Nonergodicity and growth are compatible for 1D local interaction

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Abstract. We present results of Monte Carlo simulation and chaos approximation of a class of Markov processes with a countable or continuous set of states. Each of these states can be written as a finite (finite case) or infinite in both directions (infinite case) sequence of *pluses* and *minuses* denoted by $\oplus$ and $\ominus$. As continuous time goes on, our sequence undergoes the following three types of local transformations: the first one, called *flip*, changes any minus into plus and any plus into minus with a rate β; the second, called *annihilation*, eliminates two neighbor components with a rate α whenever they are in differents states; and the third, called *mitosis*, doubles any component with a rate γ. All of them occur at any place of the sequence independently. Our simulations and approximations suggest that with appropriate positive values of α, β and γ this process has the following two properties. *Growth*: In the finite case, as the process goes on, the length of the sequence tends to infinity with a probability which tends to 1 when the length of the initial sequence tends to ∞. *Nonergodicity*: The infinite process is nonergodic and the finite process keeps most of the time at two extremes, occasionally swinging from one to the other.
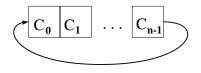
1 Introduction

Since the first studies of the Ising model, it became common among physicists to recognize the qualitative difference between the one-dimensional and the multi-dimensional cases for all multicomponent models with local interaction. This lore crystallized in the shape of the "positive rates conjecture" (see [4], pages 178, 201) and was brilliantly refuted by Peter Gács [2,3]. However, the cases when a random process with one-dimensional local interaction shows some form of nonergodicity, remain nontrivial and for this reason still attract attention; our task is to provide another case of this sort.

Our ultimate goal is to study the case which we call *infinite*. In this case, the configurations are double infinite sequences of *pluses* and *minuses*, denoted by $\oplus$ and $\ominus$, respectively. However, this case is not yet defined rigorously. In addition, every computer has a finite memory, so any computer simulation in fact is a simulation of some finite process. For these reasons along with infinite processes, we

Figure 1 *A circular C with $|C| = n$.*

deal with analogous finite processes, which are easy to define and which we in fact model.

In the finite case, to avoid complications at the ends, we use configurations called "circulars." A circular is just a finite sequence of pluses and minuses, but terms of this sequence, called components, are enumerated by remainders modulo $|C|$, where $|C|$ is the length (i.e., the number of components) of the circular, rather than natural numbers. (In the literature this is sometimes called periodic condition.) Figure 1 shows a circular C with length $n = |C|$ and components $C_0, \ldots, C_{n-1}$, whose indices $0, \ldots, n-1$ are remainders modulo n, so the index next to $n-1$ is zero.

The usual finite sequences of pluses and minuses, whose terms are indexed by natural numbers, are called *words*. The length of a word W is denoted by $|W|$. There is the empty word, denoted by Λ, whose length is zero. We say that a word $W = (a_1, a_2, \ldots, a_k)$ appears at a place i in a circular $C = (C_0, \ldots, C_n - 1)$ if $C_{i+1} = a_1, C_{i+2} = a_2, \ldots, C_{i+k} = a_k$, where the sums in the indices are modulo n. If a word W appears in a circular C and $|W| \leq |C|$, we can substitute it by another word V, thus obtaining another circular. We shall label such a substitution with $W \rightarrow V$. Our process consists in iterative applications of three concrete kinds of substitutions. Namely, as continuous time goes on, our sequence (finite or infinite) undergoes the following types of transformation:

- *Annihilation*: $(\oplus, \ominus) \rightarrow \Lambda$ and $(\ominus, \oplus) \rightarrow \Lambda$. If the states of the components with indices x and $x + 1$ are different, both disappear with a rate α independently of the other components. The components $x - 1$ and $x + 2$ become neighbours. The length of the circular decreases by two.
- *Flip*: $\oplus \rightarrow \ominus$ and $\ominus \rightarrow \oplus$. This changes the state of one component with a rate β independently of the other components. The length of the circular does not change. $\qquad$ (1)
- *Mitosis*: $\oplus \rightarrow \oplus\oplus$ and $\ominus \rightarrow \ominus\ominus$. This duplicates one component with a rate γ independently of other components. The length of the cicular increases by one.

In the finite case the text presented above may be accepted as a definition. In the infinite case the corresponding class of processes has never been rigorously defined (according to our knowledge). However, we have reasons to believe that such a

definition is possible. Processes of this sort have been mentioned in a physical context [5,6]. For similar processes with discrete time some special cases have been already studied [7–9,11] and a general definition for the discrete time case is already available [10]. Thus we take the liberty to speak about both finite-space and infinite-space versions of the processes described above and study them using Monte Carlo method and chaos approximation.

Our main results: Monte Carlo simulation and chaos approximation lead to similar pictures of ergodicity versus nonergodicity and growth versus shrinking (Figures 2 and 6). Both of them suggest that with appropriate positive values of α, β and γ our processes have the following two properties:

Growth: In the finite case the length of the circular tends to infinity with probability that tends to one, as the length of initial circular (consisting of several minuses) tends to ∞.

Nonergodicity: The infinite process is nonergodic and the finite process keeps most of the time at two extremes, occasionally swinging from one to the other.

Our work was motivated by success and failure of [11], which considered infinite processes similar to ours with these differences: time was discrete (which we deem unimportant), flip was asymmetric, that is, it turned minuses into pluses, but not vice versa (which also is unimportant for us since our initial configuration consists of minuses) and mitosis was absent (which is important). [11] proved some form of nonergodicity for that process for α small enough: if the process started with "all minuses," the percentage of pluses always remained small. This was a success and it was improved in [8] and studied numerically in [7,9]. The failure of [11] was the impossibility to present a finite analog: in the absence of mitosis, length of the sequence decreased in average and the configuration degenerated. In our work this failure is removed.

2 Monte Carlo simulation

Due to limitations of all real computers, the Monte Carlo method by its very nature always refers to some finite space case, even when the ultimate motivation is to study the infinite case. In addition, even when we study a continuous time process, its computer simulation always has discrete time. This applies to our study also. Thus, we approximate our infinite-space process with a Markov process with a countable set Ω of states, where Ω is the set of circulars of all lengths. The time t (i.e., the number of iterations of our computer simulation) is discrete and at every time step at most one transformation of the list (1), chosen at random, takes place. Thus, in each individual experiment we obtain a randomly generated sequence of circulars and the circular obtained at time t is denoted by C^t. Its x-th component is denoted by C_x^t, where $x = 0, \ldots, |C^t| - 1$. We denote by $\mathcal{M}$ the set of probability distribution, that is, the set of normalized measures on Ω, the set of circulars. We call a measure $\mu \in \mathcal{M}$ *local* if it is in fact concentrated on a finite subset

of Ω. We consider the circulars C^t as representations of the corresponding local measures $\mu^t \in \mathcal{M}$. So the sequence $C^0, C^1, C^2, \ldots$ is a "typical" trajectory of the corresponding process $\mu^0, \mu^1, \mu^2, \ldots$.

Let us denote by $\mathsf{quant}(W|C)$ the *quantity* of different places, at which a word W appears in a circular C. After that we define the *frequency* of W in C as

$$\mathsf{freq}(W|C) = \frac{\mathsf{quant}(W|C)}{|C|}. \tag{2}$$

For any $\mu \in \mathcal{M}$ we define the *frequency* of the word W according to μ as

$$\mathsf{freq}(W|\mu) = \sum_{C \in \Omega} \mathsf{freq}(W|C) \cdot \mu(C). \tag{3}$$

We were especially interested in $\mathsf{freq}(\oplus|\mu^t)$, that is, the frequency of pluses at time t. Due to limitations of our computer facilities, we could not estimate $\mathsf{freq}(\oplus|\mu^t)$ directly, so we approximated it by

$$\overline{\mathsf{freq}(\oplus|\mu^t)} \stackrel{\text{def}}{=} \frac{1}{t} \sum_{k=1}^{t} \mathsf{freq}(\oplus|C^k). \tag{4}$$

Since the rules of our process do not change when we swap plus and minus, the ergodicity of our infinite process implies that the frequency of pluses tends to $1/2$ and for its finite analog this frequency tends to $1/2$ with a probability, which tends to 1 when the length of the initial condition (consisting only of minuses) tends to ∞. To estimate the desired frequencies we used the following procedure, which we call *Imitation*. This procedure generates a sequence of circulars in the following inductive way. (Forget imprefections of computer-generated random numbers.)

Base of induction. The initial circular C^0 consists of 1000 minuses.

t-th induction step. Given a circular C^t, where $t = 0, 1, 2, \ldots$ we performed these three procedures:

The first procedure imitated the random choice of a place where to perform a transformation: a random integer number x distributed uniformly in $\{0, 1, \ldots, |C^t| - 1\}$ was generated to identify the position, where the transformation would occur.

The second procedure imitated (1): first it generated a real random number ξ distributed uniformly in $(0, 1)$. Then:

- if $\xi \in [0, \frac{\alpha}{\alpha+\beta+\gamma})$ and $C_x^t \neq C_{x+1}^t$, these components annihilated, that is, both of them disappeared.
- If $\xi \in [\frac{\alpha}{\alpha+\beta+\gamma}, \frac{\alpha+\beta}{\alpha+\beta+\gamma})$, the component C_x^t changed its state from $\ominus$ to $\oplus$ or from $\oplus$ to $\ominus$.
- If $\xi \in [\frac{\alpha+\beta}{\alpha+\beta+\gamma}, 1]$, this component underwent mitosis, that is turned into two components in the same state.

If we denote the resulting circular by C^{t+1}, we obtain the induction step of that process, which we have in mind. However, this process cannot yet be implemented on a real computer. That is why we denote the resulting circular by $(C')^t$. Due to presence of annihilation and mitosis, the length of $(C')^t$ might be different from the length of C^t; so in the course of the process the length of our circular changed randomly and usually had a tendency either to shrink all the time or to grow all the time. To prevent our process from shrinking to degeneration or growing beyond our computer possibilities we used the third procedure.

The third procedure which helped to keep C^t within range: given $(C')^t$, we generated a new circular, namely C^{t+1}, in one of the following ways.
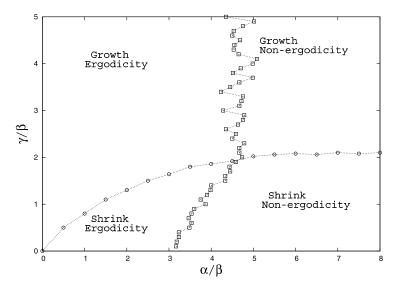
Duplication: if $|(C')^t| < N_{\min}$, where $N_{\min} = 500$, then C^{t+1} was obtained from $(C')^t$ by concatenating it with its copy and thereby duplicating its length.

Cut: if $|(C')^t| > N_{\max}$, where $N_{\max} = 15{,}000$, then C^{t+1} was obtained from $(C')^t$ deleting half of it.

Otherwise we changed nothing and obtained $C^{t+1} = (C')^t$.

When we stopped: We stopped our simulation when each one of the three transformations (1) occured at least 100,000 times. Thus the procedure Imitation is described.

To obtain the small squares on Figure 2, approximating the boundary between the regions of ergodicity and nonergodicity, we used Imitation to attribute an appropriate value to a Boolean variable denoted by E (which means "it seems to be ergodicity") as follows: if at the end of iteration the quantity $\overline{\mathrm{freq}(\oplus|\mu^t)}$ was in the

Figure 2 *White squares approximate the boundary between suggested ergodicity and suggested nonergodicity. White balls approximate the boundary between suggested shrinking and suggested growth. Compare this figure with Figure 6.*

range $(0.45, 0.55)$, we set $E = yes$; otherwise we set $E = no$. We interpreted the result $E = yes$ as a suggestion that the infinite process with the triple (α, β, γ) is ergodic; the result $E = no$ was interpreted as a suggestion that this triple produces a nonergodic process.
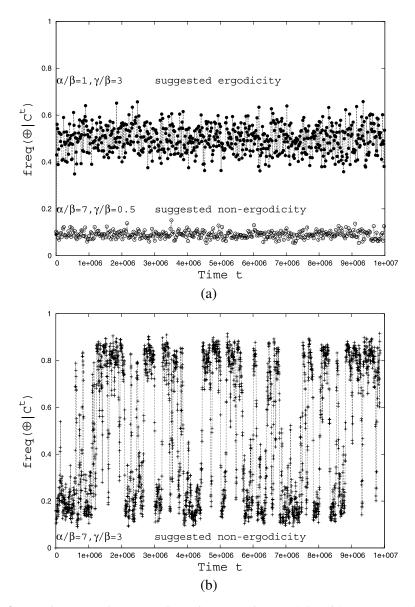
Notice that if we multiply α, β and γ by one and the same positive number, the process does not change. So only ratios of these three numbers to each other are important for us. We used Imitation within a cycle with a fixed γ/β and growing α/β: we started with $\alpha/\beta = 0.1$ and then iteratively performed Imitation and increased α/β by 0.1 and repeated this until α/β reached the value 8 or E got the value no. Thus we obtained a certain value of α/β. In fact we performed this cycle 5 times and recorded the arithmetical average of the 5 values of α/β thus obtained. All this was done for 50 values of γ/β, namely the values $0.1 \times i$ with $i = 1, \ldots, 50$. Thus we obtained 50 pairs $(\alpha/\beta, \gamma/\beta)$ represented by small squares on Figure 2.

To obtain the small circles on Figure 2, approximating the boundary between the regions of growth and shrinking, we used Imitation within a cycle with a fixed α/β and growing γ/β: we started with $\gamma/\beta = 0.1$ and then iteratively performed Imitation and increased γ/β by 0.1 and repeated this until γ/β reached the value 8 or there was none duplication in the course of performing Imitation. Thus we obtained a certain value of γ/β. In fact, we performed this cycle 5 times and recorded the arithmetical average of the 5 values of γ/β thus obtained. All this was done with 17 values of α/β, namely, the values $0.5 \times i$ with $i = 0, \ldots, 16$. Thus we obtained 17 pairs $(\alpha/\beta, \gamma/\beta)$ represented by small circles on Figure 2.

Now let us look at Figure 3(a), where we put results of two different experiments to save space. One experiment with $\alpha/\beta = 1$ and $\gamma/\beta = 3$ is represented by the points, whose vertical coordinates are between 0.3 and 0.7. In this case the frequency of pluses kept close to 1/2. We interpreted this case as an example of growth and ergodicity.

Figure 3(b) pertains to the case $\alpha/\beta = 7$ and $\gamma/\beta = 3$. It shows a different behavior: most of the time the frequency of pluses keeps either near a number close to zero or near a number close to one. In other words, in this case $\mathsf{freq}(\oplus|C^t)$ oscillates between two extremes. We interpreted this case as an example of growth and nonergodicity.

The points on Figure 3(a), whose vertical coordinates are less than 0.2, pertain to the case $\alpha/\beta = 7$ and $\gamma/\beta = 1/2$. The figure shows no swing in this case, but there may be swings too rare for our computer time. In this connection let us distinguish two cases: "real" and "ideal." The real case is that, which was actually programmed, including the third procedure at every step of iteration. In this case the length of the circular remains bounded and therefore our process is an ergodic finite Markov chain. Due to its symmetry, the limit frequencies of pluses and minuses must be equal. So, since we see the process staying for a long time with a small (less than 0.2) frequency of pluses, somewhen in the future it must stay for an equally long time with an equally small frequency of minuses and so on with

(a)

(b)

Figure 3 (a) *The points whose vertical coordinates are between* 0.3 *and* 0.7 *pertain to the case* $\alpha/\beta = 1$ *and* $\gamma/\beta = 3$. *We suggest that in this case our infinite process grows and is ergodic. The points with the vertical coordinates less than* 0.2 *pertain to the case* $\alpha/\beta = 7$ *and* $\gamma/\beta = 1/2$. *We conjecture that in this case our infinite process shrinks and is nonergodic.* (b) *Pertains to the case* $\alpha/\beta = 7$ *and* $\gamma/\beta = 3$. *We observe that* $\mathsf{freq}(\oplus|C^t)$ *spends most of the time near* 0.1 *or* 0.9, *sometimes rapidly swinging from one extreme to the other. We suggest that the corresponding infinite process grows and is nonergodic.*

swings between them. The ideal case is the same without the third procedure. In this case, if the process grows, swings may be completely absent with a positive probability. However, in the present case the process shrinks, so the swings seem to be inevitable also. Anyway, we suggest to classify this case as an example of shrinking and nonergodicity.

3 Chaos approximation

Chaos approximation is widely used in the numerical study of random processes, especially in physical context. In spite of its simplicity, its behavior often is qualitatively similar to behavior of the original process. In our case chaos approximation gives us a deterministic approximation of our process, whose behavior is shown on Figure 6. It is really similar to that on Figure 2.

Now let us describe the chaos approximation in our case. Remember our procedure of Imitation. Let us imagine that at every step of this procedure, in addition to the operations described above (before or after them—it does not matter), all the components of C^t are randomly permuted. Behavior of the resulting process essentially has only two parameters: quantity of pluses and quantity of minuses at time t, which we denote by $X(t)$ and $Y(t)$. When these quantities are large, we may approximatedly treat them as if they were real. In this approximation, we obtain a random process described by the differential equations (5):

$$\left.\begin{aligned}\frac{dX(t)}{dt} &= -\beta \cdot X(t) + \beta \cdot Y(t) + \gamma \cdot X(t) - \alpha \cdot \frac{X(t)Y(t)}{X(t)+Y(t)}, \\ \frac{dY(t)}{dt} &= -\beta \cdot Y(t) + \beta \cdot X(t) + \gamma \cdot Y(t) - \alpha \cdot \frac{X(t)Y(t)}{X(t)+Y(t)}.\end{aligned}\right\} \tag{5}$$

The last term in each formula is based on our assumption that all the components are mixed all the time, whence the neighbor components are independent from each other. Also notice that in this case multiplying α, β and γ by one and the same positive number does change the process, but in a very special way: it only slows it down or speeds it up. This allows us to use only ratios α/β and γ/β in the Figure 6.

Since the process (5) is homogeneous, we deal in fact with a two-dimensional analog of the theorem on page 7 of [1]. So we may go to other variables

$$S(t) = X(t) + Y(t) \quad \text{and} \quad B(t) = \frac{X(t) - Y(t)}{X(t) + Y(t)}. \tag{6}$$

For simplicity, sometimes we shall denote $X(t), Y(t), S(t)$ and $B(t)$ by X, Y, S and B, respectively. The following system of equations is equivalent to (5):

$$\frac{dS}{dt} = S \cdot \left(\gamma - \frac{\alpha}{2}(1 - B^2) \right), \tag{7}$$

$$\frac{dB}{dt} = B \cdot \left(\frac{\alpha}{2}(1 - B^2) - 2\beta \right). \tag{8}$$

The last equation is easy to solve explicitly, but we shall get all we need by quali-
tative arguments. Since we are especially interested in the proportion of each type
of particles, we consider also another process, which we call *normalized chaos
approximation*:

$$X_{\text{norm}}(t) = \frac{X(t)}{X(t) + Y(t)}, \qquad Y_{\text{norm}}(t) = \frac{Y(t)}{X(t) + Y(t)}. \qquad (9)$$

Then

$$X_{\text{norm}}(t) = \frac{1 + B(t)}{2} \quad \text{and} \quad Y_{\text{norm}}(t) = \frac{1 - B(t)}{2}.$$

Thus, all we need to know will follow as soon as we study behavior of $B(t)$.
So let us treat the equation (8) as a deterministic dynamical system with a space
$[-1, 1]$ and continuous time t. We call a number $B^* \in [-1, 1]$ a *fixed point* of
this system if (8) equals zero at $B = B^*$. We say that a fixed point $B^* \in [-1, 1]$
attracts a point $B \in [-1, 1]$ if the process (8) starting at $B(0) = B$ tends to B^*
when $t \to \infty$. Given a fixed point, we call its *basin of attraction* or just *basin* the
set of points attracted by it. It is easy to describe completely fixed points and their
basins for (8). The right-hand side of (8) equals zero at three (generally complex)
values of B, which we denote by

$$B_1^* = -\sqrt{1 - \frac{4\beta}{\alpha}}, \qquad B_2^* = 0, \qquad B_3^* = \sqrt{1 - \frac{4\beta}{\alpha}}. \qquad (10)$$
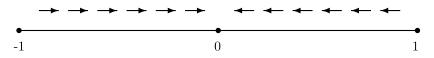
Hence follows our classification:
If $\alpha/\beta < 4$, then B_1^* and B_3^* are not real and the right-hand side of (8) is

$$\begin{cases} \text{positive when} & B \in [-1, 0), \\ \text{zero when} & B = 0, \\ \text{negative when} & B \in (0, 1]. \end{cases}$$
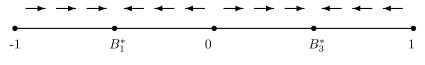
Figure 4 illustrates this.
    Therefore in this case $B(t)$ tends to zero from any initial value when $t \to \infty$.
    If $\alpha/\beta = 4$, then B_1^* and B_3^* are real and equal to zero. The signs of the right-
hand side of (8) are the same as in the previous case and $B(t)$ also tends to zero
from any initial condition when $t \to \infty$.

Figure 4   *Behavior of $B(t)$ when $\alpha/\beta < 4$.*

Figure 5   *Behavior of $B(t)$ when $\alpha > 4\beta$.*

If $\alpha/\beta > 4$, then B_1^* and B_3^* are real and $-1 < B_1^* < B_2^* = 0 < B_3^* < 1$ (remember that $\beta > 0$). So the right-hand side of (8) is

$$
\begin{cases}
\text{positive when} & B \in [-1, B_1^*), \\
\text{zero when} & B = B_1^*, \\
\text{negative when} & B \in (B_1^*, B_2^*), \\
\text{zero when} & B = B_2^* = 0, \\
\text{positive when} & B \in (B_2^*, B_3^*), \\
\text{zero when} & B = B_3^*, \\
\text{negative when} & B \in (B_3^*, 1].
\end{cases}
$$

Figure 5 illustrates this.

Therefore in this case $B(t)$ tends to B_1^* or B_2^* or B_3^* from any initial condition when $t \to \infty$, so $[-1, 1]$ is a union of three basins:

$$
\text{basin}(B_1^*) = [-1, 0), \qquad \text{basin}(B_2^*) = \{0\}, \qquad \text{basin}(B_3^*) = (0, 1].
$$

Thus the normalized chaos approximation is ergodic if $\alpha/\beta \leq 4$ and nonergodic if $\alpha/\beta > 4$.

Now we are ready to study the chaos approximation (5). Let us remember that $X(t) + Y(t) = S(t)$ and say that our dinamical system:

- *grows* if $S(t)$ tends to infinity when $t \to \infty$.
- *shrinks* if $S(t)$ tends to zero when $t \to \infty$.

Let us find out when it grows and when it shrinks.

Notice that we may rewrite (7) as

$$
\frac{d \ln S}{dt} = \gamma - \frac{\alpha}{2}(1 - B^2). \tag{11}
$$

Let us denote by $G(B)$ the right-hand side of (11).

Given two positive functions f_1 and f_2 of $t \geq 0$, let us write $f_1 \asymp f_2$ if $f_1 = \mathcal{O}(f_2)$ and $f_2 = \mathcal{O}(f_1)$.

Lemma.   *Let $B(0) \in \text{basin}(B_i^*)$, where $i \in \{1, 2, 3\}$. Then:*
   If $G(B_i^*) > 0$, then $\ln S(t) \asymp t$.
   If $G(B_i^*) = 0$, then $|\ln S(t)| = o(t)$.
   If $G(B_i^*) < 0$, then $-\ln S(t) \asymp t$.

Proof. The easiest case is when the initial value of B is a fixed point of (8). In this case $B(t) \equiv B(0)$ for all $t \geq 0$, so the right-hand side of (11) is a constant.

If $B(0)$ equals $B_2^* = 0$, then (11) turns into

$$\frac{d \ln S(t)}{dt} = \gamma - \alpha/2,$$

whence $\ln S(t) = (\gamma - \alpha/2) \cdot t + const$, so

$$S(t) = e^{(\gamma - \alpha/2) \cdot t} \text{ times a constant.}$$

Therefore our process grows if $\gamma/\alpha > 1/2$ and shrinks if $\gamma/\alpha < 1/2$.

If B_1^* and B_3^* are real and $B(0)$ equals one of them, then (11) turns into

$$\frac{d \ln S(t)}{dt} = \gamma - 2\beta,$$

whence $\ln S(t) = (\gamma - 2\beta) \cdot t + const$, so

$$S(t) = e^{(\gamma - 2\beta) \cdot t} \text{ times a constant.}$$

Therefore our process grows if $\gamma/\beta > 2$ and shrinks if $\gamma/\beta < 2$.

Now let us consider the general case: $B(0)$ is any number in $[-1, 1]$. Then $B(0)$ belongs to a basin of some B_i^*, where $i \in \{1, 2, 3\}$. Then from (11) for any $t \geq 0$

$$\ln S(T) = \ln S(0) + \int_0^T G(B(t)) \, dt, \tag{12}$$

where the integration is taken along the trajectory $(S(t), B(t))$ of our process. It is easy to prove that

$$\frac{\ln S(t)}{t} \text{ tends to } G(B(b_i)) \qquad \text{when } t \to \infty. \tag{13}$$
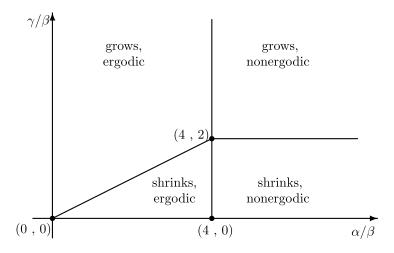
This fact is similar to the well-known fact that if a sequence has a limit, its Cesàro transformation has the same limit. So the idea of the proof is the same. Figure 6 resumes our findings.

In the special case when $X(0) = Y(0)$ we have $B(t) = 0$ for all t. But zero is a fixed point, so the process is ergodic. $\qquad \square$

Conclusion

Our main purpose was to study a class of random processes, whose states were infinite in both directions sequences of pluses and minuses. At the same time we had to deal with analogous processes, whose states were finite sequences of pluses and minuses, which we called circulars. We studied these processes using two methods: Monte Carlo and chaos approximations. These methods led us to similar results and suggested that our processes can grow and be nonergodic at the same time. So we may have found another example of 1-D nonergodicity.

Figure 6 *Classification for $X(0) \neq Y(0)$. Compare this figure with Figure 2.*

Acknowledgments

References

[1] Arnold, V. I. (1983). *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer, New York. MR0695786

[2] Gács, P. (2001). Reliable cellular automata with self-organization. *Journal of Statistical Physics* **103** 45–267. MR1828729

[3] Gray, L. F. (2001). A Reader's guide to Gács's "Positive rates" paper. *Journal of Statistical Physics* **103** 1–44. MR1828728

[4] Liggett, T. M. (1985). *Interacting Particle Systems*. Springer, New York. MR0776231

[5] Malyshev, V. (2000). Quantum grammars. Part 4.1: KMS states on quantum grammars. *Journal of Mathematical Physics* **41** 4508–4520. MR1765609

[6] Malyshev, V. (2002). Quantum evolution of words. *Theoretical Computer Science* **273** 263–269. MR1872453

[7] Ramos, A. D. (2007). Particle processes with variable length. Ph.D. thesis, Federal Univ. Pernambuco, Recife, Pernambuco, Brazil. Avaliable at http://www.de.ufpe.br/~toom/alunos/ doutorado/.

[8] Ramos, A. D. and Toom, A. (2008). An error correction. Letter to the editor. *Journal of Statistical Physics* **131** 167–168. MR2394703

[9] Ramos, A. D. and Toom, A. (2008). Chaos and Monte-Carlo approximations of the flip-annihilation process. *Journal of Statistical Physics* **133** 761–771. MR2456944

[10] Rocha, A. V. (2009). Substitution operators. Ph.D. thesis, Federal Univ. Pernambuco, Recife, Pernambuco, Brazil. Avaliable at http://www.de.ufpe.br/~toom/alunos/doutorado/.

[11] Toom, A. (2004). Non-ergodicity in a 1-D particle process with variable length. *Journal of Statistical Physics* **115** 895–924. MR2054167

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