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ASYMPTOTICS OF ONE-DIMENSIONAL FOREST FIRE PROCESSES

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We consider the so-called one-dimensional forest fire process. At each site of $\mathbb{Z}$, a tree appears at rate 1. At each site of $\mathbb{Z}$, a fire starts at rate $\lambda > 0$, immediately destroying the whole corresponding connected component of trees. We show that when λ is made to tend to 0 with an appropriate normalization, the forest fire process tends to a uniquely defined process, the dynamics of which we precisely describe. The normalization consists of accelerating time by a factor $\log(1/\lambda)$ and of compressing space by a factor $\lambda \log(1/\lambda)$. The limit process is quite simple: it can be built using a graphical construction and can be perfectly simulated. Finally, we derive some asymptotic estimates (when $\lambda \to 0$) for the cluster-size distribution of the forest fire process.

1. Introduction and main results.

1.1. *The model.* Consider two independent families of independent Poisson processes, $N = (N_t(i))_{t\geq 0, i\in\mathbb{Z}}$ and $M^\lambda = (M_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$, with respective rates 1 and $\lambda > 0$. Define $\mathcal{F}_t^{N,M^\lambda} := \sigma(N_s(i), M_s^\lambda(i), s \leq t, i \in \mathbb{Z})$. For $a, b \in \mathbb{Z}$ with $a \leq b$, we set $[\![a, b]\!] = \{a, \dots, b\}$.

DEFINITION 1. Consider a $\{0,1\}^{\mathbb{Z}}$-valued $(\mathcal{F}_t^{N,M^\lambda})_{t\geq 0}$-adapted process $(\eta_t^\lambda)_{t\geq 0}$ such that $(\eta_t^\lambda(i))_{t\geq 0}$ is a.s. càdlàg for all $i \in \mathbb{Z}$.

We say that $(\eta_t^\lambda)_{t\geq 0}$ is a λ-FFP (forest fire process) if a.s., for all $t \geq 0$ and all $i \in \mathbb{Z}$,

$$\eta_t^\lambda(i) = \int_0^t \mathbf{1}_{\{\eta_{s-}^\lambda(i)=0\}}\, dN_s(i) - \sum_{k\in\mathbb{Z}} \int_0^t \mathbf{1}_{\{k\in C_{s-}^\lambda(i)\}}\, dM_s^\lambda(k),$$

where $C_s^\lambda(i) = \varnothing$ if $\eta_t^\lambda(i) = 0$, while $C_s^\lambda(i) = [\![l_s^\lambda(i), r_s^\lambda(i)]\!]$ if $\eta_s^\lambda(i) = 1$, with

$$l_s^\lambda(i) = \sup\{k < i; \eta_s^\lambda(k) = 0\} + 1 \quad \text{and} \quad r_s^\lambda(i) = \inf\{k > i; \eta_s^\lambda(k) = 0\} - 1.$$

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Formally, we say that $\eta_t^\lambda(i) = 0$ if there is no tree at site i at time t and $\eta_t^\lambda(i) = 1$ otherwise. $C_t^\lambda(i)$ stands for the connected component of occupied sites around i at time t. Thus, the forest fire process starts from an empty initial configuration, trees appear on vacant sites at rate 1 (according to N) and a fire starts on each site at rate $\lambda > 0$ (according to M^λ), immediately burning the corresponding connected component of occupied sites.

This process can be shown to exist and to be unique (for almost every realization of N, M^λ) by using a *graphical construction*. Indeed, to build the process until a given time $T > 0$, it suffices to work between sites i which are vacant until time T [because $N_T(i) = 0$]. Interaction cannot cross such sites. Since such sites are a.s. infinitely many, this allows us to handle a graphical construction. We refer to Van den Berg and Jarai [16] (see also Liggett [13]) for many examples of graphical constructions. It should be pointed out that this construction only works in dimension 1.

1.2. *Motivation and references.* The study of self-organized critical (SOC) systems has become rather popular in physics since the end of the 1980s. SOC systems are simple models which are supposed to shed light on temporal and spatial randomness observed in a variety of natural phenomena showing *long-range correlations*, like sand piles, avalanches, earthquakes, stock market crashes, forest fires, shapes of mountains, clouds, etc. Roughly, the idea, which appears in Bak, Tang and Wiesenfeld [1] with regard to sand piles, is that of systems *growing* toward a *critical state* and relaxing through *catastrophic* events (avalanches, crashes, fires, etc.). The most classical model is the sand pile model introduced in 1987 in [1], but many variants or related models have been proposed and studied more or less rigorously, describing earthquakes (Olami, Feder and Christensen [14]) or forest fires (Henley [11], Drossel and Schwabl [6]). For surveys on the subject, see Bak, Tang and Wiesenfeld [1, 2], Jensen [12] and the references therein.

From the point of view of SOC systems, the forest fire model is interesting in the asymptotic regime $\lambda \to 0$. Indeed, fires are less frequent, but when they occur, destroyed clusters may be huge. This model has been the subject of many numerical and heuristic studies; see Drossel, Clar and Schwabl [7] and Grassberger [10] for references. However, there are few rigorous results. Even existence of the (time-dependent) process for a multidimensional lattice and given $\lambda > 0$ has been proven only recently [8, 9] and uniqueness is known to hold only for λ large enough. The existence and uniqueness of an invariant distribution (as well as other qualitative properties), even in dimension 1, have been proven only recently in [3] for $\lambda = 1$. These last results can probably be extended to the case where $\lambda \geq 1$, but the method in [3] completely breaks down for small values of λ.

The asymptotic behavior of the λ-FFP as $\lambda \to 0$ has been studied numerically and heuristically [5–7, 10]. To our knowledge, the only mathematically rigorous results are the following.

(a) Van den Berg and Jarai [16] have proven that for $t \geq 3$, $\mathbb{P}[\eta^{\lambda}_{t\log(1/\lambda)}(0) = 0] \simeq 1/\log(1/\lambda)$, thus giving some idea of the density of vacant sites. This result was conjectured by Drossel, Clar and Schwabl [7].

(b) Van den Berg and Brouwer [15] have obtained some results in the two-dimensional case concerning the behavior of clusters near the *critical time*. However, these results are not completely rigorous since they are based on a percolation-like assumption, which is not rigorously proved.

(c) Brouwer and Pennanen [4] have proven the existence of an invariant distribution for each fixed $\lambda > 0$, as well as a precise version of the following estimate which extends (a): for $\lambda \in (0, 1)$, at equilibrium, $\mathbb{P}[\#(C^{\lambda}(0)) = x] \simeq c/[x\log(1/\lambda)]$ for $x \in \{1, \dots, (1/\lambda)^{1/3}\}$. It was conjectured in [7] that this actually holds for $x \in \{1, \dots, 1/(\lambda\log(1/\lambda))\}$, but this was rejected in [16].

In this paper, we rigorously derive a limit theorem which shows that the λ-FFP converges, under rescaling, to some limit forest fire process (LFFP). We precisely describe the dynamics of the LFFP and show that it is quite simple: in particular, it is unique, can be built by using a *graphical construction* and can thus be *perfectly* simulated. Our result allows us to prove a very weak version of (c) for $x \in \{1, \dots, (1/\lambda)^{1-\varepsilon}\}$, for any $\varepsilon > 0$; see Corollary 6 below.

1.3. *Notation*. We denote by $\#(I)$ the number of elements of a set I.

For $a, b \in \mathbb{Z}$, with $a \leq b$, we set $[\![a, b]\!] = \{a, \dots, b\} \subset \mathbb{Z}$.

For $I = [\![a, b]\!] \subset \mathbb{Z}$ and $\alpha > 0$, we will set $\alpha I := [\alpha a, \alpha b] \subset \mathbb{R}$. For $\alpha > 0$, we naturally adopt the convention that $\alpha\varnothing = \varnothing$.

For $J = [a, b]$, an interval of $\mathbb{R}$, $|J| = b - a$ stands for the length of J and for $\alpha > 0$, we set $\alpha J = [\alpha a, \alpha b]$.

For $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the integer part of x.

1.4. *Heuristic scales and relevant quantities*. Our aim is to find some time scale for which tree clusters experience approximately one fire per unit of time. However, for λ very small, clusters will be very large immediately before they burn. We must thus also rescale space, in order that, immediately before burning, clusters have a size of order 1.

Time scale. Consider the cluster $C^{\lambda}_t(x)$ around some site x at time t. It is quite clear that for $\lambda > 0$ very small and t not too large, one can neglect fires so that, roughly, each site is occupied with probability $1 - e^{-t}$ and, thus, $C^{\lambda}_t(x) \simeq [\![x - X, x + Y]\!]$, where X, Y are geometric random variables with parameter $1 - e^{-t}$. As a consequence, $\#(C^{\lambda}_t(x)) \simeq e^t$ for t not too large. On the other hand, the cluster $C^{\lambda}_t(x)$ burns at rate $\lambda\#(C^{\lambda}_t(x))$ (at time t) so that we decide to accelerate time by a factor $\log(1/\lambda)$. In this way, $\lambda\#(C^{\lambda}_{\log(1/\lambda)}(x)) \simeq 1$.

Space scale. We now rescale space in such a way that during a time interval of order $\log(1/\lambda)$, something like one fire starts per unit of (space) length. Since fires occur at rate λ, our space scale has to be of order $\lambda\log(1/\lambda)$: this means that we will identify $[\![0, \lfloor 1/(\lambda\log(1/\lambda))\rfloor]\!] \subset \mathbb{Z}$ with $[0, 1] \subset \mathbb{R}$.

Rescaled clusters. We thus set, for $\lambda \in (0, 1)$, $t \geq 0$ and $x \in \mathbb{R}$, recalling Section 1.3,

$$D_t^\lambda(x) := \lambda\log(1/\lambda) C^\lambda_{t\log(1/\lambda)}\big(\lfloor x/\big(\lambda\log(1/\lambda)\big)\rfloor\big) \subset \mathbb{R}. \tag{1}$$

However, this creates an immediate difficulty: recalling that $\#(C_t^\lambda(x)) \simeq e^t$ for t not too large, we see that for each site x, $|D_t^\lambda(x)| \simeq \lambda\log(1/\lambda)e^{t\log(1/\lambda)} = \lambda^{1-t}\log(1/\lambda)$, of which the limit as $\lambda \to 0$ is 0 for $t < 1$ and $+\infty$ for $t \geq 1$.

For $t \geq 1$, there might be fires in effect and one hopes that this will make the possible limit of $|D_t^\lambda(x)|$ finite. However, fires can only reduce the size of clusters so that for $t < 1$, the limit of $|D_t^\lambda(x)|$ will really be 0. Thus, for a possible limit $|D(x)|$ of $|D^\lambda(x)|$, we should observe some paths of the following form: $|D_t(x)| = 0$ for $t < 1$, $|D_t(x)| > 0$ for some times $t \in (1, \tau)$, after which it might be killed by a fire and thus come back to 0, at which time it remains at 0 for a time interval of length 1, and so on.

This cannot be a Markov process because $|D(x)|$ always remains at 0 during a time interval of length exactly 1. We thus need to keep track of more information in order to control when it exits from 0.

Degree of smallness. As was stated previously, we hope that for $t < 1$, $|D_t^\lambda(x)| \simeq \lambda^{1-t}\log(1/\lambda) \simeq \lambda^{1-t}$. Thus, we will try to keep in mind the degree of smallness. We will define, for $\lambda \in (0, 1)$, $x \in \mathbb{R}$ and $t > 0$,

$$Z_t^\lambda(x) := \frac{\log[1 + \#(C^\lambda_{t\log(1/\lambda)}(\lfloor x/(\lambda\log(1/\lambda))\rfloor))]}{\log(1/\lambda)} \in [0, \infty). \tag{2}$$

Final description. We will study the λ-FFP via $(D_t^\lambda(x), Z_t^\lambda(x))_{x\in\mathbb{R}, t\geq 0}$. The main idea is that for $\lambda > 0$ very small:

(i) if $Z_t^\lambda(x) = z \in (0, 1)$, then $|D_t^\lambda(x)| \simeq 0$ and the (rescaled) cluster containing x is microscopic, but we control its smallness, in the sense that $|D_t^\lambda(x)| \simeq \lambda^{1-z}$ (in a very unprecise way);

(ii) if $Z_t^\lambda(x) = 1$ [we will show below that $Z_t^\lambda(x)$ will never exceed 1 in the limit $\lambda \to 0$], then the (rescaled) cluster containing x is automatically macroscopic and has a length equal to $|D_t^\lambda(x)| \in (0, \infty)$.

1.5. *The limit process.* We now describe the limit process. We want this process to be Markov and this forces us to add some variables.

We consider a Poisson measure $M(dt, dx)$ on $[0, \infty) \times \mathbb{R}$ with intensity measure $dt\,dx$. Again, we define $\mathcal{F}_t^M = \sigma(M(A), A \in \mathcal{B}([0, t] \times \mathbb{R}))$. We also define $\mathcal{I} := \{[a, b], a \leq b\}$, the set of all closed finite intervals of $\mathbb{R}$.

DEFINITION 2. A $(\mathcal{F}_t^M)_{t\geq 0}$-adapted process $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ with values in $\mathbb{R}_+ \times \mathcal{I} \times \mathbb{R}_+$ is a limit forest fire process (LFFP) if a.s., for all $t \geq 0$ and all $x \in \mathbb{R}$,

$$
(3) \qquad \begin{cases} Z_t(x) = \displaystyle\int_0^t \mathbf{1}_{\{Z_s(x)<1\}}\,ds - \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{Z_{s-}(x)=1, y\in D_{s-}(x)\}} M(ds, dy), \\ H_t(x) = \displaystyle\int_0^t Z_{s-}(x)\mathbf{1}_{\{Z_{s-}(x)<1\}} M(ds \times \{x\}) - \int_0^t \mathbf{1}_{\{H_s(x)>0\}}\,ds, \end{cases}
$$

where $D_t(x) = [L_t(x), R_t(x)]$ with

$$
L_t(x) = \sup\{y \leq x;\ Z_t(y) < 1 \text{ or } H_t(y) > 0\},
$$
$$
R_t(x) = \inf\{y \geq x;\ Z_t(y) < 1 \text{ or } H_t(y) > 0\}.
$$

A typical path of the finite box version of the LFFP (see Section 2) is drawn and commented on in Figure 2 and a simulation algorithm is explained in the proof of Proposition 8.

Let us explain the dynamics of this process. We consider $T > 0$ fixed and set $\mathcal{B}_T = \{x \in \mathbb{R}; M([0, T] \times \{x\}) > 0\}$. For each $t \geq 0$ and $x \in \mathbb{R}$, $D_t(x)$ stands for the occupied cluster containing x. We call this cluster *microscopic* if $D_t(x) = \{x\}$. We also have $D_t(x) = D_t(y)$ for all y in the interior of $D_t(x)$: if $D_t(x) = [a, b]$, then $D_t(y) = [a, b]$ for all $y \in (a, b)$.

1. *Initial condition.* We have $Z_0(x) = H_0(x) = 0$ and $D_0(x) = \{x\}$ for all $x \in \mathbb{R}$.

2. *Occupation of vacant zones.* Here, we consider $x \in \mathbb{R} \setminus \mathcal{B}_T$. We then have $H_t(x) = 0$ for all $t \in [0, T]$. If $Z_t(x) < 1$, then $D_t(x) = \{x\}$ and $Z_t(x)$ stands for the *degree of smallness* of the cluster containing x. Then $Z_t(x)$ grows linearly until it reaches 1, as described by the first term on the right-hand side of the first equation in (3). If $Z_t(x) = 1$, then the cluster containing x is macroscopic and is described by $D_t(x)$.

3. *Microscopic fires.* Here, we assume that $x \in \mathcal{B}_T$ and that the corresponding mark of M happens at some time t where $z := Z_{t-}(x) < 1$. In such a case, the cluster containing x is microscopic. We then set $H_t(x) = Z_{t-}(x)$, as described by the first term on the right-hand side of the second equation of (3), and we leave the value of $Z_t(x)$ unchanged. We then let $H_s(x)$ decrease linearly until it reaches 0; see the second term on the right-hand side of the second equation in (3). At all times where $H_s(x) > 0$, that is, during $[t, t + z)$, the site x acts like a barrier (see point 5 below).

4. *Macroscopic fires.* Here, we assume that $x \in \mathcal{B}_T$ and that the corresponding mark of M happens at some time t where $Z_{t-}(x) = 1$. This means that the cluster containing x is macroscopic and thus this mark destroys the whole component $D_{t-}(x)$. That is, for all $y \in D_{t-}(x)$, we set $D_t(y) = \{y\}$, $Z_t(y) = 0$. This is described by the second term on the right-hand side of the first equation in (3).

5. *Clusters.* Finally, the definition of the clusters $(D_t(x))_{x\in\mathbb{R}}$ becomes more clear: these clusters are delimited by zones with microscopic sites [i.e., $Z_t(y) < 1$] or by sites where there has (recently) been a microscopic fire [i.e., $H_t(y) > 0$].

1.6. *Main results*. First, we must note that it is not entirely clear that the limit process exists.

THEOREM 3. *For any Poisson measure M, there a.s. exists a unique LFFP; recall Definition 2. Furthermore, it can be constructed graphically and thus its restriction to any finite box $[0,T]\times[-n,n]$ can be perfectly simulated.*

To describe the convergence of the λ-FFP to the LFFP, we will need some more notation. Let $\mathbb{D}([0,T],E)$ denote the space of right-continuous and left-limited functions from the interval $[0,T]$ to a topological space E.

NOTATION 4. (i) For two intervals $[a,b]$ and $[c,d]$, we set $\delta([a,b],[c,d]) = |a-c|+|b-d|$. We also set, by convention, $\delta([a,b],\varnothing)=|b-a|$.

(ii) For $(x,I),(y,J)$ in $\mathbb{D}([0,T],\mathbb{R}\times\mathcal{I}\cup\{\varnothing\})$, let

$$\delta_T((x,I),(y,J)) = \sup_{[0,T]}|x(t)-y(t)| + \int_0^T \delta(I(t),J(t))\,dt.$$

We are finally in a position to state our main result.

THEOREM 5. *Consider, for all $\lambda>0$, the processes $(Z_t^\lambda(x), D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$ associated with the λ-FFP; see Definition 1 and (1), (2). Let $(Z_t(x), D_t(x), H_t(x))_{t\geq 0,x\in\mathbb{R}}$ be an LFFP, as in Definition 2.*

(a) *For any $T>0$ and any finite subset $\{x_1,\dots,x_p\}\subset\mathbb{R}$, $(Z_t^\lambda(x_i), D_t^\lambda(x_i))_{t\in[0,T],i=1,\dots,p}$ goes in law to $(Z_t(x_i),D_t(x_i))_{t\in[0,T],i=1,\dots,p}$ in $\mathbb{D}([0,T],\mathbb{R}\times\mathcal{I})^p$ as λ tends to 0. Here, $\mathbb{D}([0,\infty),\mathbb{R}\times\mathcal{I})$ is endowed with the distance δ_T; see Notation 4.*

(b) *For any finite subset $\{(t_1,x_1),\dots,(t_p,x_p)\}\subset\mathbb{R}_+\times\mathbb{R}$, $(Z_{t_i}^\lambda(x_i), D_{t_i}^\lambda(x_i))_{i=1,\dots,p}$ goes in law to $(Z_{t_i}(x_i),D_{t_i}(x_i))_{i=1,\dots,p}$ in $(\mathbb{R}\times\mathcal{I})^p$.*

Observe that the process H does not appear in the limit since for each $x\in\mathbb{R}$, a.s., for all $t\geq 0$, $H_t(x)=0$. [Of course, it is not the case that a.s., for all $x\in\mathbb{R}$, all $t\geq 0$, $H_t(x)=0$.] We obtain the convergence of D^λ to D only when integrating in time. We cannot hope for a Skorokhod convergence since the limit process $D(x)$ jumps instantaneously from $\{x\}$ to some interval with positive length, while $D^\lambda(x)$ needs many small jumps (in a very short time interval) to become macroscopic.

As a matter of fact, we will obtain a convergence in probability, using a coupling argument. Essentially, we will consider a Poisson measure $M(dt,dx)$, as in Section 1.5, and set, for $\lambda\in(0,1)$ and $i\in\mathbb{Z}$,

$$M_t^\lambda(i) = M\big([0,t/\log(1/\lambda)]\times\big[i\lambda\log(1/\lambda),(i+1)\lambda\log(1/\lambda)\big)\big).$$

Then $(M_t^\lambda(i))_{t\geq 0,i\in\mathbb{Z}}$ is an i.i.d. family of Poisson processes with rate λ.

The i.i.d. family of Poisson processes $(N_t(i))_{t\geq 0,i\in\mathbb{Z}}$ with rate 1 can be chosen arbitrarily, but we will decide to choose the same family for all values of $\lambda\in(0,1)$.

1.7. *Heuristic arguments.* We now explain roughly the reasons why Theorem 5 holds. We consider a λ-FFP $(\eta_t^\lambda)_{t\geq 0}$ and the associated process $(Z_t^\lambda(x), D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$. We assume below that λ is very small.

0. *Scales.* With our scales, there are $1/(\lambda\log(1/\lambda))$ sites per unit of length. Approximately one fire starts per unit of time per unit of length. A vacant site becomes occupied at rate $\log(1/\lambda)$.

1. *Initial condition.* We have, for all $x\in\mathbb{R}$, $(Z_0^\lambda(x), D_0^\lambda(x)) = (0,\varnothing)\simeq(0,\{x\})$.

2. *Occupation of vacant zones.* Assume that a zone $[a,b]$ (which corresponds to the zone $[\![\lfloor a/(\lambda\log(1/\lambda))\rfloor, b/(\lambda\log(1/\lambda))\rfloor]\!]$ before rescaling) becomes completely vacant at some time t [or $t\log(1/\lambda)$ before rescaling] because it has been destroyed by a fire.

(i) For $s\in[0,1)$, and if no fire starts on $[a,b]$ during $[t,t+s]$, we have $D_{t+s}^\lambda(x)\simeq[x\pm\lambda^{1-s}]$ and thus $Z_{t+s}^\lambda(x)\simeq s$ for all $x\in[a,b]$.

Indeed, $D_{t+s}^\lambda(x)\simeq[x-\lambda\log(1/\lambda)X, x+\lambda\log(1/\lambda)Y]$, where X and Y are geometric random variables with parameter $1-e^{-s\log(1/\lambda)}=1-\lambda^s$. This comes from the fact that each site of $[a,b]$ is vacant at time t and becomes occupied at rate $\log(1/\lambda)$.

(ii) If no fire starts on $[a,b]$ during $[t,t+1]$, then $Z_{t+1}^\lambda(x)\simeq 1$ and all the sites in $[a,b]$ are occupied (with very high probability) at time $t+1$. Indeed, we have $(b-a)/(\lambda\log(1/\lambda))$ sites and each of them is occupied at time $t+1$ with probability $1-e^{-\log(1/\lambda)}=1-\lambda$ so that all of them are occupied with probability $(1-\lambda)^{(b-a)/(\lambda\log(1/\lambda))}\simeq e^{-(b-a)/\log(1/\lambda)}$, which goes to 1 as $\lambda\to 0$.

3. *Microscopic fires.* Assume that a fire starts at some location x (i.e., $\lfloor x/(\lambda\log(1/\lambda))\rfloor$ before rescaling) at some time t [or $t\log(1/\lambda)$ before rescaling] with $Z_{t-}^\lambda(x)=z\in(0,1)$. The possible clusters on the left and right of x cannot then be connected during (approximately) $[t,t+z]$, but they can be connected after (approximately) $t+z$. In other words, x acts like a barrier during $[t,t+z]$.

Indeed, the fire makes vacant a zone A of approximate length λ^{1-z} around x, which thus contains approximately $\lambda^{1-z}/(\lambda\log(1/\lambda))\simeq\lambda^{-z}$ sites. The probability that a fire starts again in A after t is very small. Thus, using the same computation as in point 2(ii), we observe that $\mathbb{P}[A \text{ is completely occupied at time } t+s]\simeq(1-\lambda^s)^{\lambda^{-z}}\simeq e^{-\lambda^{s-z}}$. When $\lambda\to 0$, this quantity tends to 0 if $s<z$ and to 1 if $s>z$.

4. *Macroscopic fires.* Assume, now, that a fire starts at some place x (i.e., $\lfloor x/(\lambda\log(1/\lambda))\rfloor$ before rescaling) at some time t [or $t\log(1/\lambda)$ before rescaling] and that $Z_t^\lambda(x)\simeq 1$. Thus, $D_t^\lambda(x)$ is macroscopic (i.e., its length is of order 1 in our scales). This will thus make vacant the zone $D_t^\lambda(x)$. Such a (macroscopic) zone needs a time of order 1 to be completely occupied, as explained in point 2(ii).

5. *Clusters.* For $t\geq 0$, $x\in\mathbb{R}$, the cluster $D_t^\lambda(x)$ resembles $[x\pm\lambda^{1-z}]\simeq\{x\}$ if $Z_t^\lambda(x)=z\in(0,1)$. We then say that x is microscopic. Now, macroscopic clusters are delimited either by microscopic zones or by sites where there has been a microscopic fire (see point 3).

Comparing the arguments above to the rough description of the LFFP (see Section 1.5), our hope is that the λ-FFP resembles the LFFP for $\lambda>0$ very small.

1.8. *Decay of correlations*. A byproduct of our result is an estimate on the decay of correlations in the LFFP for finite times. We refer to Proposition 11 below for a precise statement. The main idea is that for all $T > 0$, there are constants $C_T > 0$, $\alpha_T > 0$ such that for all $\lambda \in (0, 1)$ and all $A > 0$, the values of the λ-FFP inside $[-A/(\lambda \log(1/\lambda)), A/(\lambda \log(1/\lambda))]$ are independent of the values outside $[-2A/(\lambda \log(1/\lambda)), 2A/(\lambda \log(1/\lambda))]$ during the time interval $[0, T \log(1/\lambda)]$, up to a probability smaller that $C_T e^{-\alpha_T A}$. In other words, for times of order $\log(1/\lambda)$, the range of correlations is at most of order $1/(\lambda \log(1/\lambda))$.

1.9. *Cluster size distribution*. Finally, we give results on the cluster size distribution, which are to be compared with [4, 16]; see Section 1.2 above.

COROLLARY 6. *For each $\lambda > 0$, consider a λ-FFP process $(\eta_t^\lambda)_{t\geq 0}$.*

(i) *For some $0 < c < C$, all $t \geq 5/2$ and all $0 \leq a < b < 1$,*

$$c(b-a) \leq \lim_{\lambda \to 0} \mathbb{P}\big(\#\big(C^\lambda_{t \log(1/\lambda)}(0)\big) \in [\lambda^{-a}, \lambda^{-b}]\big) \leq C(b-a).$$

(ii) *For some $0 < c < C$, some $0 < \kappa_1 < \kappa_2$, all $t \geq 3/2$ and all $B > 0$,*

$$ce^{-\kappa_2 B} \leq \lim_{\lambda \to 0} \mathbb{P}\big(\#\big(C^\lambda_{t \log(1/\lambda)}(0)\big) \geq B/\big(\lambda \log(1/\lambda)\big)\big) \leq Ce^{-\kappa_1 B}.$$

Point (i) says, roughly, that for t large enough (say at equilibrium) and for $x << 1/\lambda$ [say for $x \leq (1/\lambda)^{1-\varepsilon}$], choosing $a = \log(x)/\log(1/\lambda)$ and $b = \log(x+1)/\log(1/\lambda)$, we have

$$\mathbb{P}\big(\#(C^\lambda(0)) = x\big) \simeq \mathbb{P}\big(\#(C^\lambda(0)) \in [x, x+1]\big) \simeq \mathbb{P}\big(\#(C^\lambda(0)) \in [\lambda^{-a}, \lambda^{-b}]\big)$$
$$\simeq (b-a) \simeq \frac{1}{x \log(1/\lambda)}.$$

It is thus a very weak form of the result of [4], but it holds for a much wider class of x: here, we allow $x \leq 1/\lambda^{1-\varepsilon}$, while $x \leq 1/\lambda^{1/3}$ was imposed in [4]. Another advantage of our result is that we can prove that the limit exists in (i).

Point (ii) roughly describes the cluster size distribution of macroscopic components, that is, of components of which the size is of order $1/(\lambda \log(1/\lambda))$. Here, again, rough computations show that for $x > \varepsilon/(\lambda \log(1/\lambda))$ and for t large enough (say at equilibrium),

$$\mathbb{P}\big(\#(C^\lambda(0)) = x\big) \simeq \lambda \log(1/\lambda) e^{-\kappa x \lambda \log(1/\lambda)}.$$

Thus, there is clearly a phase transition near the *critical size* $1/(\lambda \log(1/\lambda))$; see Figure 1 for an illustration.

1.10. *Organization of the paper.* The paper is organized as follows. In Section 2, we give the proof of Theorem 3. In Section 3, we show that, in some sense, the λ-FFP can be localized in a finite box, uniformly for $\lambda > 0$. Section 4 is devoted to the proof of Theorem 5. Finally, we prove Corollary 6 in Section 5.

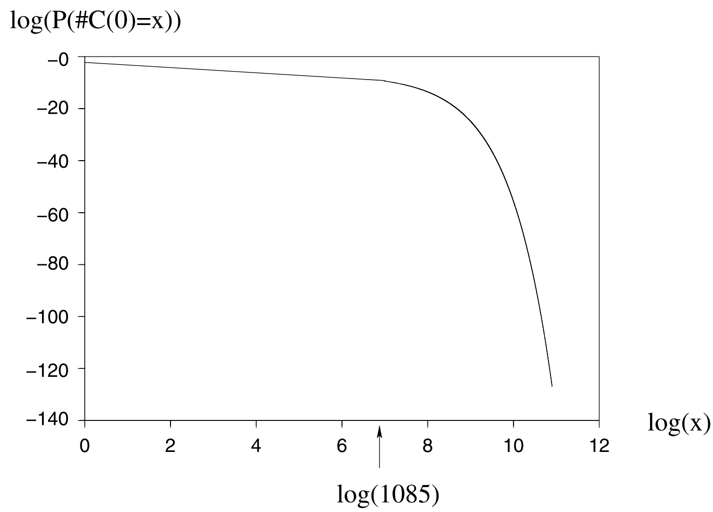

FIG. 1. *Shape of the cluster size distribution. Here*, $\lambda = 0.0001$ *and the critical size is thus* $1/(\lambda \log(1/\lambda)) \simeq 1085$. *We have drawn the approximate value* (*computed roughly just after Corollary* 6) *of* $\log(\mathbb{P}(\#(C^\lambda(0)) = x))$ *as a function of* $\log(x)$ *for* $x = 1, \dots, 54{,}250$. *We have made the curve continuous around* $x = 1085$ (*without justification*). *The curve is linear for* $x = 1, \dots, 1085$ *and nonlinear for* $x \geq 1085$.

2. Existence and uniqueness of the limit process.

The goal of this section is to show that the LFFP is well defined, unique and can be obtained from a graphical construction. First, we show that when working on a finite space interval, the LFPP is somewhat discrete.

We consider a Poisson measure $M(dt, dx)$ on $[0, \infty) \times \mathbb{R}$ with intensity measure $dt\,dx$. We define $\mathcal{F}_t^{M,A} = \sigma(M(B), B \in \mathcal{B}([0, t] \times [-A, A]))$.

DEFINITION 7. A $(\mathcal{F}_t^{M,A})_{t \geq 0}$-adapted process

$$(Z_t^A(x), D_t^A(x), H_t^A(x))_{t \geq 0, x \in [-A,A]}$$

with values in $\mathbb{R}_+ \times \mathcal{I} \times \mathbb{R}_+$ is called an A-LFFP if a.s., for all $t \geq 0$ and all $x \in [-A, A]$,

$$\begin{cases} Z_t^A(x) = \int_0^t \mathbf{1}_{\{Z_s^A(x) < 1\}}\,ds - \int_0^t \int_{[-A,A]} \mathbf{1}_{\{Z_{s-}^A(x) = 1, y \in D_{s-}^A(x)\}} M(ds, dy), \\ H_t^A(x) = \int_0^t Z_{s-}^A(x) \mathbf{1}_{\{Z_{s-}^A(x) < 1\}} M(ds \times \{x\}) - \int_0^t \mathbf{1}_{\{H_s^A(x) > 0\}}\,ds, \end{cases}$$

where $D_t^A(x) = [L_t^A(x), R_t^A(x)]$ with

$$\begin{cases} L_t^A(x) = (-A) \vee \sup\{y \in [-A, x]; Z_t^A(y) < 1 \text{ or } H_t^A(y) > 0\}, \\ R_t^A(x) = A \wedge \inf\{y \in [x, A]; Z_t^A(y) < 1 \text{ or } H_t^A(y) > 0\}. \end{cases} \tag{4}$$

A typical path of $(Z_t^A(x), D_t^A(x), H_t^A(x))_{t \geq 0, x \in [-A,A]}$ is drawn in Figure 2.

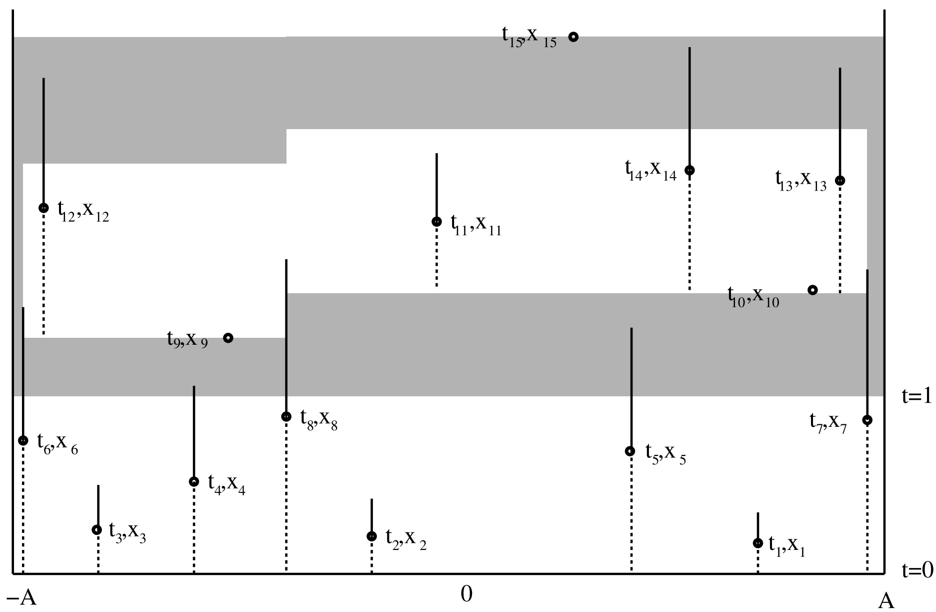

FIG. 2. *Limit forest fire process in a finite box. The filled zones represent zones in which* $Z_t^A(x) = 1$ *and* $H_t^A(x) = 0$, *that is*, *macroscopic clusters. The plain vertical segments represent the sites where* $H_t^A(x) > 0$. *In the rest of the space*, *we always have* $Z_t^A(x) < 1$. *Until time* 1, *all of the particles are microscopic. The first eight marks of the Poisson measure fall in that zone. As a consequence*, *at each of these marks*, *the process* H^A *starts. Their lifetime is equal to the instant where they have started* (*e.g.*, *the segment above* t_1, x_1 *ends at time* $2t_1$). *At time* 1, *all of the clusters where there has been no mark become macroscopic and merge together. However, this is limited by vertical segments. Here*, *at time* 1, *we have the clusters* $[-A, x_6]$, $[x_6, x_4]$, $[x_4, x_8]$, $[x_8, x_5]$, $[x_5, x_7]$ *and* $[x_7, A]$. *The segment above* (t_4, x_4) *ends at time* $2t_4$ *and thus*, *at this time*, *the clusters* $[x_6, x_4]$ *and* $[x_4, x_8]$ *merge into* $[x_6, x_8]$. *The ninth mark falls in the* (*macroscopic*) *zone* $[x_6, x_8]$ *and thus destroys it immediately. This zone* $[x_6, x_8]$ *will become macroscopic again only at time* $t_9 + 1$. *A process* H^A *then starts at* x_{12} *at time* t_{12}. *Since* $Z_{t_{12}-}^A(x_{12}) = t_{12} - t_9$ [*because* $Z_{t_9}^A(x_{12})$ *has been set to* 0], *the segment above* (t_{12}, x_{12}) *will end at time* $2t_{12} - t_9$. *On the other hand*, *the segment* $[x_8, x_7]$ *has been destroyed at time* t_{10} *and will thus remain microscopic until* $t_{10} + 1$. *As a consequence*, *the only macroscopic clusters at time* $t_9 + 1$ *are* $[-A, x_{12}]$, $[x_{12}, x_8]$ *and* $[x_7, A]$. *The zone* $[x_8, x_7]$ *then becomes macroscopic* (*but there have been marks at* x_{13}, x_{14}) *so that at time* $t_{10} + 1$, *we get the macroscopic clusters* $[-A, x_{12}]$, $[x_{12}, x_{14}]$, $[x_{14}, x_{13}]$ *and* $[x_{13}, A]$. *These clusters merge by pairs*, *at times* $2t_{12} - t_9$, $2t_{13} - t_{10}$ *and* $2t_{14} - t_{10}$, *so that we have a unique cluster* $[-A, A]$ *just before time* t_{15}, *where a mark falls and destroys the whole cluster* $[-A, A]$.

With this realization, *we have* $0 \in (x_{11}, x_{15})$ *and*, *thus*, $Z_t^A(0) = t$ *for* $t \in [0, 1]$, *then* $Z_t^A(0) = 1$ *for* $t \in [1, t_{10})$, *then* $Z_t^A(0) = t - t_{10}$ *for* $t \in [t_{10}, t_{10} + 1)$, *then* $Z_t^A(0) = 1$ *for* $t \in [t_{10} + 1, t_{15})$, *etc. We also see that* $D_t^A(0) = \{0\}$ *for* $t \in [0, 1)$, $D_t^A(0) = [x_8, x_5]$ *for* $t \in [1, 2t_5)$, $D_t^A(0) = [x_8, x_7]$ *for* $t \in [2t_5, t_{10})$, $D_t^A(0) = \{0\}$ *for* $t \in [t_{10}, t_{10} + 1)$, $D_t^A(0) = [x_{12}, x_{14}]$ *for* $t \in [t_{10} + 1, 2t_{12} - t_9)$, $D_t^A(0) = [-A, x_{14}]$ *for* $t \in [2t_{12} - t_9, 2t_{14} - t_{10})$, *etc. Of course*, $H_t^A(0) = 0$ *for all* $t \geq 0$, *but*, *for example*, $H_t^A(x_{11}) = 0$ *for* $t \in [0, t_{11})$, $H_t^A(x_{11}) = 2t_{11} - t_{10} - t$ *for* $t \in [t_{11}, 2t_{11} - t_{10})$ *and then* $H_t^A(x_{11}) = 0$ *for* $t \in [2t_{11} - t_{10}, \infty)$.

Although the following proposition is almost obvious, its proof shows the construction of the A-LFFP in an algorithmic way.

PROPOSITION 8. *Consider a Poisson measure $M(dt, dx)$ on $[0, \infty) \times \mathbb{R}$ with intensity measure $dt\,dx$. For any $A > 0$, there a.s. exists a unique A-LFFP which can be perfectly simulated.*

PROOF. We omit the superscript A in this proof. We consider the marks $(T_i, X_i)_{i\geq 1}$ of $M|_{[0,\infty)\times[-A,A]}$, where $0 < T_1 < T_2 < \cdots$. We set $T_0 = 0$ for convenience. We describe the construction via an algorithm, which also shows uniqueness, in the sense that there is no choice in the construction.

Step 0. First, we set $Z_0(x) = H_0(x) = 0$ and $D_0(x) = \{x\}$ for all $x \in [-A, A]$.

Step $n + 1$. Assume that the process has been built until T_n for some $n \geq 0$, that is, we know the values of $(Z_t(x), D_t(x), H_t(x))_{t\in[0,T_n],x\in[-A,A]}$.

We build $(Z_t(x), D_t(x), H_t(x))_{t\in(T_n,T_{n+1}),x\in[-A,A]}$ in the following way: for $t \in (T_n, T_{n+1})$ and $x \in [-A, A]$, we set $Z_t(x) = \min(1, Z_{T_n}(x) + t - T_n)$, $H_t(x) = \max(0, H_{T_n}(x) - (t - T_n))$ and define $D_t(x) = [L_t(x), R_t(x)]$, as in (4).

Next, we build $(Z_{T_{n+1}}(x), D_{T_{n+1}}(x), H_{T_{n+1}}(x))_{x\in[-A,A]}$.

(i) If $Z_{T_{n+1}-}(X_{n+1}) = 1$, then we set $H_{T_{n+1}}(x) = H_{T_{n+1}-}(x)$ for all $x \in [-A, A]$ and consider $[a, b] := D_{T_{n+1}-}(X_{n+1})$. Set $Z_{T_{n+1}}(x) = 0$ for all $x \in (a, b)$ and $Z_{T_{n+1}}(x) = Z_{T_{n+1}-}(x)$ for all $x \in [-A, A] \setminus [a, b]$. Finally, set: $Z_{T_{n+1}}(a) = 0$ if $Z_{T_{n+1}-}(a) = 1$; $Z_{T_{n+1}}(a) = Z_{T_{n+1}-}(a)$ if $Z_{T_{n+1}-}(a) < 1$; $Z_{T_{n+1}}(b) = 0$ if $Z_{T_{n+1}-}(b) = 1$; $Z_{T_{n+1}}(b) = Z_{T_{n+1}-}(b)$ if $Z_{T_{n+1}-}(b) < 1$.

(ii) If $Z_{T_{n+1}-}(X_{n+1}) < 1$, then we set $H_{T_{n+1}}(X_{n+1}) = Z_{T_{n+1}-}(X_{n+1})$, $Z_{T_{n+1}}(X_{n+1}) = Z_{T_{n+1}-}(X_{n+1})$ and $(Z_{T_{n+1}}(x), H_{T_{n+1}}(x)) = (Z_{T_{n+1}-}(x), H_{T_{n+1}-}(x))$ for all $x \in [-A, A] \setminus \{X_{n+1}\}$.

(iii) Using the values of $(Z_{T_{n+1}}(x), H_{T_{n+1}}(x))_{x\in[-A,A]}$, we finally compute the values of $(D_{T_{n+1}}(x))_{x\in[-A,A]}$. $\square$

In case (i) above, we explained precisely what is done at the boundary of burning macroscopic components. This is not so important: it does not affect the uniqueness statement, but corresponds to using a slightly different definition of the process; we could have made other choices for this.

We now prove a refined version of Theorem 3.

PROPOSITION 9. *Consider a Poisson measure $M(dt, dx)$ on $[0, \infty) \times \mathbb{R}$ with intensity measure $dt\,dx$. For $A > 0$, consider the A-LFFP $(Z^A_t(x), D^A_t(x), H^A_t(x))_{t\geq 0,x\in[-A,A]}$ constructed in Proposition 8 (using M).*

There a.s. exists a unique LFFP $(Z_t(x), D_t(x), H_t(x))_{t\geq 0,x\in\mathbb{R}}$ (corresponding to M) and, furthermore, it is such that for all $T > 0$, there are constants $\alpha_T > 0$

and $C_T > 0$ *such that for all* $A \geq 2$,

$$\mathbb{P}\big[(Z_t(x), D_t(x), H_t(x))_{t\in[0,T],x\in[-A/2,A/2]} = (Z_t^A(x), D_t^A(x), H_t^A(x))_{t\in[0,T],x\in[-A/2,A/2]}\big] \geq 1 - C_T e^{-\alpha_T A}. \tag{5}$$

PROOF. We divide the proof into several steps. We fix $T > 0$ and work on $[0, T]$.

Step 1. For $a \in \mathbb{Z}$, we define the event Ω_a in the following way (see Figure 3 for an illustration). The Poisson measure M has exactly $3n$ marks in $[0, T] \times [a, a+1]$ for some $n \geq 1$ and it is possible to call them $(T_k, X_k)_{k=1,\dots,n}$, $(\tilde{T}_k, \tilde{X}_k)_{k=1,\dots,n}$ and $(S_k, Y_k)_{k=1,\dots,n}$ in such a way that we have the following properties for all $k = 1, \dots, n$ (we set $T_0 = \tilde{T}_0 = S_0 = 0$ and $X_0 = a$, $\tilde{X}_0 = a + 1$ for convenience):

(i) T_k and $\tilde{T}_k$ belong to $(S_{k-1} + 1/2, S_{k-1} + 1)$ and $X_{k-1} < X_k < \tilde{X}_k < \tilde{X}_{k-1}$;

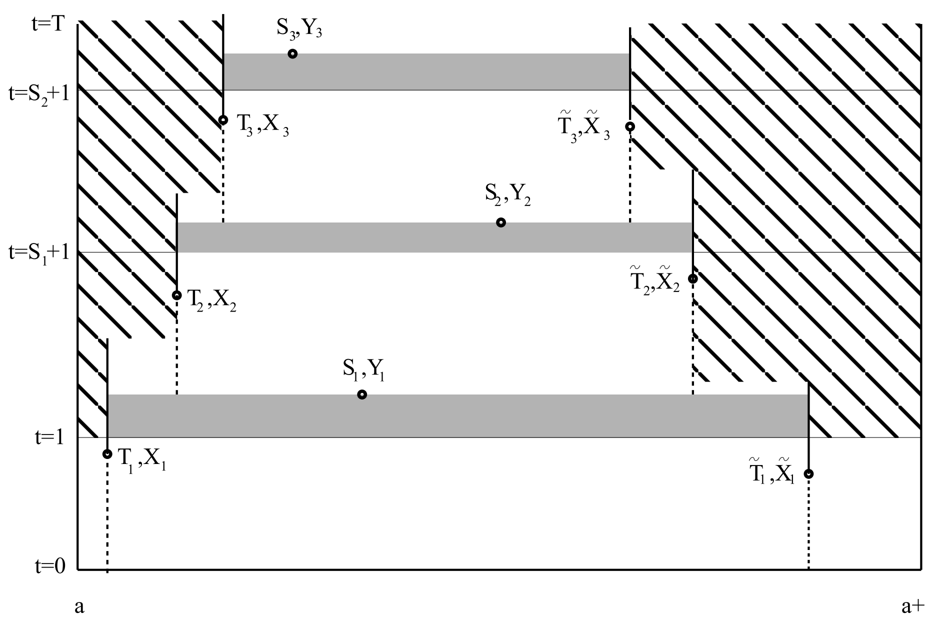

FIG. 3. *The event* Ω_a (*proof of Theorem* 3). *In hatched zones, we cannot state the values of the LFFP because one would need to know what happens outside* $[a, a+1]$. *Microscopic fires start at* (T_1, X_1) *and* $(\tilde{T}_1, \tilde{X}_1)$. *Hence, at time* S_1, *the connected component* $[X_1, \tilde{X}_1]$ *is macroscopic because* $S_1 \geq 1$ *and because during* $[1, S_1)$, *this component has not been subject to fires starting outside* $[a, a+1]$: *it is protected by* X_1 *and* $\tilde{X}_1$ *until time* $2\min(T_1, \tilde{T}_1) \geq S_1$. *As a consequence, the component* $[X_1, \tilde{X}_1]$ *is entirely killed by* (S_1, Y_1). *We then iterate the arguments until we reach the final time* T. *With such a configuration, there are always microscopic sites in* $[a, a+1]$ *during* $[0, T]$. *Indeed, during* $[0, 1)$, *all of the sites are microscopic, during* $[1, S_1)$, *the sites* X_1 *and* $\tilde{X}_1$ *are microscopic, during* $[S_1, S_1 + 1)$, *all the sites in* $[X_1, \tilde{X}_1]$ *are microscopic, etc.*

(ii) $S_k \in (S_{k-1}+1, S_{k-1}+2(T_k \wedge \tilde{T}_k - S_{k-1}))$ and $Y_k \in (X_k, \tilde{X}_k)$;
(iii) $S_n > T-1$.

Step 2. We next observe that if the LFFP exists, then, necessarily,

$$\Omega_a \subset \{\forall t \in [0,T], \exists x \in (a,a+1), H_t(x) > 0 \text{ or } Z_t(x) < 1\}.$$

Indeed, $Z_t(x) = t < 1$ for all $t \in [0,1)$ and $x \in \mathbb{R}$. Then $H_{T_1}(X_1) = Z_{T_1}(X_1) = T_1$, whence $H_t(X_1) > 0$ on $[T_1, 2T_1]$ and $H_t(\tilde{X}_1) > 0$ on $[\tilde{T}_1, 2\tilde{T}_1]$. As a consequence, we know that for all $x \in (X_1, \tilde{X}_1)$ and $t \in [1, S_1)$, we have $D_t(x) = [X_1, \tilde{X}_1]$. Since, now, $1 < S_1 < 2(T_1 \wedge \tilde{T}_1)$ and since $Y_1 \in (X_1, \tilde{X}_1)$, we deduce that $Z_{S_1}(x) = 0$ for all $x \in (X_1, \tilde{X}_1)$ and, as a consequence, $Z_t(x) = t - S_1 < 1$ for all $t \in [S_1, S_1+1)$. However, we now have $H_t(X_2) > 0$ on $[T_2, T_2 + (T_2 - S_1))$ and $H_t(\tilde{X}_2) > 0$ on $[\tilde{T}_2, \tilde{T}_2 + (\tilde{T}_2 - S_1))$. As a consequence, we know for all $x \in (X_2, \tilde{X}_2)$ and $t \in [S_1+1, S_2)$ that $D_t(x) = [X_2, \tilde{X}_2]$. Since, now, $S_1 + 1 < S_2 < S_1 + 2(T_1 \wedge \tilde{T}_1 - S_1)$ and $Y_2 \in (X_2, \tilde{X}_2)$, we deduce that $Z_{S_2}(x) = 0$ for all $x \in (X_2, \tilde{X}_2)$ and thus $Z_t(x) = t - S_2 < 1$ for all $t \in [S_2, S_2+1)$, etc.

Step 3. We deduce that for all $a \in \mathbb{Z}$, conditionally on Ω_a, clusters to the left of a are never connected (during $[0,T]$) to clusters to the right of $a+1$. Thus, on Ω_a, fires starting to the left of a do not affect the zone $[a+1, \infty)$ and fires starting to the right of $a+1$ do not affect the zone $(-\infty, a]$. Since, further, Ω_a concerns the Poisson measure M only in $[0,T] \times [a, a+1]$, we deduce that on Ω_a, the processes $(Z_t(x), D_t(x), H_t(x))_{t \geq 0, x \in [a+1,\infty)}$ and $(Z_t(x), D_t(x), H_t(x))_{t \geq 0, x \in (-\infty, a]}$ can be constructed separately.

Step 4. Clearly, $q_T = \mathbb{P}[\Omega_a]$ does not depend on a, by translation invariance (of the law of M), and obviously $q_T > 0$. Thus, a.s. there are infinitely many $a \in \mathbb{Z}$ such that Ω_a is realized. This allows a graphical construction: it suffices to work between such a's (i.e., in finite boxes), as in Proposition 8.

Step 5. Using the same arguments, we easily deduce that for $A \geq 2$, the LFFP and the A-LFFP coincide on $[-A/2, A/2]$ during $[0,T]$, provided that there are $a_1 \in [-A, -A/2-1]$ and $a_2 \in [A/2, A-1]$ with $\Omega_{a_1} \cap \Omega_{a_2}$ realized. Furthermore, since M is a Poisson measure, Ω_a is independent of Ω_b for all $a \neq b$ (with $a, b \in \mathbb{Z}$). Thus, the probability on the left-hand side of (5) is bounded below, for $A \geq 2$, by

$$1 - \mathbb{P}\Bigg[\bigcap_{a \in \mathbb{Z} \cap [-A, -A/2-1]} \Omega_a^c\Bigg] - \mathbb{P}\Bigg[\bigcap_{a \in \mathbb{Z} \cap [A/2, A-1]} \Omega_a^c\Bigg] \geq 1 - 2(1-q_T)^{A/2-2},$$

hence we have (5) with $\alpha_T = -\log(1-q_T)/2 > 0$ and $C_T = 2/(1-q_T)^2$. □

3. Localization of the FFP. We first introduce the (λ, A)-FFP. We consider two independent families of i.i.d. Poisson processes $N = (N_t(i))_{t\ge 0, i\in\mathbb{Z}}$ and $M^\lambda = (M_t^\lambda(i))_{t\ge 0, i\in\mathbb{Z}}$, with respective rates 1 and $\lambda > 0$. For $A > 0$ and $\lambda > 0$, we define

$$A_\lambda := \lfloor A/(\lambda \log(1/\lambda))\rfloor \quad \text{and} \quad I_A^\lambda := [\![-A_\lambda, A_\lambda]\!], \tag{6}$$

and we set $\mathcal{F}_t^{N,M^\lambda,A} := \sigma(N_s(i), M_s^\lambda(i), s \le t, i \in I_A^\lambda)$.

DEFINITION 10. Consider an $(\mathcal{F}_t^{N,M^\lambda,A})_{t\ge 0}$-adapted process $(\eta_t^{\lambda,A})_{t\ge 0}$ with values in $\{0,1\}^{I_A^\lambda}$, such that $(\eta_t^{\lambda,A}(i))_{t\ge 0}$ is a.s. càdlàg for all $i \in I_A^\lambda$.

We say that $(\eta_t^{\lambda,A})_{t\ge 0}$ is a (λ, A)-FFP if a.s., for all $t \ge 0$ and $i \in I_A^\lambda$,

$$\eta_t^{\lambda,A}(i) = \int_0^t \mathbf{1}_{\{\eta_{s-}^{\lambda,A}(i)=0\}}\, dN_s(i) - \sum_{k\in I_A^\lambda} \int_0^t \mathbf{1}_{\{k\in C_{s-}^{\lambda,A}(i)\}}\, dM_s^\lambda(k),$$

where $C_s^{\lambda,A}(i) = \varnothing$ if $\eta_t^{\lambda,A}(i) = 0$, while $C_s^{\lambda,A}(i) = [\![l_s^{\lambda,A}(i), r_s^{\lambda,A}(i)]\!]$ if $\eta_s^{\lambda,A}(i) = 1$, where

$$l_s^{\lambda,A}(i) = (-A_\lambda) \vee (\sup\{k < i; \eta_s^{\lambda,A}(k) = 0\} + 1),$$
$$r_s^{\lambda,A}(i) = A_\lambda \wedge (\inf\{k > i; \eta_s^{\lambda,A}(k) = 0\} - 1).$$

For $x \in [-A, A]$ and $t \ge 0$, we introduce

$$D_t^{\lambda,A}(x) = \lambda \log(1/\lambda) C_t^{\lambda,A}(\lfloor x/(\lambda\log(1/\lambda))\rfloor) \subset [-A, A], \tag{7}$$

$$Z_t^{\lambda,A}(x) = \frac{\log[1 + \#(C_t^{\lambda,A}(\lfloor x/(\lambda\log(1/\lambda))\rfloor))]}{\log(1/\lambda)} \ge 0. \tag{8}$$

We now prove the following result, which is similar to Proposition 9 for the λ-FFP.

PROPOSITION 11. *Let $T > 0$ and $\lambda \in (0,1)$. Consider two families of Poisson processes $N = (N_t(i))_{t\ge 0, i\in\mathbb{Z}}$ and $M^\lambda = (M_t^\lambda(i))_{t\ge 0, i\in\mathbb{Z}}$ with respective rates 1 and $\lambda > 0$. Let $(\eta_t^\lambda)_{t\ge 0}$ be the corresponding λ-FFP and, for each $A > 0$, let $(\eta_t^{\lambda,A})_{t\ge 0}$ be the corresponding (λ, A)-FFP. Recall (1), (2) and (7), (8). There are constants $\alpha_T > 0$ and $C_T > 0$, not depending on $\lambda \in (0,1)$, $A \ge 2$, such that [recalling (6)]*

$$\mathbb{P}[(\eta_t^\lambda(i))_{t\in[0,T\log(1/\lambda)], i\in I_{A/2}^\lambda} = (\eta_t^{\lambda,A}(i))_{t\in[0,T\log(1/\lambda)], i\in I_{A/2}^\lambda}] \ge 1 - C_T e^{-\alpha_T A},$$

$$\mathbb{P}[(Z_t^\lambda(x), D_t^\lambda(x))_{t\in[0,T], x\in[-A/2,A/2]} = (Z_t^{\lambda,A}(x), D_t^{\lambda,A}(x))_{t\in[0,T], x\in[-A/2,A/2]}] \ge 1 - C_T e^{-\alpha_T A}.$$

PROOF. The proof is similar (but more complicated) to that of Proposition 9. Consider the true λ-FFP $(\eta_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$. Temporarily assume that for $a\in\mathbb{R}$, there is an event Ω_a^λ, depending only on the Poisson processes $N_t(i)$ and $M_t^\lambda(i)$ for $t\in[0, T\log(1/\lambda)]$ and $i\in J_a^\lambda := [\![\lfloor a/(\lambda\log(1/\lambda))\rfloor, \lfloor (a+1)/(\lambda\log(1/\lambda))\rfloor]\!]$, such that:

(i) on Ω_a^λ, a.s., for all $t\in[0, T\log(1/\lambda)]$, there is some $i\in J_a^\lambda$ such that $\eta_t^\lambda(i)=0$;

(ii) there exists $q_T>0$ such that for all $a\in\mathbb{R}$ and $\lambda\in(0,1)$, we have $\mathbb{P}(\Omega_a^\lambda)\geq q_T$.

The proof is then concluded using arguments similar to Steps 3, 4, 5 of the proof of Proposition 9.

Fix some $\alpha>0$ and some $\varepsilon_T>0$ small enough, say $\alpha=0.01$ and $\varepsilon_T=1/(32T)$. Let $\lambda_T>0$ be such that for $\lambda\in(0,\lambda_T)$, we have $1<\lambda^{\alpha-1}<\epsilon_T/(\lambda\log(1/\lambda))$.

For $\lambda\in[\lambda_T,1)$ and $a\in\mathbb{R}$, we set $\Omega_a^\lambda=\{N_{T\log(1/\lambda)}(\lfloor a/(\lambda\log(1/\lambda))\rfloor)=0\}$, on which, of course, $\eta_t^\lambda(i)=0$ for all $t\in[0,T\log(1/\lambda)]$ with $i=\lfloor a/(\lambda\log(1/\lambda))\rfloor\in J_a^\lambda$. We then observe that $q'_T=\inf_{\lambda\in[\lambda_T,1)} P(\Omega_a^\lambda)=\inf_{\lambda\in[\lambda_T,1)} e^{-T\log(1/\lambda)}=(\lambda_T)^T>0$.

For $\lambda\in(0,\lambda_T)$ and $a\in\mathbb{R}$, we define the event Ω_a^λ on which points 1, 2 and 3 below are satisfied.

1. The family of Poisson processes $(M_t^\lambda(i))_{t\in[0,T\log(1/\lambda)], i\in J_a^\lambda}$ has exactly $3n$ marks for some $1\leq n\leq\lfloor T\rfloor$ and it is possible to call them $(T_k^\lambda, X_k^\lambda)_{k=1,\dots,n}$, $(\tilde T_k^\lambda,\tilde X_k^\lambda)_{k=1,\dots,n}$ and $(S_k^\lambda, Y_k^\lambda)_{k=1,\dots,n}$ in such a way that we have the following properties for all $k=1,\dots,n$ (we set $T_0^\lambda=\tilde T_0^\lambda=S_0^\lambda=0$ and $X_0^\lambda=\lfloor a/(\lambda\log(1/\lambda))\rfloor$, $\tilde X_0^\lambda=\lfloor (a+1)/(\lambda\log(1/\lambda))\rfloor$):

(1a) $X_{k-1}^\lambda<X_k^\lambda<Y_k^\lambda<\tilde X_k^\lambda<\tilde X_{k-1}^\lambda$ with $\min\{X_k^\lambda-X_{k-1}^\lambda, Y_k^\lambda-X_k^\lambda, \tilde X_k^\lambda-Y_k^\lambda, \tilde X_{k-1}^\lambda-\tilde X_k^\lambda\}\geq 4\epsilon_T/(\lambda\log(1/\lambda))$;

(1b) T_k^λ and $\tilde T_k^\lambda$ belong to $[S_{k-1}^\lambda+(\frac12+\alpha)\log(1/\lambda), S_{k-1}^\lambda+(1-\alpha)\log(1/\lambda)]$;

(1c) $S_k^\lambda\in[S_{k-1}^\lambda+(1+\alpha)\log(1/\lambda), S_{k-1}^\lambda+2(T_k^\lambda\wedge\tilde T_k^\lambda-S_{k-1}^\lambda)-\alpha\log(1/\lambda)]$;

(1d) $S_n^\lambda\geq(T-1+\alpha)\log(1/\lambda)$.

2. For $k=1,\dots,n$, we now set $\tau_k^\lambda=(S_k^\lambda-S_{k-1}^\lambda)/(2\log(1/\lambda))$, which belongs to $[(1+\alpha)/2, 1-\alpha]$, due to 1. We consider the intervals

$$
\begin{aligned}
I_k^\lambda &= [\![X_k^\lambda-\lfloor\lambda^{-\tau_k^\lambda}\rfloor, X_k^\lambda+\lfloor\lambda^{-\tau_k^\lambda}\rfloor]\!],\\
I_{k,-}^\lambda &= [\![X_k^\lambda-\lfloor\lambda^{-\tau_k^\lambda}\rfloor-\lfloor\varepsilon_T/\lambda\log(1/\lambda)\rfloor, X_k^\lambda-\lfloor\lambda^{-\tau_k^\lambda}\rfloor-1]\!],\\
I_{k,+}^\lambda &= [\![X_k^\lambda+\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1, X_k^\lambda+\lfloor\lambda^{-\tau_k^\lambda}\rfloor+\lfloor\varepsilon_T/\lambda\log(1/\lambda)\rfloor]\!],\\
L_k^\lambda &= [\![X_k^\lambda+\lfloor\lambda^{-\tau_k^\lambda}\rfloor+\lfloor\varepsilon_T/\lambda\log(1/\lambda)\rfloor+1,\\
&\qquad \tilde X_k^\lambda-\lfloor\lambda^{-\tau_k^\lambda}\rfloor-\lfloor\varepsilon_T/\lambda\log(1/\lambda)\rfloor-1]\!]
\end{aligned}
$$

and similar intervals $\tilde{I}_k^\lambda$, $\tilde{I}_{k,-}^\lambda$, $\tilde{I}_{k,+}^\lambda$, around $\tilde{X}_k^\lambda$. For all $k = 1, \dots, n$, the family of Poisson processes $(N_t(i))_{t\geq 0, i\in J_a^\lambda}$ satisfies:

(2a) $\forall i \in I_k^\lambda$, $N_{T_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) > 0$ and $\forall i \in \tilde{I}_k^\lambda$, $N_{\tilde{T}_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) > 0$;

(2b) $\exists i \in I_{k,-}^\lambda$ such that $N_{T_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) = 0$, $\exists i \in I_{k,+}^\lambda$ such that $N_{T_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) = 0$, $\exists i \in \tilde{I}_{k,-}^\lambda$ such that $N_{\tilde{T}_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) = 0$ and $\exists i \in \tilde{I}_{k,+}^\lambda$ such that $N_{\tilde{T}_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) = 0$;

(2c) $\exists i \in I_k^\lambda$ such that $N_{S_k^\lambda}(i) - N_{T_k^\lambda}(i) = 0$ and $\exists i \in \tilde{I}_k^\lambda$ such that $N_{S_k^\lambda}(i) - N_{\tilde{T}_k^\lambda}(i) = 0$;

(2d) $\forall i \in L_k^\lambda$, $N_{S_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) > 0$.

3. We finally assume that $\exists i \in L_n^\lambda$ such that $N_{T\log(1/\lambda)}(i) - N_{S_n^\lambda}(i) = 0$.

To show that on Ω_a^λ, a.s., for all $t \in [0, T\log(1/\lambda)]$, there is some $i \in J_a^\lambda$ such that $\eta_t^\lambda(i) = 0$, we proceed recursively. At time 0, all sites are vacant. Fix $k \in \{1, \dots, n\}$. Assume that for $t \leq S_{k-1}^\lambda$, there is some $i \in J_a^\lambda$ such that $\eta_t^\lambda(i) = 0$ and that at time S_{k-1}^λ, all sites in the interval L_{k-1}^λ are vacant.

Then, for $S_{k-1}^\lambda \leq t < T_k^\lambda$ (resp., $S_{k-1}^\lambda \leq t < \tilde{T}_k^\lambda$), (2b) shows that there are vacant sites in both $I_{k,+}^\lambda$ and $I_{k,-}^\lambda$ (resp., in both $\tilde{I}_{k,+}^\lambda$ and $\tilde{I}_{k,-}^\lambda$). This, together with (2a), shows that at time $T_k^\lambda-$ (resp., $\tilde{T}_k^\lambda-$), all of the sites in the intervals I_k^λ and $\tilde{I}_k^\lambda$ are occupied (no fire may burn those sites because they are protected by the vacant sites in $I_{k,+}^\lambda$, $I_{k,-}^\lambda$, $\tilde{I}_{k,+}^\lambda$, $\tilde{I}_{k,-}^\lambda$). Hence, the interval I_k^λ (resp., $\tilde{I}_k^\lambda$) becomes completely vacant at time T_k^λ (resp., $\tilde{T}_k^\lambda$). Between time T_k^λ (resp., $\tilde{T}_k^\lambda$) and time S_k^λ, since I_k^λ (resp., $\tilde{I}_k^\lambda$) is completely vacant at time T_k^λ (resp., $\tilde{T}_k^\lambda$), (2c) shows that there is a vacant site in I_k^λ (resp., $\tilde{I}_k^\lambda$).

At time $S_k^\lambda-$, the interval L_k^λ is completely occupied, by virtue of (2d) and the fact that it cannot be burnt because it is protected by vacant sites in $I_{k,+}^\lambda$ (resp., $\tilde{I}_{k,-}^\lambda$) between S_{k-1}^λ and T_k^λ (resp., $\tilde{T}_k^\lambda$), and in I_k^λ (resp., $\tilde{I}_k^\lambda$) between T_k^λ (resp., $\tilde{T}_k^\lambda$) and S_k^λ. As a consequence, since $Y_k^\lambda \in L_k^\lambda$, the interval L_k^λ becomes completely vacant at time $S_k^\lambda-$.

All of this shows that on Ω_a^λ, there are vacant sites in J_a^λ for all $t \in [0, S_n^\lambda]$ and that L_n^λ is completely vacant at time S_n^λ. Finally, 3 implies that there are vacant sites in $L_n^\lambda \subset J_a^\lambda$ during $[S_n^\lambda, T\log(1/\lambda)]$.

It remains to prove that there exists $q_T'' > 0$ such that for all $a \in \mathbb{R}$ and $\lambda \in (0, \lambda_T)$, we have $\mathbb{P}(\Omega_a^\lambda) \geq q_T''$. We separately treat the conditions 1 on M^λ and 2 on N (conditionally on M^λ) and use independence of these two families of Poisson processes to complete the proof.

First, for $\lambda \in (0, \lambda_T)$, we observe that we can construct M^λ using a Poisson measure M on $[0, \infty) \times \mathbb{R}$ with intensity $dt\,dx$ by setting, for all $i \in \mathbb{Z}$,

$$M_t^\lambda(i) = M\big([0, t/\log(1/\lambda)] \times \big[i\lambda\log(1/\lambda), (i+1)\lambda\log(1/\lambda)\big)\big).$$

Hence [since $\varepsilon_T/(\lambda\log(1/\lambda)) > 1$], the event on which M^λ satisfies 1 contains the event Ω'_a on which M has exactly $3n$ marks in $[0,T]\times[a,a+1]$, for some $1\le n\le \lfloor T\rfloor$, which can be called $(T_k,X_k)_{k=1,\dots,n}$, $(\tilde T_k,\tilde X_k)_{k=1,\dots,n}$ and $(S_k,Y_k)_{k=1,\dots,n}$ in such a way that we have the following properties (we set $T_0=\tilde T_0=S_0=0$ and $X_0=a$, $\tilde X_0=a+1$ for convenience) for all $k=1,\dots,n$:

- $\min(\{X_k-X_{k-1}, Y_k-X_k, \tilde X_k-Y_k, \tilde X_{k-1}-\tilde X_k\})>5\epsilon_T$;
- T_k and $\tilde T_k$ belong to $(S_{k-1}+1/2+\alpha, S_{k-1}+1-\alpha)$;
- $S_k\in(S_{k-1}+1+\alpha, S_{k-1}+2(T_k\wedge\tilde T_k-S_{k-1})-\alpha)$;
- $S_n\ge(T-1)+\alpha$.

We then have $\mathbb{P}(\Omega'_a)>0$ (as in the proof of Proposition 9 and since ε_T and α are sufficiently small) and this probability does not depend on a (by translation invariance of the law of M) nor on $\lambda\in(0,\lambda_T)$ (since it concerns only M).

We then use basic computations on i.i.d. Poisson processes with rate 1 to show that there is a (deterministic) constant $c>0$ such that for all $k=1,\dots,n$, all $\lambda\in(0,\lambda_T)$, conditionally on M^λ (we write $\mathbb{P}_M$ for the conditional probability w.r.t. M^λ):

- since $T_k^\lambda-S_{k-1}^\lambda\ge(\tau_k^\lambda+\alpha/2)\log(1/\lambda)$, due to (1c), and since $\#(I_k^\lambda)=2\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1$, we have

$$\begin{aligned}\mathbb{P}_M\big(\forall i\in I_k^\lambda, N_{T_k^\lambda}(i)-N_{S_{k-1}^\lambda}(i)>0\big)&=\big(1-e^{-(T_k^\lambda-S_{k-1}^\lambda)}\big)^{2\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1}\\&\ge(1-\lambda^{\tau_k^\lambda+\alpha/2})^{2\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1}\ge c\end{aligned}$$

(it tends to 1 as $\lambda\to0$) and the same computation works for $\tilde I_k^\lambda$;

- since $T_k^\lambda-S_{k-1}^\lambda\le(1-\alpha)\log(1/\lambda)$, by (1b), and since $\#(I_{k,+}^\lambda)=\lfloor\varepsilon_T/(\lambda\times\log(1/\lambda))\rfloor$, we have

$$\begin{aligned}\mathbb{P}_M\big(\exists i\in I_{k,+}^\lambda, N_{T_k^\lambda}(i)-N_{S_{k-1}^\lambda}(i)=0\big)&=1-\big(1-e^{-(T_k^\lambda-S_{k-1}^\lambda)}\big)^{\lfloor\varepsilon_T/\lambda\log(1/\lambda)\rfloor}\\&\ge1-(1-\lambda^{1-\alpha})^{\lfloor\varepsilon_T/(\lambda\log(1/\lambda))\rfloor}\ge c\end{aligned}$$

and the same computation works for $I_{k,-}^\lambda, \tilde I_{k,+}^\lambda, \tilde I_{k,-}^\lambda$;

- since $S_k^\lambda-T_k^\lambda\le(\tau_k^\lambda-\alpha/2)\log(1/\lambda)$, due to (1c) [we use the fact that $S_k^\lambda\le 2T_k^\lambda-S_{k-1}^\lambda-\alpha\log(1/\lambda)$, whence $2S_k^\lambda\le2T_k^\lambda+S_k^\lambda-S_{k-1}^\lambda-\alpha\log(1/\lambda)=2T_k^\lambda+2(\tau_k^\lambda-\alpha/2)\log(1/\lambda)$], and since $\#(I_k^\lambda)=2\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1$, we have

$$\begin{aligned}\mathbb{P}_M\big(\exists i\in I_k^\lambda, N_{S_k^\lambda}(i)-N_{T_k^\lambda}(i)=0\big)&=1-\big(1-e^{-(S_k^\lambda-T_k^\lambda)}\big)^{2\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1}\\&\ge1-(1-\lambda^{\tau_k^\lambda-\alpha/2})^{2\lfloor\lambda^{-\tau_k^\lambda}\rfloor+1}\ge c\end{aligned}$$

and this also holds for $\tilde I_k^\lambda$;

• since $S_k^\lambda - S_{k-1}^\lambda \geq (1+\alpha)\log(1/\lambda)$, thanks to (1c), and since $\#(L_k^\lambda) \leq \lfloor (1/\lambda \log(1/\lambda)) \rfloor$, we have

$$\mathbb{P}_M\big(\forall i \in L_k^\lambda, N_{S_k^\lambda}(i) - N_{S_{k-1}^\lambda}(i) > 0\big) = \big(1 - e^{-(S_k^\lambda - S_{k-1}^\lambda)}\big)^{\#(L_k^\lambda)}$$
$$\geq (1 - \lambda^{1+\alpha})^{\lfloor 1/\lambda \log(1/\lambda) \rfloor} \geq c;$$

• since $T\log(1/\lambda) - S_n^\lambda \leq (1-\alpha)\log(1/\lambda)$, by (1d), and $\#(L_n^\lambda) \geq 4\varepsilon_T/(\lambda\log(1/\lambda))$, by (1a), we have

$$\mathbb{P}_M\big(\exists i \in L_n^\lambda, N_{T\log(1/\lambda)}(i) - N_{S_n^\lambda}(i) = 0\big) = 1 - \big(1 - e^{-(T\log(1/\lambda) - S_n^\lambda)}\big)^{\#(L_n^\lambda)}$$
$$\geq 1 - (1 - \lambda^{1-\alpha})^{4\varepsilon_T/(\lambda\log(1/\lambda))} \geq c.$$

We observe that the domains $I_k^\lambda \times (S_{k-1}^\lambda, T_k^\lambda]$, $\tilde{I}_k^\lambda \times (S_{k-1}^\lambda, \tilde{T}_k^\lambda]$, $I_{k,+}^\lambda \times (S_{k-1}^\lambda, T_k^\lambda]$, $I_{k,-}^\lambda \times (S_{k-1}^\lambda, T_k^\lambda]$, $\tilde{I}_{k,+}^\lambda \times (S_{k-1}^\lambda, \tilde{T}_k^\lambda]$, $\tilde{I}_{k,-}^\lambda \times (S_{k-1}^\lambda, \tilde{T}_k^\lambda]$, $I_k^\lambda \times (T_k^\lambda, S_k^\lambda]$, $\tilde{I}_k^\lambda \times (\tilde{T}_k^\lambda, S_k^\lambda]$, $L_k^\lambda \times (S_{k-1}^\lambda, S_k^\lambda]$, for $k = 1, \dots, n$, and $L_n^\lambda \times (S_n^\lambda, T\log(1/\lambda)]$ are pairwise disjoint, thanks to 1 and to the smallness of ε_T and λ_T: we have $\lfloor \lambda^{-\tau_k^\lambda} \rfloor \leq \lambda^{\alpha-1} \leq \varepsilon_T/(\lambda\log(1/\lambda))$.

Since $n \leq T$, we deduce from all of the previous estimates the existence of a $q_T'' > 0$ such that for all $a \in \mathbb{R}$ and $\lambda \in (0, \lambda_T)$, we have $\mathbb{P}(\Omega_a^\lambda) \geq q_T''$. We complete the proof by choosing $q_T = \min(q_T', q_T'')$. □

4. Convergence proof.

The goal of this section is to prove Theorem 5.

4.1. *Coupling.* We introduce a coupling between the λ-FFP, the LFFP and their localized versions.

NOTATION 12. We consider a Poisson measure $M(dt, dx)$ on $[0,\infty) \times \mathbb{R}$ with intensity measure $dt\,dx$. We consider an independent family of Poisson processes $(N_t(i))_{t\geq 0, i\in\mathbb{Z}}$ with rate 1. For $\lambda \in (0,1)$ and $i \in \mathbb{Z}$, we set

$$M_t^\lambda(i) = M\big([0, t/\log(1/\lambda)] \times \big[i\lambda\log(1/\lambda), (i+1)\lambda\log(1/\lambda)\big)\big).$$

Then $(M_t^\lambda(i))_{t\geq 0, i\in\mathbb{Z}}$ is a family of independent Poisson processes with rate λ. For all $\lambda \in (0,1)$, we consider the λ-FFP $(\eta_t^\lambda)_{t\geq 0}$ (see Definition 1) and for all $A > 0$, we consider the (λ, A)-FFP $(\eta_t^{\lambda,A})_{t\geq 0}$ (see Definition 10) constructed with N, M^λ. We also introduce the processes $(Z_t^\lambda(x), D_t^\lambda(x))_{t\geq 0, x\in\mathbb{R}}$, as in (1), (2), and $(Z_t^{\lambda,A}(x), D_t^{\lambda,A}(x))_{t\geq 0, x\in[-A,A]}$, as in (7), (8).

We denote by $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ the LFFP constructed with M (see Definition 2) and by $(Z_t^A(x), D_t^A(x), H_t^A(x))_{t\geq 0, x\in[-A,A]}$ the A-LFFP constructed with M (see Definition 7).

4.2. *Localization*. Temporarily assume that the following result holds.

PROPOSITION 13. *Adopt Notation 12 as well as Notation 4.*
(a) *For any* $T > 0$, $A > 0$ *and* $x_0 \in (-A, A)$, *in probability, as* $\lambda \to 0$,

$$\delta_T((Z^{\lambda,A}(x_0), D^{\lambda,A}(x_0)), (Z^A(x_0), D^A(x_0))) \qquad \textit{tends to } 0.$$

(b) *For any* $t \in [0, \infty)$, $A > 0$ *and* $x_0 \in (-A, A)$, *in probability, as* $\lambda \to 0$,

$$|Z_t^{\lambda,A}(x_0) - Z_t^A(x_0)| + \delta(D_t^{\lambda,A}(x_0)), D_t^A(x_0)) \qquad \textit{tends to } 0.$$

We are now in a position to give the following proof.

PROOF OF THEOREM 5. We only prove point (a), (b) being similarly checked. Let $T > 0$ and $\{x_1, \ldots, x_n\} \subset [-B, B] \subset \mathbb{R}$ be fixed. Consider the coupling introduced in Notation 12. Proposition 13 ensures us that for any $\varepsilon > 0$ and $A > B$, we have

$$\lim_{\lambda \to 0} \mathbb{P}\left[\sum_1^n \delta_T((Z^{\lambda,A}(x_i), D^{\lambda,A}(x_i)), (Z^A(x_i), D^A(x_i))) > \varepsilon\right] = 0.$$

Now, let

$$\begin{aligned} \Omega_{A,T}^\lambda := \{&\forall i = 1, \ldots, n, \forall t \in [0, T], \\ &(Z_t^\lambda(x_i), D_t^\lambda(x_i)) = (Z_t^{\lambda,A}(x_i), D_t^{\lambda,A}(x_i)) \\ &\text{and } (Z_t(x_i), D_t(x_i)) = (Z_t^A(x_i), D_t^A(x_i))\}. \end{aligned}$$

For all $A > 2B$, we now have

$$\begin{aligned} \Omega_{A,T}^\lambda \subset \big\{&(Z_t^\lambda(x), D_t^\lambda(x))_{t\in[0,T], x\in[-A/2,A/2]} \\ &= (Z_t^{\lambda,A}(x), D_t^{\lambda,A}(x))_{t\in[0,T], x\in[-A/2,A/2]} \\ &\text{and } (Z_t(x), D_t(x))_{t\in[0,T], x\in[-A/2,A/2]} \\ &\quad = (Z_t^A(x), D_t^A(x))_{t\in[0,T], x\in[-A/2,A/2]}\big\}. \end{aligned}$$

However, Propositions 9 and 11 yield that $\mathbb{P}[(\Omega_{A,T}^\lambda)^c] \le 2C_T e^{-\alpha_T A}$. Thus, for any $A > 2B$,

$$\limsup_{\lambda \to 0} \mathbb{P}\left[\sum_1^n \delta_T((Z^\lambda(x_i), D^\lambda(x_i)), (D(x_i), Z(x_i))) > \varepsilon\right] \le 0 + 2C_T e^{-\alpha_T A}.$$

Letting A tend to infinity, we deduce that $\sum_{i=1}^n \delta_T((Z^\lambda(x_i), D^\lambda(x_i)), (D(x_i), Z(x_i)))$ tends to 0 in probability as $\lambda \to 0$, hence the result. □

4.3. *Core of the proof.* The aim of this subsection is to prove Proposition 13. We fix $T > 0$ and $A > 0$. We consider the (λ, A)-FFP and the A-LFFP coupled, as in Notation 12, and use the notation introduced in (6). Throughout this proof, we will omit the superscript A and we do not take into account the possible dependencies in A and T.

For $J = (a, b)$ [an open interval of $(-A, A)$], $\lambda \in (0, 1)$ and $\mu \in (0, 1]$, we consider

$$\begin{aligned} J_{\lambda,\mu} &= \left[\!\left[\left\lfloor \frac{a}{\lambda \log(1/\lambda)} + \frac{\mu}{\lambda \log^2(1/\lambda)} \right\rfloor, \right.\right. \\ &\qquad \left.\left. \left\lfloor \frac{b}{\lambda \log(1/\lambda)} - \frac{\mu}{\lambda \log^2(1/\lambda)} \right\rfloor \right]\!\right] \subset \mathbb{Z}, \\ \tilde{Z}_t^{\lambda,\mu}(J) &= 1 - \frac{\log(1 + \#\{k \in J_{\lambda,\mu}, \eta^\lambda_{t \log(1/\lambda)}(k) = 0\})}{\log(1 + \#(J_{\lambda,\mu}))}. \end{aligned} \tag{9}$$

Observe that $\tilde{Z}_t^{\lambda,\mu}(J) = 1$ if and only if all the sites of $J_{\lambda,\mu}$ are occupied at time $t \log(1/\lambda)$. The quantity $\tilde{Z}_t^{\lambda,\mu}(J)$ is a function of the density of vacant clusters in the (rescaled) zone J. Under some exchangeability properties, it should be closely related to the size of occupied clusters in that zone, that is, to $Z_t^\lambda(x)$ for $x \in J$.

For $x \in (-A, A)$, $\lambda \in (0, 1)$ and $\mu \in (0, 1]$, we introduce

$$\begin{aligned} x_{\lambda,\mu} &= \left[\!\left[\left\lfloor \frac{x}{\lambda \log(1/\lambda)} - \frac{\mu}{\lambda \log^2(1/\lambda)} \right\rfloor + 1, \right.\right. \\ &\qquad \left.\left. \left\lfloor \frac{x}{\lambda \log(1/\lambda)} + \frac{\mu}{\lambda \log^2(1/\lambda)} \right\rfloor - 1 \right]\!\right] \subset \mathbb{Z}, \\ \tilde{H}_t^{\lambda,\mu}(x) &= \frac{\log(1 + \#\{k \in x_{\lambda,\mu}, \eta^\lambda_{t \log(1/\lambda)}(k) = 0\})}{\log(1 + \#(x_{\lambda,\mu}))}. \end{aligned} \tag{10}$$

Here, again, $\tilde{H}_t^{\lambda,\mu}(x) = 0$ if and only if all the sites of $x_{\lambda,\mu}$ are occupied at time $t \log(1/\lambda)$. Assume that a microscopic fire starts at some x. The process $\tilde{H}_t^{\lambda,\mu}(x)$ will then allow us to quantify the duration for which this fire will be in effect.

Observe that we always have $\log(1 + \#(x_{\lambda,\mu})) \sim \log(1 + \#(J_{\lambda,\mu})) \sim \log(1/\lambda)$ as $\lambda \to 0$. Also, observe that if $\tilde{Z}_t^{\lambda,\mu}(J) = z$, then there are $(1 + \#(J_{\lambda,\mu}))^{1-z} - 1 \simeq \lambda^{z-1}$ vacant sites in $J_{\lambda,\mu}$ at time $t \log(1/\lambda)$. In the same way, $\tilde{H}_t^{\lambda,\mu}(x) = h$ says that there are $(1 + \#(x_{\lambda,\mu}))^h - 1 \simeq \lambda^{-h}$ vacant sites in $x_{\lambda,\mu}$ at time $t \log(1/\lambda)$.

We work conditionally on M. We denote by $\mathbb{P}_M$ the conditional probability given M. We recall that, conditionally on M, $(Z_t(x), D_t(x), H_t(x))_{t \in [0,T], x \in [-A,A]}$ is deterministic. We set $n = M([0, T] \times [-A; A])$, which is a.s. finite. We set $T_0 = 0$ and consider the marks $(X_q, T_q)_{1 \le q \le n}$ of M, ordered in such a way that $T_0 < T_1 < \cdots < T_n < T$.

We set $\mathcal{B}_0 = \varnothing$ and for $q = 1, \dots, n$, we consider $\mathcal{B}_q = \{X_1, \dots, X_q\}$, as well as the set $\mathcal{C}_q$ of connected components of $(-A, A) \setminus \mathcal{B}_q$ (sometimes referred to as *cells*).

Observe that, by construction, we have, for $c \in \mathcal{C}_q$ and $x, y \in c$, $Z_t(x) = Z_t(y)$ for all $t \in [0, T_{q+1})$. Thus, we can introduce $Z_t(c)$.

We consider $\lambda_\mu > 0$ (which depends on M) such that for all $\lambda \in (0, \lambda_\mu)$, we have $(X_i)_{\lambda,\mu} \neq \varnothing$ and $(X_i)_{\lambda,\mu} \cap (X_j)_{\lambda,\mu} = \varnothing$ for all $i \neq j$ with $i, j \in \{1, \dots, n\}$.

We then observe that for $\lambda \in (0, \lambda_\mu)$ and for each $q = 0, \dots, n$, $\{x_{\lambda,\mu}, x \in \mathcal{B}_q\} \cup \{c_{\lambda,\mu}, c \in \mathcal{C}_q\}$ is a partition of $[\![-\tilde{A}_{\lambda,\mu}, \tilde{A}_{\lambda,\mu}]\!]$, where $\tilde{A}_{\lambda,\mu} = \lfloor A/(\lambda \log(1/\lambda)) - \mu/(\lambda \log^2(1/\lambda))\rfloor$.

With our coupling, for the (λ, A)-FFP $(\eta_t^\lambda)_{t\geq 0}$, for each $i = 1, \dots, n$, a fire starts at the site $\lfloor X_i/(\lambda \log(1/\lambda))\rfloor$ at time $T_i \log(1/\lambda)$ and this describes all of the fires during $[0, T \log(1/\lambda)]$.

The lemma below shows some exchangeability properties inside cells [connected components of $(-A, A) \setminus \mathcal{B}_q$]. This will allow us to prove that for c a cell and $x \in c$, the size of the occupied cluster around x [described by $Z^\lambda(x)$] is closely related to the global density of occupied clusters in c [described by $\tilde{Z}^{\lambda,\mu}(c)$].

LEMMA 14. *For* $\lambda \in (0, 1)$ *and* $\mu \in (0, 1]$, *set* $\mathcal{E}_0^{\lambda,\mu} = \Omega$, *and for* $q = 1, \dots, n$, *consider the event* [*recalling Definition* 10 *and* (9)]

$$\begin{aligned}\mathcal{E}_q^{\lambda,\mu} = \big\{&\forall i = 1, \dots, q, \forall c \in \mathcal{C}_i,\ \textit{either } c_{\lambda,\mu} \subset C^\lambda_{T_i \log(1/\lambda)-}(X_i) \\ &\textit{or } \eta^\lambda_{T_i \log(1/\lambda)-}(k) = 0 \textit{ for some } \max c_{\lambda,\mu} < k < \min C^\lambda_{T_i \log(1/\lambda)-}(X_i) \\ &\textit{or } \eta^\lambda_{T_i \log(1/\lambda)-}(k) = 0 \textit{ for some } \max C^\lambda_{T_i \log(1/\lambda)-}(X_i) < k < \min c_{\lambda,\mu}\big\}.\end{aligned}$$

Conditionally on M *and* $\mathcal{E}_q^{\lambda,\mu}$, *for all* $c \in \mathcal{C}_q$, *the random variables* $(\eta^\lambda_{T_q \log(1/\lambda)}(k))_{k \in c_{\lambda,\mu}}$ *are exchangeable.*

PROOF. Let $c \in \mathcal{C}_q$, let σ be a permutation of $c_{\lambda,\mu}$ and set, for simplicity, $\sigma(i) = i$ for $i \in I_A^\lambda \setminus c_{\lambda,\mu}$ [recall (6)].

Consider the (λ, A)-FFP process $(\eta_t^\lambda)_{t\geq 0}$ constructed with M and the family of Poisson processes $(N(i))_{i \in I_A^\lambda}$. Also, consider the (λ, A)-FFP process $(\tilde{\eta}_t^\lambda)_{t\geq 0}$ constructed with M and the family of Poisson processes $(\tilde{N}(i))_{i \in I_A^\lambda}$ defined by $\tilde{N}(i) = N(\sigma(i))$.

Observe that $\mathcal{E}_{k+1}^{\lambda,\mu} \subset \mathcal{E}_k^{\lambda,\mu}$. For all $k = 0, \dots, q$, $c \subset c_k$ for some $c_k \in \mathcal{C}_k$. We will prove the following claims by induction on $k = 0, \dots, q$:

(i) if $\tilde{\mathcal{E}}_k^{\lambda,\mu}$ is the same event as $\mathcal{E}_k^{\lambda,\mu}$ corresponding to $(\tilde{\eta}_t^\lambda)_{t\geq 0}$, then $\tilde{\mathcal{E}}_k^{\lambda,\mu} = \mathcal{E}_k^{\lambda,\mu}$;

(ii) on $\mathcal{E}_k^{\lambda,\mu}$, for all $t \in [0, T_k \log(1/\lambda)]$, $\tilde{\eta}_t^\lambda(i) = \eta_t^\lambda(\sigma(i))$ for all $i \in I_A^\lambda$ [in particular, $\tilde{\eta}_t^\lambda(i) = \eta_t^\lambda(i)$ for all $i \notin c_{\lambda,\mu}$].

Of course, (i) and (ii) with $k=q$ imply the lemma. Indeed, let $\varphi:\{0,1\}^{\#(c_{\lambda,\mu})}\mapsto\mathbb{R}$. We have

$$\mathbb{E}_M[\mathbf{1}_{\mathcal{E}_q^{\lambda,\mu}}\varphi((\eta^\lambda_{T_q\log(1/\lambda)}(i))_{i\in c_{\lambda,\mu}})]=\mathbb{E}_M[\mathbf{1}_{\tilde{\mathcal{E}}_q^{\lambda,\mu}}\varphi((\tilde{\eta}^\lambda_{T_q\log(1/\lambda)}(i))_{i\in c_{\lambda,\mu}})].$$

Using (i) and (ii), we then deduce that

$$\mathbb{E}_M[\mathbf{1}_{\mathcal{E}_q^{\lambda,\mu}}\varphi((\eta^\lambda_{T_q\log(1/\lambda)}(i))_{i\in c_{\lambda,\mu}})]=\mathbb{E}_M[\mathbf{1}_{\mathcal{E}_q^{\lambda,\mu}}\varphi((\eta^\lambda_{T_q\log(1/\lambda)}(\sigma(i)))_{i\in c_{\lambda,\mu}})],$$

which proves the lemma.

First, (i) and (ii) with $k=0$ are obviously satisfied. Assume, now, that for some $k\in\{0,\dots,q-1\}$, we have (i) and (ii). Then, on $\mathcal{E}_k^{\lambda,\mu}$, for all $t\in[0,T_{k+1}\log(1/\lambda))$, $\tilde{\eta}^\lambda_t(i)=\eta^\lambda_t(\sigma(i))$ for all $i\in I^\lambda_A$. Indeed, they are equal on $[0,T_k\log(1/\lambda)]$, by assumption, and they use the same Poisson process $\tilde{N}(i)=N(\sigma(i))$ on the time interval $[T_k\log(1/\lambda),T_{k+1}\log(1/\lambda)))$.

We now check that $\mathcal{E}_{k+1}^{\lambda,\mu}=\tilde{\mathcal{E}}_{k+1}^{\lambda,\mu}$. We know that $\mathcal{E}_k^{\lambda,\mu}=\tilde{\mathcal{E}}_k^{\lambda,\mu}$ and the additional condition [at time $T_{k+1}\log(1/\lambda)-$] concerns:

• sites outside $c_{\lambda,\mu}$, for which the values of η^λ and $\tilde{\eta}^\lambda$ at time $T_{k+1}\log(1/\lambda)-$ are the same;

• the event $c_{\lambda,\mu}\subset C^\lambda_{T_{k+1}\log(1/\lambda)-}$, which is the same for η^λ and $\tilde{\eta}^\lambda$ (it can be realized only if there are no vacant sites in $c_{\lambda,\mu}$, which occurs, or not, simultaneously for η^λ and $\tilde{\eta}^\lambda$).

We now conclude that (ii) remains true at time $T_{k+1}\log(1/\lambda)$ since the zone subject to fire either:

• is disjoint with $c_{\lambda,\mu}$ so that the values of η^λ, $\tilde{\eta}^\lambda$ are left invariant in $c_{\lambda,\mu}$, while they are modified in the same way outside $c_{\lambda,\mu}$; or

• contains the whole zone $c_{\lambda,\mu}$, which is thus destroyed simultaneously for η^λ and $\tilde{\eta}^\lambda$, and the values of η^λ, $\tilde{\eta}^\lambda$ are modified in the same way outside $c_{\lambda,\mu}$. □

The next lemma shows, in some sense, that if a cell is *almost* completely occupied at time t, then it will be *really* completely occupied at time $t+$; and, if the effect of a microscopic fire is *almost* ended at time t, then it will be *really* ended at time $t+$.

LEMMA 15. *Let* $\mu\in(0,1]$. *Consider* $k\in\{0,\dots,n\}$, $c\in\mathcal{C}_k$, $x\in\mathcal{B}_k$ *and* $t\in[T_k,T_{k+1})$.

(i) *Assume that for all* $\varepsilon>0$, $\lim_{\lambda\to0}\mathbb{P}_M(\tilde{Z}^{\lambda,\mu}_t(c)<1-\varepsilon)=0$. *Then, for all* $s\in(t,T_{k+1})$, $\lim_{\lambda\to0}\mathbb{P}_M(\tilde{Z}^{\lambda,\mu}_s(c)=1)=1$.

(ii) *Assume that for all* $\varepsilon>0$, $\lim_{\lambda\to0}\mathbb{P}_M(\tilde{H}^{\lambda,\mu}_t(x)>\varepsilon)=0$. *Then, for all* $s\in(t,T_{k+1})$, $\lim_{\lambda\to0}\mathbb{P}_M(\tilde{H}^{\lambda,\mu}_s(x)=0)=1$.

PROOF. The proofs of (i) and (ii) are similar. Let us, for example, prove (i). Thus, let $T_k\le t<t+\varepsilon=s<T_{k+1}$. We start with

$$\mathbb{P}_M(\tilde{Z}^{\lambda,\mu}_{t+\varepsilon}(c)=1)\ge\mathbb{P}_M(\tilde{Z}^{\lambda,\mu}_{t+\varepsilon}(c)=1|\tilde{Z}^{\lambda,\mu}_t(c)>1-\varepsilon/2)\mathbb{P}_M(\tilde{Z}^{\lambda,\mu}_t(c)>1-\varepsilon/2),$$

so that it suffices to check that $\lim_{\lambda\to 0}\mathbb{P}_M(\tilde{Z}_{t+\varepsilon}^{\lambda,\mu}(c)=1|\tilde{Z}_t^{\lambda,\mu}(c)>1-\varepsilon/2)=1$. Let $v_t^{\lambda,\mu}$ denote the number of vacant sites in $c_{\lambda,\mu}$ (for $\eta^{\lambda}_{t\log(1/\lambda)}$). Then $\tilde{Z}_{t+\varepsilon}^{\lambda,\mu}(c)=1$ is equivalent to $v_{t+\varepsilon}^{\lambda,\mu}=0$ and one can easily check that $\tilde{Z}_t^{\lambda,\mu}(c)>1-\varepsilon/2$ implies that $v_t^{\lambda,\mu}\le(1+\#(c_{\lambda,\mu}))^{\varepsilon/2}\le(1+2A/(\lambda\log(1/\lambda)))^{\varepsilon/2}$.

Since $M((t,s]\times[-A,A])=0$ by assumption, we deduce that $M^{\lambda}_{s\log(1/\lambda)}(i)=M^{\lambda}_{t\log(1/\lambda)}(i)$ for all $i\in I_A^{\lambda}$: no fire starts during $(t\log(1/\lambda),s\log(1/\lambda)]$. Hence, each occupied site at time $t\log(1/\lambda)$ remains occupied at time $s\log(1/\lambda)$ and each vacant site at time $t\log(1/\lambda)$ becomes occupied at time $s\log(1/\lambda)$ with probability $1-e^{(t-s)\log(1/\lambda)}=1-\lambda^{\varepsilon}$. Thus,

$$\mathbb{P}_M\big(\tilde{Z}_{t+\varepsilon}^{\lambda,\mu}(c)=1|\tilde{Z}_t^{\lambda,\mu}(c)>1-\varepsilon/2\big)\ge(1-\lambda^{\varepsilon})^{(1+2A/(\lambda\log(1/\lambda)))^{\varepsilon/2}},$$

which tends to 1 as $\lambda\to 0$. □

We end our preliminaries with a last lemma, which deals with estimates concerning the time needed to occupy vacant zones.

LEMMA 16. *Let $\mu\in(0,1]$. Let $(\zeta_0^{\lambda}(i))_{i\in I_A^{\lambda}}\in\{0,1\}^{I_A^{\lambda}}$ and consider a family of i.i.d. Poisson processes $(P_t^{\lambda}(i))_{t\ge 0,i\in I_A^{\lambda}}$, with rate* $\log(1/\lambda)$, *independent of ζ_0^{λ}. Set $\zeta_t^{\lambda}(i)=\min(\zeta_0^{\lambda}(i)+P_t^{\lambda}(i),1)$.*

1. *Let $J=(a,b)\subset(-A,A)$ and $h\in[0,1]$. Set $v_t^{\lambda,\mu}=\#\{i\in J_{\lambda,\mu},\zeta_t^{\lambda}(i)=0\}$. Assume that*

$$\forall\varepsilon>0\qquad \mathbb{P}\bigg(\bigg|\frac{\log(1+v_0^{\lambda,\mu})}{\log(1+\#(J_{\lambda,\mu}))}-h\bigg|\ge\varepsilon\bigg)=0.$$

   (a) *Then, for all $T>0$ and $\varepsilon>0$,*

$$\lim_{\lambda\to 0}\mathbb{P}\bigg(\sup_{[0,T]}\bigg|\frac{\log(1+v_t^{\lambda,\mu})}{\log(1+\#(J_{\lambda,\mu}))}-(h-t)_+\bigg|\ge\varepsilon\bigg)=0.$$

   (b) *If the family $(\zeta_0^{\lambda}(i))_{i\in J_{\lambda,\mu}}$ is exchangeable, then, for all $x\in J$, $T>0$ and $\varepsilon>0$,*

$$\lim_{\lambda\to 0}\mathbb{P}\bigg(\sup_{[0,T]}\bigg|\frac{\log(1+\#(G_t^{\lambda}(x)))}{\log(1/\lambda)}-\big(1-(h-t)_+\big)\bigg|\ge\varepsilon\bigg)=0,$$

   where $G_t^{\lambda}(x)$ is the connected component of occupied sites around $\lfloor x/\lambda\log(1/\lambda)\rfloor$ in ζ_t^{λ}.

2. *Let $x\in(-A,A)$ and $h\in[0,1]$. Set $v_t^{\lambda,\mu}=\#\{i\in x_{\lambda,\mu},\zeta_t^{\lambda}(i)=0\}$. Assume that*

$$\forall\varepsilon>0\qquad \mathbb{P}\bigg(\bigg|\frac{\log(1+v_0^{\lambda,\mu})}{\log(1+\#(x_{\lambda,\mu}))}-h\bigg|\ge\varepsilon\bigg)=0.$$

Then, for all $T > 0$ and $\varepsilon > 0$,

$$\lim_{\lambda\to 0}\mathbb{P}\left(\sup_{[0,T]}\left|\frac{\log(1+v_t^{\lambda,\mu})}{\log(1+\#(x_{\lambda,\mu}))}-(h-t)_+\right|\geq\varepsilon\right)=0.$$

PROOF. The proof of part 2 is the same as that of 1(a) because $\log(1+\#(J_{\lambda,\mu}))\sim\log(1+\#(x_{\lambda,\mu}))\sim\log(1/\lambda)$ as $\lambda\to 0$. Thus, we only prove 1 and everywhere replace $\log(1+\#(x_{\lambda,\mu}))$ by $\log(1/\lambda)$ without difficulty. By assumption, for all $\varepsilon > 0$, we have $\lim_{\lambda\to 0}\mathbb{P}(v_0^{\lambda,\mu}\in(\lambda^{\varepsilon-h}-1,\lambda^{-\varepsilon-h}))=1$. We define $h_t=(h-t)_+$, $V_t^{\lambda,\mu}=\log(1+v_t^{\lambda,\mu})/\log(1/\lambda)$ and, finally, $\Gamma_t^\lambda=\log(1+\#(G_t^\lambda(x)))/\log(1/\lambda)$.

Step 1. Let $t\geq 0$ be fixed. We first show that for all $\varepsilon>0$, $\lim_{\lambda\to 0}\mathbb{P}(|V_t^{\lambda,\mu}-h_t|\geq\varepsilon)=0$. Conditionally on $v_0^{\lambda,\mu}$, the random variable $v_t^{\lambda,\mu}$ follows a binomial distribution $B(v_0^{\lambda,\mu},\lambda^t)$ because each vacant site at time 0 remains vacant with probability $e^{-t\log(1/\lambda)}=\lambda^t$.

Case $h_t>0$. Let $\varepsilon\in(0,h_t)$. We have to prove that $\mathbb{P}(v_t^{\lambda,\mu}\in(\lambda^{\varepsilon-h_t},\lambda^{-\varepsilon-h_t}))\to 1$. We know that $\lim_{\lambda\to 0}\mathbb{P}(v_0^{\lambda,\mu}\in(\lambda^{\varepsilon/2-h},\lambda^{-\varepsilon/2-h}))=1$. The Bienaymé–Chebyshev inequality implies that

$$\begin{aligned}
&P[|v_t^{\lambda,\mu}-v_0^{\lambda}\lambda^t|\leq(v_0^{\lambda,\mu}\lambda^t)^{2/3}|v_0^{\lambda,\mu}\in(\lambda^{\varepsilon/2-h},\lambda^{-\varepsilon/2-h})]\\
&\qquad\geq 1-\mathbb{E}[v_0^{\lambda,\mu}\lambda^t(1-\lambda^t)(v_0^{\lambda,\mu}\lambda^t)^{-4/3}|v_0^{\lambda,\mu}\in(\lambda^{\varepsilon/2-h},\lambda^{-\varepsilon/2-h})]\\
&\qquad\geq 1-\mathbb{E}[(v_0^{\lambda,\mu}\lambda^t)^{-1/3}|v_0^{\lambda,\mu}\in(\lambda^{\varepsilon/2-h},\lambda^{-\varepsilon/2-h})]\\
&\qquad\geq 1-(\lambda^{\varepsilon/2-h+t})^{-1/3},
\end{aligned}$$

which tends to 1 since $h_t=h-t>\varepsilon$.

However, the events

$$|v_t^{\lambda,\mu}-v_0^{\lambda,\mu}\lambda^t|\leq(v_0^{\lambda,\mu}\lambda^t)^{2/3}\quad\text{and}\quad v_0^{\lambda,\mu}\in(\lambda^{\varepsilon/2-h},\lambda^{-\varepsilon/2-h})$$

imply that $v_t^{\lambda,\mu}\in(\lambda^{\varepsilon/2-h_t}-(\lambda^{-\varepsilon/2-h_t})^{2/3},\lambda^{-\varepsilon/2-h_t}+(\lambda^{-\varepsilon/2-h_t})^{2/3})\subset(\lambda^{\varepsilon-h_t},\lambda^{-\varepsilon-h_t})$ for λ small enough, hence the result.

Case $h_t=0$. We have to show that for all $\varepsilon>0$, $\lim_{\lambda\to 0}\mathbb{P}(v_t^{\lambda,\mu}>\lambda^{-\varepsilon})=0$, and it suffices to check that $\lim_{\lambda\to 0}\mathbb{P}(v_t^{\lambda,\mu}>\lambda^{-\varepsilon}|v_0^{\lambda,\mu}<\lambda^{-\varepsilon/2-h})=0$. However,

$$\begin{aligned}
&\mathbb{P}(v_t^{\lambda,\mu}>\lambda^{-\varepsilon}|v_0^{\lambda,\mu}<\lambda^{-\varepsilon/2-h})\\
&\qquad\leq\lambda^{\varepsilon}\mathbb{E}[v_t^{\lambda,\mu}|v_0^{\lambda,\mu}<\lambda^{-\varepsilon/2-h}]=\lambda^{\varepsilon}\mathbb{E}[v_0^{\lambda,\mu}\lambda^t|v_0^{\lambda,\mu}<\lambda^{-\varepsilon/2-h}]\\
&\qquad\leq\lambda^{\varepsilon+t}\lambda^{-\varepsilon/2-h}=\lambda^{\varepsilon/2+t-h},
\end{aligned}$$

which tends to 0 since, by assumption, $t - h \geq 0$.

Step 2. We now prove that for all $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}(|\Gamma_t^\lambda - (1-h_t)| \geq \varepsilon) = 0$. It suffices to check that $\lim_{\lambda\to 0} \mathbb{P}(\#(G_t^\lambda(x)) \in (\lambda^{\varepsilon+h_t-1} - 1, \lambda^{-\varepsilon+h_t-1})) = 1$. However, we know from Step 1 that there are approximately $(1/\lambda)^{h_t}$ vacant sites in $J_{\lambda,\mu}$, and $\#(J_{\lambda,\mu}) \simeq (1/\lambda \log(1/\lambda))$. We also know that the family $(\zeta_t^\lambda(i))_{i\in J_{\lambda,\mu}}$ is exchangeable so that the vacant sites are uniformly distributed in $J_{\lambda,\mu}$ (this statement is slightly misleading: there cannot be two vacant sites at the same place). We conclude that $\#(G_t^\lambda(x)) \simeq (1/\lambda \log(1/\lambda))/(1/\lambda)^{h_t} \simeq \lambda^{h_t-1}$. This can be done rigorously without difficulty.

Step 3. We now prove 1(a), which relies on Step 1 and an ad hoc version of Dini's theorem. Let $\varepsilon > 0$. Consider a subdivision $0 = t_0 < t_1 < \cdots < t_l = T$ with $t_{i+1} - t_i < \varepsilon/2$. Using Step 1, we have $\lim_{\lambda\to 0} \mathbb{P}[\max_{i=0,\dots,l} |V_{t_i}^{\lambda,\mu} - (h-t_i)_+| > \varepsilon/2] = 0$.

Now, observe that $t \mapsto V_t^{\lambda,\mu}$ and $t \mapsto (h-t)_+$ are a.s. nonincreasing and that $t \mapsto (h-t)_+$ is Lipschitz continuous with Lipschitz constant 1.

We deduce that $\sup_{[0,T]} |V_t^{\lambda,\mu} - (h-t)_+| \leq \varepsilon/2 + \max_{i=0,\dots,l}\{|V_{t_i}^{\lambda,\mu} - (h - t_i)_+|\}$. Thus, $\mathbb{P}(\sup_{[0,T]} |V_t^{\lambda,\mu} - (h-t)_+| > \varepsilon) \leq \mathbb{P}[\max_{i=0,\dots,l} |V_{t_i}^{\lambda,\mu} - (h-t_i)_+| > \varepsilon/2]$, which completes the proof of 1(a).

Step 4. Point 1(b) is deduced from Step 2 exactly as point 1(a) was deduced from Step 1, using the fact that $t \mapsto \Gamma_t^\lambda$ and $t \mapsto 1 - h_t$ are a.s. nondecreasing. □

We may now finally tackle the following proof.

PROOF OF PROPOSITION 13. For $x \in (-A, A)$ and $t \geq 0$, we introduce $Z_t(x-) = \lim_{y\to x, y<x} Z_t(y)$ and $Z_t(x+) = \lim_{y\to x, y>x} Z_t(y)$, which represent the values of Z_t in the cells on the left and right of x. If $x \in \mathcal{B}_n$, it is at the boundary of two cells $c_-, c_+ \in \mathcal{C}_n$, and then $Z_t(x-) = Z_t(c_-)$ and $Z_t(x+) = Z_t(c_+)$.

For $x \in \mathcal{B}_n$ and $t \geq 0$, we set $\tilde{H}_t(x) = \max(H_t(x), 1 - Z_t(x), 1 - Z_t(x-), 1 - Z_t(x+))$. Observe that for the LFFP, x is *microscopic* (or *acts like a barrier*) if and only if $\tilde{H}_t(x) > 0$ and, if so, it will remain microscopic during exactly $[t, t + \tilde{H}_t(x))$. Note that, in fact, $Z_t(x)$ always equals either $Z_t(x-)$ or $Z_t(x+)$.

We consider the set of times $\mathcal{K} := \{t \in \{0, T\}$: there exists $x \in (-A, A)$ such that $\tilde{H}_t(x) = 0$ but $\tilde{H}_{t-\varepsilon}(x) > 0$ for all $\varepsilon > 0$ small enough$\}$. By construction, we see that $\mathcal{K} \subset \{1, T_i + 1, T_i + Z_{T_i-}(X_i), i = 1, \dots, n\} \subset \{1, T_i + 1, T_i + (T_i - T_j), 0 \leq j < i \leq n\}$.

We work conditionally on M, by induction on $q = 0, \dots, n$. Consider the following assumption.

$(\mathcal{H}_q)$: (i) For all $0 < \mu \leq 1$, $c \in \mathcal{C}_q$ and $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}_M(|\tilde{Z}_{T_q}^{\lambda,\mu}(c) - Z_{T_q}(c)| > \varepsilon) = 0$.

(ii) For all $x \in \mathcal{B}_q$, $0 < \mu \le 1$ and $\varepsilon > 0$, $\lim_{\lambda \to 0} \mathbb{P}_M(|\tilde{H}^{\lambda,\mu}_{T_q}(x) - \tilde{H}_{T_q}(x)| > \varepsilon) = 0$.

(iii) For all $0 < \mu \le 1$, $\lim_{\lambda \to 0} \mathbb{P}_M(\mathcal{E}^{\lambda,\mu}_q) = 1$ (recall Lemma 14).

First, $(\mathcal{H}_0)$ is obviously satisfied because $T_0 = 0$, $\mathcal{C}_0 = (-A, A)$, $\tilde{Z}^{\lambda,\mu}_0((-A, A)) = 0 = Z_0((-A, A))$, $\mathcal{B}_0 = \varnothing$ and $\mathcal{E}^{\lambda,\mu}_0 = \Omega$.

The proposition will essentially be proven if we check that for $q = 0, \dots, n-1$, $(\mathcal{H}_q)$ implies:

(a) for $c \in \mathcal{C}_q$, $0 < \mu \le 1$ and $\varepsilon > 0$, $\lim_{\lambda \to 0} \mathbb{P}_M(\sup_{[T_q, T_{q+1})} |\tilde{Z}^{\lambda,\mu}_t(c) - Z_t(c)| > \varepsilon) = 0$;

(b) for $x \in (-A, A) \setminus \mathcal{B}_q$, $\varepsilon > 0$, $\lim_{\lambda \to 0} \mathbb{P}_M(\sup_{[T_q, T_{q+1})} |Z^\lambda_t(x) - Z_t(x)| > \varepsilon) = 0$;

(c) for $x \in \mathcal{B}_q$, $t \in [T_q, T_{q+1})$, $0 < \mu \le 1$ and $\varepsilon > 0$, $\lim_{\lambda \to 0} \mathbb{P}_M(|\tilde{H}^{\lambda,\mu}_t(x) - \tilde{H}_t(x)| > \varepsilon)$;

(d) for $x \in (-A, A) \setminus \mathcal{B}_q$, $t \in (T_q, T_{q+1}) \setminus \mathcal{K}$ and $\varepsilon > 0$, $\lim_{\lambda \to 0} \mathbb{P}_M(\delta(D^\lambda_t(x), D_t(x)) > \varepsilon) = 0$;

(e) for $x \in (-A, A) \setminus \mathcal{B}_q$, $\varepsilon > 0$, $\lim_{\lambda \to 0} \mathbb{P}_M(\int_{T_q}^{T_{q+1}} \delta(D^\lambda_t(x), D_t(x))\, dt > \varepsilon) = 0$;

(f) $(\mathcal{H}_{q+1})$ holds.

We thus assume $(\mathcal{H}_q)$ for some fixed $q \in \{0, \dots, n-1\}$ and prove points (a)–(f). Below, we repeatedly use the fact that on the time interval $[T_q, T_{q+1})$, there are no fires at all in $(-A, A)$ for the LFFP and no fires at all during $[T_q \log(1/\lambda), T_{q+1} \log(1/\lambda))$ for the λ-FFP.

Set $\zeta^\lambda_0(i) = \eta^\lambda_{T_q \log(1/\lambda)}(i)$ and consider the i.i.d. Poisson processes $P^\lambda_t(i) = N_{(T_q+t)\log(1/\lambda)}(i) - N_{T_q \log(1/\lambda)}(i)$ with rate $\log(1/\lambda)$. Then, for $t \in [T_q, T_{q+1})$, $\eta^\lambda_{t \log(1/\lambda)}(i) = \min(\zeta_0(i) + P^\lambda_{t - T_q}(i), 1)$.

Point (a). Let $0 < \mu \le 1$. Let $c \in \mathcal{C}_q$. Observe that $(\mathcal{H}_q)$(i) says precisely that with $h = 1 - Z_{T_q}(c) \in [0, 1]$, $\log(1 + \#\{k \in c_{\lambda,\mu}, \zeta^\lambda_0(k) = 0\}) / \log(1 + \#(c_{\lambda,\mu}))$ tends to h in probability (for $\mathbb{P}_M$). Applying part 1(a) of Lemma 16 (with $J = c$), we get that $\sup_{[T_q, T_{q+1})} |1 - \tilde{Z}^{\lambda,\mu}_t(c) - (h - (t - T_q))_+|$ tends to 0 in probability (for $\mathbb{P}_M$). However, for $t \in [T_q, T_{q+1})$, we have $Z_t(c) = \min(Z_{T_q}(c) + (t - T_q), 1) = \min(1 - h + (t - T_q), 1) = 1 - (h - (t - T_q))_+$. Point (a) then follows.

Point (b). Now, let $x \in (-A, A) \setminus \mathcal{B}_q$. Then $x \in c$, for some $c \in \mathcal{C}_q$. Due to Lemma 14, we know that $(\zeta^\lambda_0(i))_{i \in c_{\lambda,\mu}}$ are exchangeable on $\mathcal{E}^{\lambda,1}_q$. The previous reasoning, using part 1(b) of part 1(a) of Lemma 16, shows that for all $\varepsilon > 0$, $\lim_{\lambda \to 0} \mathbb{P}_M(\mathcal{E}^{\lambda,1}_q \cap \{\sup_{[T_q, T_{q+1})} |Z^\lambda_t(x) - Z_t(x)| > \varepsilon\}) = 0$. Using $(\mathcal{H}_q)$(iii) for $\mu = 1$, we are done.

Point (c). Let $0 < \mu \le 1$. Let $x \in \mathcal{B}_q$ and set $h = \tilde{H}_{T_q}(x)$. We know by $(\mathcal{H}_q)$(ii) that $\tilde{H}^{\lambda,\mu}_{T_q}(x)$ tends to $\tilde{H}_{T_q}(x) = h$ in probability (for $\mathbb{P}_M$). Now, using part 2(a) of Lemma 16, we deduce that $\sup_{[T_q,T_{q+1})}|\tilde{H}_t^{\lambda,\mu}(x)-(h-(t-T_q))_+|$ tends to 0 in probability (for $\mathbb{P}_M$). We conclude by observing that, by construction, $\tilde{H}_t(x)=(h-(t-T_q))_+$ for $t\in[T_q,T_{q+1})$.

Point (d). Let $x\in(-A,A)\setminus\mathcal{B}_q$ and $t\in(T_q,T_{q+1})\setminus\mathcal{K}$ be fixed.

Case $Z_t(x)<1$. In this case, $D_t(x)=\{x\}$ so that $\delta(D_t(x),D_t^\lambda(x))=|D_t^\lambda(x)|$. However, from (1), (2), we get that $|D_t^\lambda(x)|\le\lambda^{1-Z_t^\lambda(x)}\log(1/\lambda)$. Since we know from (b) that $Z_t^\lambda(x)$ goes to $Z_t(x)<1$ in probability (for $\mathbb{P}_M$), we easily deduce that $|D_t^\lambda(x)|$ goes to 0 in probability (for $\mathbb{P}_M$).

Case $Z_t(x)=1$. In this case, $D_t(x)=[a,b]$ for some $a,b\in\mathcal{B}_q\cup\{-A,A\}$. We assume that $-A<a<b<A$ for simplicity, the other cases being treated in a similar way. We thus have $Z_t(c)=1$ for all $c\in\mathcal{C}_q$ with $c\subset(a,b)$, $\tilde{H}_t(y)=0$ for all $y\in\mathcal{B}_q\cap(a,b)$ and $\tilde{H}_t(a)\tilde{H}_t(b)>0$.

On the one hand, we prove that for any $\varepsilon>0$, $\lim_{\lambda\to0}\mathbb{P}_M(D_t^\lambda(x)\subset[a-\varepsilon,b+\varepsilon])=1$. Let us consider, for example, the left boundary a and prove that $\lim_{\lambda\to0}\mathbb{P}_M(D_t^\lambda(x)\subset[a-\varepsilon,A])=1$.

We have $\tilde{H}_t(a)=h_a>0$. We deduce from (c) that $\lim_{\lambda\to0}\mathbb{P}_M(\tilde{H}_t^{\lambda,1}(a)\ge h_a/2)=1$, which implies that there are vacant sites in $a_{\lambda,1}$, that is, $\lim_{\lambda\to0}\mathbb{P}_M(\exists i\in a_{\lambda,1},\eta_{t\log(1/\lambda)}(i)=0)=1$. Recalling the definition of $a_{\lambda,1}$ [see (10)], we see that this implies that $\lim_{\lambda\to0}\mathbb{P}_M(D_t^\lambda(x)\subset[a-1/\log(1/\lambda),A])=1$, hence $\lim_{\lambda\to0}\mathbb{P}_M(D_t^\lambda(x)\subset[a-\varepsilon,A])=1$ for any $\varepsilon>0$.

On the other hand, we prove that $\lim_{\lambda\to0}\mathbb{P}_M((a+1/\log(1/\lambda),b-1/\log(1/\lambda))\subset D_t^\lambda(x))=1$. Since $t\notin\mathcal{K}$, we deduce that there exists $s\in(T_q,t)$ such that $Z_s(c)=1$ for all $c\in\mathcal{C}_q$ with $c\subset(a,b)$ and $\tilde{H}_s(y)=0$ for all $y\in\mathcal{B}_q\cap(a,b)$. We deduce from (a) that for all $c\in\mathcal{C}_q$ with $c\subset(a,b)$, $\lim_{\lambda\to0}\mathbb{P}_M(\tilde{Z}_s^{\lambda,1}(c)>1-\varepsilon)=0$, whence, by Lemma 15(i), $\lim_{\lambda\to0}\mathbb{P}_M(\tilde{Z}_t^{\lambda,1}(c)=1)=1$. Similarly, we deduce from (c) that for all $y\in\mathcal{B}_q$ with $y\in(a,b)$, $\lim_{\lambda\to0}\mathbb{P}_M(\tilde{H}_s^{\lambda,1}(y)>\varepsilon)=0$, whence, by Lemma 15(ii), $\lim_{\lambda\to0}\mathbb{P}_M(\tilde{H}_t^{\lambda,1}(y)=0)=1$. As a consequence, $\lim_{\lambda\to0}\mathbb{P}_M((a+1/\log(1/\lambda),b-1/\log(1/\lambda))\subset D_t^\lambda(x))=1$.

This completes the proof of point (d).

Point (e). Point (e) follows from (d). Indeed, observe that $\delta(I,J)\le2A$ for any intervals $I,J\subset(-A,A)$. Thus, for $x\in(-A,A)\setminus\mathcal{B}_q$, (d) implies that for $t\in[T_q,T_{q+1})\setminus\mathcal{K}$, $\lim_{\lambda\to0}\mathbb{E}_M(\delta(D_t^\lambda(x),D_t(x)))=0$. Since $\mathcal{K}$ is now finite, we deduce from Lebesgue's dominated convergence theorem that $\lim_{\lambda\to0}\int_{T_q}^{T_{q+1}}\mathbb{E}_M(\delta(D_t^\lambda(x),D_t(x)))\,dt=0$, from which (e) follows.

Point (f). Here, we show that $(\mathcal{H}_{q+1})$ holds. We set $z:=Z_{T_{q+1}-}(X_{q+1})$ and separately treat the cases $z\in(0,1)$ and $z=1$. We a.s. never have $z=0$ because $Z_{T_{q+1}-}(X_{q+1}) = \min(Z_{T_q}(X_{q+1}) + (T_{q+1} - T_q), 1)$ with $Z_{T_q}(X_{q+1}) \geq 0$ and $T_{q+1} > T_q$.

Case $z \in (0, 1)$. We fix $\mu \in (0, 1]$. In that case, $D_{T_{q+1}-}(X_{q+1}) = \{X_{q+1}\}$ and for all $c \in \mathcal{C}_{q+1}$ (thus $c \subset \tilde{c}$ for some $\tilde{c} \in \mathcal{C}_q$), $Z_{T_{q+1}}(c) = Z_{T_{q+1}-}(c)$. We have $\tilde{H}_{T_{q+1}}(X_{q+1}) = \max(z, 1-z)$ and for all $x \in \mathcal{B}_q$, $\tilde{H}_{T_{q+1}}(x) = \tilde{H}_{T_{q+1}-}(x)$. Consider the event $\Omega_\alpha^\lambda = \{Z^\lambda_{T_{q+1}-}(X_{q+1}) \leq z + \alpha\}$ for some $\alpha \in (0, 1-z)$. Point (b) implies that $\lim_{\lambda\to 0} \mathbb{P}_M(\Omega_\alpha^\lambda) = 1$ (because $X_{q+1} \notin \mathcal{B}_q$).

- On Ω_α^λ, we have $\#(C^\lambda_{T_{q+1}\log(1/\lambda)-}(X_{q+1})) \leq (1/\lambda)^{z+\alpha}$ [see (2)]. Since $z+\alpha < 1$, we deduce that on Ω_α^λ, we have $\#(C^\lambda_{T_{q+1}\log(1/\lambda)-}(X_{q+1})) < \mu/(2\lambda\log^2(1/\lambda))$ (for all μ, provided that $\lambda > 0$ is small enough). Thus, on Ω_α^λ, for all $c \in \mathcal{C}_{q+1}$, there is a vacant site (strictly) between $c_{\lambda,\mu}$ and $C^\lambda_{T_{q+1}\log(1/\lambda)-}(X_{q+1})$. Hence, $\mathcal{E}_q^{\lambda,\mu} \cap \Omega_\alpha^\lambda \subset \mathcal{E}_{q+1}^{\lambda,\mu}$. Using $(\mathcal{H}_q)$(iii), we deduce that $\lim_{\lambda\to 0} \mathbb{P}_M(\mathcal{E}_{q+1}^{\lambda,\mu}) = 1$.
- This also implies that on Ω_α^λ, for all $c \in \mathcal{C}_{q+1}$, we have $\tilde{Z}^{\lambda,\mu}_{T_{q+1}}(c) = \tilde{Z}^{\lambda,\mu}_{T_{q+1}-}(c)$ and thus point (a) and $\lim_{\lambda\to 0} \mathbb{P}_M(\Omega_\alpha^\lambda) = 1$ imply that $\lim_{\lambda\to 0} \mathbb{P}_M(|\tilde{Z}^{\lambda,\mu}_{T_{q+1}}(c) - Z_{T_{q+1}}(c)| \geq \varepsilon) = 0$ for all $\varepsilon > 0$.
- For $x \in \mathcal{B}_{q+1} \setminus \{X_{q+1}\} = \mathcal{B}_q$, still on Ω_α^λ, we also have $\tilde{H}^{\lambda,\mu}_{T_{q+1}}(x) = \tilde{H}^{\lambda,\mu}_{T_{q+1}-}(x)$, thus point (c) allows us to conclude that $(\mathcal{H}_{q+1})$(ii) holds for those points x.

  We now show that $\lim_{\lambda\to 0} \mathbb{P}_M(|\tilde{H}^{\lambda,\mu}_{T_{q+1}}(X_{q+1}) - \tilde{H}_{T_{q+1}}(X_{q+1})| \geq \varepsilon) = 0$ for all $\varepsilon > 0$, which implies that $(\mathcal{H}_{q+1})$(ii) holds for $x = X_{q+1}$. Recall that $\tilde{H}_{T_{q+1}}(X_{q+1}) = \max(z, 1-z)$. Consider $c \in \mathcal{C}_q$ such that $X_{q+1} \in c$ and denote by $v_t^{\lambda,\mu}$ the number of vacant sites in $x_{\lambda,\mu}$ at time $t\log(1/\lambda)$. Point (a) implies that at time $T_{q+1}\log(1/\lambda)-$, there are around $(1/\lambda)^{1-z}$ vacant sites in $c_{\lambda,\mu}$. Thus, by exchangeability of the family $(\eta^\lambda_{T_{q+1}\log(1/\lambda)-}(i))_{i\in c_{\lambda,\mu}}$ (on the event $\mathcal{E}_q^{\lambda,\mu}$, see Lemma 14), since $x_{\lambda,\mu} \subset c_{\lambda,\mu}$ and $\#(x_{\lambda,\mu})/\#(c_{\lambda,\mu}) \simeq 1/\log(1/\lambda)$, we deduce that $v^{\lambda,\mu}_{T_{q+1}-} \simeq (1/\lambda)^{1-z}/\log(1/\lambda) \simeq (1/\lambda)^{1-z}$ on $\mathcal{E}_q^{\lambda,\mu}$. On the other hand, recalling (2), we have $\#(C^\lambda_{T_{q+1}\log(1/\lambda)-}(X_{q+1})) \simeq (1/\lambda)^z$. At time $T_{q+1}\log(1/\lambda)$, this component is destroyed. Thus, still on $\mathcal{E}_q^{\lambda,\mu}$, $v^{\lambda,\mu}_{T_{q+1}} = v^{\lambda,\mu}_{T_{q+1}-} + \#(C^\lambda_{T_{q+1}\log(1/\lambda)}(X_{q+1})) \simeq (1/\lambda)^{1-z} + (1/\lambda)^z \simeq (1/\lambda)^{\max(z,1-z)}$. We conclude that $\tilde{H}^{\lambda,\mu}_{T_{q+1}}(X_{q+1}) = \log(1 + v^{\lambda,\mu}_{T_{q+1}})/\log(\#((X_{q+1})_{\lambda,\mu})) \simeq \max(z, 1 - z) = \tilde{H}_{T_{q+1}}(X_{q+1})$. All of this can be done rigorously without difficulty and we deduce that for $\varepsilon > 0$ and all $\mu \in (0, 1]$, $\lim_{\lambda\to 0} \mathbb{P}_M(|\tilde{H}^{\lambda,\mu}_{T_{q+1}}(X_{q+1}) - \tilde{H}_{T_{q+1}}(X_{q+1})| \geq \varepsilon) = 0$.

Case $z = 1$. Let $a, b \in \mathcal{B}_q \cup \{-A, A\}$ be such that $D_{T_{q+1}-}(X_{q+1}) = [a, b]$. We assume that $a, b \in \mathcal{B}_q$, the other cases being treated in a similar way. We thus have

$h_a := \tilde{H}_{T_{q+1}-}(a) > 0$, $h_b := \tilde{H}_{T_{q+1}-}(b) > 0$. We also have $\tilde{H}_{T_{q+1}}(x) = \tilde{H}_{T_{q+1}-}(x)$ for all $x \in \mathcal{B}_q \setminus [a, b]$, $\tilde{H}_{T_{q+1}}(x) = 1$ for all $x \in \mathcal{B}_q \cap (a, b)$, $Z_{T_{q+1}}(c) = Z_{T_{q+1}-}(c)$ for all $c \in \mathcal{C}_{q+1}$ with $c \cap (a, b) = \varnothing$ and $Z_{T_{q+1}}(c) = 0$ for all $c \in \mathcal{C}_{q+1}$ with $c \subset (a, b)$.

Let $\mu \in (0, 1]$. Now, consider $\tilde{\Omega}^{\lambda,\mu}$, the event that for all $c \in \mathcal{C}_q$ such that $c \subset (a, b)$, we have $\tilde{Z}^{\lambda,\mu}_{T_{q+1}-}(c) = 1$, that $\tilde{H}^{\lambda,\mu}_{T_{q+1}-}(a) > 0$, that $\tilde{H}^{\lambda,\mu}_{T_{q+1}-}(b) > 0$ and that for all $x \in \mathcal{B}_q \cap (a, b)$, $\tilde{H}^{\lambda,\mu}_{T_{q+1}-}(x) = 0$. Then (a), (c) and Lemma 15 collectively imply that $\lim_{\lambda\to 0} \mathbb{P}_M(\tilde{\Omega}^{\lambda,\mu}) = 1$ for all $\mu \in (0, 1]$.

- We can easily check that $\mathcal{E}^{\lambda,\mu}_q \cap \tilde{\Omega}^{\lambda,\mu} \subset \mathcal{E}^{\lambda,\mu}_{q+1}$ (because for $c \in \mathcal{C}_{q+1}$ with $c \subset [a, b]$, we have $c_{\lambda,\mu} \subset C^{\lambda}_{T_{q+1}\log(1/\lambda)-}(X_{q+1})$, while for $c \in \mathcal{C}_{q+1}$ with $c \cap [a, b] = \varnothing$, the vacant sites in $a_{\lambda,\mu}$ and $b_{\lambda,\mu}$ separate $c_{\lambda,\mu}$ from $C^{\lambda}_{T_{q+1}\log(1/\lambda)-}(X_{q+1})$). As a consequence, $(\mathcal{H}_{q+1})$(iii) holds for all $\mu \in (0, 1]$.
- On $\tilde{\Omega}^{\lambda,\mu}$, we have $\tilde{Z}^{\lambda,\mu}_{T_{q+1}}(c) = 0 = Z_{T_{q+1}}(c)$ for all $c \in \mathcal{C}_{q+1}$ with $c \subset [a, b]$, and $\tilde{Z}^{\lambda,\mu}_{T_{q+1}}(c) = \tilde{Z}^{\lambda,\mu}_{T_{q+1}-}(c)$ for $c \in \mathcal{C}_{q+1}$ with $c \cap (a, b) = \varnothing$, from which $(\mathcal{H}_{q+1})$(i) easily follows [using (a)].
- We also have, still on $\tilde{\Omega}^{\lambda,\mu}$, that $\tilde{H}^{\lambda,\mu}_{T_{q+1}}(x) = 1 = \tilde{H}_{T_{q+1}}(x)$ for all $x \in \mathcal{B}_{q+1}$ with $x \in (a, b)$, and $(\mathcal{H}_{q+1})$(ii) follows for those x. For $x \in \mathcal{B}_{q+1}$ with $x \notin [a, b]$, we have $\tilde{H}^{\lambda,\mu}_{T_{q+1}}(x) = \tilde{H}^{\lambda,\mu}_{T_{q+1}-}(x)$, hence $(\mathcal{H}_{q+1})$(ii) follows by point (c).

  Finally, we have to check that $(\mathcal{H}_{q+1})$(ii) holds for $x = a$ and $x = b$. Consider, for example, the case of a. Here, we are in the situation where $Z_{T_{q+1}}(a+) = 0$ so that, of course, $\tilde{H}_{T_{q+1}}(a) = 1$. Let c be the cell containing $a+$. We know that $\tilde{Z}^{\lambda,\mu/2}_{T_{q+1}-}(c) = 1$ which, on $\tilde{\Omega}^{\lambda,\mu/2}$, implies that all sites between $a + \frac{\mu}{2\log(1/\lambda)}$ and $a + \frac{\mu}{\log(1/\lambda)}$, that is, on an interval of length $\frac{\mu}{2\log(1/\lambda)}$, are empty at time T_{q+1}, showing that a fixed proportion of $a_{\lambda,\mu}$ is empty. Recalling that $\lim_{\lambda\to 0} \mathbb{P}_M(\tilde{\Omega}^{\lambda,\mu/2}) = 1$, it readily follows that for all $\varepsilon > 0$, $\lim_{\lambda\to 0} \mathbb{P}_M(\tilde{H}^{\lambda,\mu}_{T_{q+1}}(a) > 1 - \varepsilon) = 1$. Recalling that $\tilde{H}^{\lambda,\mu}_{T_{q+1}}(a) \le 1$, we conclude that $(\mathcal{H}_{q+1})$(ii) holds for $x = a$.

Conclusion. Using points (b) and (e) above (with $q = 0, \dots, n$), plus very similar arguments on the time interval $(T_n, T]$ (during which there are no fires), we deduce that for all $x_0 \in (-A, A) \setminus \mathcal{B}_n$ and $\varepsilon > 0$,

$$\lim_{\lambda\to 0} \mathbb{P}_M\left(\sup_{[0,T]} |Z^{\lambda}_t(x_0) - Z_t(x_0)| + \int_0^T \delta(D^{\lambda}_t(x_0), D_t(x_0))\, dt \ge \varepsilon\right) = 0.$$

But, of course, for $x_0 \in (-A, A)$, we have $\mathbb{P}(x_0 \in \mathcal{B}_n) = 0$ so that

$$\lim_{\lambda\to 0} \mathbb{P}\left(\sup_{[0,T]} |Z^{\lambda}_t(x_0) - Z_t(x_0)| + \int_0^T \delta(D^{\lambda}_t(x_0), D_t(x_0))\, dt \ge \varepsilon\right) = 0.$$

It remains to prove that for $t \in [0, T]$ and $x_0 \in (-A, A)$, we have

$$\lim_{\lambda \to 0} \mathbb{P}(\delta(D_t^\lambda(x_0), D_t(x_0))) = 0.$$

Case $t \neq 1$. We deduce from point (d) above that if $x_0 \notin \mathcal{B}_n$ and $t \notin \mathcal{K}$, then we have $\lim_{\lambda \to 0} \mathbb{P}_M(\delta(D_t^\lambda(x_0), D_t(x_0))) = 0$. Since $\mathbb{P}(x_0 \in \mathcal{B}_n) = 0$ and $\mathbb{P}(t \in \mathcal{K}) = 0$ (because $t \neq 1$, recalling the definition of $\mathcal{K}$), we easily arrive at the desired conclusion.

Case $t = 1$. In this case, $t \in \mathcal{K}$, but the result still holds. Observe that $Z_1(x_0) = 1$, by construction. Consider $q \in \{0, \dots, n\}$ such that $T_q < 1 < T_{q+1}$ (with the convention that $T_0 = 0$, $T_{n+1} = T$) and consider $a, b \in \mathcal{B}_q \cup \{-A, A\}$ such that $D_1(x_0) = [a, b]$. Using the same arguments as in the proof of (d) (see Step 1), we then easily check that $\lim_{\lambda \to 0} \mathbb{P}_M(D_1^\lambda(x_0) \subset [a - \varepsilon, b + \varepsilon]) = 1$ for all $\varepsilon > 0$ (the set $\mathcal{K}$ was not considered there). We also check, as in the proof of (d) (see Step 2), that for all $y \in \mathcal{B}_q$ with $y \in (a, b)$, $\lim_{\lambda \to 0} \mathbb{P}_M(H_1^{\lambda,1}(y) = 0) = 1$ [the set under consideration there was $\mathcal{K}$, but the time 1 was not useful since 1 is a.s. not a time where some $H(x)$ reaches 0 for the first time]. Finally, we just have to prove that for all $c \in \mathcal{C}_q$ with $c \subset (a, b)$, $\lim_{\lambda \to 0} \mathbb{P}_M(\tilde{Z}_1^{\lambda,1}(c) = 1) = 1$. Thus, let $c \in \mathcal{C}_q$ with $c \subset (a, b)$ and recall that $\lim_{\lambda \to 0} \mathbb{P}_M(\mathcal{E}_q^{\lambda,1}) = 1$. However, on $\mathcal{E}_q^{\lambda,1}$, there are no death events in c_λ during the time interval $[0, \log(1/\lambda)]$, so each site of $c_{\lambda,1}$ is occupied at time $\log(1/\lambda)$ with probability $1 - \lambda$ and, hence, all the sites of $c_{\lambda,1}$ are occupied with probability $(1 - \lambda)^{\#(c_{\lambda,1})}$. Since $\#(c_{\lambda,1}) \leq 2A/(\lambda \log(1/\lambda))$, we get $\mathbb{P}_M(\tilde{Z}_1^{\lambda,1}(c) = 1 | \mathcal{E}_q^{\lambda,1}) \geq (1 - \lambda)^{2A/(\lambda \log(1/\lambda))}$, which tends to 1 as λ tends to 0. Since we know that $\lim_{\lambda \to 0} \mathbb{P}_M(\mathcal{E}_q^{\lambda,1}) = 1$, we deduce that $\lim_{\lambda \to 0} \mathbb{P}_M([a + 1/\log(1/\lambda), b - 1/\log(1/\lambda)] \subset D_1^\lambda(x_0)) = 1$.

Finally, $\lim_{\lambda \to 0} \mathbb{P}_M(\delta(D_1^\lambda(x_0), D_1(x_0)) \geq \varepsilon) = 0$ for all $\varepsilon > 0$, which was our goal. □

5. Cluster size distribution. The aim of this section is to prove Corollary 6. We will use Theorem 5, which asserts that the λ-FFP behaves like the LFFP for $\lambda > 0$ small enough. We start with preliminary results.

LEMMA 17. *Consider an LFFP* $(Z_t(x), D_t(x), H_t(x))_{t \geq 0, x \in \mathbb{R}}$. *We then have the following*:

(i) *for any* $t \in (1, \infty)$, $x \in \mathbb{R}$ *and* $z \in [0, 1)$, $\mathbb{P}[Z_t(x) = z] = 0$;

(ii) *for any* $t \in [0, \infty)$, $B > 0$ *and* $x \in \mathbb{R}$, $P[|D_t(x)| = B] = 0$;

(iii) *there are constants* $C > 0$ *and* $\kappa_1 > 0$ *such that for all* $t \in [0, \infty)$, $x \in \mathbb{R}$ *and* $B > 0$, $\mathbb{P}[|D_t(x)| \geq B] \leq Ce^{-\kappa_1 B}$;

(iv) *there are constants* $c > 0$ *and* $\kappa_2 > 0$ *such that for all* $t \in [3/2, \infty)$, $x \in \mathbb{R}$ *and* $B > 0$, $\mathbb{P}[|D_t(x)| \geq B] \geq ce^{-\kappa_2 B}$;

(v) *there exist constants* $0 < c < C$ *such that for all* $t \geq 5/2$, $0 \leq a < b < 1$ *and* $x \in \mathbb{R}$, $c(b - a) \leq \mathbb{P}(Z_t(x) \in [a, b]) \leq C(b - a)$.

PROOF. By translation invariance, it suffices to treat the case $x = 0$.

Point (i). By Definition 2, we see that for $t \in [0, 1]$, we have a.s. $Z_t(0) = t$. However, for $t > 1$ and $z \in [0, 1)$, $Z_t(0) = z$ implies that the cluster containing 0 has been killed at time $t - z$, so, necessarily, $M(\{t - z\} \times \mathbb{R}) > 0$. This happens with probability 0 since $t - z$ is deterministic.

Point (ii). Recalling Definition 2, we see that for any $t \in [0, T]$, $|D_t(0)|$ is either 0 or of the form $|X_i - X_j|$ (with $i \neq j$), where $(T_i, X_i)_{i\geq 1}$ are the marks of the Poisson measure M. As before, we easily conclude that for $B > 0$, $\mathbb{P}(|D_t(0)| = B) = 0$.

Point (iii). First, if $t \in [0, 1)$, then we have a.s. $|D_t(0)| = 0$ and the result is obvious. Next, consider $t \geq 1$. Recalling Definition 2, we see that $|D_t(0)| = |L_t(0)| + R_t(0)$. Clearly, $|L_t(0)|$ and $R_t(0)$ have the same law. For $B > 0$, $\{R_t(0) > B\} \subset \{M([t - 1/4, t] \times [0, B]) = 0\}$. Indeed, on $\{M([t - 1/4, t] \times [0, B]) > 0\}$, denote by $(\tau, X) \in [t - 1/4, t] \times [0, B]$ a mark of M. Then, either:

- $Z_{\tau-}(X) = 1$, in which case this mark starts a macroscopic fire so that $Z_\tau(X) = 0$ and $Z_s(X) = s - \tau < 1$ for all $s \in [\tau, \tau + 1)$ (since $\tau \in [t - 1/4, t]$, we clearly have $t \in [\tau, \tau + 1)$ so that $Z_t(X) < 1$ and, as a consequence, $R_t(0) \leq X \leq B$); or
- $Z_{\tau-}(X) \in (1/4, 1]$ so that $H_\tau(X) = Z_{\tau-}(X)$ and thus $H_s(X) = Z_{\tau-}(X) - (s - \tau) > 0$ for all $s \in [\tau, \tau + Z_{\tau-}(X))$ (since $\tau \in [t - 1/4, t]$ and $Z_{\tau-}(X) > 1/4$, we have $t \in [\tau, \tau + Z_{\tau-}(X))$, so $H_t(X) > 0$ and, hence, $R_t(0) \leq X \leq B$); or, finally,
- $Z_{\tau-}(X) \leq 1/4$, in which case $Z_s(X) = Z_{\tau-}(X) + (s - \tau) < 1$ for all $s \in (\tau, \tau + 1 - Z_{\tau-}(X))$ and, in particular, $Z_t(X) < 1$, hence $R_t(0) \leq X \leq B$.

As a conclusion, for all $t \geq 1$, $\mathbb{P}[R_t(0) > B] \leq \mathbb{P}[M([t - 1/4, t] \times [0, B]) = 0] = e^{-B/4}$, so $\mathbb{P}[|D_t(0)| > B] \leq \mathbb{P}[|L_t(0)| > B/2] + \mathbb{P}[R_t(0) > B/2] \leq 2e^{-B/8}$.

Point (iv). We first observe that for all (t_0, x_0) such that $M(\{t_0, x_0\}) = 1$, we have $\max(1 - Z_t(x_0), H_t(x_0)) > 0$ for all $t \in [t_0, t_0 + 1/2)$.

Indeed, if $Z_{t_0-}(x_0) = 1$, then $Z_{t_0+s}(x_0) \leq s < 1$ for all $s \in [0, 1)$. If, now, $z = Z_{t_0-}(x_0) < 1$, then $Z_{t_0+s}(x_0) = s + z < 1$ for $s \in [0, 1 - z)$ and $H_{t_0+s}(x_0) = z - s > 0$ for $s \in [0, z)$ so that $\max(1 - Z_{t_0+s}(x_0), H_{t_0+s}(x_0)) > 0$ for all $s \in [0, 1/2)$.

Once this is seen, fix $t \geq 3/2$. Consider the event $\tilde{\Omega}_{t,B} = \tilde{\Omega}^1_{t,B} \cap \tilde{\Omega}^2_t \cap \tilde{\Omega}^3_{t,B}$, where:

- $\tilde{\Omega}^1_{t,B} = \{M([t - 3/2, t] \times [0, B]) = 0\}$;
- $\tilde{\Omega}^2_t$ is the event that in the box $[t - 3/2, t] \times [-1, 0]$, M has exactly four marks, $(S_i, Y_i)_{i=1,\dots,4}$, with $Y_4 < Y_3 < Y_2 < Y_1$, $t - 3/2 < S_1 < t - 1$, $S_1 < S_2 < S_1 + 1/2$, $S_2 < S_3 < S_2 + 1/2$, $S_3 < S_4 < S_3 + 1/2$ and $S_4 + 1/2 > t$.
- $\tilde{\Omega}^3_{t,B}$ is the event that in the box $[t - 3/2, t] \times [B, B + 1]$, M has exactly four marks, $(\tilde{S}_i, \tilde{Y}_i)_{i=1,\dots,4}$, with $\tilde{Y}_1 < \tilde{Y}_2 < \tilde{Y}_3 < \tilde{Y}_4$, $t - 3/2 < \tilde{S}_1 < t - 1$, $\tilde{S}_1 < \tilde{S}_2 < \tilde{S}_1 + 1/2$, $\tilde{S}_2 < \tilde{S}_3 < S_2 + 1/2$, $\tilde{S}_3 < \tilde{S}_4 < \tilde{S}_3 + 1/2$ and $\tilde{S}_4 + 1/2 > t$.

Of course, we have $p := \mathbb{P}(\tilde{\Omega}_t^2) = \mathbb{P}(\tilde{\Omega}_{t,B}^3) > 0$ and this probability does not depend on $t \geq 3/2$ or on $B > 0$. Furthermore, $\mathbb{P}(\tilde{\Omega}_{t,B}^1) = e^{-3B/2}$. These three events being independent, we conclude that $\mathbb{P}(\tilde{\Omega}_{t,B}) \geq p^2 e^{-3B/2}$. To conclude the proof of (iv), it thus suffices to check that $\tilde{\Omega}_{t,B} \subset \{[0, B] \subset D_t(0)\}$. However, on $\tilde{\Omega}_{t,B}$, using the arguments described at the beginning of the proof of point (iv), we observe that:

- the fire starting at (S_2, Y_2) cannot affect $[0, B]$ because at time $S_2 \in [S_1, S_1 + 1/2)$, $H_{S_2}(Y_1) > 0$ or $Z_{S_2}(Y_1) > 0$, with $Y_2 < Y_1 < 0$;
- then the fire starting at (S_3, Y_3) cannot affect $[0, B]$ because at time $S_3 \in [S_2, S_2 + 1/2)$, $H_{S_3}(Y_2) > 0$ or $Z_{S_3}(Y_2) > 0$, with $Y_3 < Y_2 < 0$;
- then the fire starting at (S_4, Y_4) cannot affect $[0, B]$ because at time $S_4 \in [S_3, S_3 + 1/2)$, $H_{S_4}(Y_3) > 0$ or $Z_{S_4}(Y_3) > 0$, with $Y_4 < Y_3 < 0$;
- furthermore, the fires starting to the left of -1 during $(S_1, t]$ cannot affect $[0, B]$ because for all $t \in (S_1, t]$, there is always a site $x_t \in \{Y_1, Y_2, Y_3, Y_4\} \subset [-1, 0]$ with $H_t(x_t) > 0$ or $Z_t(x_t) < 1$;
- the same arguments apply on the right of B.

As a conclusion, the zone $[0, B]$ is not affected by any fire during $(S_1 \vee \tilde{S}_1, t]$. Since the length of this time interval is greater than 1, we deduce that for all $x \in [0, B]$, $Z_t(x) = \min(Z_{S_1 \vee \tilde{S}_1} + t - S_1 \vee \tilde{S}_1, 1) \geq \min(t - S_1 \vee \tilde{S}_1, 1) = 1$ and $H_t(x) = \max(H_{S_1 \vee \tilde{S}_1} - (t - S_1 \vee \tilde{S}_1), 0) \leq \max(1 - (t - S_1 \vee \tilde{S}_1), 0) = 0$, hence that $[0, B] \subset D_t(0)$.

Point (v). We observe, recalling Definition 2, that for $0 \leq a < b < 1$ and $t \geq 1$, we have $Z_t(0) \in [a, b]$ if and only there exists $\tau \in [t - b, t - a]$ such that $Z_\tau(0) = 0$. This happens if and only if $X_{t,a,b} := \int_{t-b}^{t-a} \int_{\mathbb{R}} \mathbf{1}_{\{y \in D_{s-}(0)\}} M(ds, dy) \geq 1$. We deduce that

$$\mathbb{P}\big(Z_t(0) \in [a, b]\big) = \mathbb{P}(X_{t,a,b} \geq 1) \leq \mathbb{E}[X_{t,a,b}] = \int_{t-b}^{t-a} \mathbb{E}[|D_s(0)|]\, ds \leq C(b - a),$$

where we have used point (iii) for the last inequality.

Next, we have $\{M([t - b, t - a] \times D_{t-b}(0)) \geq 1\} \subset \{X_{t,a,b} \geq 1\}$: it suffices to note that a.s. $\{X_{t,a,b} = 0\} \subset \{X_{t,a,b} = 0, D_{t-b}(0) \subset D_s(0) \text{ for all } s \in [t - b, t - a]\} \subset \{M([t - b, t - a] \times D_{t-b}(0)) = 0\}$. Now, since $D_{t-b}(0)$ is $\mathcal{F}_{t-b}^M$-measurable, we deduce that for $t \geq 5/2$,

$$\begin{aligned}\mathbb{P}\big(Z_t(0) \in [a, b]\big) &\geq \mathbb{P}\big[M\big((t - b, t - a] \times D_{t-b}(0)\big) > 0\big] \\ &\geq \mathbb{P}[|D_{t-b}(0)| \geq 1]\big(1 - e^{-(b-a)}\big) \geq c\big(1 - e^{-(b-a)}\big),\end{aligned}$$

where we have used point (iv) (here, $t - b \geq 3/2$) to get the last inequality. This completes the proof since $1 - e^{-x} \geq x/2$ for all $x \in [0, 1]$. □

We now may tackle the following proof.

PROOF OF COROLLARY 6. We thus consider, for each $\lambda > 0$, a λ-FFP $(\eta_t^\lambda)_{t\geq 0}$. Also, let $(Z_t(x), D_t(x), H_t(x))_{t\geq 0, x\in\mathbb{R}}$ be an LFFP.

Point (i). Using Lemma 17(v), we only need to prove that for all $0 \leq a < b < 1$ and all $t \geq 5/2$,

$$\lim_{\lambda\to 0} \mathbb{P}\big(\#\big(C^\lambda_{t\log(1/\lambda)}(0)\big) \in [\lambda^{-a}, \lambda^{-b}]\big) = \mathbb{P}\big(Z_t(0) \in [a, b]\big).$$

Recalling (2), we observe that

$$\mathbb{P}\big(\#\big(C^\lambda_{t\log(1/\lambda)}(0)\big) \in [\lambda^{-a}, \lambda^{-b}]\big) = \mathbb{P}\big(Z_t^\lambda(0) \in [a + \varepsilon(a, \lambda), b + \varepsilon(b, \lambda)]\big),$$

where $\varepsilon(z, \lambda) = \log(1 + \lambda^z)/\log(1/\lambda) \to 0$ as $\lambda \to 0$ (if $z \geq 0$).

We arrive at the desired conclusion by using Theorem 5 [which asserts that $Z_t^\lambda(0)$ goes in law to $Z_t(0)$] and Lemma 17(i) [from which $\mathbb{P}(Z_t(0) = a) = \mathbb{P}(Z_t(0) = b) = 0$].

Point (ii). Using part (iv) of Lemma 17(iii) and recalling (1), it suffices to check that for all $t \geq 3/2$ and all $B > 0$, we have

$$\lim_{\lambda\to 0} \mathbb{P}[|D_t^\lambda(0)| \geq B] = \mathbb{P}[|D_t(0)| \geq B].$$

This follows from Theorem 5 and the fact that $\mathbb{P}(|D_t(0)| = B) = 0$, thanks to Lemma 17(ii). □

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REFERENCES

[1] BAK, P., TANG, C. and WIESENFELD, K. (1987). Self-organized criticality: An explanation of $1/f$ noise. *Phys. Rev. Lett.* **59** 381–384.

[2] BAK, P., TANG, C. and WIESENFELD, K. (1988). Self-organized criticality. *Phys. Rev. A* (3) **38** 364–374. MR949160

[3] BRESSAUD, X. and FOURNIER, N. (2009). On the invariant distribution of a one-dimensional avalanche process. *Ann. Probab.* **37** 48–77. MR2489159

[4] BROUWER, R. and PENNANEN, J. (2006). The cluster size distribution for a forest-fire process on $\mathbb{Z}$. *Electron. J. Probab.* **11** 1133–1143. MR2268540

[5] DHAR, D. (2006). Theoretical studies of self-organized criticality. *Phys. A* **369** 29–70. MR2246566

[6] DROSSEL, B. and SCHWABL, F. (1992). Self-organized critical forest-fire model. *Phys. Rev. Lett.* **69** 1629–1632.

[7] DROSSEL, B., CLAR, S. and SCHWABL, F. (1993). Exact results for the one-dimensional self-organized critical forest-fire model. *Phys. Rev. Lett.* **71** 3739–3742.

[8] DÜRRE, M. (2006). Existence of multi-dimensional infinite volume self-organized critical forest-fire models. *Electron. J. Probab.* **11** 513–539. MR2242654

[9] DÜRRE, M. (2006). Uniqueness of multi-dimensional infinite volume self-organized critical forest-fire models. *Electron. Comm. Probab.* **11** 304–315. MR2266720

[10] GRASSBERGER, P. (2002). Critical behaviour of the Drossel–Schwabl forest fire model. *New J. Phys.* **4** 17.1–17.15.

[11] HENLEY, C. L. (1989). Self-organized percolation: A simpler model. *Bull. Amer. Math. Soc.* **34** 838.

[12] JENSEN, H. J. (1998). *Self-Organized Criticality*. *Cambridge Lecture Notes in Physics* **10**. Cambridge Univ. Press, Cambridge. MR1689042

[13] LIGGETT, T. M. (1985). *Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR776231

[14] OLAMI, Z., FEDER, H. J. S. and CHRISTENSEN, K. (1992). Self-organized criticality in a continuous, nonconservative cellular automaton modeling earthquakes. *Phys. Rev. Lett.* **68** 1244–1247.

[15] VAN DEN BERG, J. and BROUWER, R. (2006). Self-organized forest-fires near the critical time. *Comm. Math. Phys.* **267** 265–277. MR2238911

[16] VAN DEN BERG, J. and JÁRAI, A. A. (2005). On the asymptotic density in a one-dimensional self-organized critical forest-fire model. *Comm. Math. Phys.* **253** 633–644. MR2116731

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