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Macroscopic stability for nonfinite range kernels

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Abstract. We extend the strong macroscopic stability introduced in Bramson and Mountford [*Ann. Probab.* **30** (2002) 1082–1130] for one-dimensional asymmetric exclusion processes with finite range to a large class of one-dimensional conservative attractive models (including misanthrope process) for which we relax the requirement of finite range kernels. A key motivation is the extension of constructive hydrodynamics result of Bahadoran et al. [*Stochastic Process. Appl.* **99** (2002) 1–30, *Ann. Probab.* **34** (2006) 1339–1369, *Electron. J. Probab.* (to appear)] to nonfinite range kernels.

1 Introduction

In this note we consider a general class of (at least potentially) long-range, one-dimensional conservative attractive particle systems (which will be shortly specified). The paper is motivated by the recent series of papers Bahadoran et al. (2002, 2006) and Bahadoran et al. (2009). Here the hydrodynamic limits of various systems was established. The needed conditions were extremely general, to the point where it was not necessary to suppose that a full characterization of translation invariant equilibria had been established. Briefly the argument built on the approach of Andjel and Vares (1987, 2003) which establishes hydrodynamic limits for Riemannian initial profiles. Then a general argument was given to pass from this particular case to general initial profiles. A key part of this passage was the existence of a macroscopic stability criterion for the particle systems whereby the known behaviour of a system corresponding to a step-function profile could yield information about systems corresponding to more general (but close) initial profiles.

We now detail the processes involved. The state space is $\mathbf{X} = \{0, \dots, K\}^{\mathbb{Z}}$. The evolution consists in particles' jumps, according to the generator

$$Lf(\eta) = \sum_{x,y\in\mathbb{Z}} p(y-x)b(\eta(x),\eta(y))[f(\eta^{x,y}) - f(\eta)] \tag{1}$$

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for a local function f, where $\eta^{x,y}$ denotes the new state after a particle has jumped from x to y [i.e., $\eta^{x,y}(x) = \eta(x) - 1$, $\eta^{x,y}(y) = \eta(y) + 1$, $\eta^{x,y}(z) = \eta(z)$ otherwise], p is the particles' jump kernel, that is, $\sum_{z\in\mathbb{Z}} p(z) = 1$, and $b : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$ is the jump rate. We assume that p and b satisfy:

(A1) The greatest common divisor of the set $\{x : p(x) \neq 0\}$ equals 1 (irreducibility);
(A2) p has a finite first moment, that is, $\mu_1 = \sum_{z\in\mathbb{Z}} |z|p(z) < +\infty$, and a positive mean, that is, $0 < \mu = \sum_{z\in\mathbb{Z}} zp(z)$;
(A3) $b(0, \cdot) = 0$, $b(\cdot, K) = 0$ (no more than K particles per site), and $b(1, K - 1) > 0$;
(A4) b is nondecreasing (nonincreasing) in its first (second) argument (attractiveness).

For us the departure from the previous works mentioned is in assumption (A2), which replaces the "finite range" assumption.

Let $\mathcal{I}$ and $\mathcal{S}$ denote, respectively, the set of invariant probability measures for L, and the set of shift-invariant probability measures on $\mathbf{X}$. It was derived in Bahadoran et al. [(2006), Proposition 3.1] that

$$(\mathcal{I} \cap \mathcal{S})_e = \{\nu^\rho, \rho \in \mathcal{R}\} \tag{2}$$

with $\mathcal{R}$ a closed subset of $[0, K]$ containing 0 and K, and ν^ρ a shift-invariant measure such that $\nu^\rho[\eta(0)] = \rho$. (The index e denotes extremal elements.) The measures ν^ρ are stochastically ordered: $\rho \le \rho' \Rightarrow \nu^\rho \le \nu^{\rho'}$.

The result to be announced in the next section considers "naturally" coupled systems. It is time to detail the natural coupling in force throughout this paper. We suppose given on a space Ω a family of independent marked Poisson processes $\mathcal{N}^{x,y}$ of rate $p(y - x)\|b\|_\infty$ where $\|b\|_\infty = \max_{0\le i,j\le K}\{b(i, j)\}$ and associated to each point $t \in \mathcal{N}^{x,y}$ are uniform random variables $U(x, y, t)$ on $[0, 1]$ which are independent over all $(x, y) \in \mathbb{Z}^2$ and $t \ge 0$. We also assume that the Poisson, uniform random variables [mutually independent and independent of the previous processes $\mathcal{N}^{x,y}$ and $U(x, y, t)$] that we will need for the proofs of this note are defined on Ω. We denote by $\mathbb{P}$ the probability measure on Ω. The initial configurations are defined on a probability space $(\Omega_0, \mathbb{P}_0)$. Given an initial configuration $\eta_0(\omega_0) \in \{0, \dots, K\}^{\mathbb{Z}}$ and a realization ω of the Poisson processes and uniform random variables, we construct a process $(\eta_t : t \ge 0) := (\eta_t(\eta_0(\omega_0), \omega) : t \ge 0)$ by stipulating that the process $\eta_\cdot$ jumps from η_{t^-} to $\eta_t = \eta_{t^-}^{x,y}$ only if $t \in \mathcal{N}^{x,y}$ and $U(x, y, t) \le b(\eta_{t^-}(x), \eta_{t^-}(y))/\|b\|_\infty$. We note that through the above (Harris) graphical construction [see Bahadoran et al. (2009) for details], an evolution is constructed given any initial configuration. Thus for any two configurations η_0 and ξ_0 we have two naturally coupled processes, through basic coupling.

We now discuss the macroscopic stability property, which was introduced in Bramson and Mountford (2002). For this we introduce some notation. For two bounded measures $\alpha(dx)$, $\beta(dx)$ on $\mathbb{R}$ with compact support, we define

$$\Delta(\alpha, \beta) := \sup_{x\in\mathbb{R}} |\alpha((-\infty, x]) - \beta((-\infty, x])|. \tag{3}$$

Let $N \in \mathbb{N}$ be the scaling parameter for the hydrodynamic limit, that is, the inverse of the macroscopic distance between two consecutive sites. Let

$$\alpha^N(\eta)(dx) = N^{-1} \sum_{y\in\mathbb{Z}} \eta(y)\delta_{y/N}(dx) \in \mathcal{M}^+(\mathbb{R})$$

denote the empirical measure of a configuration η viewed on scale N, and $\mathcal{M}^+(\mathbb{R})$ denote the set of positive measures on $\mathbb{R}$ equipped with the metrizable topology of vague convergence, defined by convergence on continuous test functions with compact support.

By macroscopic stability we mean that Δ is an "almost" nonincreasing functional for a pair of coupled evolutions $(\eta_t, \xi_t : t \geq 0)$ where η_0 and ξ_0 are any two configurations with a finite number of particles, in the following sense. There exist constants $C > 0$ and $c > 0$, depending only on $b(\cdot, \cdot)$ and $p(\cdot)$, such that for every $\gamma > 0$, the event

$$\forall t > 0: \qquad \Delta(\alpha^N(\eta_t(\eta_0, \omega)), \alpha^N(\eta_t(\xi_0, \omega))) \leq \Delta(\alpha^N(\eta_0), \alpha^N(\xi_0)) + \gamma \tag{4}$$

has $\mathbb{P}$-probability at least $1 - C(|\eta_0| + |\xi_0|)e^{-cN\gamma}$, where $|\eta| := \sum_{x\in\mathbb{Z}} \eta(x)$.

The strong macroscopic stability property was introduced in Bramson and Mountford [(2002), Section 3] to determine the existence of stationary blocking measures for one-dimensional exclusion processes with a random walk kernel $p(\cdot)$ having finite range and positive mean. It was then applied to models considered in this note in Bahadoran et al. (2002, 2006, 2009) with the additional assumption that the jumps had a finite range. An essential ingredient for this property is the attractiveness of the model.

Remark 1. While in Bramson and Mountford (2002) and in the rest of this note a function Φ is used [see (7) below] to measure distance between configurations, we use Δ in the discussion above since it is more appropriate for hydrodynamics. An elementary computation shows that the statement in (8) remains unchanged whether one uses Φ or Δ.

In Section 2 we state the macroscopic stability result, and its application to strong hydrodynamics. In Section 3 we prove it, through an analysis of the evolution of labeled discrepancies. Section 4 is devoted to two properties needed for hydrodynamics of the particle system.

2 The result

We fix

$$L > 10(\mu_1 + 1). \tag{5}$$

Theorem 2. *Let $\eta^i_\cdot$, $i = 1, 2$, be two processes both generated by the same Harris system with initial configurations η^i_0, $i = 1, 2$, such that*

$$\sum_{|x|\geq LN} \big(\eta^1_0(x) + \eta^2_0(x)\big) = 0. \tag{6}$$

We set, for $t \geq 0$, $x \in \mathbb{Z}$,

$$\Phi_t(x) = \sum_{y\geq x} \big(\eta^1_t(y) - \eta^2_t(y)\big). \tag{7}$$

Then, for each $\epsilon > 0$,

$$\mathbb{P}\Big(\sup_{x\in\mathbb{Z}} \Phi_t(x) - \sup_{x\in\mathbb{Z}} \Phi_0(x) > \epsilon N\Big) \leq Ce^{-cN} \tag{8}$$

for all $t \in [0, N]$ and N, and appropriate $c > 0$ and C, depending on ϵ and L but not on N or η^i_0, $i = 1, 2$.

One can extend this type of result to initial joint configurations which agree outside interval $(-LN, LN)$ but do not necessarily satisfy Condition (6) by an approach which relies on Theorem 13, Section 3.

Theorem 2 has practical consequences to hydrodynamics. It enables us to extend the hydrodynamics derived in Bahadoran et al. (2002, 2006, 2009), for which the assumption $p(\cdot)$ finite range [i.e., there exists $M > 0$ such that $p(x) = 0$ for all $|x| > M$] was needed, to any transition kernel $p(\cdot)$ satisfying (A1), (A2). We now state this hydrodynamic result in a more general form, namely, a strong hydrodynamic limit [which was the setup in Bahadoran et al. (2009)].

Theorem 3. *Assume $p(\cdot)$ has a finite third moment $\mu_3 = \sum_{z\in\mathbb{Z}} |z|^3 p(z) < +\infty$. Let $(\eta^N_0, N \in \mathbb{N})$ be a sequence of $\mathbf{X}$-valued random variables on Ω_0. Assume there exists a measurable $[0, K]$-valued profile $u_0(\cdot)$ on $\mathbb{R}$ such that*

$$\lim_{N\to\infty} \alpha^N(\eta^N_0)(dx) = u_0(\cdot)\, dx \qquad \mathbb{P}_0\text{-a.s.}, \tag{9}$$

that is,

$$\lim_{N\to\infty} \int_{\mathbb{R}} \psi(x)\alpha^N(\eta^N_0)(dx) = \int \psi(x)u_0(x)\, dx \qquad \mathbb{P}_0\text{-a.s.},$$

for every continuous function ψ on $\mathbb{R}$ with compact support. Let $(x, t) \mapsto u(x, t)$ denote the unique entropy solution to the scalar conservation law

$$\partial_t u + \partial_x[G(u)] = 0 \tag{10}$$

with initial condition u_0, *where* G *is a Lipschitz-continuous flux function* [*defined in* (12) *below*] *determined by* $p(\cdot)$ *and* $b(\cdot,\cdot)$. *Then*, *with* $\mathbb{P}_0 \otimes \mathbb{P}$*-probability one*, *the convergence*

$$\lim_{N\to\infty} \alpha^N(\eta_{Nt}(\eta_0^N(\omega_0),\omega))(dx) = u(\cdot,t)\,dx \tag{11}$$

holds uniformly on all bounded time intervals. That is, *for every continuous function* ψ *on* $\mathbb{R}$ *with compact support*, *the convergence*

$$\lim_{N\to\infty} \int_{\mathbb{R}} \psi(x)\alpha^N(\eta_{Nt}^N)(dx) = \int \psi(x)u(x,t)\,dx$$

holds uniformly on all bounded time intervals.

While this condition on kernel $p(\cdot)$ is probably nonoptimal, we have chosen not to pursue this question, prefering to give a simple and direct argument. We recall from Bahadoran et al. [(2006), pages 1346–1347 and Lemma 4.1] the definition of the Lipschitz-continuous macroscopic flux function G. For $\rho \in \mathcal{R}$, let

$$G(\rho) = \nu^{\rho}\biggl[\sum_{z\in\mathbb{Z}} zp(z)b(\eta(0),\eta(z))\biggr]; \tag{12}$$

this represents the expectation, under the shift invariant equilibrium measure with density ρ, of the microscopic current through site 0. On the complement of $\mathcal{R}$, which is at most a countable union of disjoint open intervals, G is interpolated linearly. A Lipschitz constant V of G is determined by the rates $b(\cdot,\cdot)$, $p(\cdot)$ in (1):

$$V = 2\mu_1 \sup_{0\le a\le K, 0\le k<K} \{b(a,k) - b(a,k+1), b(k+1,a) - b(k,a)\}.$$

To obtain the above theorem by a constructive approach, one proceeds by first proving hydrodynamics for Riemann initial profiles and then by a general argument motivated by Glimm scheme obtain the general hydrodynamics by an approximation scheme. We now explain briefly how this approximation result is proved in the setup of Bahadoran et al. (2009), that is, $\mathbb{P}_0 \otimes \mathbb{P}$-a.s. convergence. Therefore all the involved processes are evolving on a common realization $(\omega_0,\omega) \in \Omega_0 \times \Omega$, that we omit from the notation for simplicity. This proof is based on an interplay of macroscopic properties for the conservation law and microscopic properties for the particle system, in particular macroscopic stability and finite propagation property, both valid at microscopic as well as at macroscopic level. The useful properties of the entropy solution $u(\cdot,t)$ to the conservation law are summarized in Bahadoran et al. [(2009), Proposition 4.1].

For $T \in \mathbb{R}^+$, the time interval $[0,T]$ is partitioned by $\{t_1,t_2,\dots\}$ into intervals of equal length. At the macroscopic level the general profile at the beginning of each time step t_k [i.e., the solution $u(\cdot,t_k)$ of the conservation law] is approximated by a step function $v_k(\cdot)$; the time and space steps are chosen so that the Riemann solutions of different spatial steps ("waves") do not interact during $[t_k,t_{k+1}]$.

Macroscopic stability for the conservation law implies that

$$\Delta\big(u(\cdot,t_{k+1})\,dx, v_k(\cdot,t_{k+1}-t_k)\,dx\big) \le \Delta(u(\cdot,t_k)\,dx, v_k(\cdot)\,dx),$$

where $v_k(\cdot,t-t_k)$ is the entropy solution of the conservation law at time t with initial condition $v_k(\cdot)$ at time t_k. We denote by $\xi^{N,k}$ the initial configuration at time Nt_k which is a "microscopic version" of v_k, and by $\xi^{N,k}_{N(t-t_k)}$ the evolved configuration at time Nt. By this we mean that

$$\lim_{N\to\infty} \Delta(\alpha^N(\xi^{N,k}), v_k(\cdot)\,dx) = 0;$$

for $k=0$, this follows from an ergodic theorem for the densities (notice that the measures ν^ρ are not necessarily product); for $k \ge 1$, this follows from Riemann hydrodynamics applied to a profile with constant density. At the microscopic level,

$$\Delta\big(\alpha^N(\eta^N_{Nt_{k+1}}), \alpha^N\big(\xi^{N,k}_{N(t_{k+1}-t_k)}\big)\big) \le \Delta(\alpha^N(\eta^N_{Nt_k}), \alpha^N(\xi^{N,k})) + \epsilon$$

with probability greater than $1 - CNe^{-cN\epsilon}$ by macroscopic stability at the particle level (i.e., Theorem 2). If we know that

$$\lim_{N\to\infty} \Delta\big(\alpha^N\big(\xi^{N,k}_{N(t_{k+1}-t_k)}\big), v_k(\cdot,t_{k+1}-t_k)\,dx\big) = 0$$

then we would have shown that the error

$$|(\Delta(\alpha^N(\eta^N_{Nt_{k+1}}), u(\cdot,t_{k+1})\,dx) - \Delta(\alpha^N(\eta^N_{Nt_k}), u(\cdot,t_k)\,dx))|$$

is small and the proof can be completed by induction on k. This last step requires patching together Riemann hydrodynamics for which one needs the finite propagation property for the particle system [which requires that $p(\cdot)$ has a finite third moment]. The bound $CNe^{-cN\epsilon}$ is not necessary for the argument.

Since the ergodic theorem for densities and the finite propagation property were stated in Bahadoran et al. (2002, 2006, 2009) for finite range transition kernels $p(\cdot)$, we state and prove their extension to nonfinite range kernels for the sake of completeness (see Section 4).

3 Discrepancies

For two processes $(\eta^1_t : t \ge 0)$ and $(\eta^2_t : t \ge 0)$ we say that there is a discrepancy at $x \in \mathbb{Z}$ at time t if $\eta^1_t(x) \ne \eta^2_t(x)$. If $\eta^1_t(x) - \eta^2_t(x) = h \in \mathbb{N}\setminus\{0\}$ we say that there are h 1/2 discrepancies at site $x \in \mathbb{Z}$ at time $t \ge 0$. We similarly speak of 2/1 discrepancies. Indeed, we do not permit different types of discrepancies to share the same site. Given Condition (6), for two processes as in Theorem 2 there are only a finite (and, given the common Harris system, decreasing since the model is attractive) number of discrepancies of either type. It will be of interest to consider the time evolution of discrepancies; to this end we will, as in Bramson and Mountford (2002), label them: for, say, 1/2 discrepancies, we will introduce the processes

$(X_t^{x,i} : t \geq 0)$ of their positions, for $x \in (-LN, LN)$ and $1 \leq i \leq K$, taking values in $\mathbb{Z} \cup \{\Delta\}$ where Δ is a graveyard site. For 2/1 discrepancies, we will introduce processes $(Y_t^y : t \geq 0)$ for y in some labeling set J, a cemetery state, Δ', such that at all times t, $\{z : \eta_t^2(z) > \eta_t^1(z)\}$ is equal to the union of the positions $Y_t^y \neq \Delta'$ with multiplicities respected, that is,

$$\sum_{z\in\mathbb{Z}} (\eta_t^2(z) - \eta_t^1(z))^+ \delta_z = \sum_{y\in J} \mathbf{1}_{\{Y_t^y \neq \Delta'\}} \delta_{Y_t^y}.$$

A decrease of discrepancies corresponds to the coalescence of a 1/2 and a 2/1 discrepancies, due to the jump of one of them to the site where the other is; in that case, we will make the label of a 1/2 discrepancy (not necessarily the one involved in the jump, see case [e] below) jump to Δ, and the label of the 2/1 discrepancy jump to Δ'.

Remark 4. The ideas to prove Theorem 2 are similar to those in Bramson and Mountford (2002), with a few differences that improve the probability of coalescence of 1/2 and 2/1 discrepancies. First, the labeling procedure in Bramson and Mountford (2002) was different: there, all η^1 particles were labeled (but none of the η^2 particles); they were called "uncoupled" when corresponding to 1/2 discrepancies, and "coupled" otherwise. Thus a coalescence of discrepancies was called a "coupling of labels." Secondly, we introduce a notion of "windows" through stopping times slightly different from those in Bramson and Mountford (2002).

We want the processes $(X_t^{x,i} : t \geq 0)$ to be such that:

(1) for all $x \in (-LN, LN)$, $i \in \{1, \dots, K\}$, if there are h 1/2 discrepancies at x at time 0, then $X_0^{x,i} = x$ for $i \leq h$, otherwise $X_0^{x,i} = \Delta$,

(2) if $s < t$ and $X_s^{x,i} = \Delta$, then $X_t^{x,i} = \Delta$,

(3) if there are h 1/2 discrepancies at time t at site z, then there exists precisely h pairs (x_j, i_j) so that $X_t^{x_j,i_j} = z$,

(4) for all (x, i) and t, the (random) point $X_t^{x,i}$ is either the position of a 1/2 discrepancy at time t or equal to Δ and

(5) for all $x \in \mathbb{Z}$, $i \in \{1, \dots, K\}$, $X_\cdot^{x,i}$ cannot jump except (possibly) at $t \in \mathcal{N}^{z,y}$ for some $z, y \in \mathbb{Z}$ (neither of which may equal $X_{t-}^{x,i}$). Equally, we insist that if some $t \in \mathcal{N}^{z,y}$ for some z, y entails no change in both processes (i.e., $\eta_{t-}^1 = \eta_t^1$, $\eta_{t-}^2 = \eta_t^2$), then there will be no movement of any of the $X_\cdot^{x,i}$ processes at t.

Of course for those five conditions to hold there can be many choices of the processes $(X_t^{x,i} : t \geq 0)_{x\in(-LN,LN),i\leq K}$. We will make a choice that is natural, tractable and serves to prove Theorem 2.

The choice of motions for the $X_\cdot^{x,i}$ is "solved" for $p(\cdot)$ a kernel of finite range [see Bramson and Mountford (2002)]. For a general $p(\cdot)$ we must be able to deal with jumps between sites x and y separated by great distances. Accordingly we distinguish between changes in the $X_{\cdot}^{x,i}$ processes occuring at $t \in \mathcal{N}^{y,z}$ for $|z-y|$ large and those contained in a Poisson process corresponding to a close pair of sites. We fix now an $\epsilon > 0$ but arbitrarily small. Associated with this ϵ we will choose an integer $m = m_\epsilon$ which will be large enough to satisfy various (increasing) properties that we will specify as our argument progresses. The rules for the evolution of the $X_{\cdot}^{x,i}$ at a point $t \in \bigcup_{y,z} \mathcal{N}^{z,y}$ will differ according to whether $t \in \mathcal{N}^{z,y}$ for $|z-y| \geq m_\epsilon$ (we call such jumps "big jumps") or not. We note that having finite systems of particles ensures that the rate at which relevant points in $\bigcup_{z,y} \mathcal{N}^{z,y}$ occur is bounded by $K(2LN+1)\|b\|_\infty$. Thus the time for jumps in the processes forms a discrete set, having no cluster points. Between these times we specify, by (5) above, that $X_{\cdot}^{x,i}$ must be constant for all $x \in \mathbb{Z}$, $i \in \{1, \ldots, K\}$.

We must now detail the motions of the $X_{\cdot}^{x,i}$ at times $t \in \mathcal{N}^{z,y}$. As noted in (5) if no particle motion results then there is no motion of the discrepancies. Furthermore if there are no 1/2 discrepancies at sites z and y then again no motion of 1/2 discrepancies results. Equally if at this instant a particle for each process moves from z to y, then there is no motion of discrepancies. This leaves two types of big jumps occuring at t to consider: $t \in \mathcal{N}^{z,y}$ for $|z-y| \geq m_\epsilon$, with a 1/2 discrepancy located either on z or on y at "time" t^-.

[a] A η^1 particle moves from z to y (but not a η^2 particle). If at time t^- there were no 1/2 discrepancies at z then necessarily by assumption (A4) we would have 2/1 discrepancies at y, thus 1/2 discrepancies neither on z nor on y, a case we have excluded here. Therefore there are 1/2 discrepancies at z at time t^-; we pick one at random, uniformly among pairs (x, i) so that $X_{t^-}^{x,i} = z$ and move this discrepancy (and its label).

[a1] If there are no $2/1$ discrepancies at y at time t^-, then the discrepancy chosen and its label jump to y.

[a2] If there are $2/1$ discrepancies at y at time t^- then the $X_{\cdot}^{x,i}$ chosen jumps to Δ and a 2/1 discrepancy is picked at random at y and its label jumps to Δ' (and so each one must remain in these states thereafter: those 1/2 and 2/1 discrepancies have coalesced).

[b] A η^2 particle moves from z to y (but not a η^1 particle).

[b1] If at time t^- there are 2/1 discrepancies at z we pick one at random, uniformly among these and move this discrepancy to Δ'. Since at time t^- there are no 1/2 discrepancies on z, there must be some on y; then one of these discrepancies is chosen uniformly at random and its label moves to Δ.

[b2] If at time t^- there is no 2/1 discrepancy at z, then (cf. [a] above) necessarily by assumption (A4) there must exist 1/2 discrepancies at y. Again we choose one of these discrepancies at random and move it (and its label) to z.

The motion of the $X_\cdot^{x,i}$s for $t \in \mathcal{N}^{z,y}$ for some $|z-y| < m_\epsilon$ is more complicated but follows along the lines of the rules introduced in Bramson and Mountford (2002).

We adopt an *ordering* $\prec$ *of labels of discrepancies* $X_\cdot^{x,i}$ so that the spatial positioning is respected but which also orders labels of discrepancies on the same site. The ordering among "active" (in a sense made precise below) discrepancies can only be changed by a big jump of size at least m_ϵ for a 1/2 discrepancy (thus the jumps described in [a], [b] above), at which point the label of the jumping discrepancy is assigned the lowest order among labels of 1/2 discrepancies currently at the new site (this choice is consistent with the upper bound for $\Delta_t^{x,i}$ obtained below equation (16) with respect to the motions described in [a], [b], as will be explained later on).

Here a difference with the preceding cases is that at a single time t many (but always a bounded number) $X_\cdot^{x,i}$s may move so that labeled 1/2 discrepancies keep their relative order.

[c] If at time t^- neither site z nor y is the location of a 1/2 discrepancy then there is no motion for any $X_\cdot^{x,i}$ at time t.

[d] If at time t^- exactly one of the sites z, y is the location of 1/2 discrepancies, while the other site is not the current position of 2/1 discrepancies, then we fix the labels at time t according to the following two requirements (we take $[z, y]$ to signify $[y, z]$ in the case where z exceeds y): first $X_t^{x,i} = X_{t^-}^{x,i}$ for all pairs (x, i) for which $X_{t^-}^{x,i}$ is outside $[z, y]$, secondly the $X_t^{x,i}$s are chosen for $X_{t^-}^{x,i} \in [z, y]$ so as to preserve order [as in Bramson and Mountford (2002), Section 3]: $X_{t^-}^{x,i} \prec X_{t^-}^{x',i'} \Rightarrow X_t^{x,i} \prec X_t^{x',i'}$.

[e] If at t^- one of the sites z, y is the location of 1/2 discrepancies and the other of $2/1$ discrepancies, then we relabel as follows:

[e1] First we randomly select an interval, called a "window" (see below) among the "active windows" that contain both z and y. Let this window be denoted $[u, v]$. Then among all pairs (x, i) with $X_{t^-}^{x,i} \in [u, v]$ we choose (again all candidates being equally likely) one (x, i) and $X_t^{x,i}$ is specified to be Δ, for the other pairs (x', i') we specify the $X_t^{x',i'}$s so that $X_\cdot^{x',i'}$s outside $[u, v]$ remain where they were while the order of $X_\cdot^{x',i'}$s within $[u, v]$ (apart from $X_\cdot^{x,i}$) is preserved. Notice that this may result in many (but a bounded number of) motions of labels: If, for example, the motion is a 2/1 discrepancy at z moving back to a 1/2 discrepancy at y, but x, the location of a 1/2 discrepancy whose label is being chosen to be sent to Δ is such that $x > z$, then labels of 1/2 discrepancies in $[y, x]$ are shifted rightward (or stay on the same site if it is the location of many labels).

[e2] It may well be that the points z and y do not belong to a single active window, in which case $|z - y| < M_0$ or $|z - y| > M_0 + m_\epsilon$ (according to the definitions of M_0 and of windows given below). In this case the 1/2 discrepancy relevant to the pair z and y at time t has its label assigned to Δ and all other 1/2

discrepancies have their position (and label) unchanged. (Thus we are back to the behaviour described in [a], [b].)

Remark 5. The relabeling enables us to get rid of the possibility of a 2/1 discrepancy being close to a 1/2 discrepancy but not having a chance of coalescing with it. Indeed, thanks to this manoeuvre, whenever a 2/1 discrepancy comes close to a 1/2 discrepancy then there is a nontrivial chance the label of the 1/2 discrepancy will be sent to Δ, while if we would have simply said that the directly affected discrepancy has its label which goes to Δ, there might exist joint configurations where a 2/1 discrepancy is close to a 1/2 discrepancy but the chance of it coalescing with that particular discrepancy is essentially zero.

It remains to describe the random intervals we call “windows.” We follow closely the slightly different definition given in Bramson and Mountford (2002).

In the following result a process on an interval I will be a process on state space $\{0, \dots, K\}^I$ which obeys the same evolution rules as before, given the Poisson processes $\mathcal{N}^{z,y}$ [and the uniform random variables $U(z, y, t)$ associated to $t \in \mathcal{N}^{z,y}$] for $z, y \in I$. We first observe that since by assumption (A1) kernel $p(\cdot)$ is irreducible, then for n large enough

$$\text{greatest common divisor } \{x : p_n(x) \neq 0\} = 1, \tag{13}$$

where the (typically sub-Markov) kernel p_n satisfies $p_n(x) = p(x)\mathbf{1}_{\{|x|\leq n\}}$. The kernel $p_n(x)$ is finite range and we have as in Bramson and Mountford [(2002), Lemma 3.1],

Lemma 6. *Let n be sufficiently large that (13) holds. For all m sufficiently large and all Harris coupled pairs of processes on $[0, m]$ evolving according to kernel $p_n(\cdot)$ and $b(\cdot, \cdot)$, $\eta^1_\cdot$ and $\eta^2_\cdot$ with initial configurations η^1_0, η^2_0 satisfying*

$$\eta^1_0(0) > \eta^2_0(0), \qquad \eta^2_0(m) > \eta^1_0(m),$$

there is a strictly positive chance c_m that there is a coalescence for the joint processes in time interval $[0, 1]$, that is, that

$$\sum_{x\in[0,m]} |\eta^1_1(x) - \eta^2_1(x)| < \sum_{x\in[0,m]} |\eta^1_0(x) - \eta^2_0(x)|.$$

This immediately yields:

Corollary 7. *There exists M_0 so that for all $M \geq M_0$ if for Harris coupled processes $\eta^1_\cdot, \eta^2_\cdot$, for $0 \leq x \leq x + M_0 \leq y \leq M$, $\eta^1_0(x) > \eta^2_0(x)$, $\eta^2_0(y) > \eta^1_0(y)$, then there is a strictly positive constant C_M so that with probability at least C_M during time interval $[0, 1]$ (uniformly over all relevant joint initial configurations):*

(i) *there is no $t \in \mathcal{N}^{u,v}$ for $u \in [0, M]$, $v \notin [0, M]$ or vice versa,*

(ii)

$$\sum_{z\in[0,M]} |\eta_1^1(z) - \eta_1^2(z)| < \sum_{z\in[0,M]} |\eta_0^1(z) - \eta_0^2(z)|.$$

Proof. Let M_0 be a sufficiently large m in the sense of Lemma 6 and n be sufficiently large in the sense of (13). Let A be the event that in time interval $[0, 1]$ there are no $t \in \mathcal{N}^{u,v}$ with either $u \in [0, M], v \notin [0, M]$ or $u \in [x, y], v \notin [x, y]$, or vice versa. Then

$$\mathbb{P}(A) \ge e^{-4\|b\|_\infty \mu_1},$$

where recall $\mu_1 = \sum_w |w| p(w) < \infty$. Furthermore event A is independent of event

$$B = (\text{there is no } t \in [0, 1] \cap \mathcal{N}^{u,v} \text{ with } u, v \in [x, y] \text{ and } |u - v| \ge n),$$

which has probability

$$\mathbb{P}(B) \ge e^{-\|b\|_\infty (M+1) \sum_{|w|\ge n} p(w)} \ge e^{-\|b\|_\infty (M+1)}.$$

Conditional on $A \cap B$, an event of probability

$$\mathbb{P}(A \cap B) \ge e^{-\|b\|_\infty (4\mu_1 + M + 1)}$$

the joint processes $((\eta_s^1, \eta_s^2) : 0 \le s \le 1)$ restricted to interval $[x, y]$ are just spatial translations of finite processes on $[0, y - x]$. The result now follows from Lemma 6. □

We now fix a M_0 (increasing m_ϵ if necessary), so that $M_0 < m_\epsilon/10$ and $M_0 > 10n$ where n is sufficiently large in the sense of Lemma 6. According to Corollary 7, the choice of M_0 is such that if a 1/2 discrepancy and a 2/1 discrepancy are separated by at least M_0 (and less than $M_0 + m_\epsilon$) then there is a definite chance that there will be a coalescence. If the separation is less than M_0, then, in principle, we can say nothing about coalescence probabilities.

We are therefore ready to define "windows," which will be space intervals of length $m_\epsilon + M_0$, on which coalescence will be favored. A window will be associated to a label of 1/2 discrepancy $X_\cdot^{x,i}$. Given $X_\cdot^{x,i}$ (with $X_0^{x,i} \neq \Delta$) we define the following stopping times: $T_0^{x,i} = 0$,

$$\sigma_{x,i} = \inf\{t \ge 0 : X_t^{x,i} = \Delta\}, \tag{14}$$

and for $j \ge 0$ (with the convention $\inf \varnothing = +\infty$)

$$\begin{aligned} S_j^{x,i} &= \inf\{t \in [T_j^{x,i}, \sigma_{x,i}) : \exists \text{ a 2/1 discrepancy in} \\ &\qquad [X_t^{x,i} + M_0, X_t^{x,i} + M_0 + m_\epsilon]\}, \\ a_j^{x,i} &= X_{S_j^{x,i}}^{x,i}, \end{aligned} \tag{15}$$

$$
\begin{aligned}
T_{j+1}^{x,i} = (S_j^{x,i}+1) \wedge \inf\{t \geq S_j^{x,i} : t \in \mathcal{N}^{u,v} &\text{ for } u \in [a_j^{x,i}, a_j^{x,i}+M_0+m_\epsilon],\\
&v \in [a_j^{x,i}, a_j^{x,i}+M_0+m_\epsilon]^c \text{ or vice versa}\}\\
\wedge \inf\Bigg\{t \geq S_j^{x,i} : \sum_{u\in[a_j^{x,i},a_j^{x,i}+M_0+m_\epsilon]} &|\eta_t^1(u)-\eta_t^2(u)|\\
&< \sum_{u\in[a_j^{x,i},a_j^{x,i}+M_0+m_\epsilon]} |\eta^1_{S_j^{x,i}}(u)-\eta^2_{S_j^{x,i}}(u)|\Bigg\}\\
\wedge \inf\{t \geq S_j^{x,i} : \exists u, v \in [a_j^{x,i}, a_j^{x,i}+M_0+m_\epsilon], &\, t \in \mathcal{N}^{u,v}, |u-v| \geq m_\epsilon\}.
\end{aligned}
$$

Times $T_{j+1}^{x,i}$ and $S_j^{x,i}$ are defined in such a way that one can use Corollary 7 to conclude that there is a positive chance of coalescence between times $S_j^{x,i}$ and $T_{j+1}^{x,i}$: If the first or third event defining $T_{j+1}^{x,i}$ does not occur then Corollary 7 can be applied. Since these two events occur with finite rates we can expect coalescence with positive probability between times $S_j^{x,i}$ and $T_{j+1}^{x,i}$.

For some (x,i) and j with $S_j^{x,i}$ finite, a $((x,i),j)$ *space window* is a space interval $[a_j^{x,i}, a_j^{x,i}+M_0+m_\epsilon]$. It is taken to be *active* during the time interval $[S_j^{x,i}, T_{j+1}^{x,i}]$, called an $((x,i),j)$ *time window*. Indeed, the presence of a 2/1 discrepancy in $[a_j^{x,i}+M_0, a_j^{x,i}+M_0+m_\epsilon]$ should favor a coalescence with a 1/2 discrepancy (cf. Remark 5). We remark that a space window is only relevant while it is active, that a point u at a time t may belong to several distinct space windows but that this number is bounded by $M_0+m_\epsilon+1$, the size of a space window.

We now define the evolution of the labels $(Y_t^y : t \geq 0)$ of 2/1 discrepancies (which will be more natural and intuitive than the processes of labels for 1/2 discrepancies). Once a process $Y_\cdot^y$ hits Δ' it must remain at this "position" ever after. We stipulate that the $Y_\cdot^y$ be a cadlag process which jumps at time t only if for some $z \in \mathbb{Z}$, $t \in \mathcal{N}^{z,Y_{t-}^y}$ or $t \in \mathcal{N}^{Y_{t-}^y,z}$. Furthermore nothing happens if at this time t both a η^1 and a η^2 particle move.

[f] If for $t \in \mathcal{N}^{z,Y_{t-}^y}$ solely a η^1 particle moves from z to Y_{t-}^y, one of the 2/1 discrepancies at Y_{t-}^y is randomly selected; then if there were 1/2 discrepancies at z at time t^-, its label moves to Δ' (this is case [a2] above when $|z-Y_{t-}^y| \geq m_\epsilon$); if there are no 1/2 discrepancies at z at time t^-, it moves to z as well as its label. If for $t \in \mathcal{N}^{z,Y_{t-}^y}$ solely a η^2 particle moves from z nothing can happen to $Y_\cdot^y$.

[g] If for $t \in \mathcal{N}^{Y_{t-}^y,z}$ only a η^1 particle moves, then there is no motion for $Y_\cdot^y$; if the motion involves uniquely a η^2 particle, then one of the 2/1 discrepancies currently at site Y_{t-}^y is moved. If at t^- there is a 1/2 discrepancy at site z then

the label of the 2/1 discrepancy chosen moves to Δ' (this is case [b1] above when $|z - Y_{t-}^y| \geq m_\epsilon$), if not the 2/1 discrepancy (and its label) move to z.

Notice therefore that there is no relabeling scheme to preserve order for the processes $(Y_t^y : t \geq 0)$.

To deal with Theorem 2, we now consider the quantity (7). Since by (6) the total number of particles is finite, $\sup_{x\in\mathbb{Z}} \Phi_t(x)$ is equal to the maximum of 0 and the maximum over $x \in \mathbb{Z}$, $i \in \{1, \dots, K\}$ of $\Phi_t(X_t^{x,i})$.

We define for $(x, i), t$ such that $X_t^{x,i} \neq \Delta$, so that $\sigma_{x,i} > t$,

$$\begin{aligned} \Delta_t^{x,i} &= \widetilde{\Phi}_t(X_t^{x,i}) - \widetilde{\Phi}_0(X_0^{x,i}) \qquad \text{with} \\ \widetilde{\Phi}_t(X_t^{x,i}) &= \sum_{(y,j)\in\mathbb{Z}\times\{1,\dots,K\}} \mathbf{1}_{\{X_t^{x,i} \prec X_t^{y,j}\}} - \sum_{y\in\mathbb{Z}:y>X_t^{x,i}} \big(\eta_t^1(y) - \eta_t^2(y)\big)^-. \end{aligned} \tag{16}$$

The quantity $\widetilde{\Phi}_t(X_t^{x,i})$ counts (the number of 1/2 discrepancies to the right of $X_t^{x,i}$ at time t minus the number of such 2/1 discrepancies) minus (the same quantity at time 0). Thus $\Delta_t^{x,i}$ is equal to the number of labels of 1/2 discrepancies $X_\cdot^{u,k}$ for which $X_0^{u,k} \prec X_0^{x,i}$ but $X_s^{x,i} \prec X_s^{u,k}$ for some $s \leq t$ and up to time t [i.e., the labels of 1/2 discrepancies that appear in $\widetilde{\Phi}_t(X_t^{x,i})$ but were not in $\widetilde{\Phi}_0(X_0^{x,i})$], plus the number of labels of 2/1 discrepancies $Y_\cdot^y$ that were "in" $\widetilde{\Phi}_0(X_0^{x,i})$ but "disappear" from $\widetilde{\Phi}_t(X_t^{x,i})$, that is, those for which $Y_0^y > X_0^{x,i}$ but $Y_s^y < X_s^{x,i}$ for some $s \leq t$ (and up to time t) plus the number of $Y_\cdot^y$ (with $Y_0^y > X_0^{x,i}$) which jumped to Δ' at a time $s \leq t$ so that $X_{s-}^{x,i} < Y_{s-}^y$ and the label $X_\cdot^{u,k}$ of the 1/2 discrepancy which jumps to Δ at time s is such that $X_{s-}^{u,k} \prec X_{s-}^{x,i}$ [notice that since $X_t^{x,i} \neq \Delta$, we have $(u, k) \neq (x, i)$].

Indeed, the ordering of labels of 1/2 discrepancies for long jumps introduced earlier ensures that the above description for $\Delta_t^{x,i}$ remains valid during such jumps.

We now consider three classes of discrepancies contributing to the above bound. The first and second classes are not exclusive but this does not concern us as we are interested in an upper bound for $\Delta_t^{x,i}$.

(a) $\Delta_t^{x,i}(a)$ counts the number of 2/1 discrepancies that jump from $(X_s^{x,i}, \infty)$ to $(-\infty, X_s^{x,i}) \cup \{\Delta'\}$ for $s \leq t$ in some $\mathcal{N}^{u,v}$ with $|u - v| \geq m_\epsilon$ plus the number of labels of 1/2 discrepancies $X_\cdot^{w,k}$ for which there exists $s \leq t$ so that $X_{s-}^{w,k} \prec X_{s-}^{x,i} = X_s^{x,i} \prec X_s^{w,k}$. It should be noted that necessarily, given the relabeling scheme in force, such a crossing must result from a jump of size greater than or equal to m_ϵ from $(-\infty, X_s^{x,i})$ to $[X_s^{x,i}, \infty)$ by $X_\cdot^{w,k}$. Thus $\Delta_t^{x,i}(a)$ has only to do with big jumps.

(b) $\Delta_t^{x,i}(b)$ counts the number of labels of 2/1 discrepancies, $Y_\cdot^y$, for which for some $s \in \mathcal{N}^{X_{s-}^{x,i},u}$, $s \leq t$, for $u \geq X_{s-}^{x,i} + m_\epsilon$ we have $X_{s-}^{x,i} < Y_{s-}^y = Y_s^y < X_s^{x,i}$ plus the number of labels of 1/2 discrepancies $X_\cdot^{u,k}$ so that for some

$s \le t$, $X_s^{x,i} \prec X_{s-}^{u,k} = X_s^{u,k} \prec X_{s-}^{x,i}$. Again in the second case, given the relabeling scheme, it must hold at such an s that $|X_t^{x,i} - X_{t-}^{x,i}| \ge m_\epsilon$ (note in this case $\partial\Delta_t^{x,i} := \Delta_t^{x,i} - \Delta_{t-}^{x,i} \le K|X_t^{x,i} - X_{t-}^{x,i}|$). Again, $\Delta_t^{x,i}(b)$ has only to do with big jumps.

(c) $\Delta_t^{x,i}(c)$ deals with times $s \le t$, $s \in \mathcal{N}^{u,v}$ with $|u-v| < m_\epsilon$, for motions described in case [e] above, with (in [e1]) or without (in [e2]) relabeling of some 1/2 discrepancies. The quantity $\Delta_t^{x,i}(c)$ counts the number of labels of 2/1 discrepancies $Y_\cdot^y$ which have not contributed to the two preceding random variables, so that $Y_0^y > X_0^{x,i}$ and at some time $s \le t$, $s \in \mathcal{N}^{u,v}$ with $|u-v| < m_\epsilon$, $X_{s-}^{x,i} < Y_{s-}^y$ and either

(i) there exists (w,k) so that $X_{s-}^{w,k} \prec X_{s-}^{x,i}$ and at time s, $X_\cdot^{w,k}$ jumps to Δ and $Y_s^y = \Delta'$ (this case includes both [e1] and [e2]), or

(ii) $X_\cdot^{x,i}$ or $Y_\cdot^y$ jumps at time s, and $Y_s^y < X_s^{x,i}$ (if $X_\cdot^{x,i}$ jumps, we are in case [d] above, hence there cannot be any 1/2 discrepancy between $X_{s-}^{x,i}$ and Y_{s-}^y).

(iii) there exists (w,k) so that $X_{s-}^{x,i} < Y_{s-}^y < X_{s-}^{w,k}$, and at time s, $X_\cdot^{w,k}$ jumps to Δ and $X_\cdot^{x,i}$ is shifted rightward.

So we have

$$\Delta_t^{x,i} \le \Delta_t^{x,i}(a) + \Delta_t^{x,i}(b) + \Delta_t^{x,i}(c).$$

We treat each term separately:

Lemma 8. *There exists $c(\epsilon) > 0$ so that for all t sufficiently large*

$$\mathbb{P}\big(\Delta_t^{x,i}(a) > \epsilon t/5; \sigma_{x,i} > t\big) \le e^{-c(\epsilon)t}.$$

Proof. We can and will suppose that m_ϵ has been fixed sufficiently large to ensure that

$$\sum_{w \ge m_\epsilon} w\big(p(w) + p(-w)\big) < \frac{\epsilon}{20\|b\|_\infty K}. \tag{17}$$

The rate at which there is a jump of a 2/1 discrepancy from $(X_s^{x,i}, \infty)$ to $(-\infty, X_s^{x,i}) \cup \{\Delta'\}$ for $s \le t$ in some $\mathcal{N}^{u,y}$ with $|u-y| \ge m_\epsilon$ is bounded by (because either solely a η^2 particle jumps, or solely a η^1 particle jumps that makes the 2/1 discrepancy move)

$$\begin{aligned} &\|b\|_\infty \sum_{y > X_s^{x,i}} \sum_{u < X_s^{x,i}, u \le y - m_\epsilon} \big(p(y-u) + p(u-y)\big) \\ &\le \|b\|_\infty \sum_{w \ge m_\epsilon} w\big(p(w) + p(-w)\big) < \frac{\epsilon}{20} \end{aligned} \tag{18}$$

and similarly for the rate for appropriate jumps of 1/2 discrepancies. Thus these jumps are stochastically bounded by a rate $\epsilon/10$ Poisson process. So $\mathbb{P}(\Delta_t^{x,i}(a) > \epsilon t/5; \sigma_{x,i} > t) \le e^{-c(\epsilon)t}$ for some $c > 0$ not depending on N. $\square$

Lemma 9. *There exists $c = c(\epsilon) > 0$ so that for all N sufficiently large and $t \in [0, N]$*

$$\mathbb{P}\left(\left|\left\{(x,i) : \Delta_t^{x,i}(b) \ge \frac{\epsilon N}{10}\right\}\right| \ge \frac{\epsilon^2}{50}N; \sigma_{x,i} > N\right) < e^{-cN},$$

where $|A|$ denotes the cardinality of set A.

Proof. Since $\Delta_t^{x,i}(b)$ is increasing in t it is sufficient to obtain the bound for $t = N$. We use, for the moment, the fact that

$$\forall t, \qquad \partial \Delta_t^{x,i}(b) \le K|X_t^{x,i} - X_{t-}^{x,i}|\mathbf{1}_{\{|X_t^{x,i} - X_{t-}^{x,i}| \ge m_\epsilon\}} \tag{19}$$

though it should be noted that this gives a poor bound if the configurations η_t^1 and η_t^2 are "close."

Observe that for (x,i) and (u,k) distinct, jumps of size larger than m_ϵ for $X_\cdot^{x,i}$ and jumps for $X_\cdot^{u,k}$ can be derived from independent Poisson processes of random but bounded rates. Since the discrepancies are chosen uniformly randomly when a Poisson clock rings at a site this claim is true for two distinct discrepancies at the same site. Thus we can bound stochastically the number of (x,i) such that [recall (17)]

$$\sum_{s \le N} |X_s^{x,i} - X_{s-}^{x,i}|\mathbf{1}_{\{|X_s^{x,i} - X_{s-}^{x,i}| \ge m_\epsilon\}} \ge \frac{\epsilon N}{10K}$$

by the number of $Z_\cdot^{x,i}$ with $Z_N^{x,i} \ge (\epsilon N)/(10K)$ where $Z_\cdot^{x,i}$ are i.i.d. random walks all starting at zero which jump only in the positive direction by $w \ge m_\epsilon$ at rate $\|b\|_\infty(p(w) + p(-w))$.

If we chose m_ϵ sufficiently large then for all (x,i)

$$\mathbb{P}\left(Z_N^{x,i} \ge \frac{\epsilon N}{10K}\right) \le \frac{\epsilon^2}{100(2L+1)K}$$

for N large by the law of large numbers and so we obtain

$$\mathbb{P}\left(\left|\left\{(x,i) : \Delta_N^{x,i}(b) \ge \frac{\epsilon N}{10}\right\}\right| \ge \frac{N\epsilon^2}{50}; \sigma_{x,i} > N\right)$$

$$\le \mathbb{P}\left(\mathrm{Binom}\big((2L+1)NK, \epsilon^2/100(2L+1)K\big) \ge \frac{N\epsilon^2}{50}\right) \le Ce^{-cN}. \qquad \square$$

It remains to treat $\Delta_t^{x,i}(c)$. As we have just seen, the variables $\Delta_t^{x,i}(a)$ and $\Delta_t^{x,i}(b)$ can be controlled by laws of large numbers applied to big jumps. $\Delta_t^{x,i}(c)$ however is associated with the jumps of reasonable magnitude. The 2/1 discrepancies contributing to $\Delta_t^{x,i}(c)$ should be split in two. For a given pair (v,k), we say a time window $[S_j^{v,k}, T_{j+1}^{v,k}]$ [or a $((v,k), j)$ space window] is "relevant" to $X_\cdot^{x,i}$ if the spatial interval $[a_j^{v,k}, a_j^{v,k} + m_\epsilon + M_0]$ contains $X_{S_j^{v,k}}^{x,i}$. We say $Y_\cdot^y$ is "associated" to a $((v,k), j)$ space window $[a_j^{v,k}, a_j^{v,k} + m_\epsilon + M_0]$ relevant to $X_\cdot^{x,i}$ if either of the following are true:

(1) $Y_{S_j^{v,k}}^y \in [a_j^{v,k}, a_j^{v,k} + m_\epsilon + M_0]$;

(2) $Y_{T_{j+1}^{v,k}}^y \in [a_j^{v,k}, a_j^{v,k} + m_\epsilon + M_0]$ (notice that if (2) occurs but not (1) then the $((v,k), j)$ space window $[a_j^{v,k}, a_j^{v,k} + m_\epsilon + M_0]$ is deactivated by the entry of $Y_\cdot^y$);

(3) $Y_{T_{j+1}^{v,k}}^y \in [X_{T_{j+1}^{v,k}}^{x,i}, X_{T_{j+1}^{v,k}}^{x,i} + m_\epsilon]$.

We say $Y_\cdot^y$ is "associated" if it is associated to one or more space windows. Otherwise $Y_\cdot^y$ is not associated. The sum over labels of 2/1 discrepancies which are associated and contribute to $\Delta_t^{x,i}(c)$ is written $\Delta_t^{x,i}(c, \text{ass})$ the contribution of nonassociated is $\Delta_t^{x,i}(c, \text{non})$.

We note that no particle can leave or enter $[a_j^{v,k}, a_j^{v,k} + m_\epsilon + M_0]$ during the time interval $[S_j^{v,k}, T_{j+1}^{v,k})$. Therefore the 2/1 discrepancies which can contribute to $\Delta_t^{x,i}(c)$ during the time interval $[S_j^{v,k}, T_{j+1}^{v,k})$ belong to $[a_j^{v,k}, a_j^{v,k} + m_\epsilon + M_0]$ and are located at sites to the right of $X_{S_j^{v,k}}^{x,i}$ which implies that $\Delta_t^{x,i}(c)$ can increase at most by $(M_0 + m_\epsilon)K$ during this time interval. Those 2/1 discrepancies which are in $[X_{(T_{j+1}^{v,k})^-}^{x,i}, X_{(T_{j+1}^{v,k})^-}^{x,i} + m_\epsilon]$ at time $(T_{j+1}^{v,k})^-$ may lead to an increase of at most $m_\epsilon K$ at time $T_{j+1}^{v,k}$. If $Y_{T_{j+1}^{v,k}}^y \in [X_{T_{j+1}^{v,k}}^{x,i}, X_{T_{j+1}^{v,k}}^{x,i} + M_0)$, then $Y_s^y \in [X_s^{x,i} + M_0, X_s^{x,i} + M_0 + m_\epsilon]$ for some $s < T_{j+1}^{v,k}$, unless it entered $[X_{T_{j+1}^{v,k}}^{x,i}, X_{T_{j+1}^{v,k}}^{x,i} + M_0)$ by a long jump of either $Y_\cdot^y$ or $X_\cdot^{x,i}$. As we can see from case (3) above, any 2/1 discrepancy which enters $[X_{T_{j+1}^{v,k}}^{x,i}, X_{T_{j+1}^{v,k}}^{x,i} + M_0)$ by a long jump will not be associated unless it also enters a window $[a_j^{v,k}, a_j^{v,k} + M_0 + m_\epsilon]$.

Let $[S_j, T_j]$, $j \in \mathbb{N}$, denote the active windows relevant to (x,i) for all possible (v,k) with reordered opening times ($S_j \le S_{j+1}$ but not necessarily $T_j \le T_{j+1}$). We are interested in finding an increasing subsequence of active windows $[S_{j_k}, T_{j_k})$ which are disjoint since we want to use the strong Markov property to claim the independence of coalescence events in such intervals to obtain our probability estimate.

Let $j_1 = 1$. Define for all $k \ge 1$,

$$j_{k+1} = \inf\{\ell > j_k : S_\ell \ge T_{j_k}\}.$$

Now we observe that $j_{k+1} \le j_k + (2(M_0 + m_\epsilon) + 1)K$ since while the window $[S_{j_k}, T_{j_k})$ is relevant to $X^{x,i}_\cdot$, then $X^{x,i}_\cdot$ can belong to at most $2(M_0 + m_\epsilon)K$ other active windows by our reordering convention. During the time interval $[S_{j_k}, T_{j_k})$, $X^{x,i}_\cdot$ remains in the corresponding space window. Therefore during this time interval $\Delta^{x,i}_\cdot(c)$ increases by at most $K(M_0 + m_\epsilon)$. It remains to consider times for $t \in [T_{j_k}, S_{j_{k+1}})$. In this interval Y particles can pass $X^{x,i}_\cdot$ by long jumps (which have already been counted) or by being first associated to a window containing $X^{x,i}_\cdot$ at some time in $[T_{j_k}, S_{j_{k+1}})$. We easily see that the number of such associated particles is bounded by $(2(M_0 + m_\epsilon))K^2(2m_\epsilon + M_0)$. Thus we can conclude that during the time interval $[S_{j_k}, S_{j_{k+1}})$, the total number of associated 2/1 discrepancies which contribute to $\Delta^{x,i}_\cdot(c)$ is bounded above by $K^2(M_0 + 3m_\epsilon)^2$.

Treating $\Delta^{x,i}_t(c, \text{ass})$ follows naturally along the same lines as with Bramson and Mountford [(2002), Proposition 3.2]: during each relevant time window interval for label $X^{x,i}_\cdot$, there is a reasonable probability that $X^{x,i}_\cdot$ jumps to Δ and during such an interval $[S_{j_k}, S_{j_{k+1}})$ the number of $Y^y_\cdot$ which are associated is bounded by $K^2(3m_\epsilon + M_0)^2$. Therefore if the event $(\Delta^{x,i}_t(c, \text{ass}) \ge \gamma N; \sigma_{x,i} > t)$ occurs for some t, then $X^{x,i}_\cdot$ goes through at least $[\gamma N]/(K^2(3m_\epsilon + M_0)^2)$ successive disjoint time windows $[S_{j_k}, S_{j_{k+1}})$ without jumping to Δ in the time interval $[0, t]$. Since the probability of jumping to Δ during a given time window is bounded below by some $c' > 0$, and jumps in successive time windows are independent, the event $(\Delta^{x,i}_t(c, \text{ass}) \ge \gamma N; \sigma_{x,i} > t)$ has a probability bounded above by

$$(1 - c')^{[\gamma N]/(K^2(3m_\epsilon + M_0)^2)}.$$

Thus we have in place of Bramson and Mountford [(2002), Proposition 3.2]

Lemma 10. *There exists $c, C \in (0, \infty)$ so that for all $t \ge 0$*

$$\mathbb{P}\big(\Delta^{x,i}_t(c, ass) \ge \gamma N; \sigma_{x,i} > t\big) \le Ce^{-c\gamma N}. \tag{20}$$

It remains to assess $\Delta^{x,i}_t(c, \text{non})$ for $t \in [0, N]$. Any $Y^y_\cdot$ particle which is initially in $(X^{x,i}_0, X^{x,i}_0 + M_0)$ may increase $\Delta^{x,i}_t(c)$ without making a long jump, if this occurs before the first time $X^{x,i}_\cdot$ enters an active window. This increase is bounded above by KM_0. Suppose that $Y^y_\cdot$ makes a contribution to this random variable. Then, by definition, $Y^y_\cdot$ is never associated with a space window relevant to $X^{x,i}_\cdot$. Since $Y^y_\cdot$ does not contribute to $\Delta^{x,i}_t(a)$ or $\Delta^{x,i}_t(b)$, it cannot traverse $X^{x,i}_\cdot$ via a jump of size greater than m_ϵ, be it of an η^1 or of an η^2 particle.

Thus the first time s that $Y^y_s \in [X^{x,i}_s, X^{x,i}_s + m_\epsilon]$ must be less than t. Given that $Y^y_\cdot$ is not associated with a space window relevant to $X^{x,i}_\cdot$, it must be the case that in fact at this point s, $Y^y_s \in [X^{x,i}_s, X^{x,i}_s + M_0]$ and that s is the moment of a jump of size at least m_ϵ either by $X^{x,i}_\cdot$ or by $Y^y_\cdot$.

From this one sees that for all $t \in [0, N]$ the contribution $\Delta^{x,i}_t(c, \text{non})$ is stochastically bounded by a random variable that is in distribution the sum of KM_0 times a Poisson random variable of parameter $\|b\|_\infty t \sum_{z\geq m_\epsilon} p(z)$ and an independent Poisson random variable of parameter $M_0\|b\|_\infty t \sum_{z\geq m_\epsilon} p(z)$.

Thus for all $t \in [0, N]$, $\mathbb{P}(\Delta_t^{x,i}(c) \geq \frac{\epsilon}{4}N; \sigma_{x,i} > N) \leq Ce^{-cN}$ for some c depending on ϵ but not on η_0^1, η_0^2 or N.

So we have [recall $\sum_{x\in\mathbb{Z}}(\eta_0^1(x) + \eta_0^2(x)) \leq 2K(2LN + 1)$], by Lemmas 8, 9 and 10,

Theorem 11. *For all $t \in [0, N]$,*

$$\mathbb{P}\left(\left|\left\{(x, i) : \Delta_t^{x,i} \geq \frac{\epsilon N}{2}\right\}\right| \geq \epsilon^2 N; \sigma_{x,i} > N\right) \leq Ce^{-cN}$$

for $C, c \in (0, \infty)$ not depending on N.

This result is close to the announced Theorem 2. The difference being that the latter deals with the supremum over all pairs (x, i) of $\Delta_t^{x,i}$, whereas Theorem 11 asserts that the number of (x, i) for which $\Delta_t^{x,i}$ is "too large" is small.

Proof of Theorem 2. We prove the result by contradiction. By the conditions on the initial configurations we necessarily have that $\sup_{x\in\mathbb{Z}} \Phi_0(x)$ is greater than or equal to zero. So, for some $t \in [0, N]$, $\sup_{x\in\mathbb{Z}} \Phi_t(x)$ to exceed $\sup_{x\in\mathbb{Z}} \Phi_0(x)$ by some ϵN, we must have that for some pair (x, i) for a 1/2 discrepancy of label $X_t^{x,i} \neq \Delta$,

$$\Delta_t^{x,i} \geq N\epsilon + \sup_{x\in\mathbb{Z}} \Phi_0(x). \tag{21}$$

We have by Theorem 11 that outside probability Ce^{-cN},

$$|\{(v, k) : \Delta_t^{v,k} \geq \epsilon N/2\}| \leq \epsilon^2 N.$$

Suppose for some (x, i) with $X_t^{x,i} \neq \Delta$,

$$\sum_{y\geq X_t^{x,i}} (\eta_t^1(y) - \eta_t^2(y)) - \sum_{y\geq X_0^{x,i}} (\eta_0^1(y) - \eta_0^2(y)) > \epsilon N.$$

This must mean that for at least $(\epsilon N)/K$ sites $y \in [X_t^{x,i}, +\infty)$, $\eta_t^1(y) > \eta_N^2(y)$. Let these points be enumerated, in order, as $X_t^{x,i} \leq y_1 < y_2 < \cdots < y_R$.

By hypothesis at most $\epsilon^2 N$ [$< (\epsilon N)/(2K)$ for ϵ small] are positions (at time t) of labels $X_t^{v,k}$ with $X_t^{v,k} \neq \Delta$, and $\Delta_t^{v,k} \geq \epsilon N/2$. Let $i_1 = \inf\{j : y_j$ is the position of a 1/2 discrepancy label with $\Delta_t^{v,k} \leq \epsilon N/2\}$, then we have

$$\sum_{y\geq X_t^{x,i}} (\eta_t^1(y) - \eta_t^2(y)) = \sum_{y=X_t^{x,i}}^{y_{i_1}-1} (\eta_t^1(y) - \eta_t^2(y)) + \sum_{y\geq y_{i_1}} (\eta_t^1(y) - \eta_t^2(y)).$$

But $i_1 \leq \epsilon^2 N + 1$ so $\sum_{y=X_t^{x,i}}^{y_{i_1}-1} (\eta_t^1(y) - \eta_t^2(y)) \leq \epsilon^2 NK$, while

$$\sum_{y \geq y_{i_1}} \left(\eta_t^1(y) - \eta_t^2(y)\right) \leq \frac{\epsilon N}{2} + \sup_{w \in \mathbb{Z}} \Phi_0(w).$$

Thus $\sum_{y \geq X_t^{x,i}} (\eta_t^1(y) - \eta_t^2(y)) < \epsilon N + \sup_{w\in\mathbb{Z}} \Phi_0(w)$, a contradiction. □

The above argument yields

Corollary 12. *For η_0^1 and η_0^2 vacant on $[-LN, LN]^c$ and satisfying for all $x \in [-LN, LN]$,*

$$\left|\sum_{y \geq x} (\eta_0^1(y) - \eta_0^2(y))\right| \leq \frac{\epsilon N}{4},$$

we have outside probability $2Ce^{-cN}$ (for C, c as in Theorem 11)

$$\forall t \in [0, N], \forall x \in \mathbb{Z}, \qquad \left|\sum_{y \geq x} (\eta_t^1(y) - \eta_t^2(y))\right| \leq \epsilon N. \tag{22}$$

From our conclusions on finite configurations, one can compare infinite configurations through the following result for two initial configurations close on a (large) finite interval, one being finite (vacant outside that interval) and the other one infinite. We wish to show that

Theorem 13. *For $\epsilon > 0$, there exists L_0 so that for $L \geq L_0$, if η_0^1 is vacant on $[-LN, LN]^c$ and η_0^2 satisfies*

$$\forall x \in [-LN, LN], \qquad \left|\sum_{y=x}^{LN} (\eta_0^1(y) - \eta_0^2(y))\right| \leq \frac{\epsilon N}{4},$$

then for C^1, c^1 depending on L, but not N, outside probability $C^1 e^{-c^1 N}$ for all interval $I \subseteq [-N, N]$ and for all $t \in [0, N]$,

$$\left|\sum_{y \in I} (\eta_t^1(y) - \eta_t^2(y))\right| \leq 3\epsilon N.$$

Proof. We split the process $\eta_\cdot^2$ in two classes of particles: first-class particles $\eta_\cdot^{2.1}$ which at time 0 were in the interval $[-LN, LN]$ and second-class particles which are in $[-LN, LN]^c$ initially. From Corollary 12 we have that outside probability Ce^{-cN}

$$\forall t \in [0, N], \forall x \in \mathbb{Z}, \qquad \left|\sum_{y > x} \left(\eta_t^1(y) - \eta_t^{2.1}(y)\right)\right| \leq \epsilon N. \tag{23}$$

This implies that for all $I \subset [-N, N]$, for all $t \in [0, N]$,

$$\left| \sum_{y \in I} \left(\eta_t^1(y) - \eta_t^{2.1}(y) \right) \right| \leq 2\epsilon N. \tag{24}$$

Thus to prove Theorem 13 it will be enough to effectively bound

$$\sum_{y \in I} \eta_t^{2.2}(y) \leq \sum_{y \in [-N,N]} \eta_t^{2.2}(y),$$

the number of second-class particle in $[-N, N]$ at time t.

The second-class particles in $[-N, N]$ can be divided in two: those which jumped into $[-LN, LN]$ at the same time as hitting $[-(L-1)N, (L-1)N]$, and those that enter $[-LN, LN]$ at a point in $[-LN, -(L-1)N) \cup ((L-1)N, LN]$.

For the first we note that the entry of particles to $[-(L-1)N, (L-1)N]$ from $[-LN, LN]^c$ has (random) rate bounded by

$$2 \sum_{w \geq N}^{\infty} \sum_{y \geq w} \|b\|_\infty \bigl(p(y) + p(-y) \bigr) \leq 2\|b\|_\infty \sum_{|w| \geq N} |w| p(w) \leq \frac{\epsilon}{10} \tag{25}$$

for N large.

So the probability that the number of such entries over the time interval $[0, t]$ exceeds $\epsilon/5N$ is less than He^{-hN} for H, h in $(0, \infty)$ not depending on N.

For the remainder we note that the rate of the entrants to $[-LN, LN]$ must be bounded by $2\mu_1$ and so outside probability $H_1 e^{-h_1 N}$ at most $4\mu_1 N$ particles enter during time interval $[0, t]$.

If a second class particle enters during time interval $[0, t]$ at $[-LN, LN] \setminus [-(L-1)N, (L-1)N]$ then for it to be in $[-N, N]$ at time t, the sum of its absolute displacements over this time interval must exceed $(L-2)N$, but for each such particle, the absolute values of the jumps are stochastically bounded by independent r.w.s. Z_N which jump over in positive direction and jump to $w > 0$ at rate $\|b\|_\infty (p(w) + p(-w))$.

Thus

$$\begin{aligned} \mathbb{P}\Bigl(& |\{\text{second class particles in } [-N, N] \text{ at time } t \text{ which entered} \\ & \text{via } [-LN, LN] \setminus [-(L-1)N, (L-1)N]\}| \geq \frac{\epsilon N}{4} \Bigr) \\ & \leq \mathbb{P}\bigl(\text{Binom}(4\mu_1 N, p) > \epsilon N/4\bigr) + H_1 e^{-h_1 N}, \end{aligned}$$

where

$$p = \mathbb{P}\bigl(Z_N \geq (L-2)N\bigr) \to 0 \qquad \text{as } N \to \infty$$

for $L - 2 > \mu_1$ [which is the case by (5)].

Thus we obtain the defined bound for N large, for all $t \in [0, N]$,

$$\mathbb{P}\left(\sum_{y=-N}^{N} \eta_t^{2.2}(y) \geq \epsilon N\right) \leq He^{-hN} + \varepsilon' e^{-h_1 N} + \varepsilon'' e^{-h'' N} \leq Re^{-rN}$$

and we are done. □

4 Remaining lemmas

We need an extension of Rezakhanlou [(2001), Lemma 4.5] to nonfinite range kernels:

Lemma 14. *Under the assumption* $\mu_1 = \sum_{z\in\mathbb{Z}} |z| p(z) < \infty$, *the measure* ν^ρ *has a.s. density* ρ, *that is*,

$$\lim_{l\to\infty} \frac{1}{2l+1} \sum_{x=-l}^{l} \eta(x) = \rho, \qquad \nu^\rho\text{-a.s.} \tag{26}$$

Proof. Let η be a random configuration with a distribution ν^ρ. We want to show (26). We consider the stationary process $\eta.$ with initial distribution ν^ρ. Since ν^ρ is translation invariant we have

$$\lim_{l\to\infty} \sum_{x=-l}^{l} \frac{1}{2l+1} \eta_0(x) = f(\eta_0)$$

for ν^ρ almost every $\eta_0 \in \mathbf{X}$, where f is a translation invariant function. We will show that $f(\eta_0) = f(\eta_1)$, $\nu^\rho \otimes \mathbb{P}$-a.s., thus showing that f is also a time invariant function. Let $n \in \mathbb{N}$ and define the event

$$A_n^l = \Bigg\{(\omega_0, \omega) \in (\Omega_0 \times \Omega): \\ \left|\sum_{x=-l}^{l} \eta_0(\omega_0)(x) - \sum_{x=-l}^{l} \eta_1(\eta_0(\omega_0), \omega)(x)\right| > \frac{2l+1}{n}\Bigg\}.$$

Now $|\sum_{x=-l}^{l} \eta_0(\omega_0)(x) - \sum_{x=-l}^{l} \eta_1(\eta_0(\omega_0), \omega)(x)|$ is the change in the net number of particles in $[-l, l]$ during the time interval $[0, 1]$. This can be written as

$$\big(J_{-l}^+(\omega_0, \omega) + J_l^+(\omega_0, \omega)\big) - \big(J_{-l}^-(\omega_0, \omega) + J_l^-(\omega_0, \omega)\big),$$

where $J_{-l}^+(\omega_0, \omega)$ is the total number of particles which jump from $\mathbb{Z} \cap (-\infty, -l-1]$ into $\mathbb{Z} \cap [-l, l]$ during time interval $[0, 1]$ and $J_{-l}^-(\omega_0, \omega)$ is the total number of particles which jump from $\mathbb{Z} \cap [-l, l]$ into $\mathbb{Z} \cap (-\infty, -l-1]$ during time interval $[0, 1]$. We have that $J_l^\pm(\omega_0, \omega)$ are defined similarly. The sum $(J_{-l}^+(\omega_0, \omega) +$

$J_l^+(\omega_0,\omega)$) is bounded above by a Poisson process with rate $\|b\|_\infty \mu_1$. The same is true for $(J_{-l}^-(\omega_0,\omega) + J_l^-(\omega_0,\omega))$. Therefore $|\sum_{x=-l}^l \eta_0(\omega_0)(x) - \sum_{x=-l}^l \eta_1(\eta_0(\omega_0),\omega)(x)|$ is bounded above by a Poisson random variable with parameter $2\|b\|_\infty \mu_1$. This implies that $\sum_{l=1}^\infty (\nu^\rho \otimes \mathbb{P})(A_n^l) < \infty$. Since this is true for all $n \in \mathbb{N}$ we have shown by Borel–Cantelli Lemma that $f(\eta_0) = f(\eta_1)$, $\nu^\rho \otimes \mathbb{P}$-a.s. Since $\nu^\rho \in (\mathcal{I} \cap \mathcal{S})_e$ this implies that f is a constant, proving the result. □

We finally state an extension to nonfinite range kernels of the *finite propagation* property at particle level [see Bahadoran et al. (2006), Lemma 5.2].

Lemma 15. *There exist constant v, and function $A(\cdot)$ [satisfying $\sum_n A(n) < \infty$], depending only on $b(\cdot,\cdot)$ and $p(\cdot)$, such that the following holds. For any $x, y \in \mathbb{Z}$, any $(\eta_0, \xi_0) \in \mathbf{X}^2$, and any $0 < t < (y-x)/(2v)$: if η_0 and ξ_0 coincide on the site interval $[x, y]$, then with $\mathbb{P}$-probability at least $1 - A(t)$, $\eta_s(\eta_0,\omega)$ and $\eta_s(\xi_0,\omega)$ coincide on the site interval $[x+vt, y-vt] \cap \mathbb{Z}$ for every $s \in [0,t]$.*

Proof. Let η_0 and ξ_0 be configurations in $\mathbf{X}$ which agree on all sites x such that $m \le x \le n$. We couple the processes starting from η_0 and ξ_0 by basic coupling so that they move together whenever they can. We define random walks $L_t(\eta_t, \xi_t)$ and $R_t(\eta_t, \xi_t)$ as follows. Initially $L_0 = m$ and $R_0 = n$. If $t \in \mathcal{N}^{z,w}$ for some sites $z < L_{t^-} \le w$, then $L_t = L_{t^-} + (w - L_{t^-} + 1)$. Similarly if $t \in \mathcal{N}^{z',w'}$ for some sites $w' < L_{t^-} \le z'$, then $L_t = L_{t^-} + (z' - L_{t^-} + 1)$. Thus L_t is a random walk moving to the right. We define R_t similarly moving to the left. From the definition of L_t and R_t it follows that if, at some time t, $a = L_t < R_t = b$, then during the time interval $[0,t]$ no particles entered or left $[a,b]$. Since $\eta_0(x) = \xi_0(x)$ for all $a \le x \le b$ it follows that $\eta_t(x) = \xi_t(x)$ for all $L_t \le x \le R_t$. Now the drift of L_t can be written as

$$v_L = \|b\|_\infty \biggl(\sum_{z<L_{t^-}} \sum_{w \ge L_{t^-}} (w - L_{t^-} + 1) p(w-z) + \sum_{w'<L_{t^-}} \sum_{z' \ge L_{t^-}} (z' - L_{t^-} + 1) p(w'-z') \biggr).$$

Using translation invariance of $p(\cdot)$, then summation by parts, the first term of v_L can be written as

$$\|b\|_\infty \sum_{j=1}^{+\infty} \sum_{i=j}^{+\infty} (i-j+1) p(i) = \|b\|_\infty \sum_{i=1}^{+\infty} p(i) \sum_{j=1}^{i} j = \|b\|_\infty \sum_{i=1}^{+\infty} \frac{i(i+1)}{2} p(i).$$

Similarly for the second term of v_L we write

$$\|b\|_\infty \sum_{j=1}^{+\infty} \sum_{i=j}^{+\infty} j p(-i) = \|b\|_\infty \sum_{i=1}^{+\infty} p(-i) \sum_{j=1}^{i} j = \|b\|_\infty \sum_{i=1}^{+\infty} \frac{i(i+1)}{2} p(-i).$$

Both terms are finite because of the moment assumptions on $p(\cdot)$, and

$$v_L = \|b\|_\infty \frac{\mu_1 + \mu_2}{2},$$

where $\mu_2 = \sum_{z\in\mathbb{Z}} z^2 p(z)$. We can proceed similarly with $R_t(\eta_t, \xi_t)$ to show that the drift v_R of R_t is $v_R = -v_L$. From the argument above it follows that if

$$\sum_{z\in\mathbb{Z}} |z|^k p(z) < \infty$$

for $k > 1$ then both L_t and R_t have finite $(k-1)$th moment. Since we have assumed that $p(\cdot)$ has finite third moment, we can conclude that R_t and L_t have finite second moment.

Therefore if we take v bigger than $v_L = -v_R$, then for all $t \geq 0$, $\mathbb{P}\{L_t \geq v_L t\} \leq A([t])$, for some function A satisfying the announced finiteness condition which depends only on $p(\cdot)$, $b(\cdot, \cdot)$ and $v - v_L$. $\square$

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