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Closed-form expressions for moments of a class of beta generalized distributions

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Abstract. For the first time, we derive explicit closed-form expressions for moments of some beta generalized distributions including the beta gamma, beta normal, beta beta, beta Student t and beta F distributions. These expressions are given as infinite weighted sums of well-known special functions for which numerical routines for computation are available.

1 Introduction

Generalized distributions have been widely studied in statistics and numerous authors have developed various classes of these distributions. Eugene, Lee and Famoye (2002) first proposed a general class of distributions for a random variable defined from the logit of the beta random variable by employing two parameters whose role is to introduce skewness and to vary tail weights. In this paper, we derive explicit closed-form expressions for the moments of this class of distributions. The expressions take the form of infinite sums of well-known special functions which are very simple to be implemented in practice for several beta generalized distributions. In fact, we derive closed-form expressions for the moments of the beta gamma, beta normal, beta beta, beta Student t and beta F distributions. The reason that we choose these special distributions (gamma, normal, beta, Student t and F) is because they are perhaps the most popular distributions in statistics and various applied areas, and they possess finite moments to ensure existence of the moments of the associated beta generalized distributions. Similar results could in principle be derived for other beta generalized distributions. These closed-form expressions can be used to obtain the same for functions of moments, for example, moment generating function, cumulant generating function, characteristic function, factorial and central moments, etc.

The calculations in this article involve some special functions, including the well-known incomplete gamma function defined by

$$\gamma(\alpha, x) = \int_0^x w^{\alpha-1} e^{-w}\, dw, \qquad \alpha > 0,$$

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the error function $\mathrm{erf}(\cdot)$ defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt,$$

the beta function ($\Gamma(\cdot)$ is the gamma function) given by

$$B(a,b) = \int_0^\infty w^{a-1}(1-w)^{b-1}\, dw = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

the incomplete beta function ratio, that is, the cumulative distribution function (cdf) of the beta distribution with parameters a and b, defined by

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x w^{a-1}(1-w)^{b-1}\, dw,$$

the confluent hypergeometric function defined by

$${}_1F_1(a;b;z) = \sum_{i=0}^\infty \frac{(a)_i z^i}{(b)_i i!},$$

where $(a)_i$ is the ascending factorial defined by (with the convention that $(a)_0 = 1$)

$$(a)_i = a(a+1)\cdots(a+i-1),$$

the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) = \sum_{i=0}^\infty \frac{(a)_i (b)_i z^i}{(c)_i i!},$$

the Lauricella function of type A [Exton (1978); Aarts (2000)] defined by

$$\begin{aligned} &F_A^{(n)}(a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n) \\ &\quad = \sum_{m_1=0}^\infty \cdots \sum_{m_n=0}^\infty \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \end{aligned} \tag{1.1}$$

and the generalized Kampé de Fériet function [Exton (1978); Mathai (1993); Aarts (2000); Chaudhry and Zubair (2002)] defined by

$$\begin{aligned} &F_{C:D}^{A:B}\big((a) : (b_1); \ldots, (b_n); (c) : (d_1); \ldots, (d_n); x_1, \ldots, x_n\big) \\ &\quad = \sum_{m_1=0}^\infty \cdots \sum_{m_n=0}^\infty \frac{((a))_{m_1+\cdots+m_n}((b_1))_{m_1}\cdots((b_n))_{m_n}}{((c))_{m_1+\cdots+m_n}((d_1))_{m_1}\cdots((d_n))_{m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \end{aligned} \tag{1.2}$$

where $a = (a_1, a_2, \ldots, a_A)$, $b_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,B})$ for $i = 1, 2, \ldots, n$, $c = (c_1, c_2, \ldots, c_C)$, $d_i = (d_{i,1}, d_{i,2}, \ldots, d_{i,D})$ for $i = 1, 2, \ldots, n$, and

$$((f))_k = ((f_1, f_2, \ldots, f_p))_k = (f_1)_k (f_2)_k \cdots (f_p)_k$$

denotes the product of ascending factorials. Numerical routines for the direct computation of functions (1.1) and (1.2) are available; see Exton (1978) and Mathematica [Trott (2006)].

The rest of the paper is organized as follows. Section 2 defines the class of beta generalized distributions. Section 3 gives general expansions for calculating the moments of beta generalized distribution as infinite weighted sums of probability weighted moments (PWMs) of the parent distribution. Closed-form expressions for moments of the beta gamma and beta normal distributions are given in Sections 4 and 5, respectively. In Sections 6, 7 and 8 we give closed-form expressions for moments of the beta beta, beta Student t and beta F distributions, respectively. Section 9 provides some numerical calculations for these moments. Section 10 ends with some conclusions.

2 The class of beta generalized distributions

Let G be the cdf of a random variable. The function $F(x)$ given by

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} \omega^{a-1}(1-\omega)^{b-1}\, d\omega, \qquad (2.1)$$

defines the cdf of the class of beta G distributions, where $a > 0$ and $b > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weights, and $I_{G(x)}(a,b)$ is the incomplete beta function ratio evaluated at $G(x)$. Application of $X = G^{-1}(V)$ to V following the beta $B(a,b)$ distribution yields X with cdf (2.1). Several authors introduced and studied particular members of this class of distributions over the last years, mainly after the works of Eugene, Lee and Famoye (2002) and Jones (2004a).

The beta normal (BN) distribution introduced by Eugene, Lee and Famoye (2002) is obtained by taking $G(x)$ to be the cdf of the normal distribution. This distribution can be unimodal and bimodal. Some expressions for the moments of the BN distribution were derived by Gupta and Nadarajah (2004a). Nadarajah and Kotz (2004) introduced the beta Gumbel distribution by taking $G(x)$ to be the cdf of the Gumbel distribution and provided closed form expressions for the moments and discussed the asymptotic distribution of the extreme order statistics and the maximum likelihood estimation procedure. Nadarajah and Gupta (2004) proposed the beta Fréchet distribution by taking $G(x)$ to be the Fréchet distribution, derived the analytical shapes of the density and hazard rate functions and calculated the asymptotic distribution of the extreme order statistics. Nadarajah and Kotz (2006) obtained the moment generating function, the first four cumulants and the asymptotic distribution of the extreme order statistics for the beta exponential distribution and examined maximum likelihood estimation of its parameters.

The probability density function (pdf) corresponding to (2.1) has a very simple form

$$f(x) = \frac{g(x)}{B(a,b)} G(x)^{a-1}\{1-G(x)\}^{b-1}, \qquad (2.2)$$

where $g(x) = dG(x)/dx$ is the density of the baseline distribution. The density $f(x)$ will be most tractable when both functions $G(x)$ and $g(x) = dG(x)/dx$ have simple analytic expressions. Except for some special choices for $G(x)$ in equation (2.1), it would appear that the density (2.2) will be difficult to deal with in generality. If $g(x)$ is a symmetric distribution around zero, then $f(x)$ will also be a symmetric distribution when $a = b$.

3 General formulae for the moments

For $b > 0$ real noninteger, we have the power series

$$\{1 - G(x)\}^{b-1} = \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} G(x)^i,$$

where the binomial coefficient is defined for any real b. From the above expansion and equation (2.2), we can express the density of the beta G as

$$f(x) = g(x) \sum_{i=0}^{\infty} w_i G(x)^{a+i-1}, \tag{3.1}$$

where

$$w_i = w_i(a, b) = \frac{(-1)^i \binom{b-1}{i}}{B(a, b)}.$$

If b is an integer, the index i in the previous sum stops at $b - 1$. If a is an integer, equation (3.1) gives the pdf of the beta G as an infinite power series expansion of cdf's of G. Otherwise, if a is real noninteger, we can expand $G(x)^{a+i-1}$ as follows:

$$G(x)^{a+i-1} = [1 - \{1 - G(x)\}]^{a+i-1} = \sum_{j=0}^{\infty} (-1)^j \binom{a+i-1}{j} \{1 - G(x)\}^j$$

and then

$$G(x)^{a+i-1} = \sum_{j=0}^{\infty} \sum_{r=0}^{j} (-1)^{j+r} \binom{a+i-1}{j} \binom{j}{r} G(x)^r.$$

Hence, we can write from equation (2.2)

$$f(x) = g(x) \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} w_{i,j,r} G(x)^r, \tag{3.2}$$

where the coefficients

$$w_{i,j,r} = w_{i,j,r}(a, b) = \frac{(-1)^{i+j+r} \binom{a+i-1}{j} \binom{b-1}{i} \binom{j}{r}}{B(a, b)}$$

are constants. Expansion (3.2), which holds for any real noninteger a, gives the pdf of the beta G as an infinite power series expansion of cdf's of G. If b is an integer, the index i in equation (3.2) stops at $b-1$. Equations (3.1) and (3.2) are used to derive closed form expressions for the moments of the beta gamma, beta normal, beta beta, beta Student t and beta F distributions valid for a integer and a real noninteger, respectively. We have

$$\sum_{i=0}^{\infty} w_i = 1 \quad \text{and} \quad \sum_{i,j=0}^{\infty}\sum_{r=0}^{j} w_{i,j,r} = 1.$$

From now on we assume X following the pdf of any parent G distribution and Y the pdf of the beta G distribution. The sth moment of Y can be expressed in terms of the (s,r)th PWM of X, say $\tau_{s,r} = E\{X^s G(X)^r\}$. For a integer, we obtain

$$\mu'_s = \sum_{r=0}^{\infty} w_r \tau_{s,r+a-1}, \tag{3.3}$$

whereas for a real noninteger we have

$$\mu'_s = \sum_{i,j=0}^{\infty}\sum_{r=0}^{j} w_{i,j,r}\tau_{s,r}. \tag{3.4}$$

Equations (3.3) and (3.4) are of very simple forms and constitute the main results of this section. Hence, we can calculate the moments of the beta G distribution in terms of infinite weighted sums of PWMs of G.

4 Moments of the beta gamma

A random variable X has a gamma $G(\alpha,\beta)$ distribution with parameters $\alpha > 0$ and $\beta > 0$ if its cdf is

$$G(x) = \frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}, \qquad x > 0.$$

For $s > 0$, we have $E(X^s) = \Gamma(s+\alpha)/\Gamma(\beta^s\alpha)$. A random variable Y has a beta gamma $BG(a,b,\alpha,\beta)$ distribution if its pdf is

$$f(x) = \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{B(a,b)\Gamma(\alpha)^{a+b-1}}\gamma(\alpha,\beta x)^{a-1}\{\Gamma(\alpha)-\gamma(\alpha,\beta x)\}^{b-1}, \qquad x > 0.$$

Some properties of the beta gamma distribution are discussed in Kong, Lee and Sepanski (2007). We obtain $\tau_{s,r}$ using the series expansion for the incomplete gamma function. We have

$$G(x) = \frac{(\beta x)^\alpha}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{(-\beta x)^m}{(\alpha+m)m!},$$

and then

$$\tau_{s,r} = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{s+\alpha-1} \exp(-\beta x) \left\{ \frac{(\beta x)^{\alpha}}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(-\beta x)^m}{(\alpha+m)m!} \right\}^r dx$$

$$= \frac{\beta^{-s}}{\Gamma(\alpha)^{r+1}} \int_0^{\infty} u^{s+\alpha-1} \exp(-u) \left\{ u^{\alpha} \sum_{m=0}^{\infty} \frac{(-u)^m}{(\alpha+m)m!} \right\}^r du.$$

The last integral can be obtained from equations (24) and (25) of Nadarajah (2008) as

$$\tau_{s,r} = \frac{\beta^{-s}\alpha^{-r}\Gamma(s+\alpha(r+1))}{\Gamma(\alpha)^{r+1}} \times F_A^{(r)}\big(s+\alpha(r+1); \alpha, \ldots, \alpha; \alpha+1, \ldots, \alpha+1; -1, \ldots, -1\big). \tag{4.1}$$

Hence, the moments of the beta gamma distribution can be written as infinite weighted sums of the Lauricella functions of type A from equations (3.3) and (4.1) for a integer and from (3.4) and (4.1) for a real noninteger, respectively.

5 Moments of the beta normal

The *BN* distribution, introduced by Eugene, Lee and Famoye (2002), is obtained by taking $G(x)$ to be the cdf of the normal distribution in equation (2.1). The density of the $BN(a, b, \mu, \sigma^2)$ distribution is given by

$$f(x) = \frac{\sigma^{-1}}{B(a,b)} \phi\left(\frac{x-\mu}{\sigma}\right) \left\{\Phi\left(\frac{x-\mu}{\sigma}\right)\right\}^{a-1} \left\{1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right\}^{b-1}, \tag{5.1}$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter, a and b are shape parameters, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal pdf and cdf, respectively. For $\mu = 0$ and $\sigma = 1$, we obtain the beta (standard) normal distribution. Plots of the beta (standard) normal for selected parameter values are given in Figure 1.

We can work with the beta standard normal distribution in generality, since we can obtain the moments of $Y \sim BN(a, b, \mu, \sigma)$ from the moments of $Z \sim BN(a, b, 0, 1)$ using $E(Y^r) = E[(\mu + \sigma Z)^r] = \sum_{t=0}^{r} \mu^{r-t} \sigma^r E(Z^r)$. The standard normal cdf can be written as

$$\Phi(x) = \frac{1}{2}\left\{1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right\}, \qquad x \in \mathbb{R}.$$

We can obtain the moments of the beta (standard) normal from equations (3.3) and (3.4) for a integer and a real noninteger, respectively, if we calculate

$$\tau_{s,r} = \int_{-\infty}^{\infty} x^s \phi(x) \Phi(x)^r \, dx$$

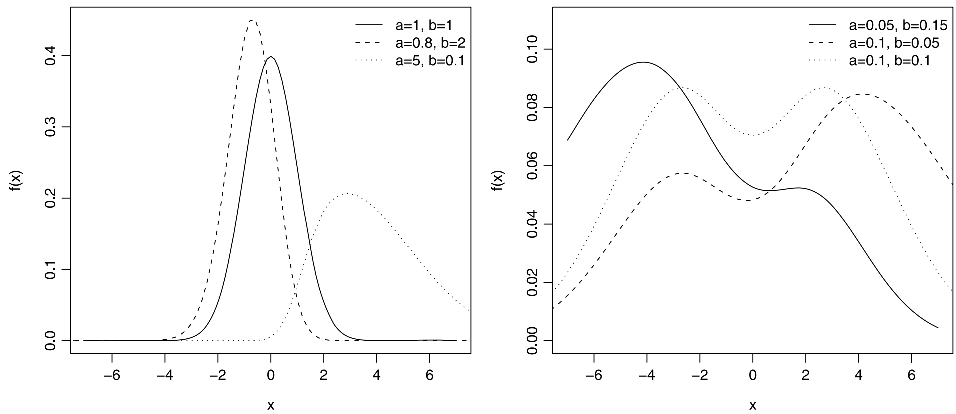

Figure 1 *Plots of the BN density for some parameter values.*

for s and r integers. Using the binomial expansion and interchanging terms, we have

$$\tau_{s,r} = \frac{1}{2^r\sqrt{2\pi}} \sum_{l=0}^{r} \binom{r}{l} \int_{-\infty}^{\infty} x^s \exp(-x^2/2)\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)^{r-l} dx.$$

Using the series expansion for the error function erf(·)

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)m!},$$

we can solve the last integral following equations (9)–(11) of Nadarajah (2008). We can obtain when $s+r-l$ is even

$$\begin{aligned}\tau_{s,r} &= 2^{s/2}\pi^{-(r+1/2)} \\ &\quad \times \sum_{\substack{l=0 \\ (s+r-l)\,\text{even}}}^{r} \binom{r}{l} 2^{-l}\pi^{l}\Gamma\left(\frac{s+r-l+1}{2}\right) \\ &\quad \times F_A^{(r-l)}\left(\frac{s+r-l+1}{2}; \frac{1}{2},\ldots,\frac{1}{2}; \frac{3}{2},\ldots,\frac{3}{2}; -1,\ldots,-1\right).\end{aligned} \tag{5.2}$$

Expressions for the integral and then for terms in $\tau_{s,r}$ vanish when $s+r-l$ is odd. Hence, equations (3.3) and (5.2) for a integer and (3.4) and (5.2) for a real noninteger can be applied to calculate the moments of the beta normal distribution.

6 Moments of the beta beta

The beta distribution is the most flexible family of distributions. It has relationships with several of the well-known univariate distributions. Beta distributions are very versatile and a variety of uncertainties can be usefully modeled by them. Many of the finite range distributions encountered in practice can be easily transformed into the standard beta distribution. In reliability and life testing experiments, many times the data are modeled by finite range distributions; see, for example, Barlow and Proschan (1975). In recent years, beta distributions have been used in modeling distributions of hydrologic variables. Many generalizations of the beta distribution involving algebraic, exponential and hypergeometric functions have been proposed in the literature; see the book of Gupta and Nadarajah (2004b) for detailed accounts.

The pdf and cdf of the beta $B(\alpha, \beta)$ distribution with parameters $\alpha > 0$ and $\beta > 0$ are simply $g(x) = x^{\alpha-1}(1-x)^{\beta-1}/B(\alpha, \beta)$ and $G(x) = I_x(\alpha, \beta) = B(\alpha, \beta)^{-1} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\, dt$ for $0 < x < 1$. In this section, an enlargement of the beta family of distributions on (0, 1) is presented. We introduce, for the first time, the so-called four parameter beta beta $BB(a, b, \alpha, \beta)$ distribution with density (for $0 < x < 1$) given by

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)B(a, b)} I_x(\alpha, \beta)^{a-1}\{1 - I_x(\alpha, \beta)\}^{b-1}. \tag{6.1}$$

The PWM of the $B(\alpha, \beta)$ distribution is

$$\tau_{s,r} = \frac{1}{B(\alpha, \beta)} \int_0^1 x^{s+\alpha-1}(1-x)^{\beta-1} I_x(\alpha, \beta)^r\, dx.$$

Using the incomplete beta function expansion for β real noninteger

$$I_x(\alpha, \beta) = \frac{x^\alpha}{B(\alpha, \beta)} \sum_{m=0}^{\infty} \frac{(1-\beta)_m x^m}{(\alpha+m)m!},$$

and the fact $(f)_k = \Gamma(f+k)/\Gamma(f)$, the last integral can be obtained from the algebraic developments made by Nadarajah [(2008), Section 5] which convert the function $I(k, l)$ defined in equation (28) to the expression (30) of his paper written in terms of the generalized Kampé de Fériet function. Hence, we obtain

$$\begin{aligned} \tau_{s,r} = {} & \alpha^{-r} B(\alpha, \beta)^{-(r+1)} B\big(\beta, s + \alpha(r+1)\big) \\ & \times F_{1:1}^{1:2}\big(\big(s + \alpha(r+1)\big) : (1-\beta, \alpha); \ldots; (1-\beta, \alpha) : \\ & \qquad \big(\beta + s + \alpha(r+1)\big) : (\alpha+1); \ldots; (\alpha+1); 1, \ldots, 1\big). \end{aligned} \tag{6.2}$$

The moments of the beta beta distribution follow immediately as infinite sums of the generalized Kampé de Fériet functions from equations (3.3) and (6.2) for a integer and from (3.4) and (6.2) for a real noninteger.

7 Moments of the beta Student t

The Student t distribution is the second most popular continuous distribution in statistics, second only to the normal distribution. The density of the Student t_ν

distribution with $\nu > 0$ degrees of freedom is (for $-\infty < x < \infty$)

$$g(x) = \frac{1}{\sqrt{\nu} B(1/2, \nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}.$$

For any real x, the cdf of the Student t_ν distribution is simply $G(x) = I_y(1/2, \nu/2)$, where $y = (x + \sqrt{x^2+\nu})/(2\sqrt{x^2+\nu})$. The density of the beta Student $BS(a, b, \nu)$ distribution with parameters ν, a and b is then given by (for any x)

$$\begin{aligned} f(x) = {} & \frac{(1 + x^2/\nu)^{-(\nu+1)/2}}{\sqrt{\nu} B(a, b) B(1/2, \nu/2)} \\ & \times \left\{ I_{(x+\sqrt{x^2+\nu})/(2\sqrt{x^2+\nu})}(1/2, \nu/2) \right\}^{a-1} \\ & \times \left\{ 1 - I_{(x+\sqrt{x^2+\nu})/(2\sqrt{x^2+\nu})}(1/2, \nu/2) \right\}^{b-1}. \end{aligned} \tag{7.1}$$

The beta Student is then symmetric around zero only when $a = b$.

Since the pdf of the Student t_ν distribution is symmetric around zero, the (s, r)th PWM of the Student t_ν distribution can be expressed as

$$\tau_{s,r} = \int_0^\infty x^s G(x)^r g(x)\, dx + (-1)^s \int_0^\infty x^s \{1 - G(x)\}^r g(x)\, dx.$$

For k, n and m positive integers, we now define

$$A(k, n, m) = \int_0^\infty x^k G(x)^{m-1} \{1 - G(x)\}^{n-m} g(x)\, dx$$

and rewrite $\tau_{s,r}$ as

$$\tau_{s,r} = A(s, r+1, r+1) + (-1)^s A(s, r+1, 1).$$

For $x \geq 0$, we have $G(x) = \frac{1}{2} + \frac{1}{2} I_{x^2/(\nu+x^2)}(1/2, \nu/2)$. Following Nadarajah (2007), setting $y = x^2/(\nu + x^2)$ and using the incomplete beta function expansion and the fact $(f)_k = \Gamma(f + k)/\Gamma(f)$, we calculate the integral $A(k, n, m)$ in terms of the generalized Kampé de Fériet function. Then, from equation (7) of his paper we can obtain $A(s, r+1, r+1)$ and $A(s, r+1, 1)$. Combining these expressions, we reach the formula

$$\begin{aligned} \tau_{s,r} = {} & \frac{\nu^{s/2}}{2^{r+1}} \sum_{\substack{p=0 \\ p \text{ even}}}^{r} \binom{r}{p} 2^{p+1} B^{-1-p}(1/2, \nu/2) B\left(\frac{\nu - s}{2}, \frac{s+p+1}{2}\right) \\ & \times F_{1:1}^{1:2}\left(\left(\frac{s+p+1}{2}\right) : \left(1 - \frac{\nu}{2}, \frac{1}{2}\right); \ldots; \left(1 - \frac{\nu}{2}, \frac{1}{2}\right); \right. \\ & \left. \left(\frac{\nu+p+1}{2}\right) : \left(\frac{3}{2}\right); \ldots; \left(\frac{3}{2}\right); 1, \ldots, 1 \right). \end{aligned} \tag{7.2}$$

Hence, the moments of the beta Student t distribution can be written as infinite weighted sums of the generalized Kampé de Fériet functions from equations (3.3) and (7.2) for a integer and from (3.4) and (7.2) for a real noninteger.

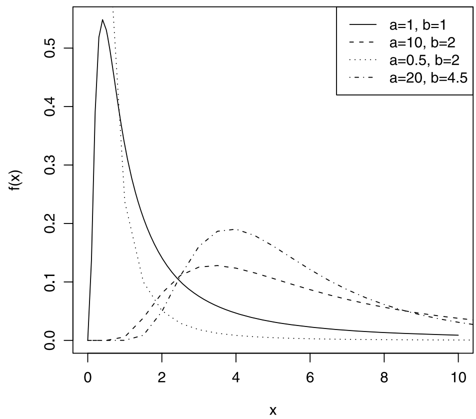

Figure 2 *Plots of the beta F density for some parameter values.*

8 Moments of the beta F

The F distribution arises frequently as the null distribution of a test statistic, especially in likelihood ratio tests, perhaps most notably in the analysis of variance. Consider the $F(2\alpha, 2\beta)$ distribution with degrees of freedom 2α and 2β and pdf and cdf for $x > 0$, $\alpha > 0$ and $\beta > 0$ given by

$$g(x) = \frac{\alpha^\alpha x^{\alpha-1}}{\beta^\alpha B(\alpha,\beta)(1+\alpha x/\beta)^{\alpha+\beta}} \tag{8.1}$$

and $G(x) = I_{\alpha x/(\alpha x+\beta)}(\alpha,\beta)$, respectively. The existence of the ordinary moments of (8.1) requires the condition that the order of the moment be smaller than β.

The density of the beta F distribution $BF(a, b, 2\alpha, 2\beta)$ with parameters $a, b, 2\alpha$ and 2β, can be written (for any $x > 0$) as

$$\begin{aligned} f(x) = {} & \frac{\alpha^\alpha x^{\alpha-1}}{\beta^\alpha B(\alpha,\beta)(1+\alpha x/\beta)^{\alpha+\beta} B(a,b)} \\ & \times I_{\alpha x/(\alpha x+\beta)}(\alpha,\beta)^{a-1}\{1 - I_{\alpha x/(\alpha x+\beta)}(\alpha,\beta)\}^{b-1}. \end{aligned} \tag{8.2}$$

The (s, r)th PWM of the $F(2\alpha, 2\beta)$ distribution can be expressed as

$$\tau_{s,r} = \frac{\alpha^\alpha}{\beta^\alpha B(\alpha,\beta)} \int_0^\infty \frac{x^{\alpha+s-1}}{(1+\alpha x/\beta)^{\alpha+\beta}} I_{\alpha x/(\alpha x+\beta)}(\alpha,\beta)^r \, dx.$$

Setting $y = \alpha x/(\alpha x + \beta)$, we obtain

$$\tau_{s,r} = \frac{\beta^s}{\alpha^s B(\alpha,\beta)} \int_0^1 y^{\alpha+s-1}(1-y)^{\beta-s-1} I_y(\alpha,\beta)^r \, dy.$$

Using the series expansion

$$I_y(\alpha, \beta) = \frac{y^{\alpha}}{B(\alpha, \beta)} \sum_{m=0}^{\infty} \frac{(1-\beta)_m y^m}{(\alpha+m)m!},$$

the last integral can be expressed for $s < \beta$ as

$$\begin{aligned}
&\int_0^1 y^{\alpha+s-1}(1-y)^{\beta-s-1}\left\{\frac{y^{\alpha}}{B(\alpha, \beta)} \sum_{m=0}^{\infty} \frac{(1-\beta)_m y^m}{(\alpha+m)m!}\right\}^r dy \\
&\quad = \int_0^1 \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \big((1-\beta)_{m_1} \cdots (1-\beta)_{m_r} \\
&\qquad\qquad \times y^{s+\alpha(r+1)+m_1+\cdots+m_r-1}(1-y)^{\beta-s-1} \\
&\qquad\qquad /\big(B(\alpha, \beta)^r (\alpha+m_1)\cdots(a+m_r)m_1!\cdots m_r!\big)\big)\, dy \\
&\quad = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} (1-\beta)_{m_1} \cdots (1-b)_{m_r} \\
&\qquad\qquad \times B\big(s+\alpha(r+1)+m_1+\cdots+m_r, \beta-s\big) \\
&\qquad\qquad /\big(B(\alpha, \beta)^r (\alpha+m_1)\cdots(\alpha+m_r)m_1!\cdots m_r!\big).
\end{aligned}$$

Using $(f)_k = \Gamma(f+k)/\Gamma(f)$ and the definition of the generalized Kampé de Fériet function in equation (1.2), we can write $\tau_{s,r}$ (for $s < b$) as

$$\begin{aligned}
\tau_{s,r} = {} & \frac{\beta^s}{\alpha^{s+r} B(\alpha, \beta)^{r+1}} B\big(\beta-s, s+\alpha(r+1)\big) \\
& \times F_{1:1}^{1:2}\big(\big(s+\alpha(r+1)\big) : (1-\beta, \alpha); \ldots; (1-\beta, \alpha); \\
& \quad \big(\beta+\alpha(r+1)\big) : (\alpha+1); \ldots; (\alpha+1); 1, \ldots, 1\big).
\end{aligned} \tag{8.3}$$

It is easy to verify that that this expression exists for $s < \beta$. Hence, the moments of the beta F distribution can be written as infinite weighted sums of the generalized Kampé de Fériet functions from equations (3.3) and (8.3) for a integer and from (3.4) and (8.3) for a real noninteger.

9 Numerical applications

In this section we provide numerical values for the moments of some beta generalized distributions. We compute Lauricella function of type A using the formula [Erdélyi (1936), page 696, equation (1)]

$$\begin{aligned}
&F_A^{(n)}[\alpha, \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n] \\
&\quad = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} \exp(-t)\, {}_1F_1(\beta_1; \gamma_1; x_1 t) \cdots {}_1F_1(\beta_n; \gamma_n; x_n t)\, dt
\end{aligned} \tag{9.1}$$

and the Generalized Kampé de Fériet function using the equation (2.1.5.15) given in Exton (1978)

$$F_{1:1}^{1:2}\big((a):(c_1,d_1);\ldots;(c_n,d_n);(c):(a+b);(f_1);\ldots;(d_n);s_1,\ldots,s_n\big)$$
$$= \frac{1}{B(a,b)}\int_0^1 x^{a-1}(1-x)^{b-1} \tag{9.2}$$
$$\times\, {}_2F_1(c_1,d_1;\,f_1;s_1x)\cdots\,{}_2F_1(c_n,d_n;\,f_n;s_nx)\,dx.$$

Establishing in-built routines for these special functions can be used to compute the moments. This can be more efficient than computing the moments by writing say some codes in SAS or R. It can also be more accurate computationally to use these in-built routines. Other representations for moments (e.g., integral representations) can be prone to rounding off errors among others. In fact, we compare the numerical moments obtained using these special functions in Mathematica scripts with those calculated from some Maple codes for direct integration of the density functions in 350 selected choices of parameters for the beta generalized distributions discussed in this section. The scripts were written and tested on Maple version 10, and Mathematica version 5.0.0.0, to obtain numerical moments of the beta generalized distributions. For rare selections of parameters (6.86%), Maple fails to calculate numerical values for the moments while Mathematica succeeds in almost all cases tested (it fails in only 2.29% of cases). Comparing the numerical values obtained with Maple and Mathematica we found that, in almost all cases, the results agree using both softwares.

The moments of five beta generalized distributions were computed using our infinite weighted sums of Lauricella and Generalized Kampé de Fériet functions by evaluating these functions from equations (9.1) and (9.2), respectively. For selected parameter values $a = 1.5$ and $b = 2.5$, Table 1 gives some numerical values for the ordinary moments ($\mu'_r, r = 1, \ldots, 4$), variance, skewness and kurtosis of the beta normal, beta gamma, beta beta, beta t and beta F distributions computed using in-built functions in Mathematica. Tables 2 and 3 do the same for $a = 2.5$ and $b = 3.5$ and $a = 0.3$ and $b = 0.9$, respectively. The parent normal ($N(0, 1)$), gamma ($G(2, 3)$), beta ($B(2, 3)$) and Student (t_6) distributions were adopted in these tables, whereas the parent F ($F(2, 4)$) distribution was considered in Tables 1 and 2 and the F ($F(2, 9)$) distribution in Table 3. Several other tables are computed by Brito (2009) in her MSc Thesis. For the beta normal distribution, our numerical results are in good agreement with the results using Maple provided by Gupta and Nadarajah (2004a) for integers a and b. Further, for all distributions considered, our numerical results for $a = b = 1$ are identical to those corresponding values of the parent distributions. The current Mathematica scripts are given in the Appendix for the beta normal and beta F distributions.

The numerical moments obtained from our Mathematica scripts fully agree with the previously reported results in the literature when both parameters a and b are

Table 1 *Moments for some beta generalized distributions for $a = 1.5$ and $b = 2.5$*

Parent distribution →	$N(0, 1)$	$G(2, 3)$	$B(2, 3)$	t_6	$F(2, 4)$
μ'_1	−0.38915	4.47390	0.31334	−0.43863	0.71655
μ'_2	0.62203	23.75800	0.11655	0.81558	1.13500
μ'_3	−0.63769	145.65200	0.04903	−1.12010	3.75720
μ'_4	1.17200	1010.94950	0.02266	2.79680	31.45660
Variance	0.47059	3.74240	0.01837	0.62319	0.62156
Skewness	−0.09098	0.81158	0.40319	−0.43840	4.18980
Kurtosis	3.05170	3.98000	2.79380	4.27950	60.55120

Table 2 *Moments for some beta generalized distributions for $a = 2.5$ and $b = 3.5$*

Parent distribution →	$N(0, 1)$	$G(2, 3)$	$B(2, 3)$	t_6	$F(2, 4)$
μ'_1	−0.24014	4.79300	0.33919	−0.26180	0.75933
μ'_2	0.35014	25.57150	0.12786	0.42072	0.96253
μ'_3	−0.23245	150.18740	0.05248	−0.33435	1.94650
μ'_4	0.37198	962.54130	0.02311	0.62776	6.36000
Variance	0.29247	2.59890	0.01281	0.35218	0.38595
Skewness	−0.04996	0.64709	0.28978	−0.19044	2.62560
Curtose	3.03800	3.64320	2.82400	3.51970	18.66460

Table 3 *Moments for some beta generalized distributions for $a = 0.3$ and $b = 0.9$*

Parent distribution →	$N(0, 1)$	$G(2, 3)$	$B(2, 3)$	t_6	$F(2, 9)$
μ'_1	−1.25901	1.27411	0.22010	−2.38624	0.59256
μ'_2	3.93174	3.37942	0.09290	24.20698	2.18504
μ'_3	−11.54703	12.64137	0.04985	−529.19846	26.82111
μ'_4	43.67801	61.14438	0.03046	18134.79921	9568.47986
Variance	2.34663	1.75606	0.04446	18.51284	1.83391
Skewness	−0.74657	0.54745	1.04880	−4.97989	9.48695
Curtose	2.79341	7.04616	3.30838	40.30453	2827.38692

integers [see Jones (2004b); Choi (2005); and references therein]. This fact builds confidence regarding the correctness of the presented scripts, and the ability of the software for analytic formulae manipulation, so that implementation of similar scripts to other density functions may be expected to produce reliable numerical values for the moments.

Graphical representation of skewness and kurtosis for some beta generalized distributions as a function of parameter a for fixed b, and as a function of parameter b for fixed a, are given in Figures 1–10. The plots of Figures 1 and 2 show that the skewness of the beta normal distribution increases when a increases (for fixed b) and decreases when b increases (for fixed a), whereas the kurtosis first decreases steadily to a minimum value and then increases when a increases for fixed b or when b increases for fixed a. The plots of Figures 3 and 4 show that the skewness of the beta gamma distribution decreases when a increases (for fixed b) or when b increases (for fixed a) and the kurtosis always decreases in both cases. Similar conclusions could be drawn for the skewness and kurtosis of the beta beta (Figures 5 and 6), beta Student (Figures 7 and 8) and beta F (Figures 9 and 10) distributions.

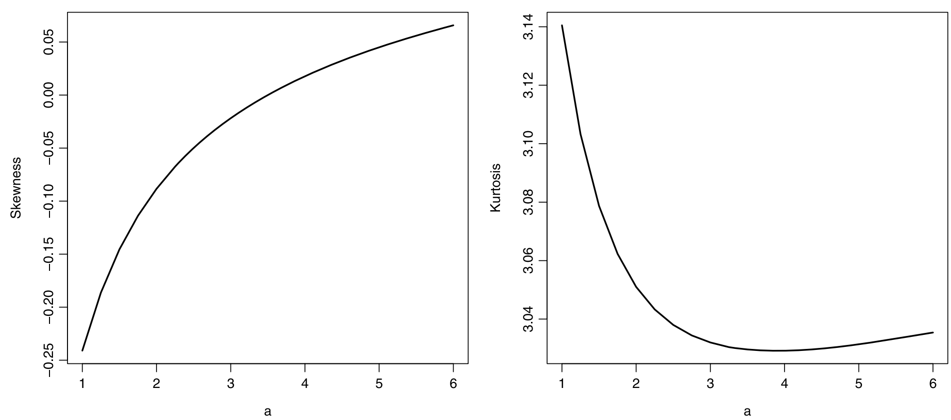

Figure 3 *Plots of the skewness and kurtosis for the beta normal* $BN(a, 3.5, 0, 1)$ *as a function of a for fixed* $b = 3.5$.

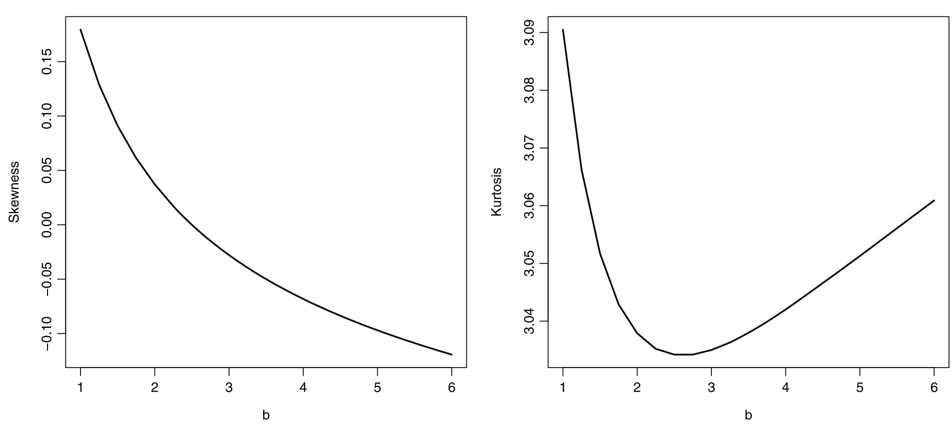

Figure 4 *Plots of the skewness and kurtosis for the beta normal* $BN(2.5, b, 0, 1)$ *as a function of b for fixed* $a = 2.5$.

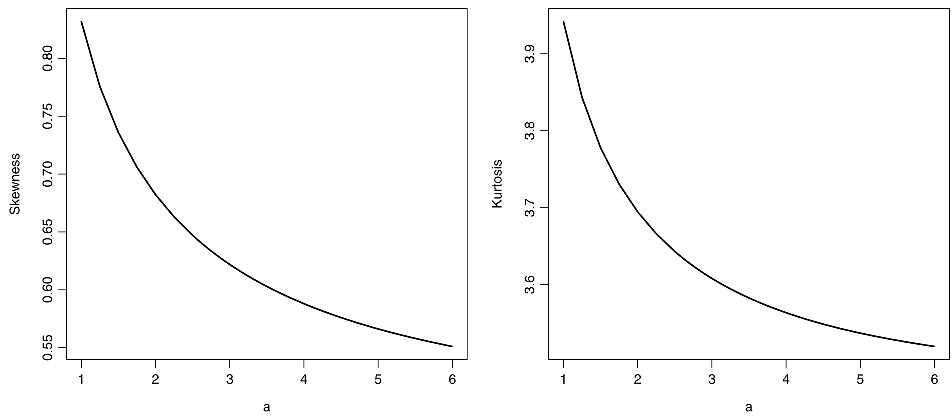

Figure 5 *Plots of the skewness and kurtosis for the beta gamma* $BG(a, 3.5, 2, 3)$ *as a function of* a *for fixed* $b = 3.5$.

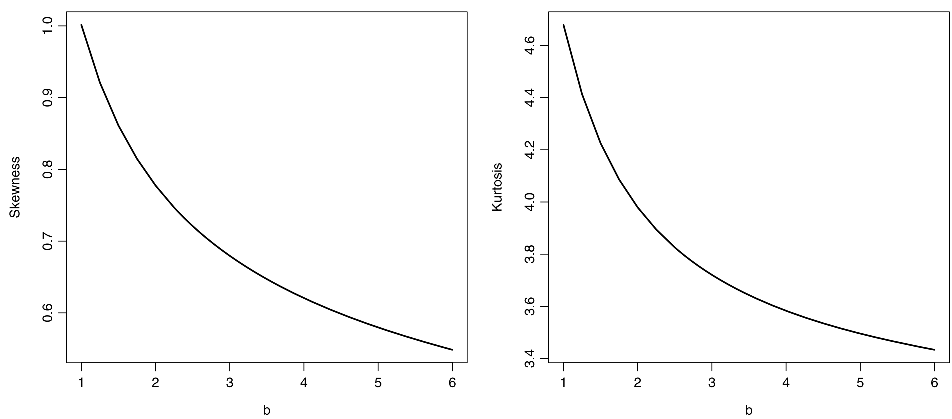

Figure 6 *Plots of the skewness and kurtosis for the beta gamma* $BG(2.5, b, 2, 3)$ *as a function of* b *for fixed* $a = 2.5$.

10 Conclusion

Beta generalized type distributions are very versatile and a variety of uncertainties can be usefully modeled by them. We derive closed-form expressions for moments of some beta type generalized distributions. These expressions are provided as infinite weighted sums of well-known functions. Specifically, the beta gamma, beta normal, beta beta, beta Student and beta F distributions are considered. The expressions are simple and extend some previously known results for the beta normal distribution. Similar expressions can be worked out for other distributions. The computer code for generating these moments are available to the reader.

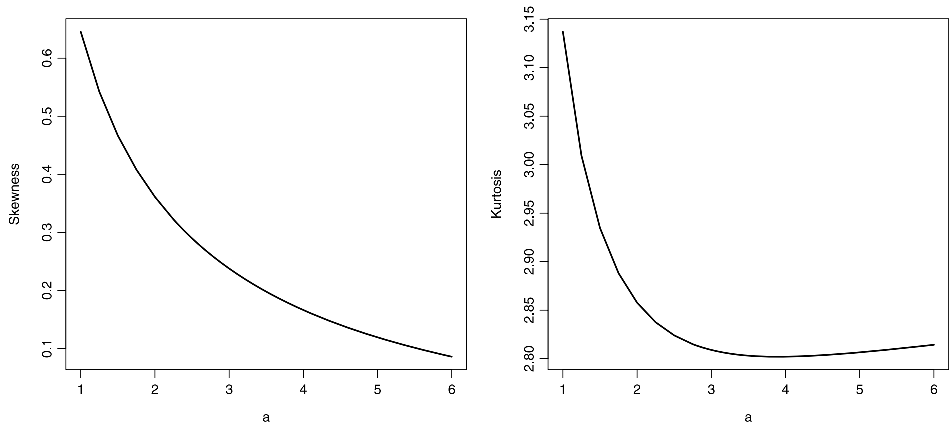

Figure 7 *Plots of the skewness and kurtosis for the beta beta* $BB(a, 3.5, 2, 3)$ *as a function of a for fixed* $b = 3.5$.

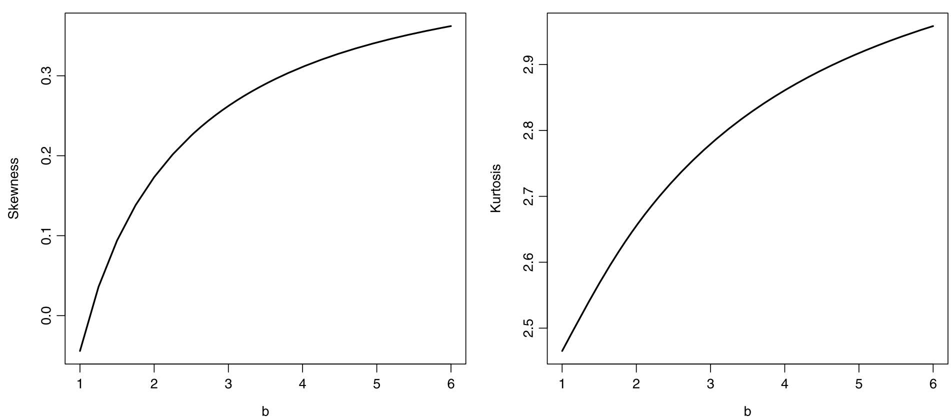

Figure 8 *Plots of the skewness and kurtosis for the beta beta* $BB(2.5, b, 2, 3)$ *as a function of b for fixed* $a = 2.5$.

Apendix

We present Mathematica scripts to calculate the moments of the beta normal and beta F distributions for a integer and real noninteger. In the sums, ∞ was substituted by 100.

Beta normal for a integer

```
"Function to calculate the moments of **Beta Normal** distribution
for -a- integer";

Clear[a,r,s,l,M,w];Clear[FBN0,MOBFi,MOBNi];Clear[beta0,gamma0];
```

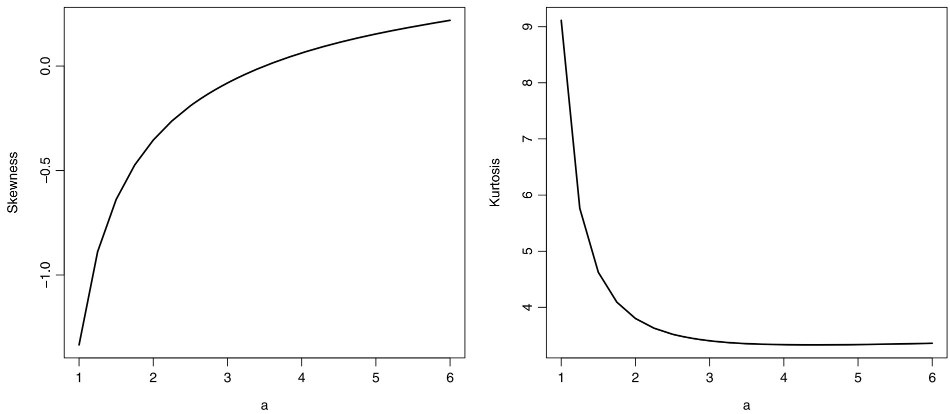

Figure 9 *Plots of the skewness and kurtosis for the beta Student BS*$(a, 3.5, 6)$ *as a function of a for fixed* $b = 3.5$.

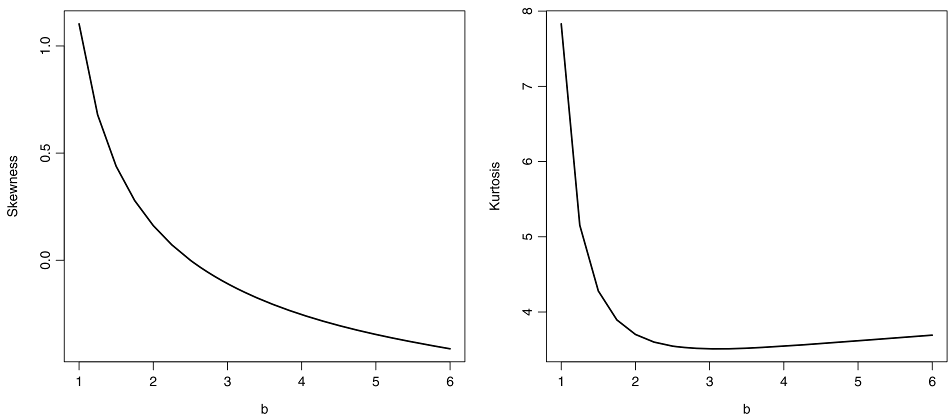

Figure 10 *Plots of the skewness and kurtosis for the beta Student BS*$(2.5, b, 6)$ *as a function of b for fixed* $a = 2.5$.

```
FBN0 = Function[{kappa,r,beta0,gamma0,x1},(Gamma[kappa])^(-1)*
NIntegrate[Exp[-t]*(t^(kappa-1))*
Product[Hypergeometric1F1[beta0,gamma0,x1*t],{m,1,r}],{t,0,Infinity}]];

MOBNi =Function[{s,a,b},Sum[
    ((((-1)^r) *Binomial[b-1,r])/Beta[a, b])*(2^(s/2))*
      ((N[Pi])^(-((r+a-1)+ 1/2)))* Sum[Binomial[(r+a-1),l]*(2^(-l))*
            ((N[Pi])^l))* Gamma[(s+(r+a-1)-l+1)/2]*
       FBN0[(s+(r+a-1)-l+1)/2,r,1/2,3/2,-1]*
        If[IntegerPart[(s+(r+a-1)-l)/2]==(s+(r+a-1)-l)/2,1,0],
         {l,0,(r+a-1)}],{r,0,100}]];
```

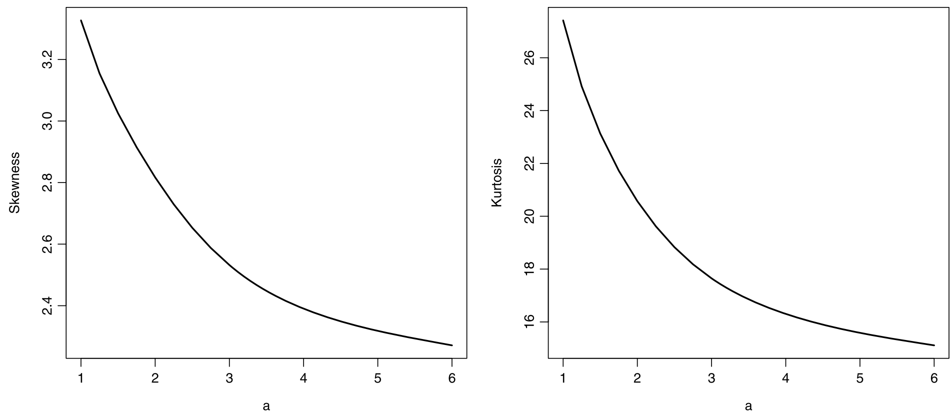

Figure 11 *Plots of the skewness and kurtosis for the beta F* $BF(a, 3.5, 4, 8)$ *as a function of a for fixed* $b = 3.5$.

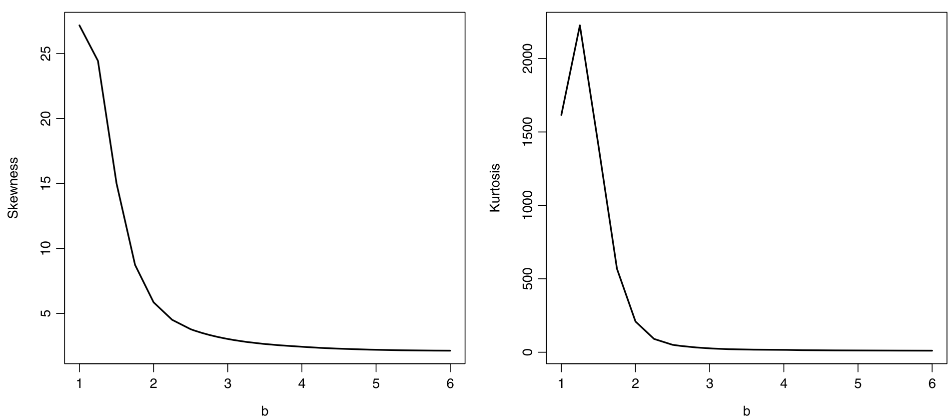

Figure 12 *Plots of the skewness and kurtosis for the beta F* $BF(2.5, b, 4, 8)$ *as a function of b for fixed* $a = 2.5$.

Beta normal for *a* real noninteger

```
"Function to calculate the moments of ** Beta Normal ** distribution
for -a- noninteger";

Clear[a,r,s,l,M,w,kappa];Clear[FBN0,FBF1,FBN1,MOBFni,MOBNni];
Clear[beta1,gamma1];

FBN1 = Function[{kappa,r,beta1,gamma1,x1},(Gamma[kappa])^(-1)*
NIntegrate[Exp[-t]*t^(kappa-1)*
Product[Hypergeometric1F1[beta1,gamma1,x1*t],{m,1,r}],{t,0,Infinity}]];

MOBNni = Function[{s,a,b},Sum[
```

```
         Sum[Sum[((((-1)^(i+j+r))*Binomial[a+i-1,j]*Binomial[b-1,i]*
         Binomial[j,r])/Beta[a,b])*(2^(s/2))*((N[Pi])^(-(r+1/2)))*
         Sum[Binomial[r,l]*(2^(-l))*((N[Pi])^l)* Gamma[(s+r-l+1)/2]*
         FBN1[(s+r-l+1)/2,r,1/2,3/2,-1]*
         If[IntegerPart[(s+r-l)/2] == (s+r-l)/2,1,0],{l 0,r}],
         {r,0,j}],{j,0,100}],{i,0,100}]];
```

Beta *F* for *a* integer

```
Clear[FBN0,FBF0];Clear[alpha0];Clear[beta0];Clear[s];Clear[a];Clear[b];

"Function to calculate the moments of **Beta F** distribution for
-a- integer";

FBF0 = Function[{z,w,r,alpha0,beta0,x},
 ((Beta[z,(w-z)])^(-1))*
  NIntegrate[(t^(z-1))*((1-t)^(w-z-1))*
   Product[Hypergeometric2F1[1-beta0,alpha0,alpha0+1,x*t],{m,1,r}],
     {t,0,1}]];

MOBFi = Function[{s,a,b,alpha0,beta0},
 If[s<beta0,Sum[(((-1)^r)*Binomial[b-1,r])/Beta[a,b])*
    ((beta0)^s)*Beta[beta0-s,s+ alpha0*(r + a)]*
     N[FBF0[(s+alpha0*(r+a)),(beta0+alpha0*(r+a)),(r+a-1),alpha0,
      beta0,1]])/(((alpha0)^(s+(r+a-1)))*(Beta[alpha0,beta0]^(r + a))),
       {r,0,100}],"**Verify if s < beta!**"]];
```

Beta *F* for *a* real noninteger

```
"Function to calculate the moments of **Beta F** distribution for
-a- noninteger";

Clear[FBF1,MOBFi,MOBFni];Clear[alpha1];Clear[beta1];Clear[a];Clear[b];

FBF1 = Function[{z,w,r,alpha1,beta1,x1},(Beta[z,(w-z)]^(-1))*
  NIntegrate[(t^(z-1))*((1-t)^(w-z-1))*
   Product[Hypergeometric2F1[1-beta1,alpha1,alpha1+1,x1*t],{m,1,r}],
    {t,0,1}]];

MOBFni = Function[{s,a,b,alpha1,beta1},
  If[s < beta1,
   Sum[Sum[Sum[((((-1)^(i+j+r))*Binomial[a+i-1,j]*Binomial[b-1,i]*
    Binomial[j,r])/Beta[a,b])*((beta1)^s/(((alpha1)^(s+r))*
     (Beta[alpha1,beta1]^(r+1))))*Beta[beta1-s,s+alpha1*(r+1)]*
       N[FBF1[(s+alpha1*(r+1)),(beta1+alpha1*(r+1)),r,alpha1,beta1,1]],
        {r,0,j}],{j,0,100}],{i,0,100}], "** Verify if s < beta! **"]].
```

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