s - b^*(s)\} \geq 2^{-n}\Big\},$$

$$\Big\{\min_{s\in[a,c]}\{X_s - b^*(s)\} \geq h\Big\} = \bigcap_{s\in\{a,c\}\cup(\mathbb{A}(b)\cap[a,c])} \{X_s \geq b^*(s) + h\} \in \mathcal{F}_c \qquad \forall h \in \mathbb{R}.$$

It remains to show that for each $c > a > 0$, the set $\mathbf{B}(a,c)$ has measure zero. Suppose, on the contrary, that $\mathbb{P}(\mathbf{B}(a,c)) > 0$ for some fixed $c > a > 0$. Fixing $t_0 \in (0,a)$, we then have

$$0 < \mathbb{P}(\mathbf{B}(a,c)) = \int_{\mathbb{R}} \mathbb{P}\big(\mathbf{B}(a,c)|X_{t_0} = z\big)\rho_0(z,t_0)\,dz.$$

Consequently, there exists a finite number $M > 0$ such that

$$\int_{-M}^{M} \mathbb{P}\big(\mathbf{B}(a,c)|X_{t_0} = z\big)\rho_0(z,t_0)\,dz > 0.$$

For each $h \in \mathbb{R}$, we consider the set

$$\mathbf{B}^h(a,c) = \Big\{\min_{s\in[a,c]}\big(X_s - b^*(s)\big) = h\Big\}.$$

For the process $\{X_t\}_{t\geq t_0}$, for each $\omega \in \{X_s = z\}$ and $h \in \mathbb{R}$, we denote by ω^h the element in $\{X_{t_0} = z + h\}$ such that $X_t(\omega^h) = h + X_t(\omega)$ $\forall t \in [t_0, \infty)$. Then

$$\min_{s\in[a,c]}\big(X_s(\omega) - b^*(s)\big) = 0 \quad \Longleftrightarrow \quad \min_{s\in[a,c]}\big(X_s(\omega^h) - b^*(s)\big) = h.$$

Assume for simplicity that we are dealing with Brownian motion. [By a change of variables, we can assume $\sigma \equiv 1$; see Section 4. Since μ is smooth and bounded,

if $\{X_t\}$ is not a Brownian motion, we can use the Girsanov theorem [Karatzas and Shreve (1996)] to change to an equivalent measure under which X_t is a Brownian motion, and the argument below can still be used to show that $\mathbb{P}(\mathbf{B}(a,c)) = 0$.] By the translation invariance of Brownian motion and the Markov property, we have

$$\mathbb{P}\big(\mathbf{B}(a,c)|X_{t_0} = z\big) = \mathbb{P}\big(\mathbf{B}^h(a,c)|X_{t_0} = z+h\big).$$

Hence,

$$\begin{aligned}\mathbb{P}(\mathbf{B}^h(a,c)) &= \int_{\mathbb{R}} \mathbb{P}\big(\mathbf{B}^h(a,c)|X_{t_0} = z+h\big)\rho_0(z+h,t_0)\,dz \\ &= \int_{\mathbb{R}} \mathbb{P}\big(\mathbf{B}(a,c)|X_{t_0} = z\big)\rho_0(z+h,t_0)\,dz \\ &\geq \min_{z\in[-M,M]} \frac{\rho_0(z+h,t_0)}{\rho_0(z,t_0)} \int_{-M}^{M} \mathbb{P}\big(\mathbf{B}(a,c)|X_{t_0} = z\big)\rho_0(z,t_0)\,dz > 0.\end{aligned}$$

Note that all elements in $\{\mathbf{B}^h(a,c)\}_{h\in\mathbb{R}}$ are disjoint and measurable. We then obtain a contradiction since Ω does not contain an uncountable disjoint union of measurable sets with positive probability. Thus, $\mathbf{B}(a,c)$ must have probability zero, for every pair (a,c) with $a > c > 0$. Consequently, the set $\{\hat{\tau} \neq \tau\}$ has probability zero. This completes the proof of Proposition 2. □

THEOREM 2. *The operator $\mathcal{P}$ defined by $\mathcal{P}[b](t) = \mathbb{P}(\tau \geq t)$ maps B_0 to P_0.*

PROOF. Suppose $b \in B_0$. Then $b^* = b^*_-$ so $p := \mathcal{P}[b] \in C((0,\infty))$. In addition,

$$\begin{aligned}\lim_{\varepsilon\searrow 0} p(\varepsilon) &= \lim_{\varepsilon\searrow 0} \mathbb{P}\Big(\bigcap_{s\in(0,\varepsilon)} \{X_s > b(s)\}\Big) \\ &= \mathbb{P}\Big(\bigcup_{\varepsilon>0}\bigcap_{s\in(0,\varepsilon)} \{X_s > b(s)\}\Big) = 1 = p(0).\end{aligned}$$

Hence, $p \in C([0,\infty))$. It remains to show that $p > 0$ on $[0,\infty)$. Since $p(0+) = p(0) = 1$, there exists $\varepsilon > 0$ such that $p(t) > 0$ for every $t \in [0,\varepsilon]$. Let $T > \varepsilon$. The upper-semi-continuity of b implies that $M := \sup_{s\in[0,T]} b(s)$ is finite. Then $\mathbb{P}(\tau > \varepsilon, X_\varepsilon > M) > 0$. Using standard results for a constant barrier $\tilde{b} \equiv M$ on the set $\{\tau > \varepsilon, X_\varepsilon > M\}$ for the time interval $[\varepsilon, T]$, we see that $\mathbb{P}(\{\tau > \varepsilon, X_s > M\ \forall s \in [\varepsilon, T]\}) > 0$. Hence, $p(T) > 0$. As T is arbitrary, we see that $p > 0$ on $[0,\infty)$, so that $p \in P_0$. □

PROPOSITION 3 (A semi-continuous dependence property of $\mathcal{P}$). *Assume that $b, b_1, b_2, \ldots$ are upper-semi-continuous functions having the property*

$$b_1 \leq b_2 \leq b_3 \leq \cdots, \qquad b(t) = \lim_{n\to\infty} b_n(t) \qquad \forall t > 0.$$

Let $p = \mathcal{P}[b]$ and $p_n = \mathcal{P}[b_n]$. Then for every $t \geq 0$, $p(t) = \lim_{n\to\infty} p_n(t)$.

PROOF. For every $t > 0$,

$$\{\tau \geq t\} = \bigcap_{s\in(0,t)} \{X_s \geq b(s)\} = \bigcap_{s\in(0,t)} \bigcap_{n\in\mathbb{N}} \{X_s \geq b_n(s)\}$$
$$= \bigcap_{n\in\mathbb{N}} \bigcap_{s\in(0,t)} \{X_s \geq b_n(s)\}.$$

Hence,

$$p(t) = \mathbb{P}\Big(\bigcap_{n\in\mathbb{N}} \bigcap_{s\in(0,t)} \{X_s \geq b_n(s)\}\Big) = \lim_{n\to\infty} \mathbb{P}\Big(\bigcap_{s\in(0,t)} \{X_s \geq b_n(s)\}\Big)$$
$$= \lim_{n\to\infty} p_n(t). \qquad \square$$

3. Approximating sequences for boundary crossing times. In this section, we use the landmark points to construct straightforward approximations that eventually, upon passing to the limit, will allow us to transfer the problem of calculating the survival probability to problems of solving partial differential equations. The advantage of studying the first time the process is strictly below the boundary is suggested by comparing the relative complexity of the expressions (2.3) and (2.7). In this case, a simple approximation to the survival probability and distribution can be developed. We approximate a real barrier b by a simple barrier b_n defined by $b_n(t) = b^*(t)$ if $t \in \mathbb{A}_n(b)$ and $b_n(t) = -\infty$ otherwise. The approximate problem then involves only the random variables $\{X_t | t \in \mathbb{A}_n(b)\}$ so that all relevant probabilities can be calculated through transition probability densities. Though it turns out that survival probabilities computed using both viewpoints are equivalent, we do not see a simple adaptation of our method that allows us to approximate $\hat{\tau}$ directly without appealing to the results in Section 2.

PROPOSITION 4. *Let $b : [0, \infty) \to [-\infty, \infty)$ be upper-semi-continuous and (τ, p, w) be defined as in (2.4)–(2.6). Let $\mathbb{A}(b) = \bigcup_{n\in\mathbb{N}} \mathbb{A}_n(b)$ be the landmark points of b, $\mathbb{A}_n(b) = \{t_n^i | i \in \mathbb{N}\}$ and*

$$\tau_n(\omega) := \min\{s \in \mathbb{A}_n(b) | X_s(\omega) < b(s)\}, \qquad p_n(t) := \mathbb{P}(\tau_n \geq t), \tag{3.1}$$

$$w_n(x, t) := \mathbb{P}(\tau_n \geq t, X_t > x). \tag{3.2}$$

Then the following hold:

1. *For all (x, t),*

$$w_n(x, t) = \int_x^\infty U_n(y, t)\, dy,$$

where, for $t \in [0, t_n^0]$, $\{\tau_n \geq t\} = \Omega$, $p_n(t) = 1$, $w_n(\cdot, t) = 1 - p_0(\cdot, t)$ and $U_n(\cdot, t) = \rho_0(\cdot, t)$, and when $t \in (t_n^k, t_n^{k+1}]$ with $k \in \mathbb{N}$, $\{\tau_n \geq t\} = \bigcap_{i=1}^k \{X_{t_n^i} \geq$

$b(t_n^i)\}$ *and*

$$U_n(x,t) = \int_{b(t_n^k)}^{\infty} U_n(y,t_n^k)\rho(y,t_n^k;x,t)\,dy,$$
$$p_n(t) = \int_{\mathbb{R}} U_n(y,t)\,dy = \int_{b(t_n^k)}^{\infty} U(y,t_n^k)\,dy. \tag{3.3}$$

2. *For every* $n \in \mathbb{N}$, $\tau_n \geq \tau_{n+1} \geq \tau$, $p_n \geq p_{n+1} \geq p$, $w_n \geq w_{n+1} \geq w$, $\rho_0 \geq U_n \geq U_{n+1} \geq 0$.
3. *There exists* $U : \mathbb{R} \times (0,\infty) \to [0,\infty)$ *such that for every* $\omega \in \Omega, t > 0$ *and* $x \in \mathbb{R}$,

$$\lim_{n\to\infty} (\tau_n(\omega), p_n(t), w_n(x,t), U_n(x,t)) = (\tau(\omega), p(t), w(x,t), U(x,t)).$$

PROOF. 1. This result follows immediately from the Chapman–Kolmogorov equations and the fact that the fundamental solution of $\mathcal{L}_1$ gives the transition densities of the Markov process X [see, e.g., Friedman (1975), Theorem I.6.5.4, page 149]. From the definition of τ_n and $\mathbb{A}_n(b)$, it is easy to see that $\{\tau_n \geq t_n^1\} = \Omega$ and $\{\tau_n \geq t\} = \bigcap_{i=1}^k \{X_{t_n^k} \geq b(t_n^i)\}$ when $t \in (t_n^k, t_n^{k+1}]$. When $t \in [0, t_n^1]$, $\{\tau_n \geq t\} = \Omega$ so the evaluation of p_n, w_n, U_n is trivial. When $t \in (t_n^i, t_n^{i+1}]$, $\mathbb{P}(\tau_n \geq t, X_t > x) = \mathbb{P}(X_{t_n^1} \geq b(t_n^1), \ldots, X_{t_n^i} \geq b(t_n^i), X_t > x)$, so using the transition probability density for the Markov process, we have $U_n(\cdot,t) = \int_{b_n(t_n^i)}^{\infty} U_n(y,t_n^i)\rho(y,t_n^i;\cdot,t)\,dy$, from which we find the corresponding w_n and p_n. The first assertion thus follows.

2. By the second part of Lemma 2.1, we have $\tau \leq \tau_{n+1} \leq \tau_n$, and therefore $p \leq p_{n+1} \leq p_n$, and $w \leq w_{n+1} \leq w_n$. It is clear from the definition that $t_n^0 \leq t_{n+1}^0$ and so $\rho_0 = U_n = U_{n+1}$ on $(0, t_{n+1}^0]$. Now suppose $U_{n+1} \leq U_n$ on $(0, t_{n+1}^k]$. Let $t \in (t_{n+1}^k, t_{n+1}^{k+1}]$. Then $t \in (t_n^j, t_n^{j+1}]$ for some j (the case $t \leq t_n^1$ is easier and handled similarly). Then

$$\begin{aligned}
U_{n+1}(x,t) &= \int_{b(t_{n+1}^k)}^{\infty} U_{n+1}(y,t_{n+1}^k)\rho(y,t_{n+1}^k;x,t)\,dy \\
&\leq \int_{b(t_{n+1}^k)}^{\infty} U_n(y,t_{n+1}^k)\rho(y,t_n^k;x,t)\,dy \\
&= \int_{b(t_{n+1}^k)}^{\infty} \rho(y,t_{n+1}^k;x,t) \int_{b(t_n^j)}^{\infty} U_n(z,t_n^j)\rho(z,t_n^j;y,t_{n+1}^k)\,dz\,dy \\
&\leq \int_{b(t_n^j)}^{\infty} U_n(z,t_n^j) \int_{-\infty}^{\infty} \rho(z,t_n^j;y,t_{n+1}^k)\rho(y,t_{n+1}^k;x,t)\,dy\,dz \\
&= \int_{b(t_n^j)}^{\infty} U_n(z,t_n^j)\rho(z,t_n^j;x,t)\,dz = U_n(x,t)
\end{aligned}$$

and $U_{n+1} \leq U_n$ by induction on k. The proof that $0 \leq U_n \leq \rho_0$ is similar. The second assertion thus follows.

3. The monotonicity of (τ_n, p_n, w_n, U_n) implies the existence of the limit as $n \to \infty$. First we show that $\lim_{n\to\infty} \tau_n = \tau$. For this, let $\omega \in \Omega$ be arbitrary. (i) If $\tau(\omega) = \infty$, then we have $\tau_n(\omega) = \infty$ for all $n \in \mathbb{N}$ so $\lim_{n\to\infty} \tau_n(\omega) = \infty = \tau(\omega)$. (ii) Suppose $\tau(\omega) < \infty$. Note that (2.3) gives

$$\begin{aligned}\{\tau \geq t\} &= \bigcap_{s\in\mathbb{A}(b)\cap(0,t)} \{X_s \geq b(s)\} \\ &= \bigcap_{n\in\mathbb{N}} \bigcap_{s\in\mathbb{A}_n(b)\cap(0,t)} \{X_s \geq b(s)\} = \bigcap_{n\in\mathbb{N}} \{\tau_n \geq t\}.\end{aligned} \tag{3.4}$$

Set $t := \lim_{n\to\infty} \tau_n(\omega)$. Then as $\tau_{n+1} \geq \tau_n \geq \tau$ for all $n \in \mathbb{N}$, we see that $\tau_n(\omega) \geq t \geq \tau(\omega)$ for every $n \in \mathbb{N}$. Consequently, $\omega \in \bigcap_{n\in\mathbb{N}}\{\tau_n \geq t\} = \{\tau \geq t\}$. Hence, we must have $\tau(\omega) = t = \lim_{n\to\infty} \tau_n(\omega)$. Combining the two cases, we obtain $\lim_{n\to\infty} \tau_n(\omega) = \tau(\omega)$ for every $\omega \in \Omega$.

Next, we consider the limits of w_n and p_n. When $t = 0$, we have $w(\cdot, 0) = w_n(\cdot, 0) = 1 - p_0(\cdot, 0)$ and $p(0) = 1 = p_n(0)$. When $t > 0$, for each $x \in \mathbb{R}$,

$$\begin{aligned}w_n(x,t) - w(x,t) &= \mathbb{P}(\tau_n \geq t, X_t > x) - \mathbb{P}(\tau \geq t, X_t > x) \\ &= \mathbb{P}(\tau_n \geq t > \tau, X_t > x) \leq \mathbb{P}(\tau < t \leq \tau_n).\end{aligned}$$

Thus,

$$\begin{aligned}&\overline{\lim_{n\to\infty}} \|w_n(\cdot,t) - w(\cdot,t)\|_{L^\infty(\mathbb{R})} \\ &\leq \overline{\lim_{n\to\infty}} |p_n(t) - p(t)| = \overline{\lim_{n\to\infty}} \mathbb{P}(\tau_n \geq \tau > t) \\ &= \mathbb{P}\Big(\bigcap_{n\in\mathbb{N}} \{\tau_n \geq t > \tau\}\Big) = 0.\end{aligned} \tag{3.5}$$

Finally, defining $U := \lim_{n\to\infty} U_n$ we complete the proof of the proposition. □

The approximating functions U_n introduced in the previous proposition are expressed in terms of the transition densities of the diffusion X. From an analytic point of view, they are obtained step by step, for $i = 1, 2, \ldots$, by solving the diffusion equations $\mathcal{L}_1 U_n = 0$ in the set $\mathbb{R} \times (t_n^i, t_n^{i+1}]$ with initial values $U(\cdot, t_n^i+) = U(\cdot, t) \cdot \chi_{(b(t_n^i),\infty)}(\cdot)$, where $\chi_A(x)$ is the indicator function of the set A. In the sequel, $\mathcal{L}$ and $\mathcal{L}_1$ are the differential operators introduced in (1.9) and (1.6), respectively. Recall the notation $Q_b := \{(x,t) | x > b(t), t > 0\}$. When $b : [0,\infty) \to [-\infty, \infty)$ is upper-semi-continuous, the set Q_b is an open set with $[b(0), \infty) \times \{0\}$ as its "initial" boundary.

PROPOSITION 5. *Let $b : [0,\infty) \to [-\infty, \infty)$ be upper-semi-continuous and (p, w) be the survival probability and survival distribution associated with b, de-*

fined in (2.4)–(2.6). *Then there exists a function* U *such that the following hold*:

$$p(t) = \int_{-\infty}^{\infty} U(y,t)\,dy = \int_{b(t)}^{\infty} U(y,t)\,dy, \tag{3.6}$$

$$w(x,t) = \int_{x}^{\infty} U(y,t)\,dy \tag{3.7}$$

$$\forall x \in \mathbb{R}, t > 0, 0 \le U \le \rho_0, 0 \le w \le 1 - p_0, \tag{3.8}$$

$$\|w(\cdot,t) + p_0(\cdot,t) - 1\|_{L^\infty(\mathbb{R})} \le 1 - p(t) \qquad \forall t > 0, \tag{3.9}$$

$$\begin{aligned} &\mathcal{L}_1 U \le 0, \qquad \mathcal{L} w \le 0 \qquad \text{in } \mathbb{R} \times (0,\infty), \\ &\mathcal{L}_1 U = 0, \qquad \mathcal{L} w = 0 \qquad \text{in } Q_b, \end{aligned} \tag{3.10}$$

where the inequalities in (3.10) *are understood in the sense of distributions*.

PROOF. Let $U = \lim_{n\to\infty} U_n$. Since $\rho_0 \ge U_n \ge U_{n+1} \ge 0$, using the Dominated Convergence theorem and the identity $w_n(x,t) = \int_x^\infty U_n(y,t)\,dy$ we obtain $w(x,t) = \int_x^\infty U(y,t)\,dy$ for every $x \in \mathbb{R}$ and $t > 0$. Since $\tau(\omega) \ge t$ implies $X_t(\omega) \ge b(t)$, we see that $w(x,t) = w(-\infty,t) = p(t)$ for every $x < b(t)$.

It is clear that $0 \le U \le \rho_0$ and $0 \le w \le 1 - p_0$. Also, for $t > 0$,

$$\begin{aligned} w(x,t) &= \mathbb{P}(X_t > x) - \mathbb{P}(\tau < t, X_t > x) \ge \mathbb{P}(X_t > x) - \mathbb{P}(\tau < t) \\ &= [1 - p_0(x,t)] - [1 - p(t)]. \end{aligned}$$

Thus, $0 \le 1 - w(x,t) - p_0(x,t) \le 1 - p(t)$ or $\|w(\cdot,t) + p_0(\cdot,t) - 1\|_{L^\infty(\mathbb{R})} \le 1 - p(t)$.

It is useful to note that for each $t > 0$, both $w_n(\cdot,t)$ and $U_n(\cdot,t)$ are smooth functions. In addition, as functions of (x,t), w_n and U_n are smooth in $\mathbb{R} \times (0,\infty) \setminus \bigcup_{i=1}^\infty (-\infty, b_n(t_n^i)] \times \{t_n^i\}$. In particular,

$$\mathcal{L}_1 U_n = 0, \qquad \mathcal{L} w_n = 0 \qquad \text{in } Q_b := \{(x,t) | x > b(t), t > 0\}.$$

Since both $\{U_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ are uniformly bounded in any compact subset of Q_b, it then follows from standard results on parabolic partial differential equations [see Friedman (1964), Theorem 3.11, page 74, and Theorem 3.15, page 80] that $w, U \in C^\infty(Q_b)$ and $\mathcal{L} w = 0$ and $\mathcal{L}_1 U = 0$ in Q_b.

The set of discontinuities of U_n and w_n is $\bigcup_{i\in\mathbb{N}} (-\infty, b(t_n^i)] \times \{t_n^i\}$. In particular,

$$\begin{aligned} &w_n(\cdot,t) = w_n(\cdot,t-), \qquad U_n(\cdot,t) = U_n(\cdot,t-), \\ &p(t) = p(t-) \qquad \forall t \in (0,\infty), \\ &U_n(x,t+) = 0, \qquad w_n(x,t+) = p_n(t) \qquad \forall x < b(t), t \in \{t_n^i\}_{i\in\mathbb{N}}. \end{aligned}$$

Denote by $\delta(\cdot - s)$ the Dirac measure with mass at s and by χ_A the characteristic function of the set A. Then in the sense of distributions, we find that

$$\begin{aligned}\mathcal{L}_1 U_n &= \sum_{i=0}^{\infty} [U_n(x,\cdot)]_{t_n^i-}^{t_n^i+} \delta(t - t_n^i) \\ &= -\sum_{i=0}^{\infty} U_n(x, t_n^i)\delta(t - t_n^i)\chi_{(-\infty, b(t_n^i))](x) \le 0 \qquad \text{in } \mathbb{R}\times(0,\infty).\end{aligned}$$

$$\begin{aligned}\mathcal{L} w_n &= \sum_{i=0}^{\infty} [w_n(x,\cdot)]_{t_n^i-}^{t_n^i+} \delta(t - t_n^i) \\ &= -\sum_{i=0}^{\infty} [w_n(x, t_n^i) - p_n(t_n^i)]\delta(t - t_n^i)\chi_{(-\infty, b(t_n^i))](x) \le 0 \qquad \text{in } \mathbb{R}\times(0,\infty).\end{aligned}$$

Sending $n \to \infty$ we find that $\mathcal{L} w \le 0$ and $\mathcal{L}_1 U \le 0$ in $\mathbb{R}\times(0,\infty)$ in the sense of distributions. This proves (3.10) and also completes the proof of the proposition. □

4. Viscosity solutions and boundary crossing probabilities. In this section, we show that the survival distribution w defined in (2.6) is the unique viscosity solution of the time dependent Kolmogorov forward equation (1.8). As mentioned above, the use of viscosity solutions is necessitated by the fact that the boundaries arising from the solution to the variational inequality for the inverse boundary crossing problem do not have sufficient regularity for us to employ classical solutions. We do note however, that when a classical solution exists, it gives the unique viscosity solution. Consequently, the classical solution of the partial differential equation (1.8), if it exists, is the survival distribution function associated with b, defined in (2.6). Since classical solutions of (1.4) are obtained from classical solutions of (1.8) via the transformation $U = -\partial w/\partial x$, we also see that a classical solution of (1.4), if it exists, is the survival probability density of the first boundary crossing problem that we want to calculate.

For simplicity, we work with b in the class B_0 so that the survival probability associated with b is continuous on $[0,\infty)$. Furthermore, we work with the function w which is monotone in the spatial variable, and smoother than U. The following definition is based on the differential inequalities/equalities in (3.10).

DEFINITION 2. Let $b \in B_0$. A *viscosity solution* (for the survival distribution) of the boundary crossing problem associated with b is a function w defined on $\mathbb{R}\times(0,\infty)$ that has the following properties:

1. $w \in C(\mathbb{R}\times(0,\infty))$; $\lim_{t\searrow 0}\|w(\cdot,t) + p_0(\cdot,t) - 1\|_{L^\infty(\mathbb{R})} = 0$; $0 \le w \le 1$;
2. $w(x,t) = w(b(t),t)\ \forall x \le b(t), t > 0$; $w(x,t) < w(b(t),t)\ \forall x > b(t), t > 0$; $\mathcal{L} w = 0$ in Q_b;

3. If for a smooth φ, point $x \in \mathbb{R}$ and time $t > \delta > 0$, the function $\varphi - w$ attains a local minimum at (x,t) on $[x-\delta, x+\delta] \times [t-\delta, t]$, then $\mathcal{L}\varphi(x,t) \le 0$.

We define one-sided time derivatives by

$$\frac{\partial^+ \phi(x,t)}{\partial t} = \lim_{\Delta t \searrow 0} \frac{\phi(x,t+\Delta t) - \phi(x,t)}{\Delta t},$$
$$\frac{\partial^- \phi(x,t)}{\partial t} = \lim_{\Delta t \searrow 0} \frac{\phi(x,t) - \phi(x,t-\Delta t)}{\Delta t}.$$

Denote by $\mathcal{L}^{\pm}$ and $\mathcal{L}_1^{\pm}$ the operator $\mathcal{L}$ and $\mathcal{L}_1$ with time derivative replaced by the above one-sided derivative. Then from the expression of U_n in (3.3), we see that $\mathcal{L}_1^- U_n = \mathcal{L}^- w_n = 0$ in $\mathbb{R} \times (t_n^i, t_n^{i+1}]$. Thus, we have

$$\mathcal{L}^- U_n(x,t) = 0, \qquad \mathcal{L}^- w_n(x,t) = 0 \qquad \forall (x,t) \in \mathbb{R} \times (0,\infty). \tag{4.1}$$

In the uniqueness proof in the following theorem it is convenient to work with the special case $\sigma \equiv 1$. This can be done without loss of generality by considering the transformation

$$\begin{aligned} \mathbf{Y}(x,t) &:= \int_0^x \frac{1}{\sigma(z,t)}\, dz \qquad \forall x \in \mathbb{R}, t \ge 0, \\ Y_t &:= \mathbf{Y}(X_t, t) \qquad \forall t \ge 0. \end{aligned} \tag{4.2}$$

The change of variables $(x,t) \to (y,t)$ via $y = \mathbf{Y}(x,t)$ is smooth and invertible. Also, by Itô's lemma,

$$dY_t = \tilde{\mu}(Y_t, t)\, dt + dB_t.$$

Here, denoting by $x = \mathbf{X}(y,t)$ the inverse of $y = \mathbf{Y}(x,t)$,

$$\tilde{\mu}(y,t) := -\int_0^x \frac{\sigma_t(z,t)}{\sigma^2(z,t)}\, dz + \frac{\mu(x,t)}{\sigma(x,t)} - \frac{1}{2}\sigma_x(x,t)\Big|_{x=\mathbf{X}(y,t)}.$$

Under the transformation, a boundary b for $\{X_t\}$ is transformed to a boundary $\tilde{b}: t \in [0,\infty) \to \mathbf{Y}(b(t),t)$ for $\{Y_t\}$. Similarly, a boundary $\tilde{b}$ for $\{Y_t\}$ is transferred back to $b: t \in [0,\infty) \to \mathbf{X}(\tilde{b}(t),t)$.

THEOREM 3. *Assume that $b \in B_0$.*

1. *The survival distribution associated with b defined in (2.6) is the unique viscosity solution for the survival distribution of the boundary crossing problem associated with b. Consequently, the survival probability $p = \mathcal{P}[b]$ can be evaluated by $p(\cdot) = w(-\infty, \cdot)$ where w is the viscosity solution of the boundary crossing problem associated with b in Definition 2.*

2. *If w is a classical* (i.e., $w + p_0 \in C(\mathbb{R} \times [0,\infty))$, $\partial_x w \in C(\mathbb{R} \times (0,\infty))$, $\partial_t w, \partial_x^2 w \in C(Q_b)$) *solution of* (1.8), *then w is the survival distribution of the boundary crossing problem associated with b. If U is a classical* (i.e., $U - \rho_0 \in C(\mathbb{R} \times [0,\infty))$, $\partial_t U, \partial_x^2 U \in C(Q_b)$) *solution of* (1.4), *then* $w(x,t) := \int_x^\infty U(y,t)\,dt$ *is the survival distribution of the boundary crossing problem.*

PROOF. *Existence*. Let $b \in B_0$ and τ, p, w be defined as in (2.4)–(2.6). Then $p = \mathcal{P}[b] \in C([0,\infty))$. We show that w is a viscosity solution in the sense of Definition 2. First, we show that w is continuous. Fix $x \in \mathbb{R}$. For any $t > s > 0$,

$$\begin{aligned} &w(x,t) - w(x,s) \\ &\quad = \mathbb{P}(\tau \geq t, X_t > x) - \mathbb{P}(\tau \geq s, X_s > x) \\ &\quad = \mathbb{P}(\tau \geq t, X_t > x) - \mathbb{P}(\tau \geq t, X_s > x) - \mathbb{P}(t > \tau \geq s, X_s > x) \\ &\quad = \mathbb{P}(\tau \geq t, X_t > x \geq X_s) - \mathbb{P}(\tau \geq t, X_s > x \geq X_t) - \mathbb{P}(t > \tau \geq s, X_s > x). \end{aligned}$$

Note that $\mathbb{P}(t > \tau \geq s, X_s > x) \leq \mathbb{P}(t > \tau \geq s) = p(s) - p(t)$ so we have

$$|w(x,t) - w(x,s)| \leq \mathbb{P}(X_s > x \geq X_t) + \mathbb{P}(X_t > x \geq X_s) + |p(s) - p(t)|.$$

Since p is continuous, sending $t \to s$ or $s \to t$ we conclude that $w(x,\cdot)$ is continuous in $(0,\infty)$. Next, for $x < y$ and $t > 0$,

$$\begin{aligned} 0 \leq w(x,t) - w(y,t) &= \mathbb{P}(\tau \geq t, y \geq X_t > x) \\ &\leq \mathbb{P}(y \geq X_t > x) = p_0(y,t) - p_0(x,t). \end{aligned}$$

Thus, $w(\cdot,t)$ is continuous, uniform in $t \in [\varepsilon,\infty)$ for any $\varepsilon > 0$. In conclusion, $w \in C(\mathbb{R} \times (0,\infty))$. Recall from (3.8) that $\|w(\cdot,t) + p_0(\cdot,t) - 1\|_{L^\infty(\mathbb{R})} \leq 1 - p(t)$. The continuity of p on $[0,\infty)$ then implies that $\lim_{t \searrow 0} \|w(\cdot,t) + p_0(\cdot,t) - 1\|_{L^\infty(\mathbb{R})} = \lim_{t \searrow 0}(1 - p(t)) = 0$. Thus, w satisfies the first requirement of being a viscosity solution.

REMARK 4.1. The continuity of the survival probability p plays a central role in the proof here. In a more general case, that is, $b \notin B_0$, w is not continuous so the definition of a viscosity solution needs to be revised. To avoid such technicalities, we take the simple case that $b \in B_0$. The work of Cheng et al. (2006) does allow discontinuous survival probabilities.

Note that $\tau(\omega) \geq t$ implies $X_t \geq b(t)$. Hence, $w(x,t) = \mathbb{P}(\tau \geq t, X_t > x) = w(b(t),t)$ when $x < b(t)$. Also, since $U = -\partial w/\partial x \geq 0$, $U \not\equiv 0$ and $\mathcal{L}_1 U = 0$ in Q_b, we have $U > 0$ in Q_b. In particular, if $U(x,t) = 0$, with $(x,t) \in Q_b$, then the strong maximum principle [Friedman (1964), Theorem 3.5, page 39] implies that $U(y,t) = 0$ for all y such that $(y,t) \in Q_b$ [and therefore all $y \in \mathbb{R}$, as it is

easy to see that $U(y,t) = 0$ for $(y,t) \notin Q_b$]. This is a contradiction, since the Dominated Convergence theorem implies that

$$0 < p(t) \le p_n(t) = \lim_{n\to\infty} \int_{-\infty}^{\infty} U_n(y,t)\,dy = \int_{-\infty}^{\infty} U(y,t)\,dy$$

with the application of Dominated Convergence justified by the bounds $\rho_0 \ge U_n \ge 0$ from part 2 of Proposition 4. Thus, $w(\cdot,t)$ is strictly decreasing in $(b(t),\infty)$, so $w(x,t) < w(b(t),t)$ for all $x > b(t)$. Finally, from (3.10), we know $\mathcal{L}w = 0$ in Q_b. Thus, w satisfies the second requirement of being a viscosity solution.

REMARK 4.2. Recall that $\mathcal{P}[b] = \mathcal{P}[b^*]$ for any boundary b; that is, under our nonconventional definition of default time and survival probability, both the original boundary b and its upper-semi-continuous envelope b^* produce the same crossing time, survival probability, and survival distribution. Here, we needed to use the upper-semi-continuous representation of the barrier so Q_b is open and $w(\cdot,t)$ is strictly decreasing for $x > b(t)$.

We now verify the third requirement for w being a viscosity solution. Assume that $\varphi - w$ attains a local minimum at (x,t) on $A := [x-\delta, x+\delta] \times [t-\delta, t]$ where φ is smooth and $t > \delta > 0$. We want to show that $\mathcal{L}\varphi(x,t) \le 0$. We follow a standard technique for viscosity solutions. First, we modify φ to a new smooth ψ so $\psi - w$ attains a strict local minimum value zero at (x,t) on A. The function is defined by

$$\psi(y,s) := \varphi(y,s) + (x-y)^4\delta^{-4} + (t-s)^2\delta^{-2} + [w(x,t) - \varphi(x,t)].$$

Then $\psi(x,t) - w(x,t) = 0$ and $\mathcal{L}\psi(x,t) = \mathcal{L}\varphi(x,t)$. That $\varphi - w$ attains a local minimum at (x,t) implies

$$\begin{aligned}
&\psi(y,s) - w(y,s)\\
&\quad = (x-y)^4\delta^{-4} + (t-s)^2\delta^{-2} + [\varphi(y,s) - w(y,s)] - [\varphi(x,t) - w(x,t)]\\
&\quad \ge (x-y)^4\delta^{-4} + (t-s)^2\delta^{-2} \qquad \forall (y,s) \in A.
\end{aligned}$$

Thus, $\psi(y,s) - w(y,s)$ attains on A a strict local minimum, being zero, at (x,t).

Using a standard viscosity solution technique, the differential inequality $\mathcal{L}\varphi(x,t) \le 0$ is obtained by comparison of φ with smooth approximations of viscosity solution candidates. Here, we choose the smooth approximations to be $\{w_n\}$ introduced in Proposition 4. For each positive integer n, let w_n be defined as in (3.2) in Proposition 4. Then w_n is upper-semi-continuous on $\mathbb{R} \times [0,\infty)$, so $\psi - w_n$ attains a local minimum, on the closed set $A = [x-\delta, x+\delta] \times [t-\delta, t]$. On the parabolic boundary of A, we have $\psi - w \ge 1$ and $w - w_n > -1$ so $\psi - w_n > 0$. At (x,t), $\psi - w_n = w(x,t) - w_n(x,t) \le 0$. Hence, the minimum is attained in $(x-\delta, x+\delta) \times (t-\delta, t]$. We denote by (x_n, t_n) an arbitrary local minimizer of $\psi - w_n$ in A.

That $\psi - w_n$ attains a local minimum at (x_n, t_n) implies that at (x_n, t_n), $\partial\psi/\partial x = \partial w_n/\partial x$, $\partial\psi/\partial t \le \partial^- w_n/\partial t$ and $\partial^2\psi/\partial x^2 \ge \partial^2 w_n/\partial x^2$. Hence, $\mathcal{L}\psi(x_n, t_n) \le \mathcal{L}^- w_n(x_n, t_n) = 0$.

In order to take the limit, we want to show that $(x_n, t_n) \to (x, t)$ as $n \to \infty$. Intuitively this is obvious since $\psi - w$ attains a strict local minimum at (x, t) and $w_n \to w$ (uniformly). Since $\psi(y, s) - w(y, s) \ge (x - y)^4\delta^{-4} + (t - s)^2\delta^{-2}$ with $(y, s) = (x_n, t_n)$, we have

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\{(x_n - x)^4\delta^{-4} + (t_n - t)^2\delta^{-2}\} \\
&\quad\le \overline{\lim_{n\to\infty}}\{\psi(x_n, t_n) - w(x_n, t_n)\} \\
&\quad= \overline{\lim_{n\to\infty}}\{[\psi(x_n, t_n) - w_n(x_n, t_n)] + [w_n(x_n, t_n) - w(x_n, t_n)]\} \\
&\quad\le \overline{\lim_{n\to\infty}}\Big\{[\psi(x, t) - w_n(x, t)] + \max_{[x-\delta, x+\delta]\times[t-\delta, t]} |w_n - w|\Big\} \\
&\quad= \psi(x, t) - w(x, t) = 0.
\end{aligned}$$

Here, we have used the uniform convergence of $w_n \to w$, derived as follows:

$$\begin{aligned}
0 &\le w_n(x, s) - w(x, s) \\
&= \mathbb{P}(\tau_n \ge s, X_s > x) - \mathbb{P}(\tau \ge s, X_s > x) \\
&= \mathbb{P}(\tau_n \ge s > \tau, X_s > x) \le \mathbb{P}(\tau_n \ge s > \tau) \\
&= \mathbb{P}(\tau_n \ge s) - \mathbb{P}(\tau \ge s) = p_n(s) - p(s).
\end{aligned}$$

Thus, we have $\|w_n(\cdot, s) - w(\cdot, s)\|_{L^\infty(\mathbb{R})} \le p_n(s) - p(s)$. Since p_n, p are continuous, and $p_n \searrow p$, the point-wise convergence of $p_n \to p$ implies local uniform convergence, that is, $\lim_{n\to\infty} \|p_n - p\|_{L^\infty([0,T])} = 0$. Thus, $\lim_{n\to\infty} \|w_n - w\|_{L^\infty(\mathbb{R}\times[0,T])} = 0$ for any $T > 0$. Hence, $\lim_{n\to\infty}(x_n, t_n) = (x, t)$. Finally, this implies $\mathcal{L}\varphi(x, t) = \mathcal{L}\psi(x, t) = \lim_{n\to\infty} \mathcal{L}\psi(x_n, t_n) \le 0$.

Uniqueness. We can assume without loss of generality that $\sigma \equiv 1$, since otherwise we can work with the process $\{Y_t\}_{t\ge 0}$ defined in (4.2). In terms of our viscosity solution, it means that we make a smooth change of variable $(x, t) \to (y, t)$ via

$$y = Y(x, t) := \int_0^x \frac{1}{\sigma(z, t)}\, dz.$$

In the new variables, we are working on the function $w(X(y, t), t)$ and the barrier is $b(X(y, t))$ where $x = X(y, t)$ is the inverse of $y = Y(x, t)$. Retaining the notation (x, t) as independent variables, we can assume that $\mathcal{L} = \partial_t - \frac{1}{2}\partial_x^2 + \mu(x, t)\partial_x$. We denote

$$\begin{aligned}
M &= \|\mu\|_{L^\infty(\mathbb{R}\times[0,\infty))} + \|\partial_x\mu\|_{L^\infty(\mathbb{R}\times[0,\infty))}, \\
R(t) &= \|\rho_0\|_{L^\infty(\mathbb{R}\times[t,\infty))} \qquad \forall t > 0.
\end{aligned}$$

We note that $R(t) < \infty$ by the standard Gaussian upper bound on the fundamental solution of $\mathcal{L}_1$ [see Friedman (1964), page 24]. Let w be the survival probability of the boundary crossing problem. Then $|\partial_x w| \le \rho_0$ is uniformly bounded in $\mathbb{R} \times [t_0, \infty)$ for any $t_0 > 0$. Let $\tilde{w}$ be an arbitrary viscosity solution of the boundary crossing problem. We want to show that $w = \tilde{w}$.

Suppose $w \ne \tilde{w}$. Then there exists $x_0 \in \mathbb{R}, t_0 > \delta > 0$ such that either $w(x_0, t_0) > \tilde{w}(x_0, t_0) + 6\delta$ or $\tilde{w}(x_0, t_0) > w(x_0, t_0) + 6\delta$. In the former case, we set $(w_1, w_2) = (w, \tilde{w})$ and in the latter case we set $(w_1, w_2) = (\tilde{w}, w)$. Then both w_1 and w_2 are viscosity solutions and

$$w_1(x_0, t_0) - w_2(x_0, t_0) > 6\delta > 0.$$

By spatial translation, we can assume, without loss of generality, that $b(t) < 0$ for all $t \in [0, t_0]$.

We now fix a constant ε satisfying

$$0 < \varepsilon, \qquad \varepsilon t_0 + \varepsilon^4 x_0^4 \le \min(\delta, 1), \qquad \varepsilon^4 + 4\varepsilon^2 M \le \varepsilon/4.$$

We need another small positive constant η determined as follows. By the second property of viscosity solutions, we can find $t_1 \in (0, t_0)$ such that $\|w_i(\cdot, t_1) + p_0(\cdot, t_1) - 1\|_{L^\infty(\mathbb{R})} < \delta$, $i = 1, 2$. Since $p_0(\cdot, t_1)$ is uniformly continuous on $\mathbb{R}$ and w_2 is continuous at (x_0, t_0), there exist $\eta_0 > 0$ such that for every $\eta \in (0, \eta_0]$, $|w_2(x_0, t_0) - w_2(x_0 + \eta, t_0)| \le \delta$ and $\|p_0(\cdot, t_1) - p_0(\cdot + \eta, t_1)\|_{L^\infty(\mathbb{R})} \le \delta$. The latter inequality implies

$$\begin{aligned}
&\|w_1(\cdot, t_1) - w_2(\cdot + \eta, t_1)\|_{L^\infty(\mathbb{R})} \\
&\quad \le \|w_1(\cdot, t_1) + p_0(\cdot, t_1) - 1\|_{L^\infty(\mathbb{R})} \\
&\qquad + \|w_2(\cdot + \eta, t_1) + p_0(\cdot + \eta, t_1) - 1\|_{L^\infty(\mathbb{R})} \\
&\qquad + \|p_0(\cdot, t_1) - p_0(\cdot + \eta, t_1)\|_{L^\infty(\mathbb{R})} \le 3\delta.
\end{aligned}$$

Now, we fix an $\eta \in (0, \eta_0]$ such that

$$0 < M\eta[R(t_1) + 4\varepsilon^2] \le \varepsilon/4.$$

Consider the continuous function

$$\Phi(x, t) = w_1(x, t) - w_2(x + \eta, t) - \varepsilon t - \varepsilon^4 x^2, \qquad x \in \mathbb{R}, t \in [t_1, t_0].$$

Note that $\Phi(x_0, t_0) = [w_1(x_0, t_0) - w_2(x_0, t_0)] + [w_2(x_0, t_0) - w_2(x_0 + \eta, t_0)] - \varepsilon t_0 - \varepsilon^4 x_0^2 \ge 6\delta - \delta - \delta = 4\delta$. On the other hand, when $t = t_1$, $\Phi(x, t_1) \le \|w_1(\cdot, t_1) - w_2(\cdot + \eta, t_2)\|_{L^\infty(\mathbb{R})} \le 3\delta$. Hence, there exists $(x_*, t_*) \in \mathbb{R} \times (t_1, t_0]$ such that Φ attains at (x_*, t_*) the global positive maximum of Φ on $\mathbb{R} \times [t_1, t_0]$:

$$\Phi(x_*, t_*) \ge \Phi(x, t) \qquad \forall x \in \mathbb{R}, t \in [t_1, t_0].$$

We consider two cases: (i) $x_* \le b(t_*) - \eta$; (ii) $x_* > b(t_*) - \eta$.

Suppose (i) $x_* \le b(t_*) - \eta$. Then we have $w_1(x_*, t_*) = w_1(b(t_*), t_*)$ and $w_2(x_* + \eta, t_*) = w_2(b(t_*), t_*)$. Consequently, since $\|w_2(\cdot, t_*)\|_{L^\infty(\mathbb{R})} = w_2(b(t_*), t_*) = w_2(x^* + \eta, t_*)$ and $x_* < b(t_*) < 0$, we obtain

$$\begin{aligned}&\Phi(b(t_*), t_*) - \Phi(x_*, t_*)\\ &\quad = \varepsilon^4\big(x_*^2 - b(t_*)^2\big) + \big(w_2(x^* + \eta, t_*) - w_2\big(b(t_*) + \eta, t_*\big)\big) > 0\end{aligned}$$

contradicting the maximality of $\Phi(x_*, t_*)$.

(ii) $x_* > b(t_*) - \eta$. Set $\varphi(x, t) = w_2(x + \eta, t) + \varepsilon t + \varepsilon^4 x^2$. Then $\varphi - w_1 = -\Phi$ attains at (x_*, t_*) a minimum over $\mathbb{R} \times [t_0, t_1]$. Since $x_* > b(t_*) - \eta$, we see that $w_2(\cdot + \eta, \cdot)$ is smooth in a neighborhood of (x_*, t_*). Then φ is smooth in a small neighborhood of (x_*, t_*) and $\varphi - w_1$ attains a local minimum at (x_*, t_*). Since w_1 is a viscosity solution, we must have $\mathcal{L}\varphi(x_*, t_*) \le 0$.

Now, we calculate $\mathcal{L}\varphi(x_*, t_*)$. Using $\mathcal{L}w_2 = 0$ in Q_b and the fact that σ is assumed to be a constant, we have

$$\begin{aligned}\mathcal{L}\varphi(x_*, t_*) &= \varepsilon - \varepsilon^4 + 2\varepsilon^4 x_* \mu(x_*, t_*)\\ &\quad + [\mu(x_*, t_*) - \mu(x_* + \eta, t_*)]\partial_x w_2(x_* + \eta, t_*).\end{aligned}$$

First, we note that $\Phi(x_*, t_*) \ge \Phi(x_0, t_0)$, so $\varepsilon t_* + \varepsilon^4 x_*^2 \le 2 + \varepsilon t_0 + \varepsilon^4 x_0^4 \le 3$. This implies that $\varepsilon^2 |x_*| \le 2$. Hence,

$$\mathcal{L}\varphi(x_*, t_*) \ge \varepsilon - \varepsilon^4 - 4\varepsilon^2 M - M\eta|\partial_x w_2(x_* + \eta, t_*)|.$$

To estimate $\partial_x w_2(x_* + \eta, t_*)$, we consider two situations.

(a) $w_2 = w$ is the survival distribution function. Then $\|\partial_x w(x_*, t_*)\| \le \rho_0(x_*, t_*) \le R(t_1)$.

(b) $w_2 = \tilde{w}$. Then w_2 is differentiable at (x_*, t_*) and $w_1(\cdot, t_*)$ is Lipschitz continuous with Lipschitz constant $\|U(\cdot, t_*)\|_{L^\infty(\mathbb{R})}$. Hence, sending $h \searrow 0$ in $[(\Phi(x_* \pm h, t_*) - \Phi(x_*, t_*)]/h \ge 0$ we derive

$$|\partial_x w_2(x_*, t_*)| \le \|\partial_x w(\cdot, t_*)\|_{L^\infty(\mathbb{R})} + 2\varepsilon^4 |x^*| \le R(t_1) + 4\varepsilon^2.$$

Thus, in either case, we have

$$\mathcal{L}\varphi(x_*, t_*) \ge \varepsilon - \varepsilon^4 - 4\varepsilon^2 M - M\eta\{R(t_1) + 4\varepsilon^2\} \ge \varepsilon/4 > 0$$

by our careful choices of ε and η. This contradicts $\mathcal{L}\varphi(x_*, t_*) \le 0$. The contradiction implies that we must have $w_1 \equiv w_2$. Thus, the viscosity solution of the boundary crossing problem is unique.

Proof of the second assertion. The equivalence of classical solutions of (1.8) and (1.4) via $U = -\partial w/\partial x$ is trivial. Here we show that a classical solution of (1.8) is a viscosity solution.

Assume that w is a classical solution of (1.8). Then $U = -\partial w/\partial x$ is a classical solution of (1.4). Applying the maximum principle to U and $\rho_0 - U$ on $\bar{Q}_b$,

we find that $0 \le U \le \rho_0$. Also, the strong maximum principle [Friedman (1964), Theorem 3.5, page 39] shows $U > 0$ in Q_b [if $U(x,t) = 0$ for $(x,t) \in Q_b$, then $U(y,s) = 0$ for all (y,s) in Q_b with $s \le t$, contradicting the initial condition at time 0]. Hence, w is monotone in x and $w(x,t) < w(b(t),t)$ for all $x > b(t), t > 0$. In addition, for each $t_0 > 0$, comparing U with the solution of $\mathcal{L}V = 0$ in $\mathbb{R} \times (t_0, \infty)$ with initial value $V(\cdot, t_0) = U(\cdot, t_0)$ we see that $U \le V$ on $\mathbb{R} \times [t_0, \infty)$ so $\int_{\mathbb{R}} U(y,t)\,dy \le \int_{\mathbb{R}} V(y,t)\,dy = \int_{\mathbb{R}} U(y,t_0)\,dy$ for every $t > t_0$. This implies that $p(t) := w(-\infty, t) = w(b(t), t)$ is a decreasing function of t.

Next, since $w + p_0 \in C(\mathbb{R} \times [0,\infty))$, $\partial_x w \le 0$, $\partial_x p_0 \le 0$, and $w(\infty, 0) = 0$ and $w(-\infty, 0) = 1$, one can show that $\lim_{t \searrow 0} \|w(\cdot,t) + p_0(\cdot,t) - 1\|_{L^\infty(\mathbb{R})} = 0$. Thus, w satisfies the first and second requirements of a viscosity solution in Definition 2.

To verify the third requirement in Definition 2, suppose φ is smooth, $x \in \mathbb{R}, t > \delta > 0$ and $\varphi - w$ attains a local minimum at (x,t) on $[x-\delta, x+\delta] \times [t-\delta, t]$. We want to show that $\mathcal{L}\varphi(x,t) \le 0$. We consider two cases: (i) $x > b(t)$ and (ii) $x \le b(t)$.

(i) Suppose $x > b(t)$. Then (x,t) is an interior point of Q_b, in which w is smooth. Since $\varphi - w$ attains a local minimum at (x,t) on $[x-\delta, x+\delta] \times [t-\delta, t]$, we have $\partial_t \varphi(x,t) \le \partial_t w(x,t)$, $\partial_x \varphi(x,t) = \partial_x w(x,t)$ and $\partial_x^2 \varphi(x,t) \ge \partial_x^2 w(x,t)$. This implies that $\mathcal{L}\varphi(x,t) \le \mathcal{L}w(x,t) = 0$.

(ii) Suppose $x \le b(t)$. Then $\partial_x \varphi(x,t) = \partial_x w(x,t) = 0$. Note that $\varphi(x-z,t) - \varphi(x,t) \ge w(x-z,t) - w(x,t) = 0$ for all $z \ge 0$. This implies, since φ is a smooth and $\partial_x \varphi(x,t) = 0$, that $\partial_{xx}^2 \varphi(x,t) \ge 0$.

To complete the proof that $\mathcal{L}\varphi(x,t) \le 0$, it suffices to show that $\partial_t \varphi \le 0$. Suppose on the contrary that $\partial_t \varphi(x,t) > 0$. Then there exists $\varepsilon \in (0, \delta)$ such that $\varphi(x, t-s) < \varphi(x,t)$ for all $s \in (0, \varepsilon]$. As $\varphi - w$ attains a local minimum at (x,t), we see that $w(x,t-s) \le w(x,t) - \varphi(x,t) + \varphi(x,t-s) < w(x,t) = p(t) \le p(t-s)$ for all $s \in (0,\varepsilon]$. Thus, $[x,\infty) \times [t-\varepsilon, t) \subset Q_b$. Since w is monotone, we also have $w(y, t-s) \le p(t) = w(x,t)$ for all $y > x$, $s \in [0,\varepsilon]$. That is, w attains at (x,t) a local maximum over the region $[x,\infty) \times [t-\varepsilon, t]$. Hence, applying Hopf's lemma [Protter and Weinberger (1967), Theorem 3.3] for w on $[x,\infty) \times (t-s, t)$, we have $w_x(x, t-) < 0$, which contradicts the definition of a classical solution that $\partial_x w(x, t-) = \partial_x w(x,t) = 0$. Thus, we must have $\partial_t \varphi(x,t) \le 0$. Together with $\partial_x \varphi(x,t) = 0$, $\partial_{xx}^2 \varphi(x,t) \ge 0$, we conclude that $\mathcal{L}\varphi(x,t) \le 0$.

Hence, w is a viscosity solution. Applying the conclusion of the first assertion, we then see that w is the survival distribution of the boundary crossing problem associated with b. □

REMARK 4.3. (i) The addition of the term $\varepsilon^4 x^2$ confines our attention in searching for the maximum of Φ to a compact set $[-2\varepsilon^{-2}, 2\varepsilon^{-2}] \times [0, t_0]$. So we indeed only need $w + p_0 \in C(\mathbb{R} \times [0,\infty))$ and $w + p_0 - 1 = 0$ on $\mathbb{R} \times \{0\}$ in the definition of viscosity solutions.

(ii) Fix $t_2 > t_1 \ge 0$. In the proof, if we take w_2 to be a solution of $\mathcal{L}w_2 = 0$ in $\mathbb{R} \times (t_1, t_2)$ subject to initial condition $w_2(\cdot, t_1) \ge w(\cdot, t_1)$. Then following the

proof we see that $\sup_{\mathbb{R}\times[t_1,t_2)}(w - w_2) > 0$ is impossible. Thus, we have $w \le w_2$ on $\mathbb{R} \times [t_1, t_2)$. This is a simple version of the comparison principle in the theory of viscosity solutions. This result will be used in the next section.

5. Viscosity solutions for the inverse boundary crossing problem. The inverse boundary crossing problem is to find b, for a given p, such that p is the survival probability associated with b. In this section, we prove that for any $p \in P_0$, from the viscosity solution of the variational inequality (1.11), we can find an unique $b \in B_0$ such that the resulting p gives the survival distribution of the first time that X crosses b. Since the forward problem maps B_0 to P_0, we study, for simplicity, the inverse problem for $p \in P_0$, though in Cheng et al. (2006) the variational inequality (1.11) was considered for more general survival functions.

5.1. *Viscosity solutions.* In general, classical solutions of the variational inequality (1.11) for the inverse problem may not exist. In Cheng et al. (2006), viscosity solutions were introduced, and it was shown that for any p satisfying

$$p(0+) = 1 \ge p(s) = p(s-) \ge p(t) \ge 0 \qquad \forall t > s > 0, \tag{5.1}$$

there exists a unique viscosity solution. From this solution, we can define a boundary b such that $Q_b = \{w < p\}$, and consider it as a candidate for the solution to the inverse boundary crossing problem. To verify that b is indeed a solution, we show that w is a viscosity solution to the direct problem (1.8), and then appeal to Theorem 3 to see that w and p give the survival distribution of the first time that X crosses b.

When $p \in P_0$, we know a priori that the unique solution of the variational inequality is continuous so many technicalities in Cheng et al. (2006) regarding the definition, existence, and uniqueness of viscosity solutions can be ignored. In particular, the viscosity solution introduced in Cheng et al. (2006) can be reformulated (removing those specifics that take care of discontinuities) as follows.

DEFINITION 3. Let $p \in P_0$ be given. A *viscosity solution for the survival distribution of the inverse boundary crossing problem* associated with p is a function w defined on $\mathbb{R} \times (0, \infty)$ that has the following properties:

1. $w + p_0 \in C(\mathbb{R} \times (0, \infty))$, $\lim_{t \searrow 0} \| p_0(\cdot, t) + w(\cdot, t) - 1 \|_{L^\infty(\mathbb{R})} = 0$;
2. $0 \le w(x, t) \le p(t)$, $\forall (x, t) \in \mathbb{R} \times (0, \infty)$ and $\mathcal{L}w(x, t) = 0$ in the set $Q := \{(x, t) | t > 0, w(x, t) < p(t)\}$;
3. if $x \in \mathbb{R}$ and $t > \delta > 0$, and φ is a smooth function such that $\varphi - w$ attains at (x, t) a local minimum on $[x - \delta, x + \delta] \times [t - \delta, t]$, then $\mathcal{L}\varphi(x, t) \le 0$.

A *viscosity solution of the inverse boundary crossing problem* associated with p is the function b given by

$$b(t) := \inf\{x \in \mathbb{R} | w(x, t) < p(t)\} \qquad \forall t > 0, \tag{5.2}$$

where w is a viscosity solution for the survival distribution of the inverse boundary crossing problem associated with p. If there is a unique viscosity solution, we denote $b = \mathcal{B}[p]$.

The remainder of this section is devoted to a proof of the main result of the paper, Theorem 1, stated in the Introduction.

PROOF OF THEOREM 1. The fourth assertion follows from the first assertion and the following facts which are easy to verify: a classical solution of (1.11) is automatically a viscosity solution, and if (U, b) is a classical solution of (1.4), then (w, b) with w defined by $w(x,t) = \int_x^\infty U(y,t)\,dy$ is a classical solution of (1.11). We divide the proof of the first three assertions into several parts.

Existence and uniqueness of a viscosity solution. The proof of the existence of a unique viscosity solution, together with the properties (1.14), is the major result of Cheng et al. (2006) and hence is omitted here.[7] It is important to note that, by the monotonicity of w in the spatial variable and the definition of b in (5.2), we have

$$Q := \{(x,t)|t > 0, w(x,t) < p(t)\} = Q_b := \{(x,t)|t > 0, x > b(t)\}.$$

Weak regularity of the free boundary. The regularity of the free boundary $b = \mathcal{B}[p]$ defined by (5.2) was not discussed in Cheng et al. (2006). Here, under the assumption that $p \in P_0$, we establish a very basic regularity result on b. In particular, we show that $b \in B_0$.

We begin by showing that $b(t) < \infty$ for every $t > 0$. Indeed, if $b(t) = \infty$, then by the definition $b(t) = \inf\{x|w(x,t) < p(t)\}$ we see that $w(x,t) = p(t)$ for all $x \in \mathbb{R}$. Since $\lim_{x\to\infty} w(x,t) = 0$ (recalling $w \le 1 - p_0$), we see that $w(\cdot,t) \equiv 0$. This contradicts the assumption that $p \in P_0$, since $p \in P_0$ guarantees $p(s) > 0$ for all $s \in [0,\infty)$.

Next, we show that $X_0 \ge b^*(0) = \overline{\lim}_{t\searrow 0} b(t)$ almost surely. To see this, we use the estimate $w \le 1 - p_0$ to derive $p(t) = w(b(t),t) \le 1 - p_0(b(t),t)$. This implies that $\overline{\lim}_{t\searrow 0} p_0(b(t),t) \le \lim_{t\to 0}(1 - p(t)) = 0$ since $p \in P_0$ gives $p \in C([0,\infty))$ and $p(0) = 1$. Now suppose to the contrary that $\mathbb{P}(X_0 < b^*(0)) > 0$. Then there exists $\delta > 0$ and $\varepsilon > 0$ such that $p_0(b^*(0) - 2\delta, 0) = \mathbb{P}(X_0 \le b^*(0) - 2\delta) > 2\varepsilon$. Consequently, there exists $t_0 > 0$ such that $p_0(b^*(0) - \delta, t) = \mathbb{P}(X_t < b^*(0) - \delta) > \varepsilon$ for all $t \in [0, t_0]$. However, by the definition of $b^*(0)$, there exists a sequence $t_k \searrow 0$ such that $\lim_{k\to\infty} b(t_k) = b^*(0)$. For all sufficiently large k, we have $b(t_k) > b^*(0) - \delta$ which implies that $p_0(b(t_k), t_k) \ge p(b^*(0) - \delta, t_k) > \varepsilon$, so

[7] In Cheng et al. (2006), it was assumed that $\mathbb{P}(X_0 = 0) = 1$. The same techniques can be applied to prove existence and uniqueness for more general initial distributions considered here.

we obtain $\overline{\lim}_{t\searrow 0} p_0(b(t),t) \geq \varepsilon$. This contradicts $\lim_{t\searrow 0} p_0(b(t),t) = 0$. Hence, we must have $\mathbb{P}(X_0 < b^*(0)) = 0$, that is, $X_0 \geq b^*(0)$ a.s.

The next step is to show that $b := \mathcal{B}[p]$ is upper-semi-continuous on $[0,\infty)$. First of all, the definition $b(0) := \limsup_{t\searrow 0} b(t)$ implies that b is upper-semi-continuous at $t = 0$. Next, let $t > 0$ be arbitrary. We consider two cases: (i) $b(t) > -\infty$, (ii) $b(t) = -\infty$.

(i) Suppose $b(t) > -\infty$. Fix any $\varepsilon > 0$. Then $p(t) - w(b(t)+\varepsilon, t) > 0$. By continuity, $p(s) - w(b(t)+\varepsilon, s) > 0$ for all s in a neighborhood of t. This implies that $b(s) < b(t) + \varepsilon$ for all s in a neighborhood of t. Consequently, $\overline{\lim}_{s\to t} b(s) \leq b(t) + \varepsilon$. As $\varepsilon > 0$ is arbitrary, we have $\overline{\lim}_{s\to t} b(s) \leq b(t)$.

(ii) Suppose $b(t) = -\infty$. Then for any $M > 0$, we have $p(t) - w(-M, t) > 0$. Consequently, $p(s) - w(-M, s) > 0$ for all s sufficiently close to t. Hence, $b(s) < -M$ for all s sufficiently close to t. This implies that $\overline{\lim}_{s\to t} b(s) \leq -M$. As M can be made arbitrarily large, we hence see that $\lim_{s\to t} b(s) = -\infty = b(t)$.

In conclusion, $b : [0,\infty) \to [-\infty,\infty)$ is upper-semi-continuous.

Let $Q := \{w < p\} := \{(x,t) \in \mathbb{R}\times(0,\infty) | w(x,t) < p(t)\}$ and $Q_b := \{(x,t) \in \mathbb{R}\times(0,\infty) | x > b(t)\}$. Then $Q = Q_b$ and $U := -\partial_x w > 0$ in Q_b. Indeed, $Q = Q_b$ follows from the definition of $b = \mathcal{B}[p]$ in (5.2) and the monotonicity of w in the spatial variable [see Cheng et al. (2006)]. In addition, since b is upper-semi-continuous and bounded above in any compact interval in $[0,\infty)$, any points (x_1,t_1), (x_2,t_2), $t_1 \leq t_2$ in Q_b can be connected by a smooth curve $x = h(t)$ in Q_b defined on $t \in [t_1,t_2]$ such that $h(t_i) = x_i$. Applying the strong maximum principle [Friedman (1964), Theorem 3.5, page 39] to $U := -\partial_x w$ ($U \geq 0$ by the monotonicity of w), we conclude that $U > 0$ in $Q = Q_b$ [an elementary argument shows there cannot exist a $t_2 > 0$ such that $U(\cdot,t_1) \equiv 0$ in Q_b for all $t_1 \leq t_2$].

Next, we show that $b = b^* = b^*_-$. Upper-semi-continuity ($b = b^*$) was shown above, so it remains to prove that $b(t) = \overline{\lim}_{s\nearrow t} b(s) =: b^*_-(t)$ for every $t > 0$. Let $t > 0$, and suppose $b(t) \neq b^*_-(t)$. Then $b(t) = b^*(t) > b^*_-(t)$. Set $\delta = [b(t) - b^*_-(t)]/4$. By the definition of $b^*_-(t)$, we can find $\varepsilon > 0$ such that $b(s) < b^*_-(t) + \delta$ for all $s \in [t-2\varepsilon, t)$. Then $D := [b^*_-(t)+\delta, b(t)] \times [t-2\varepsilon, t)$ is a subset of $Q := \{w < p\}$. Since we know that $\mathcal{L}_1 U = 0$ and $U = -\partial w/\partial x > 0$ in Q. We can apply the Harnack inequality on the cube $(b^*_-(t)+\delta, b(t)) \times (t-2\varepsilon, t)$ to conclude that there exists a positive constant $\eta > 0$ such that $U > \eta$ in $[b^*_-(t)+2\delta, b(t)-\delta] \times (t-\varepsilon, t)$. Consequently,

$$\begin{aligned} &w\big(b^*_-(t)+2\delta, s\big) - w\big(b(t)-\delta, s\big) \\ &\quad = \int_{b^*_-(t)+2\delta}^{b(t)-\delta} U(y,s)\,dy \geq [b(t) - b^*_-(t) - 3\delta]\eta \\ &\quad = \delta\eta \qquad \forall s \in (t-\varepsilon, t). \end{aligned}$$

Sending $s \nearrow t$ we then conclude that $w(b^*+2\delta, t) \geq w(b(t)-\delta, t) + \delta\eta \geq p(t) + \delta\eta$, which violates the requirement that $w \leq p$ for a viscosity solution.

Hence, we must have $b(t) = b^*_-(t)$. In summary, $b = b^* = b^*_-$. This also implies that

$$b(t) = \overline{\lim_{s \to t}} b(s) = \overline{\lim_{s \nearrow t}} b(s) \geq \overline{\lim_{s \searrow t}} b(s).$$

Finally, to show that $b \in B_0$, it remains to show that the survival probability $\tilde{p} := \mathcal{P}[b]$ associated with b has the property $\lim_{t \searrow 0} \tilde{p}(t) = 1$. For this, we consider the sequence $\{w_n\}$, associated with b, defined in Proposition 4. It follows from a (viscosity solution) comparison principle, applied iteratively to $\mathbb{R} \times (t_n^i, t_n^{i+1}]$ ($t_n^0 := 0$) for $i = 0, 1, \ldots$, that $w_n \geq w$; see Remark 4.3. Taking the limit, we find that $w \leq \lim_{n \to \infty} w_n$. This implies that $p(t) = w(-\infty, t) \leq \lim_{n \to \infty} w_n(-\infty, t) = \tilde{p}(t)$. Since our assumption $p \in P_0$ implies that $\lim_{t \searrow 0} p(t) = 1$, we also know that $\lim_{t \searrow 0} \tilde{p}(t) = 1$. Thus, we have shown that $b \in B_0$.

Verification that the boundary derived from the variational inequality has the required crossing time distribution. Given $p \in P_0$, let $b = \mathcal{B}[p]$ be the boundary derived from the unique solution of the variational inequality (1.11). We need to show that $\mathcal{P}[b] = p$, that is, that b is truly a solution of the inverse boundary crossing problem. Summarizing, this means that we want to show that $(\mathcal{P} \circ \mathcal{B})[p] = p$ for every $p \in P_0$.

Let w be the unique *viscosity* solution for survival distribution of the inverse problem associated with p as given in Definition 3. Define $b = \mathcal{B}[p]$ as in (5.2). Let $\tilde{p} = \mathcal{P}[b]$. We want to show that $\tilde{p} = p$. It is enough to show that w is a viscosity solution of the survival probability for the boundary crossing problem associated with b, since in this case part 1. of Theorem 3 yields that $\tilde{p}(t) = w(-\infty, t)$, while taking limits as x goes to $-\infty$ in (1.14) gives that $p(t) = w(-\infty, t)$. By checking the Definitions 3 and 2 of viscosity solutions, one readily sees that w being a viscosity solution in the sense of Definition 3 implies that w is indeed the viscosity solution in the sense of Definition 2, provided that $Q := \{w < p\} = Q_b := \{(x, t) | x > b(t), t > 0\}$. But this last property is immediate from the monotonicity of w. We thus conclude that w is indeed the viscosity solution of the survival distribution of the boundary crossing problem associated with b. Consequently, $\mathcal{P}[b](t) = w(-\infty, t) = p(t)$, so we have $\mathcal{P}[b] = p$ and $p = \mathcal{P}[b] = (\mathcal{P} \circ \mathcal{B})[p]$.

Uniqueness of the solution of the inverse boundary crossing problem in the class B_0. For a given $p \in P_0$, we have shown that $\mathcal{B}[p]$ is a solution of the original inverse boundary crossing problem. Here we show that there is indeed only one such b in the class B_0. To show this, it suffices to show that $(\mathcal{B} \circ \mathcal{P})[b] = b$ for every $b \in B_0$, since this implies that if $\mathcal{P}[\tilde{b}] = p$ then $\tilde{b} = (\mathcal{B} \circ \mathcal{P})[\tilde{b}] = \mathcal{B}[p]$.

Let $\tilde{b} \in B_0$. Define (τ, p, w) as in (2.4)–(2.6). That is, $p = \mathcal{P}[\tilde{b}]$ and w are the survival probability and distribution of the boundary crossing problem associated with $\tilde{b}$. Since $\tilde{b}$ is upper-semi-continuous we can derive from Proposition 5 and

the strong maximum principle that $U := -\partial_x w > 0$ in $Q_{\tilde{b}}$ (see the proof of Theorem 3). This implies that $Q_{\tilde{b}} \subset \{w < p\}$. Also, since $w(x,t) = \mathbb{P}(\tau \geq t, X_t > x)$ we know that $w(x,t) = p(t)$ when $x \leq \tilde{b}(t)$. Thus, $Q_{\tilde{b}} = \{w < p\}$.

By Theorem 3, w is a *viscosity* solution of the survival probability distribution of the boundary crossing problem in the sense of Definition 2, associated with $\tilde{b}$. By checking the definition of a viscosity solution of the variational inequality associated with p (Definition 3), we find that w is indeed a viscosity solution associated with $p = \mathcal{P}[\tilde{b}]$. Now, according to the definition of $\mathcal{B}[p]$ in (5.2), $\mathcal{B}[p](t) = \inf\{x | w(x,t) < p(t)\}$. Since $Q_{\tilde{b}} = \{w < p\}$, we see that $\mathcal{B}[p] = \tilde{b}$. Thus $\tilde{b} = \mathcal{B}[p] = (\mathcal{B} \circ \mathcal{P})[\tilde{b}]$ for every $\tilde{b} \in B_0$. This completes the proof of Theorem 1. □

5.2. *Continuity of the free boundary in the inverse boundary crossing problem.* In this subsection, we investigate the continuity of the free boundary $b = \mathcal{B}[p]$ for the inverse boundary crossing problem for $p \in P_0$. We already know that b is upper-semi-continuous, and since $b = b^*_-$, it cannot "jump up." For b to be continuous, we need to prevent it from "jumping down." Note the fact that if p is a constant in an open interval, then $b = -\infty$ in that interval. Hence, to eliminate steep drops of b we require a lower bound on the rate of decrease of p. We consider the following:

$$L(p, T_1, T_2) := \inf_{T_1 \leq s < t \leq T_2} \frac{p(s) - p(t)}{t - s} \qquad \forall 0 \leq T_1 < T_2.$$

The following proposition gives a sufficient condition for the boundary to be continuous, in the case that X is a standard Brownian motion.

PROPOSITION 6. *Suppose that X is a standard Brownian motion, that is, $\mu \equiv 0$, $\sigma \equiv 1$, and $p_0(x,0) = \chi_{[0,\infty)}(x)$. Let $p \in P_0$ and $b = \mathcal{B}[p]$.*

1. *If $L(p, T_1, T_2) > 0$ for some positive T_1, T_2 with $T_1 < T_2$, then $b = \mathcal{B}[p]$ is continuous on (T_1, T_2).*

2. *Assume that $L(p, 0, T) > 0$ for every $T > 0$. Then $b \in C([0,\infty))$.*

PROOF. 1. Let $t_1 \geq 0$ be arbitrary. Define

$$\tilde{w}(x,t) = \frac{w(b(t_1) + x, t + t_1)}{p(t_1)}, \qquad \tilde{b}(t) = b(t_1 + t) - b(t_1),$$
$$\tilde{p}(t) = \frac{p(t + t_1)}{p(t_1)}.$$

Then $(\tilde{w}, \tilde{b})$ is the solution of the inverse problem with initial value $\tilde{w}(\cdot, 0)$ and survival probability $\tilde{p}$. This statement follows by an immediate application of the definitions. Note that $\tilde{w}(x,t) = \mathbb{P}(X_{t+t_1} > x, \tau \geq t + t_1 | \tau \geq t_1)$. $\tilde{p}(t) = \mathbb{P}(\tau \geq t + t_1 | \tau \geq t_1)$. The conditional probabilities and the boundary from time t_1 on are

the same as the solution of the inverse problem started with the initial position equal to the conditional distribution of X given that $\tau \geq t_1$.

Now let $(\underline{w}, \underline{b})$ be the solution of the inverse problem with initial data $\chi_{(-\infty,0)}$ and survival probability $\tilde{p}$. Note that $\underline{w}(x,0) = 1 = \tilde{w}(x,0)$ for $x < 0$ and $\underline{w}(x,0) = 0 \leq w(x,0)$ for $x \geq 0$. Hence, $\underline{w}(\cdot,0) \leq \tilde{w}(\cdot,0)$. It then follows from a comparison principle [cf. the proof of Lemma 4.2 in Cheng et al. (2006)] that $\underline{w} \leq \tilde{w}$ and that $\underline{b} \leq \tilde{b}$. Again, this is obvious from the probabilistic interpretation of the problem. The boundary $\underline{b}$ is the one that produces the hitting distribution $\tilde{p}$ when the process starts at $b(t_1)$ at time t_1. The boundary $\tilde{b}$ produces the same hitting distribution with the process started at the conditional distribution of X_{t_1} given that $\tau \geq t_1$. Since in this case we must have $X_{t_1} \geq b(t_1)$, we have that the boundaries $\tilde{b}$ and $\underline{b}$ produce the same hitting distribution for the process X, with $\tilde{b}$ arising from X starting at a higher point with probability 1. Therefore, we must have $\tilde{b} \geq \underline{b}$.

Thus, for $0 < t \leq 1/2$,

$$b(t_1 + t) - b(t_1) = \tilde{b}(t) \geq \underline{b}(t) \geq -[1 + O(1)]\sqrt{-2t \log[1 - \tilde{p}(t)]}$$

by the estimate on line 16, page 867 of Cheng et al. (2006). Upon noting that

$$\begin{aligned} |\log[1 - \tilde{p}(t)]| &= |\log[p(t_1 + t) - p(t_1)] - \log p(t_1)| \\ &= \big|\log\big(t\dot{p}(t_1 + \theta)\big)\big| + O(1) \\ &= |\log t| + O(1) = O(1)|\log t|, \end{aligned}$$

where

$$\dot{p}(t) := \limsup_{s \nearrow t} \frac{p(t) - p(s)}{t - s} \in [0, \infty]$$

we find that there exists a constant $C(t_1)$ such that

$$b(t + t_1) - b(t_1) \geq -C(t_1)\sqrt{t|\log t|} \qquad \forall t \in (0, 1/2].$$

Now pick any $t \in (T_1, T_2)$. Let $\{t_i\}_{i=1}^{\infty}$ be a sequence in $[T_1, t)$ such that $\lim_{i\to\infty} t_i = t$ and $\lim_{i\to\infty} b(t_i) = b(t)$ [recalling $b(t) = b^*_-(t) := \overline{\lim}_{s \nearrow t} b(s)$]. Then setting $h_i = [t - t_i]/2$, we have

$$b(t_i + h) > b(t_i) - c\sqrt{|h_i \log h_i|} \qquad \forall h \in [h_i, 3h_i].$$

This implies that

$$\inf_{s \in [t - h_i, t + h_i]} b(s) \geq b(t_i) - c\sqrt{h_i |\log h_i|},$$

so that

$$\underline{\lim}_{s \to t} b(s) \geq b(t_i) - c\sqrt{h_i |\log h_i|}.$$

Sending $i \to \infty$ we then obtain $\underline{\lim}_{s \to t} b(s) \geq b(t)$. Thus, b is lower-semi-continuous in (T_1, T_2). Since b is also upper-semi-continuous, we see that b is continuous in (T_1, T_2).

2. By the first assertion, we know that b is continuous in $(0, \infty)$. At $t = 0$, since $p_0(x, 0) > 0$ for all $x > 0$, the proof in Cheng et al. [(2006), Lemma 4.5, page 865] implies that there exists a positive constant C that depends on $L(p, 0, 1/2)$ such that

$$b(t) \geq -C\sqrt{t|\log t|} \qquad \forall t \in [0, 1/2].$$

This implies that $\underline{\lim}_{t \searrow 0} b(t) \geq 0$.

We recall that $1 - w(x, t) - p_0(x, t) \geq 0$. Evaluating this inequality at $x = b(t)$ gives $p_0(b(t), t) \leq 1 - p(t)$ for all $t > 0$. Since $p_0(x, 0) > 0$, sending $t \searrow 0$ we conclude that $\overline{\lim}_{t \searrow 0} b(t) \leq 0$. Thus, $b(0) := \lim_{t \searrow 0} b(t) = 0$. This completes the proof. $\square$

Note that if the hitting time density $-\dot{p}$ is everywhere strictly positive, then we obtain that $b = \mathcal{B}[p]$ is continuous. In particular, this criterion is satisfied by the boundary arising from the exponential distribution and hence provides the solution to the inverse boundary crossing problem as originally proposed by A. N. Shiryayev.

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