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LARGE DEVIATIONS OF THE FRONT IN A ONE-DIMENSIONAL MODEL OF $X + Y \to 2X$

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We investigate the probabilities of large deviations for the position of the front in a stochastic model of the reaction $X + Y \to 2X$ on the integer lattice in which Y particles do not move while X particles move as independent simple continuous time random walks of total jump rate 2. For a wide class of initial conditions, we prove that a large deviations principle holds and we show that the zero set of the rate function is the interval $[0, v]$, where v is the velocity of the front given by the law of large numbers. We also give more precise estimates for the rate of decay of the slowdown probabilities. Our results indicate a gapless property of the generator of the process as seen from the front, as it happens in the context of nonlinear diffusion equations describing the propagation of a pulled front into an unstable state.

1. Introduction. We consider a microscopic model of a one-dimensional reaction-diffusion equation, with a propagating front representing the passage from an unstable equilibrium to a stable one. It is defined as an interacting particle system on the integer lattice $\mathbb{Z}$ with two types of particles: X particles, that move as independent, continuous time, symmetric, simple random walks with total jump rate $D_X = 2$; and Y particles, which are inert and can be interpreted as random walks with total jump rate $D_Y = 0$. Initially, each site $x = 0, -1, -2, \dots$ bears a certain number $\eta(x) \geq 0$ of X particles [with at least one site x such that $\eta(x) \geq 1$], while each site $x = 0, 1, \dots$ bears a fixed number a of particles of type Y (with $1 \leq a < +\infty$). When a site $x = 1, 2, \dots$ is visited by an X particle for the first time, all the Y particles located at site x are instantaneously turned into X particles, and start moving. The *front* at time t is defined as the rightmost site that has been visited by an X particle up to time t, and is denoted by r_t, with the convention $r_0 := 0$. This model can be interpreted as an infection process, where the X and Y particles represent ill and healthy individuals, respectively. It can also be interpreted as a combustion reaction, where the X and Y particles correspond to heat units and reactive molecules, respectively, modeling the combustion of a propellant into a stable stationary state. We will denote this model the $X + Y \to 2X$

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front propagation process with jump rates D_X and D_Y. Within the physics literature, a number of studies have been done both numerically and analytically of this process for different values of D_X and D_Y and of corresponding variants where the infection of a Y particle by an X particle at the same site is not instantaneous, drawing analogies with continuous space–time nonlinear reaction-diffusion equations having uniformly traveling wave solutions [9, 14–16, 18, 22]. A particular well-known example is the F-KPP equation studied by Fisher [11] and Kolmogorov, Petrovsky and Piscounov [13].

Mathematically not too much is known. For the case $D_Y = 0$, when $\sum_{x\le 0}\exp(\theta x)\eta(x) < +\infty$ for a small enough $\theta > 0$, a law of large numbers with a deterministic speed $0 < v < +\infty$ not depending on the initial condition is satisfied (see [5], Section 4.1, page 7):

$$\lim_{t\to+\infty} t^{-1} r_t = v \qquad \text{a.s.} \tag{1}$$

In [5] (Theorems 1 and 2), it was proved that the fluctuations around this speed satisfy a functional central limit theorem and that the marginal law of the particle configuration as seen from the front converges to a unique invariant measure as $t \to \infty$. Furthermore, a multi-dimensional version of this process on the lattice $\mathbb{Z}^d$, with an initial configuration having one X particle at the origin and one Y particle at every other site was studied in [1, 21], proving an asymptotic shape theorem as $t \to \infty$ for the set of visited sites (Theorems 1.1 in [1] and [21]). A similar result was proved by Kesten and Sidoravicius [12] (Theorem 1) for the case $D_X = D_Y > 0$ with a product Poisson initial law. In particular, in dimension $d = 1$ they prove a law of large numbers for the front as in (1). For the case $D_X \ne D_Y > 0$, even the problem of establishing whether the front is ballistic or not in dimension $d = 1$, remains open (see [12]).

Within a certain class of one-dimensional nonlinear diffusion equations having uniformly traveling wave solutions describing the passage from an unstable to a stable state, it has been observed that for certain initial conditions the velocity of the front at a given time has a rate of relaxation toward its asymptotic value which is algebraic (see [9, 18] and the physics literature references therein). These are the so called *pulled* fronts, whose speed is determined by a region of the profile linearized about the unstable solution. For the F-KPP equation, Bramson [3] proved that the speed of the front at a given time is below its asymptotic value and that the convergence is algebraic. In general, the slow relaxation is due to a gapless property of a linear operator governing the convergence of the centered front profile toward the stationary state. A natural question is whether such a behavior can be observed in the $X + Y \to 2X$ front propagation type processes. Deviations from the law of large numbers of a larger size than those given by central limit theorem should shed some light on such a question: in particular it would be reasonable to expect a large deviations principle with a degenerate rate function, reflecting a slow convergence of the particle configuration as seen from equilibrium toward the unique invariant measure [5] (page 2, line –3). In this paper, we investigate for the case $D_Y = 0$ the large time asymptotics of the distribution of r_t/t,

$$\mathbb{P}\left[\frac{r_t}{t} \in \cdot\right].$$

Our main result is that a full large deviations principle holds, with a degenerate rate function on the interval $[0, v]$, when the initial condition satisfies the following growth condition.

ASSUMPTION (G). For all $\theta > 0$,

$$\sum_{x \le 0} \exp(\theta x)\eta(x) < +\infty. \tag{2}$$

THEOREM 1 (Large deviations principle). *There exists a rate function* $I : [0, +\infty) \to [0, +\infty)$ *such that, for every initial condition satisfying* (G),

$$\limsup_{t\to+\infty} \frac{1}{t} \log \mathbb{P}\left[\frac{r_t}{t} \in C\right] \le -\inf_{b\in C} I(b) \qquad \text{for } C \subset [0, +\infty) \text{ closed},$$

and

$$\liminf_{t\to+\infty} \frac{1}{t} \log \mathbb{P}\left[\frac{r_t}{t} \in G\right] \ge -\inf_{b\in G} I(b) \qquad \text{for } G \subset [0, +\infty) \text{ open}.$$

Furthermore, I *is identically zero on* $[0, v]$, *positive*, *convex and increasing on* $(v, +\infty)$.

It is interesting to notice that the rate function I is independent of the initial configuration of X particles within the class (G): the large deviations of the empirical distribution function of the process as seen from the front appear to exhibit a uniform behavior for such initial conditions. Furthermore, this result seems to be in agreement with the phenomenon of slow relaxation of the velocity in the so-called pulled reaction diffusion equations. In [9], a nonlinear diffusion equation of the form

$$\partial_t \phi = \partial_x^2 \phi + f(\phi) \tag{3}$$

is studied where f is a function chosen so that $\phi = 0$ is an unstable state and the equation develops pulled fronts. It is argued that for steep enough initial conditions, the velocity relaxes algebraically toward the asymptotic speed, providing an explicit expansion up to order $O(1/t^2)$. Such a nonexponential decay is explained by the fact that the linearization of (3) around the uniformly translating front, gives a linear equation for the perturbation governed by a gapless Schrödinger operator. The position of the front in the $X + Y \to 2X$ particle system can be decomposed as $r_t = \int_0^t Lg(\eta_s)\,ds + M_t$, where L is the generator of the centered dynamics, g is an explicit function and M_t is a martingale. The fact that under Assumption (G) the zero set of the large deviations principle of Theorem 1 is the interval $[0, v]$ is an indication that the symmetrization of L is a gapless operator.

The second result of this paper gives more precise estimates for the probability of the slowdown deviations. Let

$$U(\eta) := \limsup_{x\to-\infty} \frac{1}{\log|x|} \log\left(\sum_{y=0}^{x} \eta(y)\right), \qquad u(\eta) := \liminf_{x\to-\infty} \frac{1}{\log|x|} \log\left(\sum_{y=0}^{x} \eta(y)\right)$$

and

$$s(\eta) := \min(1, U(\eta)).$$

For the statement of the following theorem, we will write U, u, s instead of $U(\eta), u(\eta), s(\eta)$.

THEOREM 2 (Slowdown deviations estimates). *Let η be an initial condition satisfying (G). Then the following statements are satisfied.*

(a) *For all $0 \le c < b < v$, as t goes to infinity,*

$$\mathbb{P}\left[c \le \frac{r_t}{t} \le b\right] \ge \exp\left(-t^{s/2+o(1)}\right). \tag{4}$$

(b) *In the special case where $\eta(x) \ge a$ for all $x \le 0$, one has that, for every $0 \le b < v$, as t goes to infinity,*

$$\mathbb{P}\left[\frac{r_t}{t} \le b\right] \le \exp\left(-t^{1/3+o(1)}\right). \tag{5}$$

(c) *When $u < +\infty$, as t goes to infinity,*

$$\exp\left(-t^{U/2+o(1)}\right) \le \mathbb{P}[r_t = 0] \le \exp\left(-t^{u/2+o(1)}\right). \tag{6}$$

In the case of a homogeneous initial configuration, like $d_- \le \eta(y) \le d_+$ for all $y \le 0$, with $1 \le d_- \le d_+ < +\infty$, or when $(\eta(y))_{y\le 0}$ forms a realization of an i.i.d. family of random variables with finite positive expectation, the above results take a simpler form since $u = U = s = 1$. As a consequence, $\exp(-t^{1/2})$ turns out to be the actual order of magnitude for $\mathbb{P}[r_t = 0]$, and a lower bound for $\mathbb{P}[c \le \frac{r_t}{t} \le b]$ when $0 \le c < b < v$. Note that even in such a homogeneous case, there is a discrepancy between the lower bound (4) and the upper bound (5) (more on this question at the end of Section 5). On the other hand, one may notice that the slowdown probabilities considered in (4) and in (6) exhibit distinct behaviors when $u > 1$. Furthermore, the results contained in Theorems 1 and 2 should be compared with the case of the random walk in random environment with positive or zero drift [19, 20].

A natural question is whether it is possible to relax (G) in Theorem 1. It appears that even if (G) is but mildly violated, the slowdown behavior is not in accordance with that described by Theorem 1. Moreover, if (G) is strongly violated, the law of large numbers with asymptotic velocity v breaks down, so that the speedup part of Theorem 1 cannot hold either.

THEOREM 3. *The following properties hold*:

(i) *Assume there is a* $\theta > 0$, *such that*

$$\liminf_{x\to-\infty} \eta(x)\exp(\theta x) = +\infty.$$

Then there exists $b > 0$ *such that*

$$\limsup_{t\to+\infty} \frac{1}{t}\log \mathbb{P}\left[\frac{r_t}{t} \le b\right] < 0.$$

(ii) *There exists* $\theta' > 0$ *and* $v' > v$ *such that*, *when*

$$\liminf_{x\to-\infty} \eta(x)\exp(\theta' x) = +\infty,$$

then

$$\mathbb{P}\left[\liminf_{t\to+\infty} \frac{r_t}{t} \ge v'\right] = 1.$$

It is important to stress that the proof of Theorem 1 would not be much simplified if we considered initial conditions with only a finite number of particles. Indeed, condition (G) is an assumption which delimits sensible initial data. To prove Theorem 1, we first establish that for initial conditions consisting only of a single particle at the origin, for all $b \ge 0$, the limit

$$\lim_{t\to+\infty} t^{-1}\log \mathbb{P}(r_t \ge bt) \tag{7}$$

exists. The proof of this fact relies on a soft argument based on the sub-additivity property of the hitting times. On the other hand, it is not difficult to show that for b large enough the decay of $\mathbb{P}(r_t \ge bt)$ is exponentially fast. Nevertheless, showing this for b arbitrarily close to but larger than the speed v is a subtler problem. For example, it is not clear how the standard sub-additive arguments could help. Our main tool to tackle this problem is the regeneration structure of the process defined in [5], Section 3.[1] To overcome the fact that the regeneration times and positions have only polynomial tails, we couple the original process with one where the X particles have a small bias to the right, so that they jump to the right with probability $1/2+\epsilon$ for some small $\epsilon > 0$, and the position of the front in the biased process dominates that of the front in the original process. We then use the regeneration structure to study the biased model and how it relates to the original one as ϵ tends to zero. In particular, if v_ϵ is the speed of the biased front, we establish via uniform bounds on the moments of the regeneration times and positions that

$$\lim_{\epsilon\to 0^+} v_\epsilon = v.$$

Furthermore, we show that the regeneration times and positions of the biased model have exponentially decaying tails. Combining these arguments, proves that the limit in (7) is positive for any $b > v$. We then establish that this limit exists and

[1]There is an error in the presentation of the regeneration structure in [5]. Here we explain how to correct it.

has the same value for all initial conditions satisfying (G) by exploiting a comparison argument.

To show that the rate function vanishes on $[0, v]$ [and more precisely (5)], we first consider initial conditions having a uniformly bounded number of particles per site. In this case, it is essentially enough to observe that the probability that the front remains at zero up to time t is bounded from below by $(1/\sqrt{t})^{t^{1/2+o(1)}}$, since there are at most of the order of $t^{1/2+o(1)}$ random walks that yield a nonnegligible contribution to this event. Similar estimates on hitting times of random walks are used to prove (6) and Theorem 3, while more refined arguments are needed to establish (4) for arbitrary initial conditions within the class (G). On the other hand, the proof of the upper bound for the slowdown probabilities (5) in Theorem 2 is more involved, and relies on arguments using the sub-additivity property and the positive association of the hitting times, together with estimates on their tails and their correlations, refining an idea already used in [21] (page 10, line –5) in a similar context.

The rest of the paper is organized as follows. In Section 2, we give a formal definition of the model and introduce its basic structural properties, including sub-additivity and monotonicity of hitting times. In Section 3, we explain how Theorem 1 is proved, building on results proved in other sections. Section 4 is devoted to the proof of the fact that speedup large deviations events have exponentially small probabilities. Section 5 contains our estimates on slowdown probabilities, with the proofs of Theorems 2 and 3. Several appendices contain proofs that are not included in the core of the paper.

2. Construction and basic properties. Throughout the sequel, we will use the convention $\inf \varnothing = +\infty$.

2.1. *Construction of the process.*

2.1.1. *Configuration space.* Any Y particle in the initial configuration of the system may be labeled by a pair $(x, i) \in \mathbb{Z} \times [\![1, a]\!]$, where x stands for the location of the particle, and i for the index of the particle among the a particles of type Y located at x. Then, to each X particle produced from the later transformation of one of these Y particles into an X, we attach the label (x, i) of the corresponding Y particle, and call it the birthplace of the X particle. As for particles which are already of type X in the initial configuration, we think of them as having been produced in the past from the transformation a Y particle which too bore a label $(x, i) \in \mathbb{Z} \times [\![1, a]\!]$ with the same meaning as above, so that these X particles too have a birthplace.

Using birthplaces to index X particles, we see that a configuration of X particles achieved at some point during the evolution of our system, may be represented as a triple $w = (F, r, A)$, where $r \in \mathbb{Z}$, A is a nonempty subset of $\mathbb{Z} \times [\![1, \dots, a]\!]$ such that $\max\{x; (x,i) \in A\} \le r$, and $F : A \to]\!]-\infty, \dots, r]\!]$ is a map. The set A stands for the set of birthplaces of X particles which are present in the configuration. The number r stands for the rightmost position ever visited by a particle of type X, and this explains the requirement that r has to be larger than the rightmost location at which an X particle was born. Then, $F(x,i)$ stands for the current position of the X particle born at (x,i).

Such a configuration carries more information than just the number of X particles at each site, and any distribution of X particles on $]\!]-\infty, 0]\!]$ with a finite number of particles at each site can be encoded by a triple $w = (F, r, A)$.

Given such a triple w, let η be the map defined on $]\!]-\infty, r]\!]$ so that $\eta(x)$ is the number of particles at site x of the configuration w. Hence,

$$\eta(x) := \#\{(y,i) \in A; F(y,i) = x\}.$$

For every $\theta > 0$, let

$$f_\theta(w) := \sum_{(x,i)\in A} \exp\bigl(\theta\bigl(F(x,i) - r\bigr)\bigr) = \sum_{x \le r} \eta(x) \exp\bigl(\theta(x-r)\bigr).$$

Let then

$$\mathbb{L} := \{w; f_\theta(w) < +\infty\}.$$

We turn $\mathbb{L}$ into a metric space by using the distance defined as follows: for $w = (F, r, A)$ and $w' = (F', r', A')$,

$$d(w, w') := |r - r'| + \sum_{(x,i)\in\mathbb{Z}\times[\![1,a]\!]} \bigl|\mathbf{1}\bigl((x,i) \in A\bigr) \exp\bigl(\theta\bigl(F(x,i) - r\bigr)\bigr) - \mathbf{1}\bigl((x,i) \in A'\bigr) \exp\bigl(\theta\bigl(F'(x,i) - r'\bigr)\bigr)\bigr|.$$

The metric space (L, d) is a Polish space. We let $\mathcal{D}$ denote the space of càdlàg functions from $[0, +\infty)$ to $\mathbb{L}$ equipped with the Skorohod topology and the corresponding Borel σ-field.

2.1.2. *Explicit construction of the process.* For our purposes, we have to define on the same probability space not only the original model, but also models including random walks with an arbitrary bias defined through a parameter $\epsilon \in [0, 1/2)$.

In the sequel, we assume that we have a reference probability space $(\Omega, \mathcal{F}, \mathbb{P})$ giving us access to an i.i.d. family of random variables

$$[(\tau_n(u,i), W_n(u,i)); n \ge 1, u \in \mathbb{Z}, 1 \le i \le a],$$

such that, for all (n,u,i), $\tau_n(u,i)$ has an exponential(2) distribution, and $W_n(u,i)$ has the uniform distribution on $[0,1]$, and $\tau_n(u,i)$ and $W_n(u,i)$ are independent.

For every $n \ge 1$, $(x,i) \in \mathbb{Z} \times [\![1,a]\!]$ and $\epsilon \in [0, 1/2)$, we let

$$\varepsilon_n(x,i,\epsilon) := 2\bigl(\mathbf{1}\bigl(W_n(x,i) \le 1/2 + \epsilon\bigr)\bigr) - 1.$$

Let $(Y_t^\epsilon(x,i))_{t\geq 0}$ be the continuous-time random walk started at $Y_0^\epsilon(x,i) := 0$, whose sequence of time steps is $(\tau_n(x,i))_{n\geq 1}$, and whose sequence of space increments is $(\varepsilon_n(x,i,\epsilon))_{n\geq 0}$.

Now, for every $\epsilon \in [0,1/2)$ and $w = (F,r,A) \in \mathbb{L}$, we define on $(\Omega,\mathcal{F},\mathbb{P})$ a collection of random variables $(X_t)_{t\geq 0} = (F_t, r_t, A_t)_{t\geq 0}$, which describes the time-evolution of the configuration of particles starting from an initial configuration given by w, and using the random walks Y^ϵ defined above. Most often in the sequel, the dependence with respect with w and ϵ is not explicitly mentioned when there is no ambiguity. When we have to stress this dependence, we use notation such as X_t^ϵ, $X_t(w)$, or $X_t^\epsilon(w)$, and accordingly for F_t, r_t and A_t.

The construction of the process is done inductively on intervals of the form $[T(u), T(u+1))$, where $T(u)$ denotes the hitting time of $u \in \mathbb{Z}$ by the front. Let us first consider the initial condition and the trajectories of particles of type X that are present in it. By definition, the initial value of the front is given by $r_0 := r$, and the set of (birthplaces of) X particles in the initial condition by $A_0 := A$. By convention, we set $T(r_0) := 0$, and, for every $(x,i) \in A_0$, the trajectory after time zero of the X particle with birthplace (x,i) is given by $F_t(x,i) := F(x,i) + Y_t^\epsilon(x,i)$ for all $t \geq 0$. Given $u \geq r_0$, assume that we have already defined the time $T(u)$, the set $A_{T(u)}$ of (birthplaces of) X particles present in the system at time $T(u)$, and the trajectories of these particles. We then let $T(u+1)$ denote the hitting time of $u+1$ by the front of X particles present in the system at time $T(u)$, or, more formally:

$$T(u+1) := \inf\{t > T(u);\ \text{there is an } (x,i) \in A_{T(u)} \text{ such that } F_t(x,i) = u+1\}.$$

Between time $T(u)$ and $T(u+1)$, no new X particle appears, and the position of the front does not move, so that we set $r_t := r_{T(u)}$ and $A_t := A_{T(u)}$ for $t \in (T(u), T(u+1))$. Then, at time $T(u+1)$, the front hits $u+1$, so that $r_{T(u+1)} := u+1$, and the Y particles present at site $u+1$ are instantaneously turned into X particles. These newly born X particles have to be added into the set $A_{T(u+1)}$, so that we set $A_{T(u+1)} := A_{T(u)} \cup \{u+1\} \times [\![1,a]\!]$, and the trajectories after time $T(u+1)$ of these particles are then defined by $F_t(u+1,i) := (u+1) + Y_t^\epsilon(u+1,i)$ for all $t \geq T(u+1)$.

From the results in [5] (Section 6)—where only the case $\epsilon = 0$ is treated, but it is immediate to adapt them to the present setting—the following results hold. For any $\epsilon \in [0,1/2)$ and $w \in \mathbb{L}$, almost surely with respect to $\mathbb{P}$:

- for every $u \geq 0$, $T(u) < T(u+1) < +\infty$, and there is exactly one X particle hitting u at time $T(u)$;
- $\lim_{u\to+\infty} T(u) = +\infty$;
- for all $t \geq 0$, the configuration X_t of particles at time t, belong to the set $\mathbb{L}$;
- the map $t \mapsto X_t$ belongs to the set $\mathcal{D}$ of càdlàg functions on $\mathbb{L}$.

In the sequel, we use the notation $\mathbb{Q}_w^\epsilon$ to denote the probability distribution of the random process $(X_t^\epsilon)_{t\geq 0}$ starting from the initial configuration w, viewed as a random element of $\mathcal{D}$. Again, as in [5] (Corollary 7), we can prove that:

PROPOSITION 1. *For any* $\epsilon\in[0,1/2)$ *and* $\theta>0$, *the family of probability measures* $(\mathbb{Q}^{\epsilon}_w)_{w\in\mathbb{L}}$ *defines a strong Markov process on* $\mathbb{L}$.

In the sequel, we use $\mathbb{E}$ to denote expectation with respect to $\mathbb{P}$ of random variables defined on $(\Omega,\mathcal{F})$. The notation $\mathbb{E}^{\epsilon}_w$ is used to denote the expectation with respect to $\mathbb{Q}^{\epsilon}_w$ of random variables defined on $\mathcal{D}$ equipped with its Borel σ-field.

2.2. *Properties of hitting times*. For $w=(F,r,A)\in\mathbb{L}$, $\epsilon\in[0,1/2)$ and $u\ge r$, the random variable $T(u)=T^{\epsilon}_w(u)$ has been defined in the previous section as the first time that the front touches site u, given that the initial condition is w. The definition was by induction, but one can check as well that, $\mathbb{P}$-a.s., one has

$$T^{\epsilon}_w(u):=\inf\{t>0;r^{\epsilon}_t=u\}.$$

For all $u,v\in\mathbb{Z}$ such that $u<v$, $1\le i\le a$, and $\epsilon\in[0,1/2)$, let

$$\mathbb{A}(u,i,v):=\inf\left\{\sum_{k=1}^{m}\tau_k(u,i);u+\sum_{k=1}^{m}\varepsilon_k(u,i,\epsilon)=v,m\ge 1\right\}. \tag{8}$$

This represents the first time that the random walk born at (u,i) hits site v (assuming that the walk starts at u at time zero).

PROPOSITION 2. *Let* $w=(F,r,A)\in\mathbb{L}$.

(i) *For all* $u>r$ *and* $\epsilon\in[0,1/2)$, $\mathbb{P}$-*a.s.*

$$T^{\epsilon}_w(u)=\inf\sum_{j=1}^{L-1}\mathbb{A}(x_j,i_j,x_{j+1}),$$

where the infimum is taken over all finite sequences with $L\ge 2$, $x_1,\dots,x_L\in\mathbb{Z}$ *and* $i_1,\dots,i_{L-1}$ *such that* $x_1=F(y_1,i_1)$ *for some* $(y_1,i_1)\in A$, $r<x_2<\cdots<x_{L-1}<u$, $x_L=u$, $i_2,\dots,i_{L-1}\in[\![1,a]\!]$.

(ii) *For all* $u>r$ *and* $\epsilon\in[0,1/2)$, *the following identity holds* $\mathbb{P}$-*a.s.*:

$$T^{\epsilon}_w(u)=\inf_{w'}T^{\epsilon}_{w'}(u),$$

where the infimum runs over all configurations w' *consisting*[2] *of a single particle chosen among those in* w.

(iii) *For all* $r<u<v$ *and* $\epsilon\in[0,1/2)$, *the following sub-additivity property holds* $\mathbb{P}$-*a.s.*:

$$T^{\epsilon}_w(v)\le T^{\epsilon}_w(u)+T^{\epsilon}_{w'}(v),$$

[2]More formally, these are the configurations of the form (F',r',A') with $A'=\{(x,i)\}$, $F'(x,i)=F(x,i)$, $r'=r$ and $(x,i)\in A$.

where w' is the configuration obtained by adding to w all the X particles born between time 0 and $T(u)$, located at their original birthplaces.[3]

(iv) *For any $0 \le \epsilon_1 \le \epsilon_2 < 1/2$, and all $u > r$, $\mathbb{P}$-almost surely, $T_w^{\epsilon_1}(u) \ge T_w^{\epsilon_2}(u)$.*

PROOF. The proof of (i) is quite similar to that of Lemma 2.1 in [21], and so is the proof that (iii) is a consequence of (i). Then (ii) is a consequence of (i).

As for (iv), this is a consequence of the characterization in (i) and of the fact that, for every $(x,i) \in \mathbb{Z} \times [[1,a]]$ and $n \ge 1$, $\varepsilon_n(x,i,\epsilon_1) \le \varepsilon_n(x,i,\epsilon_2)$. $\square$

A consequence of (iv) in the above proposition is the following result.

COROLLARY 1. *For any $w \in \mathbb{L}$, $0 \le \epsilon_1 \le \epsilon_2 < 1/2$, $\mathbb{P}$-almost surely, for all $t \ge 0$, $r_t^{\epsilon_1}(w) \le r_t^{\epsilon_2}(w)$.*

3. Proof of the large deviations principle for $t^{-1}r_t$. In this section, we always have $\epsilon = 0$, and the notation T_w, r_w, etc. are used to denote the corresponding T_w^0, r_w^0, etc. without further mention.

We shall repeatedly deal with configurations consisting of a single particle at a site, so we introduce the following notation: for $u \in \mathbb{Z}$, δ_u is the configuration formed by a single particle located at its birthplace $(u,1)$. More formally, $\delta_u := (F, r, A)$ with $A := \{(u,1)\}$, $r := u$, $F(u,1) := u$.

PROPOSITION 3. *There exists a convex function $J : (0,+\infty) \to [0,+\infty)$ such that, for all $b \in (0,+\infty)$,*

$$\lim_{n\to+\infty} n^{-1} \log \mathbb{P}\big(T_{\delta_0}(n) \le bn\big) = -J(b).$$

PROOF. For any $b > 0$, and all $n \ge 1$, one can check that $\mathbb{P}(T_{\delta_0}(n) \le bn) > 0$. Then let $u_n(b) := \log \mathbb{P}(T_{\delta_0}(n) \le bn)$. Observe that, by subadditivity [part (iii) of Proposition 2], $T_{\delta_0}(n+m) \le T_{\delta_0}(n) + T_{w'}(n+m)$, where w' is obtained by adding to δ_0 all the particles born between time 0 and $T_{\delta_0}(n)$, located at their original birthplaces. Now, by part (ii) of Proposition 2, $T_{w'}(n+m) \le T_{\delta_n}(n+m)$, since the infimum characterizing $T_{w'}(n+m)$ runs over a larger set than the infimum characterizing $T_{\delta_n}(n+m)$. As a consequence, $T_{\delta_0}(n+m) \le T_{\delta_0}(n) + T_{\delta_n}(n+m)$. We deduce that, for all $m, n \ge 1$, and all $b, c > 0$,

$$\{T_{\delta_0}(n) \le bn\} \cap \{T_{\delta_n}(n+m) \le cm\} \subseteq \{T_{\delta_0}(n+m) \le bn + cm\}. \tag{9}$$

[3]More formally, this means that w' is of the form (F', r', A') with $A' := A \cup [[r+1, r+u]] \times [[1,a]]$, $r' := r + u$, and $F'(x,i) := F(x,i)$ when $(x,i) \in A$, and $F'(x,i) := x$, otherwise.

Now, observe that $T_{\delta_0}(n)$ and $T_{\delta_n}(n+m)$ are independent random variables, since their definitions involve disjoint sets of independent random walks. As a consequence,

$$\begin{aligned}(10)\qquad &\mathbb{P}\bigl(\{T_{\delta_0}(n) \le bn\} \cap \{T_{\delta_n}(n+m) \le cm\}\bigr)\\ &\quad = \mathbb{P}\bigl(T_{\delta_0}(n) \le bn\bigr)\mathbb{P}\bigl(T_{\delta_n}(n+m) \le cm\bigr).\end{aligned}$$

From the above two relations (9), (10), and the fact that, by translation invariance of the model, $T_{\delta_0}(m)$ and $T_{\delta_n}(n+m)$ possess the same distribution, we deduce that, for all $m, n \ge 1$, and all $b, c > 0$,

$$(11)\qquad u_{n+m}\left(\frac{bn+cm}{n+m}\right) \ge u_n(b) + u_m(c).$$

Applying inequality (11) above with $c = b$, we deduce that the sequence $(u_n(b))_{n\ge1}$ is super-additive. Since $u_n(b) \le 0$ for all $n \ge 1$, we deduce from the standard subadditive lemma that there exists a nonnegative real number $J(b)$ such that $\lim_{n\to+\infty} n^{-1}u_n(b) = -J(b)$. Moreover, by definition, $b \mapsto u_n(b)$ is nondecreasing, and so $b \mapsto J(b)$ is nonincreasing.

To establish that J is convex, consider b, c, such that $0 < b < c$, $t \in (0,1)$, $k \ge 1$, and apply (11) with $n_k := \lceil kt \rceil$ and $m_k := \lfloor k(1-t) \rfloor$. For large enough k, $\frac{bn_k+cm_k}{n_k+m_k} \le tb + (1-t)c$, so that $u_{n_k+m_k}(tb+(1-t)c) \ge u_{n_k}(b) + u_{m_k}(c)$. Taking the limit as k goes to infinity, we deduce that $J(tb+(1-t)c) \le tJ(b) + (1-t)J(c)$. □

PROPOSITION 4. *The function J defined in Proposition 3 is identically zero on $[v^{-1}, +\infty)$, positive and decreasing on $(0, v^{-1})$.*

The proof of the above proposition makes use of the following result, which is the main result of Section 4.

PROPOSITION 5. *If we start with an initial condition with $r = 0$ and exactly a particles at each site $x \le 0$, then, for any $c > v$,*

$$\limsup_{t\to+\infty} t^{-1} \log \mathbb{P}(r_t \ge ct) < 0.$$

PROOF PROPOSITION 4. We shall use the notation $\mathcal{I}_0$ to denote a specific initial condition with a particles at each site left of the origin: let $\mathcal{I}_0$ be of the form (F, r, A) with $r := 0$, $A :=]\!]-\infty, 0]\!] \times [\![1, a]\!]$, $F(x,i) := x$ for each $(x,i) \in A$. For $n \ge 1$, (ii) of Proposition 2 implies that $T_{\mathcal{I}_0}(n) \le T_{\delta_0}(n)$ $\mathbb{P}$-a.s. In view of the identity $\{T(n) \le bn\} = \{r_{bn} \ge n\}$, we deduce that

$$\mathbb{P}\bigl(T_{\delta_0}(n) \le bn\bigr) \le \mathbb{P}\bigl(r_{bn}(\mathcal{I}_0) \ge n\bigr).$$

From Proposition 5, we deduce that J is positive on $(0, v^{-1})$. On the other hand, by the law of large numbers (1), we see that J must be identically 0 on $(v^{-1}, +\infty)$.

The function J being convex on $(0,+\infty)$, it is also continuous, so that $J(v^{-1})=0$. Moreover, as we have already noted, J is nonincreasing. These facts imply that J is decreasing on $(0, v^{-1})$. □

Let I be defined by $I(b) := bJ(b^{-1})$ for $b>0$ and $I(0):=0$. From the previous results on J, one can deduce the following.

COROLLARY 2. *The function I is identically zero on $[0,v]$, positive, increasing and convex on $(v,+\infty)$.*

PROOF. Only the convexity of I is not obvious. Note that, since J is convex, $b\mapsto J(b^{-1})$ is convex on $(0,+\infty)$ as the composition of two convex functions. Then, since $b\mapsto J(b^{-1})$ is also increasing and positive, the convexity of $b\mapsto bJ(b^{-1})$ on $(0,+\infty)$ follows. □

PROPOSITION 6. *Assume that the initial condition w satisfies $r=0$ and (G). Then, for all $b>0$,*

$$\lim_{n\to+\infty} n^{-1}\log \mathbb{P}\bigl(T_w(n)\le bn\bigr) = -J(b),$$

where J is the function defined in Proposition 3.

The proof of the proposition makes use of the following lemma.

LEMMA 1. *Let $w=(F,r,A)\in\mathbb{L}$. For all $t\ge 0$, and all $\gamma>0$,*

$$\mathbb{P}\Bigl(\max_{(x,i)\in A}\sup_{s\in[0,t]} F_s(x,i)\ge r+\gamma t\Bigr)\le f_\theta(w)\exp[-g_\gamma(\theta)t],$$

where

$$g_\gamma(\theta) := \gamma\theta - 2(\cosh\theta - 1).$$

PROOF. For all $K\in[\![-\infty,0]\!]$, let

$$G_K := \bigcup_{(x,i)\in A;\, F(x,i)\ge K}\Bigl\{\sup_{s\in[0,t]} F_s(x,i) > r+\gamma t\Bigr\}.$$

Clearly, $K_1\le K_2$ implies that $G_{K_2}\subset G_{K_1}$, and $\bigcup_{K\in]\!]-\infty,0]\!]} G_K = G_{-\infty}$, whence $\mathbb{P}(G_{-\infty}) = \lim_{K\to-\infty}\mathbb{P}(G_K)$. Now observe that, given K, the process $(M_s)_{s\ge 0}$ defined by

$$M_s := \sum_{(x,i)\in A;\, F(x,i)\ge K} \exp\bigl(\theta\bigl(F_s(x,i)-r\bigr) - 2(\cosh\theta-1)s\bigr)$$

is a càdlàg martingale. Then note that $G_K \subset \{\sup_{s\in[0,t]} M_s \geq \exp(g_\gamma(\theta)t\}$, and apply the martingale maximal inequality to deduce that

$$\mathbb{P}\Big(\sup_{s\in[0,t]} M_s \geq \exp(g_\gamma(\theta)t)\Big) \leq \sum_{(x,i)\in A;\, F(x,i)\geq K} \exp\big[\theta\big(F(x,i)-r\big) - g_\gamma(\theta)t\big].$$

The right-hand side of the above inequality is bounded above, for every value of K, by $f_\theta(w)\exp[-g_\gamma(\theta)t]$. The conclusion follows. □

PROOF OF PROPOSITION 6. Consider $0 < b < v^{-1}$, and fix $\theta > 0$. Choose $\gamma > 0$ large enough so that

$$g_\gamma(\theta)b > J(b).$$

Denote by $w = (F, r, A)$ the initial condition, and consider the set $B_n := \{(x, i); F(x,i) \leq -\lceil \gamma bn\rceil\}$. Let $m_n := \sum_{(x,i)\in B_n} \exp(\theta(F(x,i) - \lceil\gamma bn\rceil))$. Now let $\Xi_n := \inf\{s \geq 0; \exists (x,i) \in B_n, F_s(x,i) = 0\}$. We see that $\Xi_n \leq bn$ implies that $\sup_{(x,i)\in B_n} \sup_{s\in[0,bn]} F_s(x,i) \geq 0$. Thanks to Lemma 1 and translation invariance of the model, we deduce that

$$\mathbb{P}(\Xi_n \leq bn) \leq m_n \exp(-g_\gamma(\theta)bn). \tag{12}$$

From the fact that w satisfies (G), we obtain that, for all $\varphi > 0$, $y \leq 0$, $\#\{(x,i) \in A; F(x,i) = y\} \leq f_\varphi(w)\exp(-\varphi y)$. As a consequence, whenever $\varphi < \theta$, we have that

$$m_n \leq f_\varphi(w)\big(1 - \exp(\varphi - \theta)\big)^{-1} \exp(\varphi\lceil\gamma bn\rceil). \tag{13}$$

Now consider $(x,i) \in A \setminus B_n$, so that $F(x,i) > -\lceil\gamma bn\rceil$. Let w' denote the configuration consisting in a single particle located at (x,i), or, more formally, let w' be of the form (F', r', A') with $r' := x$, $A' := \{(x,i)\}$) and $F'(x,i) := (x,i)$. By a coupling argument, we see that, since $F(x,i) \leq 0$,

$$\mathbb{P}(T_{w'} \leq bn) \leq \mathbb{P}(T_{\delta_0} \leq bn). \tag{14}$$

Moreover, according to (G),

$$\#A \setminus B_n \leq f_\varphi(w)\exp(\varphi\lceil\gamma bn\rceil). \tag{15}$$

Now, by (ii) of Proposition 2,

$$\{\Xi_n > bn\} \cap \{T_w(n) \leq bn\} \subset \Big\{\inf_{w'} T_{w'} \leq bn\Big\},$$

where the infimum runs over all configurations w' formed by a single particle located at some $(x,i) \in A \setminus B_n$. We deduce from (12), (13), (14), (15) and the union bound that

$$\begin{aligned}&\mathbb{P}\big(T_w(n) \leq bn\big)\\ &\quad\leq f_\varphi(w)e^{\varphi\lceil\gamma bn\rceil}\big[\big(1 - e^{(\varphi-\theta)}\big)^{-1}\exp(-g_\gamma(\theta)bn) + \mathbb{P}(T_{\delta_0} \leq bn)\big].\end{aligned} \tag{16}$$

Now, according to Proposition 3,

$$\lim_{n\to+\infty} n^{-1}\log\mathbb{P}\big(T_{\delta_0}(n)\le bn\big) = -J(b).$$

Since we have chosen γ so that $g_\gamma(\theta)b > J(b)$, we deduce from (16) that

$$\limsup_{n\to+\infty} n^{-1}\log\mathbb{P}\big(T_w(n)\le bn\big) \le -J(b)+\varphi\gamma b.$$

Since $\varphi>0$ is arbitrary, we deduce that

$$\limsup_{n\to+\infty} n^{-1}\log\mathbb{P}\big(T_w(n)\le bn\big) \le -J(b). \tag{17}$$

On the other hand, consider a given $(x,i)\in A$, and let again w' denote the configuration with a single particle located at (x,i). Clearly,

$$\mathbb{P}\big(T_w(n)\le bn\big) \ge \mathbb{P}\big(T_{w'}(n)\le bn\big).$$

Now consider $\tilde{\tau}=\inf\{s\ge 0; F_s(x,i)=0\}$. Clearly, $\tilde{\tau}$ is a.s. finite, and, conditional upon $\tilde{\tau}$, $T_{w'}(n)-\tilde{\tau}$ has the (unconditional) distribution of $T_{\delta_0}(n)$. Choosing any M such that $\mathbb{P}(\tilde{\tau}\le M)>0$, one has that $\mathbb{P}(T_{w'}(n)\le bn)\ge \mathbb{P}(\tilde{\tau}\le M)\mathbb{P}(T_{\delta_0}(n)\le bn-M)$. Taking an arbitrary $c>b$, we deduce that

$$\liminf_{n\to+\infty} n^{-1}\log\mathbb{P}\big(T_w(n)\le bn\big) \ge -J(c).$$

By continuity of J, we conclude that

$$\liminf_{n\to+\infty} n^{-1}\log\mathbb{P}\big(T_w(n)\le bn\big) \ge -J(b).$$

The above inequality, together with (17), concludes the proof. $\square$

PROOF OF THEOREM 1. Consider a nonempty closed subset $F\subset[0,+\infty)$, and let $b:=\inf F$. Assume that $b\le v$. We have that $\inf_F I=0$, so the upper bound of the LDP for F is always satisfied. Assume now that $b>v$. One has that $\mathbb{P}(t^{-1}r_t\in F)\le\mathbb{P}(r_t\ge\lceil tb\rceil)=\mathbb{P}(T(\lceil tb\rceil)\le t)$. Proposition 6 entails that $\lim_{t\to+\infty} t^{-1}\log\mathbb{P}(T(\lceil tb\rceil\le t)\le -I(b)$, so that the upper bound of the LDP holds for F since I is nondecreasing.

Consider now an open set $G\subset(v,+\infty)$. For every $b\in G$, there exists an interval $[b,c)\subset G$. By the large deviations upper bound, we know that $\limsup_{t\to+\infty} t^{-1}\log\mathbb{P}(r_t\ge bt)\le -I(b)$ and that $\limsup_{t\to+\infty} t^{-1}\log\mathbb{P}(r_t\ge ct)\le -I(c)$. By strict monotonicity of I on $(v,+\infty)$, we have that $I(b)<I(c)$, so we can conclude that $\liminf_{t\to+\infty} t^{-1}\log\mathbb{P}(bt\le r_t<ct)\ge -I(b)$. As a consequence, $\liminf_{t\to\infty}\frac{1}{t}\mathbb{P}(t^{-1}r_t\in G)\ge -I(b)$. Since this holds for an arbitrary $b\in G$, the lower bound of the LDP for G follows.

Consider now a nonempty open set $G\subset[0,+\infty)$ such that $G\cap[0,v]\neq\varnothing$. Then $\inf_G I=0$. On the other hand, there is a nonempty interval of the form $[c,b)\subset G\cap[0,v]$. In Section 5, we prove that, under (G),

$$\liminf_{t\to\infty} t^{-1}\log\mathbb{P}\left[c\le\frac{r_t}{t}\le b\right]=0. \tag{18}$$

Applying inequality (18), we see that $\liminf t^{-1}\log\mathbb{P}(t^{-1}r_t\in G)=0$, so that the lower bound of the LDP holds. □

4. Speedup probabilities. The main result in this section is Proposition 5: when $\epsilon=0$, starting from an initial condition with exactly a particles at each site $x\le 0$,

$$\text{for any } b>v \qquad \limsup_{t\to+\infty} t^{-1}\log\mathbb{P}(r_t\ge bt)<0. \tag{19}$$

In the following discussion, we always assume that we are under such an initial condition, without explicitly mentioning this assumption.

Our strategy for proving Proposition 5 is to exploit the renewal structure already used in [5] to prove the CLT. However, this renewal structure leads to random variables (renewal time κ, and displacement of the front at a renewal time r_κ, see the precise definitions below) whose tails have polynomial decay, and asymptotic exponential bounds such as (19) cannot be derived from such random variables. Whether it is possible to modify the definition of the renewal structure so as to obtain random variables enjoying an exponential decay of the tails, as required for a direct proof of Proposition 5, is unclear, and instead we make use of a different idea. Indeed, we apply the renewal structure defined in [5] (Section 3) to a perturbation of the original model, one in which the random walks have a small bias to the right. Again, a law of large numbers holds.

PROPOSITION 7. *For all small enough $\epsilon\ge 0$, there exists $0<v_\epsilon<+\infty$ such that*

$$\lim_{t\to\infty} t^{-1}r_t^\epsilon=v_\epsilon, \qquad \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P}).$$

The interest of introducing a bias to the right is that, reworking the estimates of [5] (Section 5) in this context, we can show that for any small value of the bias parameter $\epsilon>0$, exponential decay of the tail of the renewal times holds, so that the following result can be proved.

PROPOSITION 8. *There exists $\epsilon_0>0$ such that, for any $\epsilon\in(0,\epsilon_0]$, for any $b>v_\epsilon$,*

$$\limsup_{t\to+\infty} t^{-1}\log\mathbb{P}(r_t^\epsilon\ge bt)<0.$$

On the other hand, it is shown in Corollary 1 above that, as expected, biasing the random walks to the right cannot decrease the position of the front, so that at each time t, a comparison holds between the position of the front in the original model and in the model with a bias. We deduce that:

PROPOSITION 9. *For any* $\epsilon \in [0, 1/2)$ *and* $t \geq 0$, *and all* $x \in \{1, 2, \dots\}$,

$$\mathbb{P}(r_t^0 \geq x) \leq \mathbb{P}(r_t^\epsilon \geq x).$$

As a consequence, we can prove that (19) holds for all b such that there exists an $\epsilon \in (0, \epsilon_0]$ for which $v_\epsilon < b$. Noting that v_ϵ is a nondecreasing function of ϵ, we see that the following result would make our strategy work for all $b > v$.

PROPOSITION 10.

$$\lim_{\epsilon \to 0+} v_\epsilon = v. \tag{20}$$

It is indeed natural to expect such a continuity property to hold, but proving it seems to require substantial work.

Indeed, write

$$\begin{aligned} v_\epsilon &= \lim_{t \to +\infty} t^{-1}\mathbb{E}(r_t^\epsilon), \\ v &= \lim_{t \to +\infty} t^{-1}\mathbb{E}(r_t^0). \end{aligned} \tag{21}$$

For fixed t, it is possible (using the dominated convergence theorem) to prove that

$$\lim_{\epsilon \to 0+} \mathbb{E}(r_t^\epsilon) = \mathbb{E}(r_t^0). \tag{22}$$

Hence, to prove identity (20), it is enough to prove that

$$\lim_{\epsilon \to 0+} \lim_{t \to +\infty} t^{-1}\mathbb{E}(r_t^\epsilon) = \lim_{t \to +\infty} \lim_{\epsilon \to 0+} t^{-1}\mathbb{E}(r_t^\epsilon).$$

Our strategy for proving Proposition 10 is based on the observation that, if some sort of uniformity with respect to $\epsilon \in [0, \epsilon_0]$ is achieved in (21), then the limits with respect to $\epsilon \to 0+$ and to $t \to +\infty$ in (21) and (22) can be exchanged. It was proved in [5] (Section 5) that the second moments of the random variables defined by the renewal structure (renewal time κ, and displacement of the front at a renewal time r_κ, see the precise definitions below) are finite in the case $\epsilon = 0$. Reworking these estimates, we obtain uniform upper bounds (with respect to $\epsilon \in [0, \epsilon_0]$) for the second moments of these variables, and are thus able to prove that the required uniformity in (21) holds.

4.1. *Random variables on* $\mathcal{D}$. It will be convenient in Sections 4.3, 4.4 and Appendix C to work with random variables defined on the canonical space of càdlàg trajectories $\mathcal{D}$, rather than on Ω. In fact, each random variable (generically denoted by L in this discussion) on Ω introduced in Section 2 for the definition of the process, can be written as $L = \hat{L}((X_t^\epsilon(w))_{t\geq 0})$, where $\hat{L}$ is a corresponding random variable on $\mathcal{D}$. To avoid unduly complicated notation, we shall use the same notation to denote L and $\hat{L}$. This should not introduce ambiguities, for our notation concerning probabilities on Ω and $\mathcal{D}$ are distinct, and for, except in Lemma 4 below, we make exclusive use of random variables defined on $\mathcal{D}$.

Specifically, we define the random variables X_t, F_t, r_t and A_t on $\mathcal{D}$ by writing a generic càdlàg trajectory on $\mathbb{L}$ under the form $(X_t)_{t\geq 0}$, with $X_t = (F_t, r_t, A_t)$ for all $t \geq 0$. Moreover, for $u \in \mathbb{Z}$, we let $T(u) := \inf\{s \geq 0; r_s = u\}$ if $y \geq r_0 + 1$, and $T(u) := 0$, otherwise. For $(x, i) \in \mathbb{Z} \times [\![1, a]\!]$, we also define $Y_t(x, i) := F_{T(x)+t}(x, i) - F_{T(x)}$ if $T(x) < +\infty$, and $Y_t(x, i) := 0$, otherwise.

As a consequence of these definitions, the probability distributions of X_t, $F_t(x, i), r_t, A_t, T(u)$ with respect to the probability measure $\mathbb{Q}_w^\epsilon$, are, respectively, the same as the probability distributions of $X_t^\epsilon(w)$, $F_t^\epsilon(w)(x, i)$, $r_t^\epsilon(w)$, $A_t^\epsilon(w)$, $T_w^\epsilon(u)$, $Y_t^\epsilon(x, i)$ with respect to $\mathbb{P}$. In particular, with respect to $\mathbb{Q}_w^\epsilon$, the processes $(Y_t(x, i))_{t\geq 0}$ form a family of independent nearest-neighbor random walks on $\mathbb{Z}$ with jump rate 2 and step distribution $(1/2 + \epsilon)\delta_{+1} + (1/2 - \epsilon)\delta_{-1}$.

For $z \in \mathbb{Z}$, and $w = (F, r, A) \in \mathbb{L}$, define $\phi_z(w)$ by

$$\phi_z(w) := \sum_{(x,i)\in A\cap]\!]-\infty,z]\!]\times[\![1,a]\!]} \exp\bigl(\theta\bigl(F(x, i) - r\bigr)\bigr)$$

and for $z_1 < z_2 \in \mathbb{Z}$, let

$$m_{z_1,z_2}(w) := \sum_{(x,i)\in A\cap[\![z_1+1,z_2]\!]\times[\![1,a]\!]} \mathbf{1}\bigl(F(x, i) \in [\![z_1 + 1, z_2]\!]\bigr).$$

We use the notation θ_s to denote the canonical time-shift on $\mathcal{D}$. We denote by $(\mathcal{F}_t^\epsilon)_{t\geq 0}$ the usual augmentation of the natural filtration on $\mathcal{D}$ with respect to the Markov family $(\mathbb{Q}_w^\epsilon)_{w\in\mathbb{L}}$.

4.2. *An elementary speedup estimate.* The following lemma is stated in [5] (Lemma 10) in the case $\epsilon = 0$, and its adaptation to the more general case $\epsilon \in [0, 1/2)$ is straightforward.

LEMMA 2. *Let* $\lambda(\epsilon) := 2(\cosh\theta - 1) + 4\epsilon\sinh\theta + a(1 + 2\epsilon)\exp\theta$ *and* $c_\gamma(\epsilon) := \gamma\theta - \lambda(\epsilon)$. *For all* $\epsilon \in [0, 1/2)$, $w \in \mathbb{L}$, *and* $t \geq 0$,

$$\mathbb{Q}_w^\epsilon(r_t - r_0 \geq \gamma t) \leq \phi_{r_0}(w)\exp(-c_\gamma(\epsilon)t).$$

4.3. *Definition of the renewal structure.* We follow the definition of the renewal structure in [5] (Section 3). Consider a parameter

$$M := 4(a + 9). \tag{23}$$

Let $\nu_0 := 0$ and ν_1 be the first time one of the random walks $(r_0 + Y_s(r_0, i))_{s\geq 0}; i \in [\![1, a]\!]$, hits the site $r_0 + 1$ [the random walks $(Y_s(x, i))$ are defined in Section 4.1]. Next, define ν_2 as the first time one of the random walks $(z + Y_s(z, i))_{s\geq 0}; z \in [\![r_0, r_0 + 1]\!], i \in [\![1, a]\!]$, hits the site $r_0 + 2$. In general, for $k \geq 2$, we define ν_k

as the first time one of the random walks $(z + Y_s(z,i))_{s\geq 0}$; $z \in [\![r_0 \vee (r_0 + k - M), r_0 + k - 1]\!]$, $i \in [\![1,a]\!]$, hits the site $r_0 + k$. For $n \in \mathbb{N}$, let

$$\tilde{r}_t := r_0 + n \qquad \text{if } \sum_{k=0}^{n} \nu_k \leq t < \sum_{k=0}^{n+1} \nu_k.$$

The construction corresponding to $\tilde{r}_t$ with $M = \infty$ will be denoted by $\bar{r}_t$. Note that $\tilde{r}_t \leq r_t$. The following proposition (see Lemma 1 from [5]) shows that the so-called auxiliary front $\tilde{r}_t$ can be used to estimate the position of the front r_t.

PROPOSITION 11. *For every $\epsilon \in [0,1/2)$, $\theta > 0$ and $w \in \mathbb{L}$, the following holds $\mathbb{Q}_w^\epsilon$-almost surely*:

$$\textit{for every } t \geq 0 \qquad \tilde{r}_t \leq r_t.$$

Now, observe that for any $w = (F, r, A)$ such that $r \times [\![1,a]\!] \subset A$ and $F(r,i) = r$ for all $i \in [\![1,a]\!]$, with respect to $\mathbb{Q}_w^\epsilon$, for each $j \in [\![1, M-1]\!]$, the random variables $(\nu_i)_{i\geq 1}$ are a.s. finite, and that the random variables $\{\nu_{Mk+j} : k \geq 1\}$ are i.i.d. and have finite expectation since[4] $M \geq 3$. We deduce that a.s. (see also [4], Lemmas 1 and 6)

$$\lim_{t\to\infty} \tilde{r}_t/t =: \alpha(\epsilon) > 0.$$

First, note that $\alpha(\epsilon)$ does not depend on θ nor on w since the distribution of the random walks $(Y_s(x,i))_{s\geq 0}$ with respect to $\mathbb{Q}_w^\epsilon$ does not. Moreover, $\alpha(\epsilon)$ is a non-decreasing function of ϵ by a coupling argument.

Now consider $\epsilon_0 < 1/2, \theta > 0, \alpha_1, \alpha_2 > 0$ such that

$$\begin{cases} 0 < \alpha_1 < \alpha_2 < \alpha(0), \\ \theta^{-1}\big(2(\cosh\theta - 1) + 4\epsilon_0 \sinh\theta\big) < \alpha_1, \\ 4\epsilon_0 < \alpha_1. \end{cases} \tag{24}$$

In the sequel, we always assume that $\epsilon \in [0, \epsilon_0]$.

Let us define the following random variables on $\mathcal{D}$:[5]

$$\begin{cases} U := \inf\{t \geq 0; \tilde{r}_t - r_0 < \lfloor \alpha_2 t \rfloor\}, \\ V := \inf\Big\{t \geq 0; \max\limits_{z \in [\![r_0 - L + 1, r_0 - 1]\!]} F_t(z,i) > \lfloor \alpha_1 t \rfloor + r_0\Big\}, \\ W := \inf\big\{t \geq 0; \phi_{r_0 - L}(X_t) \geq e^{\theta(\lfloor \alpha_1 t\rfloor - (r_t - r_0))}\big\}. \end{cases}$$

[4]The hitting time of a site by a single symmetric random walk has a tail decaying roughly as $t^{-1/2}$. Taking into account M independent such random walks yields a tail decaying as $t^{-M/2}$, which corresponds to an integrable random variable as soon as $M \geq 3$.

[5]In [5], U was defined in terms of $\tilde{r}_t$ instead of $\bar{r}_t$, rendering the event $\{U = \infty\}$ not measurable with respect to the information up to time κ.

Note that, for all ϵ, U, V, W are stopping times with respect to $(\mathcal{F}_t^\epsilon)_{t\geq 0}$, and that they are mutually independent with respect to $\mathbb{Q}_w^\epsilon$.

Let

$$D := \min(U, V, W).$$

Now let $p > 0$ be such that

$$p \exp(\theta) < 1$$

and L such that $L^{1/4}$ is an integer and

$$L^{1/4} \geq M + 1 \quad \text{and} \quad a \exp(-L\theta)\big(1 - \exp(-\theta)\big)^{-1} < p. \tag{25}$$

For $x \in \mathbb{Z}$, define J_x as the smallest integer $j \geq 1$ such that the following two conditions are satisfied:

- $\phi_{x+(j-1)L}(X_{T(x+jL)}) \leq p$;
- $m_{x+jL-L^{1/4},x+jL}(X_{T(x+jL)}) \geq aL^{1/4}/2$.

Let $S_0 := 0$ and $D_0 := 0$. Then define for $k \geq 0$,

$$S_{k+1} := T(r_{D_k} + J_{r_{D_k}} L), \qquad D_{k+1} := D \circ \theta_{S_{k+1}} + S_{k+1}.$$

Finally, let $K := \inf\{k \geq 1 : S_k < \infty, D_k = \infty\}$, and define the *regeneration time*

$$\kappa := S_K.$$

Note that κ is *not* a stopping time with respect to $(\mathcal{F}_t^\epsilon)_{t\geq 0}$.

4.4. *Properties of the renewal structure.* Throughout this section, we assume that $\theta, \alpha_1, \alpha_2, \epsilon_0$ satisfy the assumptions listed in Section 4.3. We use the notation $a\delta_0$ to denote an initial configuration consisting in exactly a particles at site $x = 0$, and $\mathcal{I}_0$ to denote an initial configuration with exactly a particles at each site $x \leq 0$.

PROPOSITION 12. *The following properties hold*:

(i) *There exist* $0 < C, L^* < +\infty$ *not depending on* ϵ (*but possibly depending on the choice of* $\theta, \alpha_1, \alpha_2, \epsilon_0$) *such that*, *for* $L := L^*$, *and all* $\epsilon \in [0, \epsilon_0]$,

$$\mathbb{E}_{\mathcal{I}_0}^\epsilon(\kappa^2) \leq C, \qquad \mathbb{E}_{a\delta_0}^\epsilon(\kappa^2 | U = +\infty) \leq C,$$

$$\mathbb{E}_{\mathcal{I}_0}^\epsilon(r_\kappa^2) \leq C, \qquad \mathbb{E}_{a\delta_0}^\epsilon(r_\kappa^2 | U = +\infty) \leq C.$$

(ii) *For all* $0 < \epsilon \leq \epsilon_0$, *there exist* $0 < C', L', t < +\infty$, *depending on* ϵ, *such that for* $L := L'$,

$$\mathbb{E}_{\mathcal{I}_0}^\epsilon\big(\exp(t\kappa) \leq C', \mathbb{E}_{a\delta_0}^\epsilon(\exp(t\kappa))|U = +\infty\big) \leq C',$$

$$\mathbb{E}_{\mathcal{I}_0}^\epsilon(\exp(tr_\kappa)) \leq C' \mathbb{E}_{a\delta_0}^\epsilon\big(\exp(tr_\kappa)|U = +\infty\big) \leq C'.$$

Proposition 12 provides the key estimates needed for the proof of the main results in this section. Most of the technical work needed to prove it consists in a reworking of the estimates in [5] (Section 5), either proving that, for each positive value of the bias parameter ϵ, exponential estimates can be obtained instead of the polynomial ones derived in [5], or that the polynomial estimates already obtained in [5] can be made uniform with respect to $\epsilon \in [0, \epsilon_0]$. The proofs go along the lines of [5], and are deferred to Appendix C. In the sequel, we always assume that $L := L^*$ or $L := L'$. As a consequence of Proposition 12 we see that for all $\epsilon \in [0, \epsilon_0]$, $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(0 < \kappa < +\infty) = 1$ and $\mathbb{Q}^{\epsilon}_{a\delta_0}(0 < \kappa < +\infty | U = +\infty) = 1$.

We then define inductively the whole sequence of renewal times $(\kappa_i)_{i\geq 0}$ by $\kappa_1 := \kappa$ and for $i \geq 1$, $\kappa_{i+1} := \kappa_i + \kappa \circ \theta_{\kappa_i}$.

As in [5] (Corollary 1), the following proposition can be proved.

PROPOSITION 13. *The following properties hold*:

(i) *Under* $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}$, $\kappa_1, \kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \dots$ *are independent, and* $\kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \dots$ *are identically distributed with law identical to that of* κ *under* $\mathbb{Q}^{\epsilon}_{a\delta_0}(\cdot | U = +\infty)$.

(ii) *Under* $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}$, $r_{\kappa_1}, r_{\kappa_2} - r_{\kappa_1}, r_{\kappa_3} - r_{\kappa_2}, \dots$ *are independent, and* $r_{\kappa_2} - r_{\kappa_1}, r_{\kappa_3} - r_{\kappa_2}, \dots$ *are identically distributed with law identical to that of* r_κ *under* $\mathbb{Q}^{\epsilon}_{a\delta_0}(\cdot | U = +\infty)$.

We now give the proofs of Propositions 7, 8 and 10.

PROOF OF PROPOSITION 7. First, note that the $\mathbb{P}$-a.s. convergence stated in Proposition 7 follows from the integrability of renewal times by a standard argument. To prove that the convergence also takes place in $L^1(\mathbb{P})$, we note that, from Lemma 2 above, it stems that $\mathbb{E}^{\epsilon}_{\mathcal{I}_0}(r_t) < +\infty$ for all t and that the family of random variables $(t^{-1} r_t)_{t\geq 1}$ is uniformly integrable with respect to $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}$. The convergence in $L^1(\mathbb{P})$ then follows from the $\mathbb{P}$-a.s. convergence. $\square$

PROOF OF PROPOSITION 8. Fix $0 < \epsilon \leq \epsilon_0$, and let $L := L'$. For all $t \geq 0$, define $a(t) := \sup\{n \geq 1; \kappa_n \leq t\}$, with the convention that $\sup \varnothing = 0$. From Propositions 12 and 13, we deduce that, $a(t) < +\infty$ a.s. for all $t \geq 0$ and that $\lim_{t\to+\infty} a(t) = +\infty$ a.s. Using the fact that the map $t \mapsto r_t$ is nondecreasing, we have that $r_t \leq r_{\kappa_{a(t)+1}}$. Now observe that, for any $0 < \epsilon \leq \epsilon_0$, any $b > v_\epsilon$, and any $0 < c < +\infty$, by the union bound,

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(r_t \geq bt) \leq \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(a(t) \geq \lfloor ct \rfloor\big) + \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(r_{\kappa_{\lfloor ct \rfloor+1}} \geq bt\big).$$

Note that $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(a(t) \geq \lfloor ct \rfloor) \leq \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(\kappa_{\lfloor ct \rfloor} \leq t)$, and observe that, by Cramér's theorem (see, e.g., [6], Theorem 2.2.3) for the i.i.d. nonnegative sequence $(\kappa_{i+1} - \kappa_i)_{i\geq 1}$ and Proposition 12 for κ_1, whenever $c^{-1} < \mathbb{E}^{\epsilon}_{a\delta_0}(\kappa | U = +\infty)$, $\limsup_{t\to+\infty} t^{-1} \log \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(\kappa_{\lfloor ct \rfloor} \leq t) < 0$. On the other hand, writing $r_{\kappa_{\lfloor ct \rfloor+1}} =$

$r_{\kappa_1} + \sum_{i=1}^{\lfloor ct \rfloor+1} (r_{\kappa_{i+1}} - r_{\kappa_i})$, and using Proposition 12 and Cramér's theorem as above, we have that, as soon as $b/c > \mathbb{E}^{\epsilon}_{a\delta_0}(r_\kappa | U = +\infty)$, $\limsup_{t\to+\infty} t^{-1} \times \log \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(r_{\kappa_{\lfloor ct \rfloor+1}} \geq bt) < 0$.

The proof of the law of large numbers for r^{ϵ}_t given above (proof of Proposition 7) used Kingman's subadditive ergodic theorem. However, from the existence of a renewal structure for which κ and r_κ have finite expectation (Propositions 12 and 13), we can deduce that the asymptotic speed in the law of large numbers is in fact given by

$$v_\epsilon = \frac{\mathbb{E}^{\epsilon}_{a\delta_0}(r_\kappa | U = +\infty)}{\mathbb{E}^{\epsilon}_{a\delta_0}(\kappa | U = +\infty)}. \tag{26}$$

As a consequence, if $b > v_\epsilon$, we see that we can choose a $c > 0$ such that $c^{-1} < \mathbb{E}^{\epsilon}_{a\delta_0}(\kappa | U = +\infty)$ and $b/c > \mathbb{E}^{\epsilon}_{a\delta_0}(r_\kappa | U = +\infty)$. $\square$

LEMMA 3. *There exists* $0 < c < +\infty$ *such that, for all* $\epsilon \in [0, \epsilon_0]$,

$$\mathbb{E}^{\epsilon}_{a\delta_0}(\kappa | U = +\infty) \geq c.$$

PROOF. Use the fact that, by definition, $\kappa \geq T(1)$, so that $\mathbb{E}^{\epsilon}_{a\delta_0}(\kappa | U = +\infty) \geq \mathbb{E}^{\epsilon}_{a\delta_0}(T(1)\mathbf{1}(U = +\infty))$. Now, by coupling, $\mathbb{Q}^{\epsilon}_{a\delta_0}(U = +\infty) \geq \mathbb{Q}^{0}_{a\delta_0}(U = +\infty)$ for all $\epsilon \in [0, \epsilon_0]$. By coupling again, for all $u > 0$, $\mathbb{Q}^{\epsilon}_{a\delta_0}(T(1) \geq u) \geq \mathbb{Q}^{\epsilon_0}_{a\delta_0}(T(1) \geq u)$. Now, since $\mathbb{Q}^{\epsilon_0}_{a\delta_0}(T(1) > 0) = 1$, we can find $u > 0$ small enough so that $\mathbb{Q}^{\epsilon_0}_{a\delta_0}(T(1) \geq u) \geq 1 - (1/2)\mathbb{Q}^{0}_{a\delta_0}(U = +\infty)$. Putting the previous inequalities together, we see that, for all $\epsilon \in [0, \epsilon_0]$, $\mathbb{Q}^{0}_{a\delta_0}(T(1) \geq u, U = +\infty) \geq (1/2)\mathbb{Q}^{0}_{a\delta_0}(U = +\infty)$. The conclusion follows. $\square$

The following proposition contains the uniform convergence estimate that is required for the proof of Proposition 10. Broadly speaking, the idea is to control the convergence speed with second moment estimates on the renewal structure, so that uniform estimates on these moments yield uniform estimates on the convergence speed.

PROPOSITION 14. *For all* $\zeta > 0$, *there exists* $t_\zeta \geq 0$ *such that, for all* $t \geq t_\zeta$ *and all* $\epsilon \in [0, \epsilon_0]$,

$$v_\epsilon \leq \mathbb{E}^{\epsilon}_{\mathcal{I}_0}(t^{-1} r_t) + \zeta.$$

PROOF. Assume that $L := L^*$. Let $0 < \lambda < 1$ be given, and let

$$m(t, \epsilon) := \lfloor (1 - \lambda) t \bigl(\mathbb{E}^{\epsilon}_{a\delta_0}(\kappa | U = +\infty)\bigr)^{-1} \rfloor.$$

In the rest of the proof, we write m instead of $m(t, \epsilon)$ for the sake of readability. Note that, in view of Proposition 12, for all $\epsilon \in [0, \epsilon_0]$, $\mathbb{E}^{\epsilon}_{a\delta_0}(\kappa | U = +\infty) \leq C^{1/2}$, so that $m \geq 1$ as soon as $t \geq C^{1/2}(1 - \lambda)^{-1}$, which does not depend on ϵ.

We now reuse the random variables $a(t)$ defined in the proof of Proposition 8 above. Using the fact that $t \mapsto r_t$ is nondecreasing, we see that $r_t \geq r_{\kappa_{a(t)}}$. Moreover, $r_{\kappa_{a(t)}} \geq r_{\kappa_{a(t)}} \mathbf{1}(a(t) \geq m)$, and $r_{\kappa_{a(t)}} \mathbf{1}(a(t) \geq m) \geq r_{\kappa_m} \mathbf{1}(a(t) \geq m)$ when $m \geq 1$. Taking expectations, we deduce that, when $m \geq 1$,

$$\mathbb{E}^{\epsilon}_{\mathcal{I}_0}(t^{-1} r_t) \geq \mathbb{E}^{\epsilon}_{\mathcal{I}_0}(t^{-1} r_{\kappa_m}) - \mathbb{E}^{\epsilon}_{\mathcal{I}_0}\big(t^{-1} r_{\kappa_m} \mathbf{1}\big(a(t) < m\big)\big). \tag{27}$$

Consider the first term in the right-hand side of (27) above, and observe that

$$\mathbb{E}^{\epsilon}_{\mathcal{I}_0}(r_{\kappa_m}) = \mathbb{E}^{\epsilon}_{\mathcal{I}_0}(r_\kappa) + (m-1)\mathbb{E}^{\epsilon}_{a\delta_0}(r_\kappa | U = +\infty).$$

From Proposition 12, $\mathbb{E}^{\epsilon}_{\mathcal{I}_0}(r_\kappa) \leq C^{1/2}$ for all $\epsilon \in [0, \epsilon_0]$. Moreover, from identity (26), $(\mathbb{E}^{\epsilon}_{a\delta_0}(r_\kappa | U = +\infty))(\mathbb{E}^{\epsilon}_{a\delta_0}(\kappa | U = +\infty))^{-1} = v_\epsilon$. We deduce that, as t goes to infinity, uniformly with respect to $\epsilon \in [0, \epsilon_0]$,

$$\mathbb{E}^{\epsilon}_{\mathcal{I}_0}(r_{\kappa_m}) = (1-\lambda) t v_\epsilon + O(1). \tag{28}$$

Consider now the second term in the right-hand side of (27). By Schwarz's inequality,

$$\mathbb{E}^{\epsilon}_{\mathcal{I}_0}\big(t^{-1} r_{\kappa_m} \mathbf{1}\big(a(t) < m\big)\big) \leq (\mathbb{E}^{\epsilon}_{\mathcal{I}_0}[(t^{-1} r_{\kappa_m})^2])^{1/2} \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(a(t) < m\big)^{1/2}. \tag{29}$$

From Propositions 12 and 13, one can check that

$$\mathbb{E}^{\epsilon}_{\mathcal{I}_0}[r^2_{\kappa_m}] \leq C m^2. \tag{30}$$

On the other hand, one has that $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(a(t) < m) \leq \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(\kappa_m \geq t)$. From Propositions 12 and 13, the variance of κ_m with respect to $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}$ is bounded above by Cm, so that we can use the Bienaymé–Chebyshev's inequality to prove that, whenever $t > \mathbb{E}^{\epsilon}_{\mathcal{I}_0}(\kappa_m)$,

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(a(t) < m\big) \leq Cm\big(t - \mathbb{E}^{\epsilon}_{\mathcal{I}_0}(\kappa_m)\big)^{-2}. \tag{31}$$

Now, using Proposition 12 as in the proof of (28) above, we can prove that, as t goes to infinity, uniformly with respect to $\epsilon \in [0, \epsilon_0]$,

$$\mathbb{E}^{\epsilon}_{\mathcal{I}_0}(\kappa_m) = (1-\lambda)t + O(1).$$

Putting the above identity together with (29), (30) and (31), we deduce that, as t goes to infinity, uniformly with respect to $\epsilon \in [0, \epsilon_0]$,

$$\mathbb{E}^{\epsilon}_{\mathcal{I}_0}\big(t^{-1} r_{\kappa_m} \mathbf{1}\big(a(t) < m\big)\big) \leq C m^{3/2}\big(\lambda t^2 + O(t)\big)^{-1}.$$

In view of Lemma 3, we have that $m \leq c^{-1} t$ for all $\epsilon \in [0, \epsilon_0]$, so we can conclude that, uniformly with respect to $\epsilon \in [0, \epsilon_0]$,

$$\lim_{t \to +\infty} \mathbb{E}^{\epsilon}_{\mathcal{I}_0}\big(t^{-1} r_{\kappa_m} \mathbf{1}\big(a(t) < m\big)\big) = 0. \tag{32}$$

Plugging (28) and (32) in (27), we finally deduce that, as t goes to infinity, uniformly with respect to $\epsilon \in [0, \epsilon_0]$,

$$\mathbb{E}^{\epsilon}_{\mathcal{I}_0}(t^{-1} r_t) \geq (1-\lambda) v_\epsilon + o(1).$$

The conclusion of the proposition follows by noting that, since $v_\epsilon \leq v_{\epsilon_0}$, $(1-\lambda) v_\epsilon \geq v_\epsilon - \lambda v_{\epsilon_0}$. $\square$

LEMMA 4. *For all $t \geq 0$,*

$$\lim_{\epsilon \to 0+} \mathbb{E}(r_t^\epsilon) = \mathbb{E}(r_t^0).$$

PROOF. Consider a given $t \geq 0$. By Proposition 16 in Appendix B, with $\mathbb{P}$ probability one, we can find a (random) $K \leq 0$ such that $\sup\{F_s^{\epsilon_0}(x,i); s \in [0,t]x < K, i \in [[1,a]]\} \leq 0$, so that $\sup\{F_s^\epsilon(x,i); s \in [0,t]x < K, i \in [[1,a]]\} \leq 0$ for all $\epsilon \in [0,\epsilon_0]$. As a consequence, for all $\epsilon \in [0,\epsilon_0]$, with probability one, $r_t^\epsilon(\mathcal{I}_0) = r_t^\epsilon(w(K))_{s \geq 0})$, where $w(K)$ is the configuration defined by $A = \{K,\dots,0\} \times [[1,a]]$, $r = 0$ and $F(x,i) = x$ for all $(x,i) \in A$.

Since, for every $\epsilon \in [0,\epsilon_0]$, with probability one $r_t^\epsilon \leq r_t^{\epsilon_0}$, we see that the value of r_t^ϵ is entirely determined by the trajectories up to time t of the random walks born at sites (x,i) with $K \leq x \leq r_t^{\epsilon_0}$. With probability one again, we are dealing with a finite number of random walks, and a finite number of steps. We now see that, for all ϵ small enough, these trajectories are identical to what they are for $\epsilon = 0$, so that $r_t^\epsilon = r_t^0$. Since $0 \leq r_t^\epsilon \leq r_t^{\epsilon_0}$ and $r_t^{\epsilon_0}$ is integrable w.r.t. $\mathbb{P}$, we can use the dominated convergence theorem to deduce the conclusion. $\square$

PROOF OF PROPOSITION 10. Let $\zeta > 0$, and, following Proposition 14, consider a t_ζ such that, for all $t \geq t_\zeta$ and all $\epsilon \in [0,\epsilon_0]$,

$$v_\epsilon \leq \mathbb{E}_{\mathcal{I}_0}^\epsilon(t^{-1}r_t) + \zeta.$$

Consider now, thanks to Proposition 7, a $t \geq t_\zeta$ such that $\mathbb{E}_{\mathcal{I}_0}^0(t^{-1}r_t) \leq v + \zeta$. Now, thanks to Lemma 4, we know that, for all ϵ small enough,

$$\mathbb{E}_{\mathcal{I}_0}^\epsilon(t^{-1}r_t) \leq \mathbb{E}_{\mathcal{I}_0}^0(t^{-1}r_t) + \zeta.$$

Putting together the above inequalities, we deduce that, for all ϵ small enough, $v_\epsilon \leq v + 3\zeta$. Since $v_\epsilon \geq v$, the conclusion follows. $\square$

Now Proposition 5 follows from Propositions 8, 9 and 10, as explained in the beginning of this section.

5. Slowdown large deviations. In all this section, we work under the assumption that $\epsilon = 0$, and the dependence of various quantities with respect to ϵ is thus not explicitly mentioned.

Given an initial configuration w such that $w = 0$, remember that $\eta(x)$ counts the number of particles that are located at site x in the configuration, and let, for $x \leq 0$

$$H(x) := \sum_{y=0}^{x} \eta(y). \tag{33}$$

For $x \geq 0$ and $t \geq 0$, let $(\zeta_t)_{t\geq 0}$ denote a continuous time simple symmetric random walk starting at 0 with total jump rate 2. Let

$$\bar{G}_t(x) := P\Big(\sup_{s\in[0,t)} \zeta_s < x\Big), \qquad G_t(x) := P(\zeta_t \geq x).$$

In the sequel, we will use the fact that for fixed $t \geq 0$, $G_t(\cdot)$ is nondecreasing and $\bar{G}_t(\cdot)$ is nonincreasing, and that, thanks to the reflection principle,

$$1 - \bar{G}_t(x) = 2G_t(x) - P(\zeta_t = x). \tag{34}$$

5.1. *Proof of Theorem* 2(a) *and* (c). We start with the proof of Theorem 2(c). The fact that $r_t = 0$ means that no particle in the initial configuration hits 1 before time t. Both the upper and lower bounds can then be understood heuristically as follows. Since we consider simple symmetric random walks, for large t, the constraint of not hitting 1 before time t has a cost only for particles within a distance of order $t^{1/2}$ of the origin. Now these particles perform independent random walks, and their number has an order of magnitude lying between $t^{u/2}$ and $t^{U/2}$.

We start with the lower bound. When $U = +\infty$, the inequality holds trivially, so we assume in the sequel that $U < +\infty$. The event $t^{-1}r_t = 0$, implies that none of the random walks corresponding to particles in the initial condition w hit 1 before time t. By independence of the random walks, the corresponding probability equals

$$\prod_{x=0}^{-\infty} \bar{G}_t(-x+1)^{\eta(x)}.$$

Now let $b_1 > 0$ be such that $1 - 2s \geq \exp(-4s)$ for all $0 \leq s \leq b_1$. From (34), we see that for any $t \geq 0$ and $y \leq 0$, $\bar{G}_t(-y+1) \geq 1 - 2G_t(-y+1)$. By the central limit theorem, we can find t_0 and $K > 0$ such that, for all $t \geq t_0$ and $y \leq -Kt^{1/2}$, $G_t(-y+1) \leq b_1$.

Let $k_t := \lceil Kt^{1/2} \rceil$. Then, for all $t \geq t_0$,

$$\prod_{x=-k_t}^{-\infty} \bar{G}_t(-x+1)^{\eta(x)} \geq \exp\left(-4 \sum_{x=-k_t}^{-\infty} \eta(x) G_t(-x+1)\right).$$

Now, by definition of G_t,

$$\begin{aligned}
\sum_{x=0}^{-\infty} \eta(x) G_t(-x+1) &= E\left(\sum_{x=0}^{-\infty} \eta(x) \mathbf{1}(\zeta_t \geq -x+1)\right) \\
&= E\left[\mathbf{1}(\zeta_t \geq 1)\left(\sum_{x=0}^{-\zeta_t+1} \eta(x)\right)\right] \\
&= E\big[\mathbf{1}(\zeta_t \geq 1)\big(H(-\zeta_t+1)\big)\big].
\end{aligned}$$

By assumption, $H(x) \le |x|^{U+o(1)}$. Hölder's inequality yields that

$$E\big[\mathbf{1}(\zeta_t \ge 1)\big(H(-\zeta_t + 1)\big)\big] \le t^{U/2+o(1)}.$$

We deduce that for all $t \ge t_0$

$$\prod_{x=-k_t}^{-\infty} \bar{G}_t(x)^{\eta(x)} \ge \exp\big(-t^{U/2+o(1)}\big). \tag{35}$$

Now, for $-k_t < y \le 0$, observe that $\bar{G}_t(-y+1) \ge \bar{G}_t(1)$. As a consequence,

$$\prod_{x=0}^{-k_t+1} \bar{G}_t(-x+1)^{\eta(x)} \ge \bar{G}_t(1)^{H(-k_t+1)}.$$

But there exists $c_4 > 0$, such that, for large enough t, $\bar{G}_t(1) \ge c_4 t^{-1/2}$. Using again the fact that $H(x) \le |x|^{U+o(1)}$, one can deduce that $\bar{G}_t(1)^{H(-k_t+1)} \ge \exp(-t^{U/2+o(1)})$, whence

$$\prod_{x=0}^{-k_t+1} \bar{G}_t(-x+1)^{\eta(x)} \ge \exp\big(-t^{U/2+o(1)}\big). \tag{36}$$

From (35) and (36), we deduce that

$$\mathbb{P}(t^{-1} r_t \le 0) \ge \exp\big(-t^{U/2+o(1)}\big).$$

Now, let us prove the upper bound when $u < +\infty$. Using an argument similar to the one used in the proof of the lower bound above, we obtain that

$$\mathbb{P}(t^{-1} r_t = 0) \le \exp\big(-E\big[\mathbf{1}(\zeta_t \ge 1)\big(H(-\zeta_t + 1)\big)\big]\big).$$

One can then deduce that

$$E\big[\mathbf{1}(\zeta_t \ge 1)\big(H(-\zeta_t + 1)\big)\big] \ge t^{u/2+o(1)}$$

and the upper bound follows.

We now turn to the proof of Theorem 2(a). The idea of the proof when $s(\eta) = 1$ is to combine the following two arguments. First, for $b > 0$, it costs nothing to prevent all the particles in the initial condition from hitting $\lfloor bt \rfloor$ up to time t. Intuitively, this result comes from the fact that hitting $\lfloor bt \rfloor$ before time t has an exponential cost for any particle in the initial condition within distance $O(t)$ of the origin, and, due to (G), there is a subexponentially large number of such particles.

Second, in the worst case where all the particles attached to sites $1 \le x \le bt$ are turned into X particles instantaneously at time zero, the cost of preventing all these particles from hitting bt up to time t is of order $\exp(-t^{1/2+o(1)})$, due to the lower bound in (6) proved above, The actual proof is in fact more complex since we want to consider probabilities of the form $\mathbb{P}(ct \le r_t \le bt)$, and not only $\mathbb{P}(r_t \le bt)$, and deal also with the case $s(\eta) < 1$.

We state two lemmas before giving the proof.

LEMMA 5. *Consider an initial condition* $w = (F, 0, A)$ *satisfying* (G). *Then, for all* $b > 0$, *and all* $\varphi > 0$,

$$\mathbb{P}\Big[\max_{(x,i)\in A}\sup_{0\le s\le t} F_s(x,i) \ge bt\Big] \le f_\varphi(w)\exp\big[t\big(\cosh(2\varphi)-1\big)\big]G_t(\lfloor bt\rfloor)^{1/2}.$$

PROOF. The probability we are looking at is the probability that at least one of the random walks corresponding to particles in w exceeds bt before time t. By the union bound, this probability is smaller than

$$\sum_{x=0}^{-\infty}\eta(x)\big(1-\bar{G}_t(-x+\lfloor bt\rfloor)\big) \le \sum_{x=0}^{-\infty} 2\eta(x)G_t(-x+\lfloor bt\rfloor).$$

Now observe that by definition of G_t,

$$\begin{aligned}\sum_{x=0}^{-\infty}\eta(x)G_t(-x+\lfloor bt\rfloor) &= E\left(\sum_{x=0}^{-\infty}\eta(x)\mathbf{1}(\zeta_t \ge -x+\lfloor bt\rfloor)\right)\\ &= E\left[\mathbf{1}(\zeta_t\ge\lfloor bt\rfloor)\left(\sum_{x=0}^{-\zeta_t+\lfloor bt\rfloor}\eta(x)\right)\right]\\ &= E\big[\mathbf{1}(\zeta_t\ge\lfloor bt\rfloor)\big(H(-\zeta_t+\lfloor bt\rfloor)\big)\big].\end{aligned}$$

From (G), we deduce that, for all $\varphi > 0$, $H(x) \le f_\varphi(w)\exp(-\varphi x)$ for all $x \le 0$. As a consequence, when $\zeta_t \ge \lfloor bt\rfloor$, $H(-\zeta_t+\lfloor bt\rfloor) \le H(-\zeta_t) \le f_\varphi(w)\exp(\varphi\zeta_t)$. Applying Schwarz's inequality, we see that

$$E\big[\mathbf{1}(\zeta_t\ge\lfloor bt\rfloor)\big(H(-\zeta_t+\lfloor bt\rfloor)\big)\big] \le \mathbb{P}(\zeta_t\ge\lfloor bt\rfloor)^{1/2} f_\varphi(w)E[\exp(2\varphi\zeta_t)]^{1/2}.$$

Now note that $E[\exp(2\varphi\zeta_t)] = \exp[2(\cosh(2\varphi)-1)t]$. $\square$

LEMMA 6. *Consider an initial condition* $w = (F, 0, A)$ *satisfying* (G). *Then, for all* $\varphi > 0$

$$\mathbb{E}\Big[\sum_{(x,i)\in A_t}\exp\big(\varphi\big(F_t(x,i)-r_t\big)\big)\Big] \le \exp\big[2\big(\cosh(\varphi)-1\big)t\big]f_\varphi(w) + a\mathbb{E}(r_t).$$

PROOF. Write $\sum_{(x,i)\in A_t} = \sum_{(x,i)\in A} + \sum_{(x,i)\in A_t\setminus A}$. For $(x,i)\in A$, observe that $\exp(\varphi(F_t(x,i)-r_t)) \le \exp(\varphi F_t(x,i))$ and that $\mathbb{E}[\exp(\varphi F_t(x,i))] = \exp[\varphi x + 2(\cosh(\varphi)-1)t]$. As a consequence,

$$\mathbb{E}\Big[\sum_{(x,i)\in A}\exp\big(\varphi\big(F_t(x,i)-r_t\big)\big)\Big] \le \exp\big[2\big(\cosh(\varphi)-1\big)t\big]f_\varphi(w). \tag{37}$$

On the other hand, observe that $A_t\setminus A = \{1,\dots,r_t\}\times\{1,\dots,a\}$. Since it is always true that $F_t(x,i) \le r_t$,

$$\sum_{(x,i)\in A_t\setminus A}\exp\big(\varphi\big(F_t(x,i)-r_t\big)\big) \le ar_t. \tag{38}$$

The result follows from putting together (37) and (38). □

PROOF OF THEOREM 2(a). Let $\alpha, \delta \in (0, 1)$ be such that $c < v(1 - \alpha) < b$, $c < (1 - \alpha)(1 - \delta)v < (1 - \alpha)(1 + \delta)v < b$, and define $\gamma := b - (1 - \alpha)(1 + \delta)v$, $\beta_+ := (1 - \alpha)(1 + \delta)$ and $\beta_- := (1 - \alpha)(1 - \delta)$.

For each $t > 0$, define $B_t := \{\beta_- vt \le r_{(1-\alpha)t} \le \beta_+ vt\}$,

$$C_t := \bigcup_{(x,i)\in A_{(1-\alpha)t}} \Big\{ \sup_{s\in[(1-\alpha)t,t]} F_s(x, i) \le r_{(1-\alpha)t} + \gamma t \Big\},$$

$$D_t := \bigcup_{(x,i)\in[\![r_{(1-\alpha)t}, \lfloor bt \rfloor]\!]\times[\![1,a]\!]} \Big\{ \sup_{s\in[0,\alpha t]} x + Y_s(x, i) \le bt \Big\}.$$

Observe that

$$B_t \cap C_t \cap D_t \subset \{ct \le r_t \le bt\}. \tag{39}$$

Indeed, thanks to the choice of δ, B_t implies that $r_{(1-\alpha)t} \ge ct$, so that $r_t \ge ct$. On the other hand, since $r_{(1-\alpha)t} < bt$ on B_t, the event $B_t \cap \{r_t > bt\}$ implies that either a particle born before time $(1 - \alpha)t$ at a position $x \le r_{(1-\alpha)t}$, or a particle born between time $(1 - \alpha)t$ and t at a position $r_{(1-\alpha)t} < x < bt$, exceeds bt at a time between $t(1 - \alpha)$ and t. The former possibility is ruled out by $B_t \cap C_t$, since on $B_t \cap C_t$, $r_t \le r_{(1-\alpha)t} + \gamma t \le bt$. The latter possibility is ruled out by D_t.

Now define

$$l(t) := \exp\big[2\big(\cosh(\varphi) - 1\big)(1 - \alpha)t\big] f_\varphi(w) + a\mathbb{E}\big(r_{(1-\alpha)t}\big)$$

and

$$H_t := \Big\{ \sum_{(x,i)\in A_{(1-\alpha)t}} \exp\big(\varphi\big(F_{(1-\alpha)t}(x, i) - r_{(1-\alpha)t}\big)\big) \le 2l(t) \Big\}.$$

By Lemma 6 and Markov's inequality, for all $t \ge 0$, $\mathbb{P}(H_t) \ge 1/2$. Moreover, by the law of large numbers (1), $\lim_{t\to+\infty} \mathbb{P}(B_t) = 1$. We deduce that there exists a t_0 such that, for all $t \ge t_0$, $\mathbb{P}(B_t \cap H_t) \ge 1/4$. Let us call $\mathcal{F}_t$ the σ-algebra generated by the history of the particle system up to time t. Observe that B_t and H_t belong to $\mathcal{F}_{(1-\alpha)t}$, and by Lemma 5, on H_t,

$$\mathbb{P}\big(C_t^c | \mathcal{F}_{(1-\alpha)t}\big) \le 2l(t) \exp\big[\alpha t\big(\cosh(2\varphi) - 1\big)\big] G_{\alpha t}(\lfloor \gamma t \rfloor)^{1/2}.$$

We deduce that

$$\mathbb{P}(B_t \cap H_t \cap C_t^c) \le 2l(t) \exp\big[\alpha t\big(\cosh(2\varphi) - 1\big)\big] G_{\alpha t}(\lfloor \gamma t \rfloor)^{1/2}. \tag{40}$$

Moreover, we see that, by coupling, if $\mathcal{I}_0$ denotes an initial configuration with exactly a particles per site at the left of the origin,

$$\mathbb{P}\big(D_t | \mathcal{F}_{(1-\alpha)t}\big) \ge \mathbb{P}\big(r_{\alpha t}(\mathcal{I}_0) = 0\big),$$

so that

$$\mathbb{P}(B_t \cap H_t \cap D_t) \geq (1/4)\mathbb{P}\big(r_{\alpha t}(\mathcal{I}_0) = 0\big) \geq \exp\big(-t^{1/2+o(1)}\big), \tag{41}$$

where the last inequality is due to the lower bound in (6). By standard large deviations bounds for the simple random walk, there exists $\zeta(\alpha, \gamma) > 0$ depending only on γ and α such that, as t goes to infinity, $\liminf_{t\to+\infty} t^{-1} \log G_{\alpha t}(\lfloor \gamma t \rfloor) = -\zeta(\alpha, \gamma)$. Furthermore, $\lim_{t\to+\infty} t^{-1} \log(2l(t) \exp[\alpha t(\cosh(2\varphi) - 1)]) = \xi(\alpha, \varphi)$, where $\xi(\alpha, \varphi) := \alpha(\cosh(2\varphi) - 1) + 2(\cosh(\varphi) - 1)(1 - \alpha)$. We see that, choosing φ small enough, $\xi(\alpha, \varphi) < \zeta(\alpha, \gamma)/2$. For such a φ, (40) and (41) show that $\mathbb{P}(B_t \cap H_t \cap C_t^c) = o(\mathbb{P}(B_t \cap H_t \cap D_t))$, so that $\mathbb{P}(B_t \cap H_t \cap D_t \cap C_t) \geq \exp(-t^{1/2+o(1)})$. It then follows from (39) that $\mathbb{P}(ct \leq r_t \leq bt) \geq \exp(-t^{1/2+o(1)})$, so we are done when $s(\eta) = 1$.

Now, let us consider $(x, i) \in A$. Define $\tau = \inf\{s \geq 0; F_s(x, i) = 0\}$, and let w' denote the configuration consisting in a single particle located at (x, i), or, more formally, let w' be of the form (F', r', A') with $r' := x$, $A' := \{(x, i)\})$ and $F'(x, i) := (x, i)$. Remember the notation β_+ and β_- introduced at the beginning of the present proof, then let

$$\begin{aligned}
K_t &:= \{(1 - \beta_+)t \leq \tau \leq (1 - \beta_-)t\},\\
L_t &:= \{ct \leq r_{\beta_- t+\tau}(w') \leq r_{\beta_+ t+\tau}(w') \leq bt\},\\
L'_t &:= \{ct \leq r_{\beta_- t}(\delta_0) \leq r_{\beta_+ t}(\delta_0) \leq bt\},\\
M_t &:= \big\{\text{for all } (y, j) \in A \setminus \{(x, i)\} \text{ and all } s \in [0, t], F_s(y, j) \leq 0\big\}.
\end{aligned}$$

Observe that, on M_t, $r_t(w) = r_t(w')$. Moreover, $K_t \cap L_t \subset \{ct \leq r_t(w') \leq bt\}$. As a consequence,

$$M_t \cap K_t \cap L_t \subset \{ct \leq r_t(w) \leq bt\}. \tag{42}$$

But according to the lower bound of Theorem 2(b), $\mathbb{P}(M_t) \geq \exp(-t^{U/2+o(1)})$. On the other hand, conditional upon τ, $r_{s+\tau}(w')$ has the (unconditional) distribution of $r_s(\delta_0)$, for all $s \geq 0$. As a consequence, $\mathbb{P}(K_t \cap L_t) = \mathbb{P}(K_t)\mathbb{P}(L'_t)$, and, by the law of large numbers (1), $\lim_{t\to+\infty} \mathbb{P}(L'_t) = 1$. Moreover, it is seen from elementary estimates on hitting times by a simple symmetric continuous time random walk that $\liminf_{t\to+\infty} t^{-1/2}\mathbb{P}(K_t) > 0$. Finally, M_t being defined in terms of random walks that do not enter the definition of K_t and L_t, we deduce that M_t is independent from $K_t \cap L_t$. We finally deduce that $\mathbb{P}(M_t \cap K_t \cap L_t) \geq \exp(-t^{U/2+o(1)})$, and the result follows from (42). $\square$

5.2. *Proof of Theorem* 3. Exactly as in the proof of the the upper bound (6) of Theorem 2(c) given above, we can prove that

$$\mathbb{P}\big(t^{-1} r_t(w) \leq bt\big) \leq \exp\big(-E[\mathbf{1}(\zeta_t \geq \lceil bt \rceil) H(-\zeta_t + \lceil bt \rceil)]\big).$$

One can check that, for small enough $b>0$,

$$\liminf_{t\to+\infty} t^{-1}\log E\bigl[\mathbf{1}(\zeta_t \ge \lceil bt\rceil)\exp\bigl(\theta(\zeta_t-\lceil bt\rceil)\bigr)\bigr]>0. \tag{43}$$

This proves (i). We now prove (ii). Again, one can check that, for all $b>0$, there exists $\theta>0$ such that (43) holds. Choosing $b>v$, the result follows.

5.3. *Proof of Theorem* 2(b). Note that by coupling, it is enough to prove the result with an initial condition consisting of exactly a particles per site $x\le 0$. We shall need to consider translated versions of such an initial condition, so we define, for all $u\in\mathbb{Z}$, the configuration $\mathcal{I}_u$ to be of the form (F,r,A) with $r:=u$, $A:=\rrbracket-\infty,u\rrbracket\times\llbracket 1,a\rrbracket$, $F(x,i):=x$ for each $(x,i)\in A$. Hence, we will establish that for all $0<b<v$, and all $\alpha>0$, as $t\to+\infty$,

$$\mathbb{P}[r_t(\mathcal{I}_0)\le bt]\le\exp\bigl(-t^{1/3+o(1)}\bigr).$$

Using the fact that $\mathbb{P}(T_{\mathcal{I}_0}(\lfloor bt\rfloor)\ge t)\le\mathbb{P}(r_t(\mathcal{I}_0)\le bt)\le\mathbb{P}(T_{\mathcal{I}_0}(\lceil bt\rceil)\ge t)$, one can see that (5) is equivalent to the following result.

PROPOSITION 15. *For every $c>v^{-1}$, as n goes to infinity,*

$$\mathbb{P}\bigl(T_{\mathcal{I}_0}(n)\ge cn\bigr)\le\exp\bigl(-n^{1/3+o(1)}\bigr).$$

Our strategy for proving Proposition 15 can be sketched as follows. Given $m\ge 1$, let $\chi_j:=T_{\mathcal{I}_{mj}}(m(j+1))$. By subadditivity, we have that

$$T_{\mathcal{I}_0}(n)\le\sum_{j=0}^{\lfloor n/m\rfloor}\chi_j,$$

so that

$$\mathbb{P}\bigl(T_{\mathcal{I}_0}(n)\ge cn\bigr)\le\mathbb{P}\Biggl(\sum_{j=0}^{\lfloor n/m\rfloor}\chi_j\ge(mc)\lfloor n/m\rfloor\Biggr). \tag{44}$$

Now, by translation invariance, for all $j\ge 0$, χ_j and $\chi_0=T_{\mathcal{I}_0}(m)$ have the same distribution, and it can be shown that

$$\lim_{m\to+\infty} m^{-1}\mathbb{E}(T_{\mathcal{I}_0}(m))=v^{-1}.$$

Hence, given $c>v^{-1}$ we can always find $m\ge 1$ such that $mc>\mathbb{E}(\chi_0)$, so that the right-hand side of (44) is the probability of a large deviation above the mean for the sum $\sum_{j=0}^{\lfloor n/m\rfloor}\chi_j$. We then seek to apply large deviations bounds for i.i.d. variables in order to estimate this probability. Of course, the random variables $(\chi_j;\,j\ge 0)$ are *not* independent, but the dependency between $(\chi_j;\,j\le j_1)$ and $(\chi_j;\,j\ge j_2)$ is weak when j_2-j_1 is large. Indeed, for given j, χ_j mostly depends on the behavior of the random walks born at sites close to mj. We implement this idea by using a technique already exploited in [21] (page 10, line –5) in a similar context. Given $\ell \geq 1$, we define a family $(\chi'_j; j \geq 0)$ of hitting times as follows: χ'_j uses the same random walks as χ_j for particles born at sites (x, i) with $mj - m\ell < x < m(j+1)$, but uses fresh independent random walks for particles born at sites (x, i) with $x \leq mj - m\ell$. We can then prove that the following properties hold:

(a) for all $j \geq 0$, the family $(\chi'_{j+p(\ell+1)}; p \geq 0)$ is i.i.d.;
(b) when ℓ is large, the probability that $\chi'_j = \chi_j$ is close to 1.

We can thus obtain estimates on the right-hand side of (44) by estimating separately the probability that $\chi'_j = \chi_j$ for all $j \in [\![0, \lfloor m/n \rfloor]\!]$, and the probability that $\sum_{j=0}^{\lfloor n/m \rfloor} \chi'_j \geq (mc)\lfloor n/m \rfloor$. Now, thanks to property (a) above, this last sum can be split evenly into $\ell + 1$ subsums of i.i.d. random variables distributed as $\chi_0 = T_{\mathcal{I}_0}(m)$. Controlling the tail of $T_{\mathcal{I}_0}(m)$ then allows us to apply large deviation bounds for i.i.d. variables separately to each of these subsums. In fact, the proof of (5) is a bit more subtle, since it also makes use of a positive association property, but we do not go into the details here (see Remark 3 below).

5.4. *Proof of Proposition* 15. Observe that the subadditivity property [part (iii)] of Proposition 2 reads as:

$$\text{for all } n, m \geq 0 \qquad T_{\mathcal{I}_0}(n+m) \leq T_{\mathcal{I}_0}(n) + T_{\mathcal{I}_n}(m).$$

We deduce that, for $c > v^{-1}$,

$$\mathbb{P}\big(T_{\mathcal{I}_0}(n) \geq cn\big) \leq \mathbb{P}\left(\sum_{j=0}^{\lfloor n/m \rfloor} \chi_j \geq cn\right).$$

In Steps 1 and 2 below, m and ℓ denote fixed positive integers, while α denotes a fixed real number $0 < \alpha < 1$. For the sake of readability, the dependence with respect to these numbers is usually not mentioned explicitly in the notation. Only in Step 3 have the values of m, ℓ and α to be specified.

5.4.1. *Step* 1: *Comparison with a sum of i.i.d. random variables*. Assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is such that we have access to an i.i.d. family of random variables

$$\big[(\tau_k^j(u, i), \varepsilon_k^j(u, i)); j \geq 0, k \geq 1, u \in \mathbb{Z}, i \in [\![1, a]\!]\big],$$

independent from the random variables

$$\big[(\tau_k(u, i), \varepsilon_k(u, i, 0)); k \geq 1, u \in \mathbb{Z}, i \in [\![1, a]\!]\big],$$

used in the construction of the process (see Section 2), and such that, for all j, k, u, i, $\tau_k^j(u, i)$ has an exponential(2) distribution while $\mathbb{P}(\varepsilon_k^j(u, i) = \pm 1)$ equals $1/2$, and $\tau_k^j(u, i)$ and $W_k^j(u, i)$ are independent.

Now, for all and $j\geq 0$, all $u,v\in\mathbb{Z}$ such that $u<v$, and $i\in[\![1,a]\!]$, define a random variable $\mathbb{B}_j(u,i,v)$ in the same way as $\mathbb{A}(u,i,v)$ in (8) of Section 2, but using $(\tau_k^j(u,i),\varepsilon_k^j(u,i))$ instead of $(\tau_k(u,i),\varepsilon_k(u,i,0))$. Specifically, let

$$\mathbb{B}_j(u,i,v):=\inf\left\{\sum_{k=1}^m\tau_k^j(u,i);u+\sum_{k=1}^m\varepsilon_k^j(u,i)=v,m\geq 1\right\}.$$

Let also

$$\mathbb{C}_j(u,i,v):=\begin{cases}\mathbb{B}_j(u,i,v), & \text{if } u\leq mj-m\ell,\\ \mathbb{A}(u,i,v), & \text{if } u>mj-m\ell.\end{cases}$$

We then proceed to define χ'_j in the same way as $\chi_j=T_{\mathcal{I}_{mj}}(m(j+1))$ is characterized in Proposition 2(i), but using $\mathbb{C}_j(u,i,v)$ instead of $\mathbb{A}(u,i,v)$. Since we deal several times in the sequel with variants of such a construction, we now introduce the following general definition: a sequence $(\mathbf{x},\mathbf{i})=(x_1,\dots,x_L,i_1,\dots,i_{L-1})$ with $L\geq 2$, $x_1,\dots,x_L\in\mathbb{Z}$ and $i_1,\dots,i_{L-1}\in[\![1,a]\!]$, is said to be (u,v)-admissible if $L\geq 2$, $x_1\leq u$, $u<x_2<\cdots<x_{L-1}<v$, and $x_L=v$. Given such a sequence $(\mathbf{x},\mathbf{i})$ and a map $\mathbb{D}=\mathbb{D}(x,i,y)$ (such as $\mathbb{D}=\mathbb{A}$ or $\mathbb{C}_j$), we define the notation

$$\mathbb{D}(\mathbf{x},\mathbf{i}):=\sum_{g=1}^{L-1}\mathbb{D}(x_g,i_g,x_{g+1}).$$

When applied to χ_j, Proposition 2(i) reads as

$$\chi_j=\inf\mathbb{A}(\mathbf{x},\mathbf{i}),$$

where the infimum is taken over all finite $(mj,m(j+1))$-admissible sequences. Accordingly, we let

$$\chi'_j:=\inf\mathbb{C}_j(\mathbf{x},\mathbf{i}),$$

where the infimum is taken over all finite $(mj,m(j+1))$-admissible sequences. Clearly, χ'_j and χ_j have the same distribution. Moreover, we have the following lemma, whose proof is immediate.

LEMMA 7. *For every $j\geq 0$, the family of random variables $(\chi'_{j+p(\ell+1)};p\geq 0)$ is i.i.d.*

We now study the event $\{\chi'_j=\chi_j\}$. To this end, let

$$J_j:=\inf\mathbb{C}_j(\mathbf{x},\mathbf{i}),\qquad K_j:=\inf\mathbb{A}(\mathbf{x},\mathbf{i}),$$

where in both cases the infimum is taken over all finite $(mj,m(j+1))$-admissible sequences that satisfy the additional assumption $x_1\leq mj-m\ell$. Let also

$$L_j:=\inf\mathbb{A}(\mathbf{x},\mathbf{i}),$$

where the infimum is taken over all finite $(mj, m(j+1))$-admissible sequences that satisfy the additional assumption $x_1 > mj - m\ell$.

Observe that, $\chi'_j = \min(J_j, L_j)$ and that $\chi_j = \min(K_j, L_j)$. As a consequence,

$$\{\min(J_j, K_j) \geq L_j\} \subset \{\chi'_j = \chi_j\}.$$

For $\alpha > 0$, we now define

$$D(j) := \{\min(J_j, K_j) < \alpha(m\ell)^2\}$$

and

$$\Upsilon(j) := \{L_j \geq \alpha(m\ell)^2\},$$

so that

$$\Upsilon(j)^c \cap D(j)^c \subset \{\chi'_j = \chi_j\}. \tag{45}$$

LEMMA 8. *There exist* $\lambda_1, \lambda_2, \lambda_3 > 0$, *not depending on* m, ℓ, α, *such that*

$$\mathbb{P}(D(j)) \leq \lambda_1 \exp(-\lambda_2 \alpha(m\ell)^2) + \lambda_3 \alpha(m\ell)^2 G_{\alpha(m\ell)^2}(m\ell) =: \lambda.$$

PROOF. Consider the random walks born at sites (x, i) for $x \leq mj - \alpha(m\ell)^2$. By Lemma 1 choosing $\gamma = 1$ and $\theta > 0$ small enough so that $g_\gamma(\theta) > 0$, we obtain the existence of $\lambda_1 > 0$ and $\lambda_2 > 0$ such that the probability that any of the walks born at a site (x, i) with $x \leq mj - \alpha(m\ell)^2$ hits mj before time $\alpha(m\ell)^2$ is $\leq \lambda_1 \exp(-\lambda_2 \alpha(m\ell)^2)$. On the other hand, for $mj - \alpha(m\ell)^2 < x \leq mj - m\ell$, the probability that a walk started at x hits mj before time $\alpha(m\ell)^2$ is less than the corresponding probability for the walk started at $mj - m\ell$, that is, $1 - \bar{G}_{\alpha(m\ell)^2}(m\ell)$. In turn, this probability is less than $2G_{\alpha(m\ell)^2}(m\ell)$. A union bound over all the corresponding events yields the result. $\square$

LEMMA 9. *There exist* $V_1, V_2 > 0$, *not depending on* ℓ, α, *but depending on* m, *such that for all* j,

$$\mathbb{P}(\Upsilon(j)) \leq V_1 \exp(-V_2 \alpha^{1/2} \ell).$$

PROOF. By translation invariance, we can assume that $j = 0$. Let $t = \alpha(m\ell)^2$. Since $\Upsilon(0)$ implies that no random walk born at a site $-m\ell + 1 \leq x \leq 0$ hits 1 before time $\alpha(m\ell)^2$, one has that $\mathbb{P}(\Upsilon(0)) = \prod_{x=-m\ell+1}^{0} \bar{G}_t(1-x)^a$. Since $0 \leq \alpha \leq 1$, we see that $t^{1/2} \leq m\ell$, so that $\mathbb{P}(\Upsilon(0)) \leq \prod_{x=-\lfloor t^{1/2} \rfloor + 1}^{0} \bar{G}_t(1-x)^a$. Using monotonicity of $\bar{G}_t$, we deduce that $\mathbb{P}(\Upsilon(0)) \leq \bar{G}_t(\lfloor t^{1/2} \rfloor)^{a \lfloor t^{1/2} \rfloor}$.

By the central limit theorem, $\lim_{t \to +\infty} G_t(\lfloor t^{1/2} \rfloor) > 0$, so that, since $\bar{G}_t \leq 1 - G_t$, $\limsup_{t \to +\infty} \bar{G}_t(\lfloor t^{1/2} \rfloor) < 1$. As a consequence, we can find $c > 0$, and $t_0 \geq 0$ such that, for all $t \geq t_0$, $\bar{G}_t(\lfloor t^{1/2} \rfloor) \leq 1 - c$. For $t \geq t_0$, we deduce that $P(\Upsilon(0)) \leq (1-c)^{a \lfloor t^{1/2} \rfloor}$. For $t \leq t_0$, we see that we can find a large enough V_1 such that $P(\Upsilon(0)) \leq V_1 (1-c)^{a \lfloor t^{1/2} \rfloor}$, using only the trivial bound $\mathbb{P}(\Upsilon(0)) \leq 1$. $\square$

LEMMA 10. *For all $t \geq 0$, the events $\{\sum_{j=0}^{\lfloor n/m \rfloor} \chi_j \geq cn\}$ and $\bigcup_{j=0}^{\lfloor n/m \rfloor} D(j)$ are negatively associated.*

PROOF. For any integer $N \geq 1$, let

$$\mathbb{A}^N(u, i, v) := \inf\left\{\sum_{k=1}^{m} \tau_k(u, i); u + \sum_{k=1}^{m} \varepsilon_k(u, i, \epsilon) = v, 1 \leq m \leq N\right\}.$$

Similarly, let

$$\mathbb{B}_j^N(u, i, v) := \inf\left\{\sum_{k=1}^{m} \tau_k^j(u, i); u + \sum_{k=1}^{m} \varepsilon_k^j(u, i) = v, 1 \leq m \leq N\right\}$$

and let

$$\mathbb{C}_j^N(u, i, v) := \begin{cases} \mathbb{B}_j^N(u, i, v), & \text{if } u \leq mj - m\ell, \\ \mathbb{A}^N(u, i, v), & \text{if } u > mj - m\ell. \end{cases}$$

Now let $\chi_{j,N} := \inf \mathbb{A}^N(\mathbf{x}, \mathbf{i})$, where the infimum is taken over all finite $(mj, m(j+1))$-admissible sequences that satisfy the additional assumption $x_1 \geq -N$. Similarly, let $J_{j,N} := \inf \mathbb{C}_j^N(\mathbf{x}, \mathbf{i})$, and $K_{j,N} := \inf \mathbb{A}^N(\mathbf{x}, \mathbf{i})$, where in both cases the infimum is taken over all finite $(mj, m(j+1))$-admissible sequences that satisfy the additional assumptions $x_1 > mj - m\ell$ and $x_1 \geq -N$.

Observe that $\mathbb{P}$-almost surely, for all $j \geq 0$, the sequence $(\chi_{j,N})_{N \geq 1}$ is $\mathbb{P}$-a.s. constant after a certain rank, and that its limiting value is χ_j. Similarly, $\mathbb{P}$-almost surely, the sequences $(J_{j,N})_{N \geq 1}$ and $(K_{j,N})_{N \geq 1}$ are $\mathbb{P}$-a.s. constant after a certain rank, and their respective limits are J_j and K_j.

Then let $S_{q,N} := \sum_{p=0}^{q} \chi_{p(\ell+1),N}$ and $D(j, N) := \{\min(J_{j,N}, K_{j,N}) < \alpha(m \times \ell)^2\}$. Now let $g_1 := \mathbf{1}(S_q \geq t)$, $g_2 := \mathbf{1}(\bigcup_{p=0}^{q} D(p(\ell+1)))$, and $g_{1,N} := \mathbf{1}(S_{q,N} \geq t)$ and $g_{2,N} := \mathbf{1}(\bigcup_{p=0}^{q} D(p(\ell+1), N))$.

Note that $(g_{1,N})_{N \geq 1}$ is a bounded sequence of random variables, that is, $\mathbb{P}$-a.s. constant after a certain rank, and converging to g_1 as K goes to infinity. The same holds for $(g_{2,N})_{N \geq 1}$ and g_2. Now, for every N, $g_{1,N}$ and $g_{2,N}$ are functions of a finite number of the random variables $(-\varepsilon_n(x, i), -\varepsilon_n^j(x, i, 0), \tau(x, i), \tau^j(x, i); n \geq 1, x \in \mathbb{Z}, i \in [\![1, a]\!])$. Moreover, one can check from the definitions that, with respect to these random variables, $g_{1,N}$ is nonincreasing, while $g_{2,N}$ is nondecreasing. Since these random variables are independent, we deduce that $\mathbb{E}(-g_{1,N} g_{2,N}) \geq \mathbb{E}(-g_{1,N})\mathbb{E}(g_{2,N})$ (see, e.g., [10], Theorem 2.1). Taking the limit as $N \to +\infty$, and using the dominated convergence theorem, we obtain the result. $\square$

Now let us define three events X, Y, Z by

$$X := \bigcup_{j=0}^{\lfloor n/m \rfloor} D(j), \qquad Y := \bigcap_{j=0}^{\lfloor n/m \rfloor} (D(j)^c \cap \Upsilon(j)^c), \qquad Z := \bigcup_{j=0}^{\lfloor n/m \rfloor} \Upsilon(j)$$

and observe that $\Omega \subset X \cup Y \cup Z$. Let then

$$\Phi := \left\{ \sum_{j=0}^{\lfloor n/m \rfloor} \chi_j \geq cn \right\}.$$

By the union bound, $\mathbb{P}(\Phi) \leq \mathbb{P}(\Phi \cap X) + \mathbb{P}(\Phi \cap Y) + \mathbb{P}(\Phi \cap Z)$. Now, according to Lemmas 8 and 10 we see that

$$\mathbb{P}(\Phi \cap X) \leq \mathbb{P}(\Phi) \times (\lfloor n/m \rfloor + 1)\lambda.$$

From (45), we see that

$$\mathbb{P}(\Phi \cap Y) \leq \mathbb{P}\left(\sum_{j=0}^{\lfloor n/m \rfloor} \chi'_j \geq cn \right).$$

From Lemma 9, we see that,

$$\mathbb{P}(\Phi \cap Z) \leq (\lfloor n/m \rfloor + 1) V_1 \exp(-V_2 \alpha^{1/2} \ell).$$

This leads to the following bound:

$$\rho(n)\mathbb{P}(\Phi) \leq \mathbb{P}\left(\sum_{j=0}^{\lfloor n/m \rfloor} \chi'_j \geq cn \right) + (\lfloor n/m \rfloor + 1) V_1 \exp(-V_2 \alpha^{1/2} m\ell). \tag{46}$$

where $\rho(n) := 1 - (\lfloor n/m \rfloor + 1)\lambda$.

Using the independence properties of the random variables χ'_j (Lemma 7), and the union bound, we see that the following inequality holds:

$$\mathbb{P}\left(\sum_{j=0}^{\lfloor n/m \rfloor} \chi'_j \geq cn \right) \leq (\ell + 1) P\left(R_1 + \cdots + R_{k(n)} \geq \frac{cn}{\ell + 1} \right),$$

where $R_1, R_2, \ldots$ denote i.i.d. copies of χ_0, and where $k(n) := 1 + \lfloor \frac{n/m-1}{\ell+1} \rfloor$.

5.4.2. *Step* 2: *Large deviations estimates for i.i.d. random variables*. We start with a bound on the tail of χ_0.

LEMMA 11. *There exist $\beta_1, \beta_2 > 0$ (depending on m) such that, for all $t \geq 0$,*

$$\mathbb{P}(\chi_0 \geq t) \leq \beta_1 \exp(-\beta_2 \lfloor t^{1/2} \rfloor).$$

PROOF. Observe that the event $\chi_0 \geq t$ implies that none of the random walks born at a site (x, i) with $x \leq 0$ has hit m before time t. As a consequence, $\mathbb{P}(\chi_0 \geq t) \leq \prod_{x=0}^{-\lfloor t^{1/2} \rfloor} \bar{G}_t(-x + m)^a$. Using monotonicity of $\bar{G}_t$, we deduce that $\mathbb{P}(\chi_0 \geq t) \leq G_t(m + \lfloor t^{1/2} \rfloor)^{-a\lfloor t^{1/2} \rfloor)}$. Reusing the notation of the proof of Lemma 9, we see that, for all $t \geq t_0$, $\mathbb{P}(\chi_0 \geq t) \leq (1 - c)^{-a\lfloor t^{1/2} \rfloor)}$. Now, for $t \leq t_0$, we can find β_1 such that, using only the trivial bounds $-\lfloor t^{1/2} \rfloor \geq 0$ and $\mathbb{P}(\chi_0 \geq t) \leq 1$, $\mathbb{P}(\chi_0 \geq t) \leq \beta_1 (1 - c)^{-a\lfloor t^{1/2} \rfloor}$ for all $t \in [0, t_0]$. $\square$

REMARK 1. The lower bound (4) shows that the upper bound of Lemma 11 yields the right order of magnitude for the tail of χ_0.

The probabilities of large deviations above the mean for sums of i.i.d. random variables with an $\exp(-t^{1/2})$ decay of the tail are described by the following lemma, whose proof is deferred to Appendix A.

LEMMA 12. *Let $(R_j)_{j\geq 1}$ be a sequence of i.i.d. nonnegative random variables with common distribution μ. Let $M := \int x\, d\mu(x)$. Assume that there exist $\beta_1, \beta_2 > 0$ such that for every $x \geq 0$*

$$\mu([x, +\infty)) \leq \beta_1 \exp(-\beta_2 x^{1/2}). \tag{47}$$

Then $M < +\infty$ and for all $f > M$, there exists $h > 0$ and n_0 such that if $n \geq n_0$

$$P\big(n^{-1}(R_1 + \cdots + R_n) \geq f\big) \leq \exp(-hn^{1/2}).$$

5.4.3. *Step* 3: *Conclusion*. Lemma 12 above can be applied to probabilities of large deviations of the form $P(R_1 + \cdots + R_k \geq kb)$, where $b > \mathbb{E}(\chi_0)$, and our goal is to control probabilities of the form $P(R_1 + \cdots + R_{k(n)} \geq \frac{cn}{\ell+1})$, where $c > v^{-1}$. First, one can check from the definition that

$$\frac{cn}{\ell + 1} \geq k(n)cm\left(1 + \frac{m(\ell + 1)}{n}\right)^{-1}. \tag{48}$$

Then, observe that Kingman's subadditive ergodic theorem (see, e.g., [8], Theorem 6.4.1) can be applied to the sequence of random variables $(T_{\mathcal{I}_u}(v))_{u\leq v}$. Indeed, these variables are nonnegative, integrable (Lemma 11), and satisfy the required distributional translation invariance properties. We deduce that

$$\lim_{m\to+\infty} m^{-1}\mathbb{E}(T_{\mathcal{I}_0}(m)) = v^{-1}.$$

As a consequence, for all $c > v^{-1}$, we can find $m \geq 1$ large enough so that

$$cm > \mathbb{E}(T_{\mathcal{I}_0}(m)) = \mathbb{E}(\chi_0). \tag{49}$$

In the sequel, we assume that m is chosen such that (49) holds. Now let us choose $\ell := \ell_n = n^{1/3}$. Taking into account Lemmas 11, 12, (48) and (49), we now see that, as n goes to infinity, there exists a constant $h_1 > 0$ such that

$$P\left(R_1 + \cdots + R_{k(n)} \geq \frac{cn}{\ell + 1}\right) = O(\exp(-h_1 n^{1/3})). \tag{50}$$

Now, for $0 < \zeta < 1/2$, let us choose $\alpha := \alpha_n = n^{-\zeta}$, and consider inequality (46). With our definitions, $\alpha_n^{1/2}(m\ell_n) = mn^{1/3-\zeta/2}$ while $m\ell_n = mn^{1/3}$. As a consequence, a moderate deviations bound for the simple random walk (see, e.g., [6], Theorem 3.7.1) yields that $G_{\alpha_n(m\ell_n)^2}(m\ell_n + 1) = O(\exp(-h_2 n^{\zeta}))$ for some constant $h_2 > 0$, whence the fact that $\rho(n) = 1 + o(1)$. Using (50), we see that inequality (46) entails that, for large n,

$$\mathbb{P}(\Phi) \le O(\exp(-h_3 n^{1/3-\zeta/2})).$$

Since ζ can be taken arbitrarily small, the conclusion of Proposition 15 follows.

REMARK 2. In view of (4) and (5), we see that our upper and lower bounds on slowdown probabilities do not match. One may wonder whether it is possible to improve upon either of these bounds so as to find the exact order of magnitude of slowdown large deviations probabilities. What we can prove (the details are not given here) is that the $\exp(-n^{1/3+o(1)})$ bound in Proposition 15 gives the best order of magnitude that can be reached by following our proof strategy based on subadditivity. Indeed, despite the fact that each χ_j has a tail decaying roughly as $\exp(-t^{1/2})$, so that the probabilities of large deviations above the mean would be of order $\exp(-n^{-1/2})$ if these random variables were independent, the positive dependence between these variables makes such large deviations much more likely, with probabilities of order $\exp(-n^{1/3})$.

REMARK 3. One may wonder whether the use of association (see Lemma 10) is really needed in the proof. Indeed, a simpler approach would be to bound the probability of the event X by $\mathbb{P}(\bigcup_{j=0}^{\lfloor n/m \rfloor} D(j))$. By properly choosing α_n and ℓ_n, we could make this probability of the order of $\exp(-n^{1/3+o(1)})$, compared to the $\exp(-h_2 n^{-\zeta})$ obtained in the proof of Proposition 15. However, such a choice interferes with the other bounds used in the proof [making α_n smaller increases the probability of $\Upsilon(j)$]. The best order of magnitude we could obtain with that simpler method is $\exp(-n^{2/7+o(1)})$.

APPENDIX A: LARGE DEVIATIONS OF I.I.D. RANDOM VARIABLES WITH $\exp(-t^{1/2})$ TAILS

Neither the result stated in Lemma 12 nor the idea of its proof are new, but we could not find a reference providing both a statement suited to our purposes and a short proof, so we chose to give the details here.

We refer to the papers [7, 17] for a review of results concerning large deviations of random variables with subexponential tails, and to Theorem 4.1 in [2] for an example of a result from which Lemma 12 may be derived.

LEMMA 13. *For every $\nu > 0$, as $x \to +\infty$,*

$$\int_x^{+\infty} \exp(-\nu u^{1/2})\, du = O[\exp(-(\nu/2)x^{1/2})].$$

PROOF. Observe that there exists $d_1 > 0$ such that, for every $u \ge 1$, $u^{1/2} \times \exp(-(\nu/2)u^{1/2}) \le d_1$. As a consequence, $\exp(-\nu u^{1/2}) \le d_1 u^{-1/2} \exp(-(\nu/2) \times$

$u^{1/2}$), so that

$$\int_x^{+\infty} \exp(-\nu u^{1/2})\,du \le d_1 \int_x^{+\infty} u^{-1/2}\exp(-(\nu/2)u^{1/2})\,du.$$

The right-hand side of the above inequality is then equal to $d_1(4/\nu)\exp(-(\nu/2)\times x^{1/2})$. $\square$

PROOF OF LEMMA 12. Let A and c be as in the statement of the lemma. And let G be defined by $G(x) := \mu([x,+\infty))$.

Let A_n be the following event: $A_n := \bigcap_{1\le i\le n}\{R_i \le n\}$. By the union bound, $P(A_n^c) \le n\mu([n,+\infty))$, so that, by assumption (47) above and Lemma 13 below,

$$P(A_n^c) = O[n\exp(-(\beta_2/2)n^{1/2})]. \tag{51}$$

We now apply the Cramér bound for i.i.d. random variables possessing finite exponential moments (see, e.g., [6], Theorem 2.2.3) to the i.i.d. bounded random variables $R_{i,n}$ defined by $R_{i,n} := \min(R_i, n)$. For every $\lambda > 0$, the following inequality holds:

$$P\big(n^{-1}(R_{1,n}+\cdots+R_{n,n}) \ge f\big) \le \exp[-n\lambda f][E\exp(\lambda R_{1,n})]^n. \tag{52}$$

Let $\lambda_n := (\beta_2/3)n^{-1/2}$ and $K_n := n^{1/4}$. By definition $E\exp(\lambda_n R_{1,n}) = \int_{[0,n)} \exp(\lambda_n x)\,d\mu(x) + \exp(\lambda_n n)\mu([n,+\infty))$. Let us split the above integral into $\int_{[0,n)} = \int_{[0,K_n)} + \int_{[K_n,n)}$. Fix a real number $\alpha > 0$. Since $\lambda_n K_n$ goes to zero as n goes to infinity, we have, for all large enough n (depending on α), an inequality of the following form: for every $x \in [0, K_n)$, $\exp(\lambda_n x) \le 1 + (1+\alpha)\lambda_n x$. Taking the integral in this inequality, we obtain that, for all large enough n,

$$\int_{[0,K_n)} \exp(\lambda_n x)\,d\mu(x) \le \mu([0,K_n)) + (1+\alpha)\lambda_n \int_{[0,K_n)} x\,d\mu(x).$$

Since α is arbitrary in the above argument, we see that

$$\int_{[0,K_n)} \exp(\lambda_n x)\,d\mu(x) \le \mu([0,K_n)) + \big(1+o(1)\big)\lambda_n \int_{[0,K_n)} x\,d\mu(x). \tag{53}$$

By definition, $M = \int_{[0,K_n)} x\,d\mu(x) + \int_{[K_n,+\infty)} x\,d\mu(x)$. Integration by parts yields that $\int_{[K_n,+\infty)} x\,d\mu(x) = -[xG(x)]_{K_n}^{+\infty} + \int_{[K_n,+\infty)} G(x)\,dx$. Assumption (47) above says that $G(x) \le A\exp(-\beta_2 x^{1/2})$. As a consequence, $-[xG(x)]_{K_n}^{+\infty} \le AK_n\exp(-\beta_2 K_n^{1/2})$. Moreover, Lemma 13 yields that $\int_{[K_n,+\infty)} G(x)\,dx = O[\exp(-(\beta_2/2)K_n^{1/2})]$.

Putting the above estimates together, and using the definitions of λ_n and K_n, the above estimates clearly imply that $\int_{[K_n,+\infty)} x\,d\mu(x) = o(\lambda_n)$. Similarly, $\mu([K_n,+\infty)) = o(\lambda_n)$. As a consequence, inequality (53) above yields that

$$\int_{[0,K_n)} \exp(\lambda_n x)\,d\mu(x) \le 1 + \big(1+o(1)\big)M\lambda_n.$$

We now study $\int_{[K_n,n)} \exp(\lambda_n x)\, d\mu(x)$. Integration by parts says that $\int_{[K_n,n)} \exp(\lambda_n x)\, d\mu(x) = -[\exp(\lambda_n x) G(x)]_{K_n}^n + \int_{K_n}^n \lambda_n \exp(\lambda_n x) G(x)\, dx$. Observe that, with our definitions of λ_n and K_n, for every $x \in [\![0, n]\!]$, $\lambda_n x \le (\beta_2/3) x^{1/2}$. As a consequence, $\exp(\lambda_n x) G(x) \le A \exp(-(2\beta_2/3) x^{1/2})$. This estimate, together with Lemma 13, yields that, as n goes to infinity, $\int_{K_n}^n \exp(\lambda_n x) \times G(x)\, dx = o(1)$. Similarly, $[\exp(\lambda_n x) G(x)]_{K_n}^n = o(\lambda_n)$. As a consequence, as n goes to infinity, $\int_{[K_n,n)} \exp(\lambda_n x)\, d\mu(x) = o(\lambda_n)$. Similarly, $\exp(\lambda_n n) \mu([n, +\infty)) = o(\lambda_n)$.

Finally, we obtain the following estimate: $E \exp(\lambda_n R_{1,n}) = 1 + \lambda_n m(1 + o(1))$. As n goes to infinity, an expansion yields that $[E \exp(\lambda_n R_{1,n})]^n = \exp(n M \lambda_n (1 + o(1)))$ From Cramér's inequality (52), we obtain that

$$P\bigl(n^{-1}(R_{1,n} + \cdots + R_{n,n}) \ge f\bigr) \le \exp\bigl(-n\lambda_n (f - M)\bigl(1 + o(1)\bigr)\bigr). \tag{54}$$

Now, on the event A_n, $R_i = R_{i,n}$ for all $1 \le i \le n$.

As a consequence, $P(n^{-1}(R_1 + \cdots + R_n) \ge f) \le P(n^{-1}(R_{1,n} + \cdots + R_{n,n}) \ge f) + P(A_n^c)$.

The statement of the lemma now follows from the bound (51) on $P(A_n^c)$ and the large deviations bound (54) for $R_{1,n} + \cdots + R_{n,n}$. $\square$

APPENDIX B: NEGLIGIBILITY OF REMOTE PARTICLES

PROPOSITION 16. *For any $w \in \mathbb{L}$, $\epsilon \in [0, 1/2)$, and any $t \ge 0$, with $\mathbb{P}$ probability one*,

$$\lim_{K \to -\infty} \sup_{s \in [0,t]} \sum_{(x,i) \in A; x \le r+K} \exp\bigl(\theta\bigl(F_s(x,i) - r\bigr)\bigr) = 0.$$

PROOF. For all x, i, t, let $C_{x,i,t} := \exp(\theta(F_t(x,i) - r))$. For $k \in [\![-\infty, 0]\!]$, let also

$$H_{K,k}(s) := \sum_{(x,i) \in A; r+K+k < x \le r+K} C_{x,i,s}.$$

Now let $\gamma := [2(\cosh\theta - 1) + 4\epsilon \sinh\theta]$, and observe that, for every $(x,i) \in A$, $(C_{x,i,s} \exp(-\gamma s))_{s \ge 0}$ is a càdlàg martingale. As a consequence, so is $(H_{K,k}(t) \times \exp(-\gamma t))_{t \ge 0}$ for all $k \in]\!]-\infty, 0]\!]$, and we have the following inequality, valid for all $\lambda > 0$:

$$\mathbb{P}\Bigl(\sup_{s \in [0,t]} H_{K,k}(s) \exp(-\gamma s) > \lambda\Bigr) \le \lambda^{-1} \mathbb{E}(H_{K,k}(0)).$$

Since $\mathbb{E}(H_{K,k}(0)) = \sum_{(x,i) \in A; x \in]\!]r+K+k, r+K]\!]} \exp(\theta(F(x,i) - r))$, we deduce that

$$\begin{aligned} &\mathbb{P}\Bigl(\sup_{s \in [0,t]} H_{K,k}(s) > \lambda\Bigr) \\ &\quad \le \lambda^{-1} \exp(\gamma t) \sum_{(x,i) \in A; x \in]\!]r+K+k, r+K]\!]} \exp\bigl(\theta\bigl(F(x,i) - r\bigr)\bigr). \end{aligned} \tag{55}$$

Now observe that, for every s, the sequence $(H_{K,k}(s))_{k=0,-1,\dots}$ is nondecreasing since we are summing nonnegative terms. As a consequence, $\mathbb{P}(\sup_{s\in[0,t]} H_{K,-\infty}(s) > \lambda)$ equals $\mathbb{P}(\bigcup_{k=0}^{-\infty} \sup_{s\in[0,t]} H_{K,k}(s) > \lambda)$, which is the probability of the union of a nondecreasing sequence of events, and so is equal to $\lim_{k\to-\infty} \mathbb{P}(\sup_{s\in[0,t]} H_{K,k}(s) > \lambda)$. As a consequence, by (55),

$$\mathbb{P}\Big(\sup_{s\in[0,t]} H_{K,-\infty}(s) > \lambda\Big) \le \lambda^{-1}\exp(\gamma t) \sum_{(x,i)\in A; x\le r+K} \exp\big(\theta\big(F(x,i)-r\big)\big). \tag{56}$$

Now observe that, for every s, the sequence $(\sum_{(x,i)\in A; x\le r+K} C_{x,i,s})_{K=0,-1,\dots}$ is nonincreasing, since we are summing nonnegative terms. As a consequence, $\lim_{K\to-\infty}\sup_{s\in[0,t]} H_{K,-\infty}(s)$ exists, and $\mathbb{P}(\lim_{K\to-\infty}\sup_{s\in[0,t]} H_{K,-\infty}(s) > \lambda)$ equals $\mathbb{P}(\bigcap_{K\le 0}\sup_{s\in[0,t]} H_{K,-\infty}(s) > \lambda)$, which is the probability of the intersection of a nonincreasing sequence of events, and so is equal to the limit $\lim_{K\to-\infty}\mathbb{P}(\sup_{s\in[0,t]} H_{K,-\infty}(s)) > \lambda)$. From inequality (56), we see that this last expression equals zero. $\square$

APPENDIX C: ESTIMATES ON THE RENEWAL STRUCTURE

In this section, we work with random variables defined on the space of trajectories $\mathcal{D}$, as explained in Section 4.1. The definitions related to the renewal structure are given in Section 4.3. In the sequel, every constant C_i or δ_i appearing in the estimates is implicitly assumed to depend on the quantities $a, \theta, \epsilon_0, \alpha_1, \alpha_2, p, L, \epsilon$ (see Section 4.3), unless there is a special mention that dependence with respect to some of these parameters is absent. The notation $(\xi_s^\epsilon)_{s\ge 0}$ stands for a nearest-neighbor random walk on $\mathbb{Z}$ with jump rate 2 and step distribution $(1/2+\epsilon)\delta_{+1} + (1/2-\epsilon)\delta_{-1}$, started at zero. The probability measure governing $(\xi_s^\epsilon)_{s\ge 0}$ is denoted by P. We use the shorthand $M' := M/4 - 1$, which is an integer number according to (23). We also use the notation

$$\mathbb{L}_1 := \{w = (F, r, A) \in \mathbb{L}; r\times [\![1, a]\!] \subset A, F(r,i) = r \text{ for all } 1\le i\le a\}.$$

For every $x\in\mathbb{Z}$, let $M_t(x,i) := \sup_{0\le s\le t} F_s(x,i)$. Let also, for $z\in\mathbb{Z}$,

$$\psi_z(t) := \sum_{(x,i); x\le z, (x,i)\in A_t} \exp\big(\theta\big(M_t(x,i) - r_t\big)\big). \tag{57}$$

Let $\mu_\epsilon := \theta\alpha_1 - 2(\cosh\theta - 1) - 4\epsilon\sinh\theta$, and observe that, for all $\epsilon\in[0,\epsilon_0]$, $\mu_\epsilon \ge \mu_{\epsilon_0}$, and that, according to (24), $\mu_{\epsilon_0} > 0$.

LEMMA 14 (See Lemma 2 in [5]). *There exists $C_1 < +\infty$ not depending on ϵ or L such that, for all $\epsilon\in[0,\epsilon_0]$ and all $w = (F, r, A)\in\mathbb{L}$,*

$$\mathbb{Q}_w^\epsilon(t < W < +\infty) \le C_1 \phi_{r-L}(w)\exp(-\mu_\epsilon t).$$

PROOF. Without loss of generality, we assume $r = 0$. Let us first note that

$$\mathbb{Q}_w^\epsilon[t < W < \infty] \le \mathbb{Q}_w^\epsilon\big[\cup_{s\ge t}\{\phi_{-L}(X_s) \ge e^{\theta(\lfloor\alpha_1 s\rfloor - r_s)}\}\big].$$

By the fact that $s \mapsto M_s(x,i)$ is nondecreasing, and the union bound, we deduce that

$$\mathbb{Q}_w^\epsilon[t < W < \infty] \le \sum_{n=\lfloor t\rfloor}^{+\infty} \mathbb{Q}_w^\epsilon\Bigg[\sum_{(x,i)\in A\cap]\!]-\infty,-L]\!]\times[\![1,a]\!]} e^{\theta M_{n+1}(x,i)} \ge e^{\theta\lfloor\alpha_1 n\rfloor}\Bigg].$$

Using the Markov inequality, we obtain that

$$\begin{aligned} &\mathbb{Q}_w^\epsilon[t < W < \infty] \\ &\quad\le \sum_{n=\lfloor t\rfloor}^{+\infty} \exp(-\theta\lfloor\alpha_1 n\rfloor) \sum_{(x,i)\in A\cap]\!]-\infty,-L]\!]\times[\![1,a]\!]} \mathbb{E}_w^\epsilon\big(e^{\theta M_{n+1}(x,i)}\big). \end{aligned}\tag{58}$$

For $(x,i) \in A$, write $F_s(x,i)$ as the independent sum a symmetric nearest neighbor random walk on $\mathbb{Z}$ with rate $2 - 4\epsilon$, and a Poisson process with rate 4ϵ. Since the Poisson process is nondecreasing, the supremum of its values over the time-interval $[0,s]$ is just the value at time s. Consider now the symmetric random walk part. Calling G_1 the distribution function of the supremum of its values over the time-interval $[0,s]$, and G_2 the distribution function of the value at time s, the reflection principle entails that $1 - G_1 \le 2(1 - G_2)$. Integration by parts then yields that $\int_{F(x,i)}^{+\infty} e^{\theta z}\,dG_1(z) \le 2\int_{F(x,i)}^{+\infty} e^{\theta z}\,dG_2(z) = 2\exp(\theta F(x,i))\exp(2(\cosh\theta - 1)s)$. Since $M_s(x,i)$ is bounded above by the sum of the suprema of the Poisson process and of the symmetric random walk, these two suprema being independent, we deduce that

$$\mathbb{E}_w^\epsilon\big(e^{\theta M_s(x,i)}\big) \le 2\exp(\theta F(x,i))\exp\big(s[2(\cosh\theta - 1) + 4\epsilon\sinh\theta]\big).$$

Plugging the last identity into (58) and summing, we finish the proof of the lemma. $\square$

Define for $t \ge 0$, and $z \le r_0$,

$$N_z(t) := e^{\theta r_t - [2(\cosh\theta - 1) - 4\epsilon\sinh\theta]t}\phi_z(X_t).\tag{59}$$

LEMMA 15 (See Lemma 3 in [5]). *For all $\epsilon \in [0,\epsilon_0]$, and all $w = (F,r,A) \in \mathbb{L}$, the family $(N_z(t))_{t\ge 0}$ is a càdlàg $(\mathcal{F}_t^\epsilon)_{t\ge 0}$-martingale with respect to $\mathbb{Q}_w^\epsilon$.*

PROOF. Let us remark that

$$N_z(t) = \sum_{(x,i)\in A, x\le z} e^{\theta F_t(x,i) - [2(\cosh\theta - 1) - 4\epsilon\sinh\theta]t}.$$

Now, each one of the terms in the above sum is an $(\mathcal{F}_t^\epsilon)_{t\geq 0}$-martingale. Furthermore, since $\phi_z(w) < +\infty$, the martingales

$$\sum_{(x,i)\in A, -n\leq x\leq z} e^{\theta F_t(x,i)-[2(\cosh\theta-1)-4\epsilon\sinh\theta]t},$$

converge in $L^1(\mathbb{Q}_w^\epsilon)$ to $N_z(t)$ as $n\to\infty$. Thus, $(N_z(t))_{t\geq 0}$ is an $(\mathcal{F}_t^\epsilon)_{t\geq 0}$-martingale. That the paths are càdlàg is a consequence of $(X_s)_{s\geq 0}$ being càdlàg. □

LEMMA 16 (See Lemma 4 in [5]). *For every* $\epsilon\in[0,\epsilon_0]$ *and* $w=(F,r,A)\in\mathbb{L}$,

$$\mathbb{Q}_w^\epsilon[W<\infty]\leq \exp(\theta)\phi_{r-L}(w).$$

PROOF. See [5]. □

LEMMA 17 (See Lemma 5 in [5]). *There exist* $0<C_2, C_3<+\infty$ *not depending on* ϵ *or* L *such that, for all* $\epsilon\in[0,\epsilon_0]$, $w=(F,r,A)\in\mathbb{L}$ *and* $t\geq 0$,

$$\mathbb{Q}_w^\epsilon[t<V<\infty]\leq LC_2\exp(-C_3 t).$$

PROOF. Without loss of generality, assume that $r=0$. Then $\mathbb{Q}_w^\epsilon(t<V<+\infty)$ is bounded above by the probability that one of the random walks born at a site between $-L+1$ and -1 is at the right of $\lfloor\alpha_1 s\rfloor$ at some time $s\geq t$. By coupling, we see that the worst case is when all the walks start at zero, in which case, by the union bound, the probability is less than aL times the probability for a single random walk started at zero to exceed $\lfloor\alpha_1 s\rfloor$ at some time $s\geq t$. Let $\tau := \inf\{s\geq t; \xi_s^\epsilon\geq\lfloor\alpha_1 s\rfloor\}$.

Using the fact that $(\exp(\theta\xi_s^\epsilon-[2(\cosh\theta-1)-4\epsilon\sinh\theta]s))_{s\geq 0}$ is a martingale, and applying Doob's stopping theorem, we obtain the bound $P(\tau<+\infty)\leq \exp(\theta)\exp(-\mu_\epsilon t)$. The result follows. □

LEMMA 18 (See Lemma 6 in [5]). *There exists* $\delta_1>0$ *not depending on* ϵ *such that, for all* $\epsilon\in[0,\epsilon_0]$ *and* $w=(F,r,A)\in\mathbb{L}$,

$$\mathbb{Q}_w^\epsilon[V<\infty]\leq 1-\delta_1.$$

PROOF. Without loss of generality, we can assume that $r=0$. Note that the probability $\mathbb{Q}_w^\epsilon[V<\infty]$ is upper bounded by the probability that a random walk within a group of aL independent ones all initially at site $x=0$, at some time $t\geq 0$ is at the right of $\lfloor\alpha_1 t\rfloor$. But this probability is $1-f(\epsilon)^{aL}$, where $f(\epsilon):=P(\text{for all } s\geq 0, \xi_s^\epsilon\leq\lfloor\alpha_1 s\rfloor)$. By coupling, observe that f is a nonincreasing function of ϵ. For $\epsilon=\epsilon_0$, the asymptotic speed of the walk is $4\epsilon_0$. Since, from (24) $\alpha_1>4\epsilon_0$, a consequence of the law of large numbers is that $f(\epsilon_0)>0$. This ends the proof. □

LEMMA 19 (See Lemma 7 in [5] and [4]). *There exists* $0 < C_4 < +\infty$ *not depending on* ϵ *or* L *such that for all* $\epsilon \leq \epsilon_0$ *and* $w = (F, r, A) \in \mathbb{L}_1$, *and all* $t > 0$,

$$\mathbb{Q}_w^\epsilon[t < U < \infty] \leq C_4 t^{-M'}.$$

PROOF. The proof given in [4] for $\epsilon = 0$ is based on tail estimates on the random variables $(\nu_k)_{k\geq 0}$. By coupling, for all $\epsilon \in [0, 1/2)$, and every $s \geq 0$, $\mathbb{Q}_w^\epsilon(\nu_k \geq s) \leq \mathbb{Q}_w^0(\nu_k \geq s)$. Thus, the estimate in [4] is in fact uniform over ϵ. □

LEMMA 20. *There exists* $0 < C_5 < +\infty$ *not depending on* ϵ *or* L *such that for all* $\epsilon \in [0, \epsilon_0]$ *and all* $t > 0$,

$$\mathbb{Q}_{\mathcal{I}_0}^\epsilon\left[\bigcup_{s\geq t} r_s < \lfloor \alpha_1 s\rfloor\right] \leq C_5 t^{-M'}.$$

PROOF. Since we start with the initial condition $\mathcal{I}_0$, we can define a modified auxiliary front $(\tilde{r}_s')_{s\geq 0}$ by replacing the random variables $(\nu_k)_{k\geq 0}$ used in the definition of $(\tilde{r}_s)_{s\geq 0}$ by the random variables $(\nu_k')_{k\geq 0}$ defined as follows. Let $\nu_0' := 0$ and, for $k \geq 1$, ν_k' is the first time one of the random walks $\{(z + Y_s(z, i))_{s\geq 0}; (r_0 + k - M) \leq z \leq r_0 + k - 1, 1 \leq i \leq a\}$, hits the site $r_0 + k$. With this definition, $\tilde{r}_s' \leq r_s$ for all $s \geq 0$, and, for each $1 \leq j \leq M - 1$, the random variables $\{\nu_{Mk+j}' : k \geq 0\}$ are i.i.d. with finite moment of order $M/2$, whereas this is only true for $\{\nu_{Mk+j} : k \geq 1\}$. The argument of [4] used to prove Lemma 19 can then be adapted to prove the present result. Alternatively, one can invoke Lemma 38. □

LEMMA 21 (See Lemma 7 in [5] and [4]). *For every* $\epsilon \in (0, 1/2]$, *there exist* $0 < C_6(\epsilon), C_7(\epsilon) < +\infty$ *not depending on* L *such that, for every* $w = (F, r, A) \in \mathbb{L}_1$, *and every* $t > 0$,

$$\mathbb{Q}_w^\epsilon[t < U < \infty] \leq C_6(\epsilon)\exp(-C_7(\epsilon)t).$$

PROOF. We observe that, for a given $\epsilon > 0$, ν_k has an exponentially decaying tail due to the positive bias of the random walks $(Y_s(x, i))_{s\geq 0}$. Using standard large deviations estimates rather than moment estimates in the proof of Lemma 19, we get the result. □

Using a similar argument, we can prove the following lemma.

LEMMA 22. *For all* $\epsilon \in (0, \epsilon_0]$, *there exist* $0 < C_8(\epsilon), C_9(\epsilon) < +\infty$ *not depending on* L *such that, for all* $t > 0$,

$$\mathbb{Q}_{\mathcal{I}_0}^\epsilon\left[\bigcup_{s\geq t} r_s < \lfloor \alpha_1 s\rfloor\right] \leq C_8(\epsilon)\exp(-C_9(\epsilon)t).$$

LEMMA 23 (See Lemma 7 in [5] and Lemma 11 in [4]). *There exists $\delta_2 > 0$ not depending on ϵ such that, for all $\epsilon \in [0, \epsilon_0]$, $w = (F, r, A) \in \mathbb{L}_1$, and $t > 0$,*

$$\mathbb{Q}_w^\epsilon[U < \infty] \le 1 - \delta_2.$$

PROOF. By coupling, we see that $\mathbb{Q}_w^\epsilon[U < \infty]$ is a nonincreasing function of ϵ. Thus, the estimate for $\epsilon = 0$ proved in [4] is enough. □

LEMMA 24 (See Lemma 9 in [5]). *Let β be such that $0 < \beta < \alpha(0)$. Then there exists $0 < C_{10} < \infty$ not depending on ϵ or L such that, for all $\epsilon \in [0, \epsilon_0]$, the following properties hold for all $w = (F, r, A) \in \mathbb{L}$:*

(a) *If $r = 0$ and $w \in \mathbb{L}_1$, and $n \ge 1$,*

$$\mathbb{Q}_w^\epsilon[T(n) > n/\beta] \le C_{10} n^{-a/2}.$$

(b) *Assume that $r = 0$, $m_{-L^{1/4},0}(w) \ge aL^{1/4}/2$ and $n \ge 1$. Then*

$$\mathbb{Q}_w^\epsilon[T(n) > n/\beta] \le (C_{10} L^{1/4} n^{-1/2})^{aL^{1/4}/2} + C_{10} n^{-M'}.$$

(c) *Assume that $r = 0$. For all $k \ge M$ and $n \ge 1$, we have,*

$$\mathbb{Q}_w^\epsilon[T(n+k) - T(k) > n/\beta] \le C_{10} n^{-M'}.$$

PROOF. The proof given in [5] for $\epsilon = 0$ is based on tail estimates for the random variables $(\nu_k)_{k \ge 0}$ and for hitting times of symmetric random walks, so that, by coupling, the estimates proved in [5] are in fact uniform over ϵ. □

LEMMA 25. *Let β be such that $0 < \beta < \alpha(0)$. Then there exists $0 < C_{11} < \infty$ not depending on ϵ or L such that, for all $\epsilon \in [0, \epsilon_0]$, for all $w = (F, r, A) \in \mathbb{L}$ such that $r = 0$ and $m_{-L^{1/4},0}(w) \ge aL^{1/4}/2$, for all $n \ge 1$,*

$$\mathbb{Q}_w^\epsilon[T(nL) > nL/\beta] \le C_{11}(nL^{1/2})^{-M'}.$$

PROOF. Consequence of Lemma 24(b), using the first inequality in (25). □

LEMMA 26 (See Lemma 9 in [5]). *For all $\epsilon \in (0, 1/2)$ and β such that $0 < \beta < \alpha(0)$, there exist $0 < C_{12}(\epsilon), C_{13}(\epsilon) < \infty$ not depending on L such that: for every $w = (F, r, A) \in \mathbb{L}_1$, and $n \ge 1$,*

$$\mathbb{Q}_w^\epsilon[T(n) > n/\beta] \le C_{12}(\beta, \epsilon) \exp(-C_{13}(\beta, \epsilon) n).$$

PROOF. Stems from the exponential decay of the tail of ν_k, as in Lemma 21. □

COROLLARY 3 (See Corollary 2 in [5]). *There exists* $0 < C_{14}, C_{15} < \infty$ *not depending on* ϵ *or* L *such that, for all* $\epsilon \in [0, \epsilon_0]$, *all* $w = (F, r, A) \in \mathbb{L}_1$ *such that* $\phi_{r-L}(w) \le p$, *and all* $t > 0$,

$$\mathbb{Q}_w^\epsilon(t < D < \infty) \le C_{14}\big(t^{-M'} + L\exp(-C_{15}t)\big).$$

COROLLARY 4 (See Corollary 2 in [5]). *For every* $\epsilon \in (0, 1/2]$, *there exist* $0 < C_{16}(\beta, \epsilon), C_{17}(\beta, \epsilon) < \infty$ *not depending on L such that, for all* $w = (F, r, A) \in \mathbb{L}_1$ *such that* $\phi_{r-L}(w) \le p$, *and for all* $t > 0$,

$$\mathbb{Q}_w^\epsilon(t < D < \infty) \le LC_{16}(\beta, \epsilon)\exp(-C_{17}(\beta, \epsilon)t).$$

COROLLARY 5 (See Corollary 2 in [5]). *There exists* $0 < \delta_3 < \infty$ *such that, for all* $\epsilon \in [0, \epsilon_0]$, *and all* $w = (F, r, A) \in \mathbb{L}_1$ *such that* $\phi_{r-L}(w) \le p$,

$$\mathbb{Q}_w^\epsilon(D < \infty) \le 1 - \delta_3.$$

PROOF OF COROLLARIES 3, 4 AND 5. See [5]. $\square$

LEMMA 27 (See Lemma 11 in [5]). *There exists* $0 < C_{18}, C_{19} < +\infty$ *not depending on* ϵ *or* L *such that, for all* $\epsilon \in [0, \epsilon_0]$, *all* $w = (F, r, A) \in \mathbb{L}_1$ *such that* $\phi_{r-L}(w) \le p$, *and all* $t > 0$,

$$\mathbb{Q}_w^\epsilon(r_D - r > t, D < +\infty) \le C_{18}\big(t^{-M'} + L\exp(-C_{19}t)\big).$$

LEMMA 28 (See Lemma 11 in [5]). *For every* $\epsilon \in (0, \epsilon_0]$, *there exist* $0 < C_{20}(\epsilon), C_{21}(\epsilon) < +\infty$ *not depending on L such that, for all* $w = (F, r, A) \in \mathbb{L}_1$ *such that* $\phi_{r-L}(w) \le p$, *and for all* $t > 0$,

$$\mathbb{Q}_w^\epsilon(r_D - r > t, D < +\infty) \le LC_{20}(\epsilon)\exp(-C_{21}(\epsilon)t).$$

PROOFS OF LEMMAS 27 AND 28. Consider $\gamma_0 > 0$ large enough so that

$$c_{\gamma_0}(\epsilon_0, \theta) > 0. \tag{60}$$

Observe that then $c_{\gamma_0}(\epsilon) \ge c_{\gamma_0}(\epsilon_0)$ for all $\epsilon \in (0, \epsilon_0]$. Now by the union bound and the fact that $(r_s)_s$ is nondecreasing, $\mathbb{Q}_w^\epsilon(r_D - r > t, D < +\infty) \le \mathbb{Q}_w^\epsilon(r_{t\gamma_0^{-1}} - r > t, D \le t\gamma_0^{-1}) + \mathbb{Q}_w^\epsilon(t\gamma_0^{-1} < D < +\infty)$. Moreover, note that, by definition, $\phi_r(0) \le \phi_{r-L}(0) + aL$. Then apply Lemma 2 and Corollaries 3 and 4. $\square$

LEMMA 29 (See Lemma 12 in [5]). *Consider* $w = (F, r, A) \in \mathbb{L}_1$ *such that* $\phi_{r-L}(w) \le p$. *Then, for all* $\epsilon \in [0, \epsilon_0]$, $\mathbb{Q}_w^\epsilon$*-a.s. on the event* $\{D < \infty\}$ *we have,*

$$\phi_{r-L}(D) \le e^\theta.$$

PROOF. See [5]. $\square$

COROLLARY 6 (See Corollary 3 in [5]). *There exists* $0 < C_{22} < +\infty$ *not depending on* ϵ *or* L, *such that, for all* $\epsilon \in [0, \epsilon_0]$, *and all* $w = (F, r, A) \in \mathbb{L}_1$ *satisfying* $\phi_{r-L}(w) \le p$,

$$\mathbb{E}_w^\epsilon[\phi_{r_D}(D), D < \infty] \le C_{22} L.$$

PROOF. See [5]. $\square$

LEMMA 30 (See Lemma 13 in [5]). *There is a constant* $0 < C_{23} < +\infty$ *not depending on* ϵ *or* L, *such that, for all* $\epsilon \in [0, \epsilon_0]$, *and all* $w = (F, r, A) \in \mathbb{L}_1$:

(a) $\mathbb{Q}_w^\epsilon(m_{r, r+L^{1/4}}(X_{T(r+L^{1/4})}) < aL^{1/4}/2) \le C_{23} L^{-a/8}$;
(b) $\mathbb{Q}_w^\epsilon(m_{r_D+L-L^{1/4}, r_D+L}(X_{T(r_D+L)}) < aL^{1/4}/2) \le C_{23} L^{-aM'/8(M'+1)}$.

PROOF. Without loss of generality, assume that $r = 0$. For the sake of readability, let $n := L^{1/4}$. We start with the proof of (a).

Choose $4\epsilon_0 < \beta < \alpha(0)$. Then,

$$\begin{aligned}\mathbb{Q}_w^\epsilon\left[m_{0,n}(X_{T(n)}) < \frac{an}{2}\right] &\le \mathbb{Q}_w^\epsilon\left[m_{0,n}(X_{T(n)}) < \frac{an}{2}, T(n) \le \frac{n}{\beta}\right] \\ &\quad + \mathbb{Q}_w^\epsilon\left[T(n) > \frac{n}{\beta}\right].\end{aligned} \tag{61}$$

Note that the event $\{m_{0,n}(X_{T(n)}) < an/2, T(n) \le n/\beta\}$ is contained in the event that at least one particle born at any of the sites $\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1, \dots, n$ hits some site $x \le 0$ in a time shorter than or equal to n/β. Hence, we can conclude that,

$$\begin{aligned}&\mathbb{Q}_w^\epsilon\left[m_{0,n}(X_{T(n)}) < \frac{an}{2}, T(n) \le \frac{n}{\beta}\right] \\ &\qquad \le a(n+1-\lfloor n/2 \rfloor) P[\Lambda_{n/\beta}^\epsilon \le -\lfloor n/2 \rfloor],\end{aligned} \tag{62}$$

where $\Lambda_t^\epsilon := \inf_{0 \le s \le t} \xi_s^\epsilon$.

Noting that, by coupling, $P[\Lambda_{n/\beta}^\epsilon \le -n/2]$ is nonincreasing as a function of ϵ, we can assume that $\epsilon = 0$.

Now, by the reflection principle, $P[\Lambda_{n/\beta}^0 \le -n/2] \le 2P[\xi_{n/\beta}^0 \le -n/2]$. Hence, from inequality (62), we see that $\mathbb{Q}_w^\epsilon[m_{0,n}(X_{T(n)}) < an/2, T(n) \le \frac{n}{\beta}]$ is bounded above by $a(n+1)P[\xi_{n/\beta}^0 \le -n/2]$. By a standard large deviations argument, for every $t \ge 0$ and positive integer x, $P[\xi_t^0 \ge x] \le e^{-tg(x/t)}$, where $g(u) > 0$ for all $u > 0$. Hence, $a(n+1)P[\xi_{n/\beta}^0 \le -n/2] \le a(n+1)\exp\{-\frac{n}{\beta} g(\beta/2)\}$. Finally, using part (a) of Lemma 24 to bound the second term of inequality (61) and using the fact that $a(n+1)\exp\{-\frac{n}{\beta} g(\beta/2)\} \le 1/n^{a/2}$ for n large enough, we conclude the proof of (a).

Now for (b), $\mathbb{P}_w[m_{r_D+L-n,r_D+L}(X_{T(r_D+L)}) < an/2]$ is upper bounded by,

$$\sum_{k:1\le k\le m} \mathbb{Q}_w^\epsilon[m_{k+L-n,k+L}(X_{T(k+L)}) < an/2] + \mathbb{Q}_w^\epsilon[r_D > m, D < \infty].$$

Letting $m := L^{a/(8(M'+1))}$, and using part (a) and Lemma 27, we obtain the result. $\square$

Throughout the sequel, to shorten the expressions, we use D as an upper index to denote quantities shifted by D. On the event $\{D < \infty\}$, let

$$T'(nL) := T(r_D + nL) - D, \qquad m'_{z_1,z_2} := m_{r_D+z_1,r_D+z_2},$$
$$\psi'_z(t) := \psi_{r_D+z}(t) \circ \theta_D, \qquad X'_t := X_{D+t},$$

where θ_D denotes time-shifting of the trajectories by an amount of D.

LEMMA 31 (See Lemma 14 in [5]). *For every $0 < \beta < \alpha(0)$, there exists $0 < C_{24} < \infty$ not depending on ϵ, L, such that for all $\epsilon \in [0, \epsilon_0]$, and all $w = (F, r, A) \in \mathbb{L}$ such that $m_{r-L^{1/4},r}(w) \ge aL^{1/4}/2$, and $\phi_{r-L}(w) \le p$, and for all natural $n \ge 1$,*

$$\mathbb{Q}_w^\epsilon\left[T'(nL) > \frac{nL}{\beta}, D < \infty\right] \le C_{24}(nL^{1/2})^{-M'+1}.$$

PROOF. Without loss of generality, we can assume that initially $r = 0$. Note that $\mathbb{Q}_w^\epsilon[T'(nL) > \frac{nL}{\beta}, D < \infty]$ is upper-bounded by

$$\sum_{k:1\le k\le L^{1/2}n} \mathbb{Q}_w^\epsilon\left[T'(nL) > \frac{nL}{\beta}, r_D = k, D < \infty\right] \\ + \mathbb{Q}_w^\epsilon[r_D > nL^{1/2}, D < \infty]. \tag{63}$$

Now, on the event $\{D < \infty\}$ we have that $T(r_D) \le D$ so that $T'(nL) \le T(r_D + nL) - T(r_D)$. Hence,

$$\mathbb{Q}_w^\epsilon\left[T'(nL) > \frac{nL}{\beta}, r_D = k, D < \infty\right] \le \mathbb{Q}_w^\epsilon\left[T(k+nL) - T(k) > \frac{nL}{\beta}\right].$$

Now, by part (c) of Lemma 24, for all $k \ge M$ we have $\mathbb{Q}_w^\epsilon[T(k+nL) - T(k) > \frac{nL}{\beta}] \le \frac{C_{10}}{(nL)^{M'}}$. On the other hand, for $1 \le k \le M-1$, $\mathbb{Q}_w^\epsilon[T(k+nL) - T(k) > \frac{nL}{\beta}] \le \mathbb{Q}_w^\epsilon[T(M+nL) > \frac{nL}{\beta}]$.

Now let $\beta < \beta' < \alpha(0)$. Observe that, when $nL^{1/2} \ge M(\beta'/\beta - 1)^{-1}$, $(nL + M)/\beta' \le nL/\beta$, so that $\mathbb{Q}_w^\epsilon[T(M+nL) > \frac{nL}{\beta}] \le \mathbb{Q}_w^\epsilon[T(M+nL) > \frac{nL+M}{\beta'}]$.

Thus, by Lemma 25, since $m_{r-L^{1/4},r}(w) \ge aL^{1/4}/2$, we know that

$$\mathbb{Q}_w^\epsilon\left[T(M+nL) > \frac{nL+M}{\beta'}\right] \le (C_{11}(nL^{1/2})^{-M'}). \tag{64}$$

When $nL^{1/2} \le M(\beta'/\beta - 1)^{-1}$, the same bound holds, with a possibly larger constant, using only the trivial inequality $\mathbb{Q}_w^\epsilon(\cdot) \le 1$. Using Lemma 27 to estimate the second term of display (63), and combining with (64), we finish the proof. □

LEMMA 32 (See Lemma 14 in [5]). *For every $\epsilon \in (0, \epsilon_0)$ and $0 < \beta < \alpha(0)$, there exist $0 < C_{25}(\beta, \epsilon), C_{26}(\beta, \epsilon) < \infty$ not depending on L, such that for all $w = (F, r, A) \in \mathbb{L}_1$ such that $\phi_{r-L}(w) \le p$, for all natural $n \ge 1$,*

$$\mathbb{Q}_w^\epsilon\left[T'(nL) > \frac{nL}{\beta}, D < \infty\right] \le C_{25}(\beta, \epsilon) \exp(-C_{26}(\beta, \epsilon)nL).$$

PROOF. Consider $\ell > 0$ such that $\beta(1+\ell) < \alpha(0)$.

As in the proof of the previous lemma, $\mathbb{Q}_w^\epsilon[T'(nL) > \frac{nL}{\beta}, D < \infty]$ is upper-bounded by

$$\begin{aligned}
&\sum_{k:\, 1 \le k \le \lfloor \ell nL \rfloor} \mathbb{Q}_w^\epsilon\left[T'(nL) > \frac{nL}{\beta}, r_D = k, D < \infty\right] \\
&\qquad + \mathbb{Q}_w^\epsilon[r_D > \lfloor \ell nL \rfloor, D < \infty].
\end{aligned} \tag{65}$$

By Lemma 28, $\mathbb{Q}_w^\epsilon[r_D > \lfloor \ell nL \rfloor, D < \infty] \le LC_{20}(\epsilon) \exp(-C_{21}(\epsilon)\lfloor \ell nL \rfloor)$. On the other hand, for $1 \le k \le \lfloor \ell nL \rfloor$, $\mathbb{Q}_w^\epsilon[T(k+nL) - T(k) > \frac{nL}{\beta}] \le \mathbb{Q}_w^\epsilon[T(\lfloor nL(1+\ell)\rfloor) > \frac{nL}{\beta}]$. By Lemma 26, $\mathbb{Q}_w^\epsilon[T(\lfloor nL(1+\ell)\rfloor) > \frac{nL}{\beta}] \le C_{12}(\beta(1+\ell), \epsilon) \times \exp(-C_{13}(\beta(1+\ell), \epsilon)\lfloor nL(1+\ell)\rfloor)$. □

Remember the definition of $\mu_\epsilon := \theta\alpha_1 - 2(\cosh\theta - 1) - 4\epsilon\sinh\theta$, and the fact that, for all $\epsilon \in [0, \epsilon_0]$, $\mu_\epsilon \ge \mu_{\epsilon_0} > 0$. We shall have ample use in the sequel of the notation $h(n) := p^{-1}2^{n+1}\exp(-\mu_{\epsilon_0} nL/\alpha_1)$.

LEMMA 33 (See Lemma 16 in [5]). *Consider $w = (F, r, A) \in \mathbb{L}$ such that $r = 0$. Then for all $\epsilon \in [0, \epsilon_0]$, the following properties hold:*

(a) *For every $n \ge 1$, we have*

$$\mathbb{Q}_w^\epsilon[\psi_0(T(n)) > 2^{-n}p, T(n) < nL/\alpha_1] \le \phi_0(w)h(n). \tag{66}$$

(b) *For every $k \ge 1$ and $n \ge k$, we have a.s.*

$$\begin{aligned}
&\mathbb{Q}_w^\epsilon\big[\psi_k(T(n)) - \psi_{k-L}(T(n)) > 2^{-(n-k)}p, \\
&\qquad T(n) - T(k) < (n-k)L/\alpha_1 | \mathcal{F}_{T(k)}^\epsilon\big] \\
&\quad \le aLh(n-k).
\end{aligned}$$

PROOF. See [5]. □

COROLLARY 7 (See Corollary 4 in [5]). *There exists $0 < C_{27} < +\infty$ not depending on ϵ or L such that, for all $w = (F, r, A) \in \mathbb{L}$, for all $\epsilon \in [0, \epsilon_0]$, $\lambda > 0$,*

$n \geq 1$,

$$\mathbb{Q}_w^{\epsilon}[\psi_0'(T_{nL}') > p2^{-n}, T'(nL) \leq nL/\alpha_1, D < +\infty] \leq C_{27}Lh(n).$$

PROOF. See [5]. □

COROLLARY 8 (See Corollary 5 in [5]). *There exists* $0 < C_{28} < +\infty$ *not depending on* ϵ *or* L *such that, for all* $\epsilon \in [0, \epsilon_0]$, *and all* $w = (F, r, A) \in \mathbb{L}_1$ *such that* $m_{r-L^{1/4},r}(w) \geq aL^{1/4}/2$,

$$\mathbb{Q}_w^{\epsilon}\big[\{\psi_0'(T'(L)) > p\} \cup \{m'_{L-L^{1/4},L}(X'_{T'(L)}) < aL^{1/4}/2\}, D < +\infty\big]$$
$$\leq C_{28}L^{-aM'/(8(M'+1))}.$$

PROOF. See [5]. □

LEMMA 34 (See Lemma 15 in [5]). *Let* $q \geq 1$ *be an integer. Consider two sequences* $(a_k)_{k\geq 1}$ *and* $(c_k)_{k\geq 1}$ *of nonnegative real numbers such that* $\sum_{k=1}^{\infty} a_k < 1$ *and such that*

$$c_1 \leq a_1, \tag{67}$$

and for every $m \geq 2$ *we have that*

$$c_m \leq a_m + \sum_{k=1}^{m-1} a_{m-k}c_k. \tag{68}$$

For all integers $q \geq 0$, *let* $A_q := \sum_{k=1}^{+\infty} a_k k^q$ *and* $C_q := \sum_{k=1}^{+\infty} c_k k^q$. *For* $t \geq 0$, *let* $\mathcal{A}(t) := \sum_{k=1}^{+\infty} a_k \exp(tk)$ *and* $\mathcal{C}(t) := \sum_{k=1}^{+\infty} c_k \exp(tk)$. *The following properties hold*:

(a) *Assume that* $q \geq 1$ *is such that* $A_q < +\infty$. *Then* $C_k < +\infty$ *for all* $1 \leq k \leq q$, *and*

$$C_q \leq (1 - A_0)^{-1}\left(A_q + \sum_{k=1}^{q} \binom{q}{k} C_{q-k}A_k\right).$$

(b) *Assume that* $\mathcal{A}(t_0) < +\infty$ *for some* $t_0 > 0$. *Then* $\mathcal{A}(t) < 1$ *for all small enough* $t > 0$ *and, for all such* t,

$$\mathcal{C}(t) \leq \big(1 - \mathcal{A}(t)\big)^{-1}\mathcal{A}(t).$$

PROOF. Part (a) is proved in [5]. As for part (b), observe that the power series $a(z) := \sum_{k=1}^{+\infty} a_k z^k$ has a convergence radius $\geq \exp(t_0)$. As a consequence, the map $t \mapsto a(\exp(t))$ is well defined and continuous for $t \leq t_0$. For $t = 0$, $a(\exp(t)) = \sum_{k=1}^{+\infty} a_k < 1$ by assumption. By continuity, $a(\exp(t)) < 1$ for all $t > 0$ small enough.

Summing (67) and (68), we see that, for all $m\geq 1$ and $t\geq 0$, $\sum_{i=1}^{m} c_i\exp(ti)\leq a_1\exp(t)+\sum_{i=2}^{m}(a_i\exp(ti)+\sum_{k=1}^{i-1}a_{i-k}c_k\exp(ti))$, so that $\sum_{i=1}^{m}c_i\exp(ti)\leq\sum_{i=1}^{m}a_i\exp(ti)+\sum_{k=1}^{m-1}c_k\exp(tk)(\sum_{i=k+1}^{m}a_{i-k}\exp(t(i-k)))$. As a consequence, $\sum_{i=1}^{m-1}c_i\exp(ti)\leq\mathcal{A}(t)+\mathcal{A}(t)\sum_{k=1}^{m-1}c_k\exp(tk))$. When $\mathcal{A}(t)<1$, we deduce that $\sum_{i=1}^{m-1}c_i\exp(ti)\leq(1-\mathcal{A}(t))^{-1}\mathcal{A}(t)$. Letting m go to infinity, we conclude the proof. $\square$

LEMMA 35. *Let $(O,\mathcal{H},\mathbb{T})$ be a probability space, and $(\mathcal{H}_n)_{n\geq 1}$ be a nondecreasing sequence of sub-σ-algebras of $\mathcal{H}$. Let $(B_n)_{n\geq 1}$, $(A_k^n)_{n\geq 2,k\in[\![0,n-1]\!]}$ and $(B'_n)_{n\geq 2}$ be sequences of events in $\mathcal{H}$ such that the following properties hold*:

(i) *for all $n\geq 1$, $B_n\in\mathcal{H}_n$*
(ii) *for all $n\geq 2$, $B_n\subset B_{n-1}\cap(B'_n\cup A_0^n\cup A_1^n\cup\cdots\cup A_{n-1}^n)$.*

Now assume that we have defined a sequence $(a_n)_{n\geq 1}$ of nonnegative real numbers enjoying the following properties:

1. $\mathbb{T}(B_1)\leq a_1$;
2. *for all $n\geq 2$, $\mathbb{T}(B'_n|\mathcal{H}_{n-1})\leq a_1$ a.s.*;
3. *for all $n\geq 3$, $\mathbb{T}(A_{n-1}^n|\mathcal{H}_{n-2})\leq a_2$ a.s.*;
4. *for all $n\geq 2$, $\mathbb{T}(A_0^n)\leq a_n/2$ a.s.*;
5. *for all $n\geq 2$, $\mathbb{T}(A_1^n)\leq a_n/2$ a.s.*;
6. *for all $n\geq 4$ and all $2\leq k\leq n-2$, $\mathbb{T}(A_k^n|\mathcal{H}_{k-1})\leq a_{n-k+1}$ a.s.*;

then, letting $c_n:=\mathbb{T}(B_n)$ for all $n\geq 1$, the inequalities (67) and (68) are satisfied by the two sequences $(a_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$.

PROOF. First, observe that inequality (67) is a mere consequence of assumption (1). Assume now that $n\geq 2$. By the union bound,

$$\mathbb{T}(B_n)\leq\sum_{k=0}^{n-1}\mathbb{T}(A_k^n,B_{n-1})+\mathbb{T}(B'_n,B_{n-1}).\tag{69}$$

Now, since $B_{n-1}\in\mathcal{H}_{n-1}$, assumption (2) entails that $\mathbb{T}(B'_n,B_{n-1})\leq a_1\times\mathbb{T}(B_{n-1})$.

On the other hand, (4) and (5) imply that $\mathbb{T}(A_0^n)+\mathbb{T}(A_1^n)\leq a_n$.

When $n=2$, we deduce from (69) that $\mathbb{T}(B_n)\leq\mathbb{T}(A_0^n)+\mathbb{T}(A_1^n)+\mathbb{T}(B'_n,B_{n-1})$, so that $\mathbb{T}(B_n)\leq a_n+a_1\mathbb{T}(B_{n-1})$, and so (68) is proved for $n=2$.

Assume now that $n\geq 3$. Since by assumption $B_{n-1}\subset B_{n-2}$, $\mathbb{T}(A_{n-1}^n,B_{n-1})\leq\mathbb{T}(A_{n-1}^n,B_{n-2})$. Now, thanks to assumption (3) and to the fact that $B_{n-2}\in\mathcal{H}_{n-2}$, $\mathbb{T}(A_{n-1}^n,B_{n-2})\leq a_2\mathbb{T}(B_{n-2})$.

For $n=3$, we deduce from (69) that $\mathbb{T}(B_n)\leq\mathbb{T}(A_0^n)+\mathbb{T}(A_1^n)+\mathbb{T}(A_{n-1}^n,B_{n-1})+\mathbb{T}(B'_n,B_{n-1})$, so that $\mathbb{T}(B_n)\leq a_n+a_2\mathbb{T}(B_{n-2})+a_1\mathbb{T}(B_{n-1})$, and so (68) is proved for $n=3$.

Assume now that $n \geq 4$. For $2 \leq k \leq n-2$, the fact that $B_{n-1} \subset B_{k-1}$ implies that $\mathbb{T}(A_k^n, B_{n-1}) \leq \mathbb{T}(A_k^n, B_{k-2})$. Since $B_{k-1} \in \mathcal{H}_{k-1}$, assumption (6) entails that $\mathbb{T}(A_k^n, B_{k-1}) \leq a_{n-k+1}\mathbb{T}(B_{k-1})$.

As a consequence, plugging the previous estimates into inequality (69), we obtain that

$$\mathbb{T}(B_n) \leq a_n + a_2\mathbb{T}(B_{n-2}) + a_1\mathbb{T}(B_{n-1}) + \sum_{k=2}^{n-2} a_{n-k+1}\mathbb{T}(B_{k-1}),$$

which is exactly (68). $\square$

LEMMA 36 (See Lemma 17 in [5]). *There exists* $0 < L_0 < +\infty$ *not depending on* ϵ *such that, for all* $L \geq L_0$ *there exists* $0 < C_{29} < +\infty$ *not depending on* ϵ, *such that for all* $\epsilon \in [0, \epsilon_0]$, *the following properties hold*:

(a) *For all* $n \geq 1$, $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(J_0 \geq n) \leq C_{29} n^{3-M'}$.

(b) *For all* $w = (F, r, A) \in \mathbb{L}_1$ *such that* $m_{r-L^{1/4},r}(w) \geq aL^{1/4}/2$, *and* $\phi_{r-L}(w) \leq p$, *we have that, for all* $n \geq 1$, $\mathbb{Q}^{\epsilon}_{w}(J_{r_D} \geq n, D < +\infty) \leq C_{29} n^{3-M'}$.

(c) *For all* $n \geq 1$, $\mathbb{Q}^{\epsilon}_{a\delta_0}(J_0 \geq n, U > T(nL)) \leq C_{29} n^{3-M'}$.

In the sequel, we use $\mathcal{F}_t$ instead of $\mathcal{F}_t^{\epsilon}$ to alleviate notation.

PROOF OF PART (a). For all $n \geq 1$, let

$$B_n := \bigcap_{i=1}^{n} \{\psi_{(i-1)L}(T(iL)) > p\} \cup B_i',$$

$$B_i' := \{m_{iL-L^{1/4},iL}(X_{T(iL)}) < aL^{1/4}/2\}.$$

Since $\phi_z(X_t) \leq \psi_z(t)$, the following inequality holds:

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(J_0 > n) \leq \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(B_n). \tag{70}$$

For $n \geq 2$ and $1 \leq k \leq n-1$, let

$$\Delta_k^n := \psi_{kL}(T(nL)) - \psi_{(k-1)L}(T(nL))$$

and let

$$A_0^n := \{\psi_0(T(nL)) > 2^{-(n-1)}p\}, \qquad A_k^n := \{\Delta_k^n > 2^{-(n-k)}p\}.$$

We now prove that the assumptions (i) and (ii) of Lemma 35 are satisfied, with $(O, \mathcal{H})$ being the space $\mathcal{D}$ equipped with the cylindrical σ-algebra, and probability $\mathbb{Q}^{\epsilon}_{w}$, and $\mathcal{H}_n := \mathcal{F}_{T(nL)}$ for all $n \geq 1$.

Assumption (i) is immediate. Note that, for $n \geq 2$, $\psi_{(n-1)L}(T(nL)) = \psi_0(T(n \times L)) + \sum_{k=1}^{n-1} \Delta_k^n$. Since $2^{-(n-1)} + \sum_{k=1}^{n-1} 2^{-(n-k)} = 1$, we have that

$$\{\psi_{(n-1)L}(T(nL)) > p\} \subset \{\psi_0(T(nL)) > 2^{-(n-1)}p\} \cup \left[\bigcup_{k=1}^{n-1} \{\Delta_k^n > 2^{-(n-k)}p\}\right],$$

so that (ii) is established.

We now look for a sequence $(a_n)_{n\geq 1}$ such that assumptions (1)–(6) of Lemma 35 are satisfied. Assume that $n\geq 2$. By the strong Markov property and Lemma 24(c), using the fact that, by (25), $L\geq M$, we have for any $1\leq k\leq n-1$, a.s.

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(T(nL)-T(kL)\geq (n-k)L/\alpha_1|\mathcal{F}_{T((k-1)L)}\big)\leq C_{10}\big((n-k)L\big)^{-M'}.$$

By the strong Markov property again, and Lemma 33(b),

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big[\Delta^n_k>2^{-(n-k)}p, T(nL)-T(kL)\leq (n-k)L/\alpha_1|\mathcal{F}_{T((k-1)L)}\big]\leq aLh(n-k).$$

We deduce that, for $n\geq 2$, and $1\leq k\leq n-1$, a.s.

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(A^n_k|\mathcal{F}_{T((k-1)L)}\big)\leq C_{10}\big((n-k)L\big)^{-M'}+aLh(n-k). \tag{71}$$

Similarly, using Lemma 25, which is possible since $m_{-L^{1/4},0}(\mathcal{I}_0)\geq aL^{1/4}/2$, we have that

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(T(nL)\geq nL/\alpha_1\big)\leq C_{11}(nL^{1/2})^{-M'}.$$

On the other hand, by Lemma 33(a), we have that

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}[\psi_0(T(nL))>2^{-n}p, T(nL)\leq nL/\alpha_1]\leq \phi_0(\mathcal{I}_0)h(n).$$

We deduce that

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(A^n_0)\leq C_{11}(nL^{1/2})^{-M'}+\phi_0(\mathcal{I}_0)h(n). \tag{72}$$

Now, for $n\geq 2$, by part (a) of Lemma 30, the strong Markov property, the fact that $(n-1)L\leq nL-L^{1/4}$ and that there are at least a particles at the rightmost visited site at time $T(nL-L^{1/4})$, a.s.

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(B'_n|\mathcal{F}_{T((n-1)L)}\big)\leq C_{23}L^{-a/8}. \tag{73}$$

Finally, observe that, by the union bound, $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(B_1)$ is upper bounded by

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(\psi_0(T(L))>p, T(L)\leq L/\alpha_1\big)+\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(T(L)>L/\alpha_1\big)$$
$$+\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(m_{L-L^{1/4},L}\big(X_{T(L)}\big)<aL^{1/4}/2\big).$$

Thanks to Lemmas 24(a), 33(a) and 30(a), we obtain that

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(B_1)\leq \phi_0(\mathcal{I}_0)h(1)+C_{10}L^{-a/2}+C_{23}L^{-a/8}. \tag{74}$$

Now we see that, by inequalities (73) and (74), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1:=\phi_0(\mathcal{I}_0)h(1)+C_{10}L^{-a/2}+C_{23}L^{-a/8}.$$

Now, for $m\geq 2$, let

$$a_m:=2\big[C_{10}\big((m-1)L\big)^{-M'}+aLh(m-1)+C_{11}(mL^{1/2})^{-M'}+\phi_0(\mathcal{I}_0)h(m)\big].$$

Inequalities (71) and (72) entail assumptions (3), (4), (5), (6) of Lemma 35. Note that the sequence $(a_m)_{m\geq 1}$ depends on ϵ_0 but not on ϵ. Moreover, observe that, for large enough L (not depending on ϵ), $\sum_{m=1}^{+\infty} a_m m^{M'-3} < +\infty$. On the other hand, as L goes to infinity, $\sum_{m=1}^{+\infty} a_m$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m\geq 1}$. Part (a) of Lemma 36 then follows from Lemma 34. $\square$

PROOF OF PART (b). We use exactly the same strategy as for part (a).

For all $n \geq 1$, let

$$B_n := \bigcap_{i=1}^{n} \{\psi'_{((i-1)L)}(T'(iL)) > p, D < +\infty\} \cup B'_i,$$

$$B'_i := \{m'_{iL-L^{1/4},iL}(X'_{T'(iL)}) < aL^{1/4}/2, D < +\infty\}.$$

Since $\phi_z(X_t) \leq \psi_z(t)$, the following inequality holds:

$$\mathbb{Q}_w^{\epsilon}(J_{r_D} > n, D < +\infty) \leq \mathbb{Q}_w^{\epsilon}(B_n). \tag{75}$$

For $n \geq 2$ and $1 \leq k \leq n-1$, on $\{D < +\infty\}$, let

$$\Delta_k^n := \psi'_{kL}(T'(nL)) - \psi'_{(k-1)L}(T'(nL))$$

and let

$$A_0^n := \{\psi'_0(T'(nL)) > 2^{-(n-1)}p, D < +\infty\},$$

$$A_k^n := \{\Delta_k^n > 2^{-(n-k)}p, D < +\infty\},$$

for $1 \leq k \leq n-1$.

We now prove that the assumptions (i) and (ii) of Lemma 35 are satisfied, with $(O, \mathcal{H}, \mathbb{T})$ being the space $\mathcal{D}$ equipped with the cylindrical σ-algebra, and probability $\mathbb{Q}_w^{\epsilon}$, and $\mathcal{H}_n := \mathcal{F}_{T(r_D+nL)} =: \mathcal{F}'_{T'(nL)}$ for all $n \geq 1$.

Assumption (i) is immediate. Note that, for $n \geq 2$, on $\{D < +\infty\}$ $\psi'_{(n-1)L}(T'(nL)) = \psi_0(T'(nL)) + \sum_{k=1}^{n-1} \Delta_k^n$. Since $2^{-(n-1)} + \sum_{k=1}^{n-1} 2^{-(n-k)} = 1$, we have that, on $\{D < +\infty\}$,

$$\{\psi'_{((n-1)L)}(T'(nL)) > p\} \subset \{\psi'_0(T'(nL)) > 2^{-(n-1)}p\} \cup \left[\bigcup_{k=1}^{n-1} \{\Delta_k^n > 2^{-(n-k)}p\}\right],$$

so that (ii) is established.

We now look for a sequence $(a_n)_{n\geq 1}$ such that assumptions (1)–(6) of Lemma 35 are satisfied. Assume that $n \geq 2$. By the strong Markov property and Lemma 24(c), using the fact that, by (25), $L \geq M$, we have for any $1 \leq k \leq n-1$, on the event $\{D < +\infty\}$, a.s.

$$\mathbb{Q}_w^{\epsilon}(T'(nL) - T'(kL) \geq (n-k)L/\alpha_1 | \mathcal{F}'_{T'((k-1)L)}) \leq C_{10}((n-k)L)^{-M'}.$$

By the strong Markov property again, and Lemma 33(b), we have that, on $\{D<+\infty\}$, a.s.

$$\mathbb{Q}_w^{\epsilon}[\Delta_k^n > 2^{-(n-k)}p, T'(nL)-T'(kL)\le (n-k)L/\alpha_1|\mathcal{F}'_{T'((k-1)L)}] \le aLh(n-k).$$

We deduce that, for $n\ge 2$, and $1\le k\le n-1$, on $\{D<+\infty\}$, a.s.

$$\mathbb{Q}_w^{\epsilon}(A_k^n|\mathcal{F}'_{T'((k-1)L)}) \le C_{10}((n-k)L)^{-M'} + aLh(n-k). \tag{76}$$

Similarly, using Lemma 31, which is possible since $m_{-L^{1/4},0}(w)\ge aL^{1/4}/2$ and $\phi_{r-L}(w)\le p$, we have that

$$\mathbb{Q}_w^{\epsilon}(T'(nL)\ge nL/\alpha_1, D<+\infty) \le C_{24}(nL^{1/2})^{-M'+1}.$$

On the other hand, by Corollary 7, we have that

$$\mathbb{Q}_w^{\epsilon}[\psi_0(T'(nL))>2^{-n}p, T'(nL)\le nL/\alpha_1] \le C_{27}Lh(n).$$

We deduce that

$$\mathbb{Q}_w^{\epsilon}(A_0^n) \le C_{24}(nL^{1/2})^{-M'+1} + C_{27}Lh(n). \tag{77}$$

Now, for $n\ge 2$, by part (a) of Lemma 30, the strong Markov property, the fact that $(n-1)L\le nL-L^{1/4}$, and that there are at least a particles at the rightmost visited site at time $T(r_D+nL-L^{1/4})$, on $\{D<+\infty\}$, a.s.

$$\mathbb{Q}_w^{\epsilon}(B'_n|\mathcal{F}'_{T'((n-1)L)}) \le C_{23}L^{-a/8}. \tag{78}$$

Finally, observe that, by Corollary 8,

$$\mathbb{Q}_w^{\epsilon}(B_1) \le C_{28}L^{-aM'/(8(M'+1))}. \tag{79}$$

Now we see that, by inequalities (78) and (79), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1 := C_{23}L^{-a/8} + C_{28}L^{-aM'/(8(M'+1))}.$$

Now, for $m\ge 2$, let

$$a_m := 2[C_{10}((m-1)L)^{-M'} + aLh(m-1) + C_{24}(mL^{1/2})^{-M'+1} + C_{27}Lh(m)].$$

Inequalities (76) and (77) entail assumptions (3), (4), (5), (6) of Lemma 35.

Note that the sequence $(a_m)_{m\ge 1}$ depends on ϵ_0 but not on ϵ. Moreover, observe that, for large enough L (not depending on ϵ), $\sum_{m=1}^{+\infty} a_m m^{M'-3} < +\infty$. On the other hand, as L goes to infinity, $\sum_{m=1}^{+\infty} a_m$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m\ge 1}$. Part (b) of Lemma 36 then follows from Lemma 34. $\square$

PROOF OF PART (c). For all $n \geq 1$, let

$$B_n := \bigcap_{i=1}^{n} \{\psi_{(i-1)L}(T(iL)) > p, U > T(iL)\} \cup B_i',$$

$$B_i' := \{m_{iL-L^{1/4},iL}(X_{T(iL)}) < aL^{1/4}/2, U > T(iL)\}.$$

Since $\phi_z(X_t) \leq \psi_z(t)$, the following inequality holds:

$$\mathbb{Q}_w^{\epsilon}(J_0 > n, U > T(nL)) \leq \mathbb{Q}_{\mathcal{I}_0}^{\epsilon}(B_n). \tag{80}$$

For $n \geq 2$ and $1 \leq k \leq n-1$, let

$$\Delta_k^n := \psi_{kL}(T(nL)) - \psi_{(k-1)L}(T(nL))$$

and let

$$A_0^n := \{\psi_0(T(nL)) > 2^{-(n-1)}p, U > T(nL)\},$$

$$A_k^n := \{\Delta_k^n > 2^{-(n-k)}p, U > T(nL)\}$$

for $1 \leq k \leq n-1$.

We now prove that the assumptions (i) and (ii) of Lemma 35 are satisfied, with $(O, \mathcal{H})$ being the space $\mathcal{D}$ equipped with the cylindrical σ-algebra and probability $\mathbb{Q}_{a\delta_0}^{\epsilon}$, and $\mathcal{H}_n := \mathcal{F}_{T(nL)}$. Assumption (i) is immediate. Assumption (ii) is proved as in (a).

We now look for a sequence $(a_n)_{n\geq 1}$ such that assumptions (1)–(6) of Lemma 35 are satisfied.

Assume that $n \geq 2$. Exactly as in part (a), we can prove that, for $n \geq 2$, and $1 \leq k \leq n-1$, a.s.

$$\mathbb{Q}_{a\delta_0}^{\epsilon}(A_k^n | \mathcal{F}_{T((k-1)L)}) \leq C_{10}((n-k)L)^{-M'} + aLh(n-k). \tag{81}$$

Now, note that, on A_0^n, one has $T(nL) \leq (nL+1)/\alpha_2$ since $U > T(nL)$, whence $T(nL) \leq nL/\alpha_1$ when $L \geq \alpha_1/(\alpha_2 - \alpha_1)$.

On the other hand, by Lemma 33(a), we have that

$$\mathbb{Q}_{a\delta_0}^{\epsilon}[\psi_0(T(nL)) > 2^{-n}p, T(nL) \leq nL/\alpha_1] \leq \phi_0(a\delta_0)h(n).$$

We deduce that

$$\mathbb{Q}_{a\delta_0}^{\epsilon}(A_0^n) \leq \phi_0(a\delta_0)h(n). \tag{82}$$

Exactly as in (a), a.s.

$$\mathbb{Q}_{a\delta_0}^{\epsilon}(B_n' | \mathcal{F}_{T((n-1)L)}) \leq C_{23}L^{-a/8}. \tag{83}$$

Finally, observe that, by the union bound, $\mathbb{Q}_{a\delta_0}^{\epsilon}(B_1)$ is upper bounded by

$$\mathbb{Q}_{a\delta_0}^{\epsilon}(\psi_0(T(L)) > p, T(L) \leq L/\alpha_1)$$
$$+ \mathbb{Q}_{a\delta_0}^{\epsilon}(T(L) > L/\alpha_1) + \mathbb{Q}_{\mathcal{I}_0}^{\epsilon}(m_{L-L^{1/4},L}(X_{T(L)}) < aL^{1/4}/2).$$

Thanks to Lemmas 24, 30(a) and 33(a), we obtain that

$$\mathbb{Q}^{\epsilon}_{a\delta_0}(B_1)\le \phi_0(a\delta_0)h(1)+C_{10}L^{-a/2}+C_{23}L^{-a/8}. \tag{84}$$

Now we see that, by inequalities (83) and (84), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1:=\phi_0(a\delta_0)h(1)+C_{10}L^{-a/2}+C_{23}L^{-a/8}.$$

Now, for $m\ge 2$, let

$$a_m:=2\big[C_{10}\big((m-1)L\big)^{-M'}+aLh(m-1)+\phi_0(a\delta_0)h(m)\big].$$

Inequalities (81) and (82) entail assumptions (3), (4), (5), (6) of Lemma 35.

Note that the sequence $(a_m)_{m\ge1}$ depends on ϵ_0 but not on ϵ. Moreover, observe that, for large enough L (not depending on ϵ), $\sum_{m=1}^{+\infty}a_m m^{M'-3}<+\infty$. On the other hand, as L goes to infinity, $\sum_{m=1}^{+\infty}a_m$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m\ge1}$. Part (c) of Lemma 36 then follows from Lemma 34. □

LEMMA 37 (See Lemma 17 in [5]). *For every $\epsilon\in(0,\epsilon_0]$, there exists $L_1(\epsilon)<+\infty$ such that, for all $L\ge L_1(\epsilon)$, there exists $0<C_{30}(\epsilon), C_{31}(\epsilon)<+\infty$ such that the following properties hold*:

(a) *For all $n\ge1$, $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(J_0\ge n)\le C_{30}(\epsilon)\exp(-C_{31}(\epsilon)n)$.*

(b) *For all $w\in\mathbb{L}_1$ such that $m_{r-L^{1/4},r}(w)\ge aL^{1/4}/2$, and $\phi_{r-L}(w)\le p$, we have that, for all $n\ge1$, $\mathbb{Q}^{\epsilon}_{w}(J_{r_D}\ge n, D<+\infty)\le C_{30}(\epsilon)\exp(-C_{31}(\epsilon)n)$.*

(c) *For all $n\ge1$, $\mathbb{Q}^{\epsilon}_{a\delta_0}(J_0\ge n, U>T(nL))\le C_{30}(\epsilon)\exp(-C_{31}(\epsilon)n)$.*

PROOF OF PART (a). We use exactly the same definitions as in the proof of part (a) of Lemma 36, except that we look for a different sequence $(a_n)_{n\ge1}$ such that assumptions (1)–(6) of Lemma 35 are satisfied. Assume that $n\ge2$. By the strong Markov property and Lemma 26, we have that, for any $1\le k\le n-1$, a.s.

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(T(nL)-T(kL)\ge(n-k)L/\alpha_1|\mathcal{F}_{T((k-1)L)}\big)\le C_{12}(\epsilon)\exp\big(-C_{13}(\epsilon)(n-k)L\big).$$

As in the proof of Lemma 36, a.s.

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big[\Delta^n_k>2^{-(n-k)}p, T(nL)-T(kL)\le(n-k)L/\alpha_1|\mathcal{F}_{T((k-1)L)}\big]\le aLh(n-k).$$

We deduce that, for $n\ge2$, and $1\le k\le n-1$, a.s.

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(A^n_k|\mathcal{F}_{T((k-1)L)}\big)\le C_{12}(\epsilon)\exp\big(-C_{13}(\epsilon)(n-k)L\big)+aLh(n-k). \tag{85}$$

By Lemma 26 again,

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\big(T(nL)\ge nL/\alpha_1\big)\le C_{12}(\epsilon)\exp(-C_{13}(\epsilon)nL).$$

On the other hand, as in the proof of Lemma 36,

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}[\psi_0(T(nL))>2^{-n}p, T(nL)\le nL/\alpha_1]\le\phi_0(\mathcal{I}_0)h(n).$$

We deduce that

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(A^n_0) \le C_{12}(\epsilon)\exp(-C_{13}(\epsilon)nL) + \phi_0(\mathcal{I}_0)h(n). \tag{86}$$

Now, for $n \ge 2$, as in the proof of Lemma 36, a.s.

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\bigl(B'_n|\mathcal{F}_{T((n-1)L)}\bigr) \le C_{23}L^{-a/8}. \tag{87}$$

Similarly,

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(B_1) \le \phi_0(\mathcal{I}_0)h(1) + C_{10}L^{-a/2} + C_{23}L^{-a/8}. \tag{88}$$

Now we see that, by inequalities (87) and (88), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1 := \phi_0(\mathcal{I}_0)h(1) + C_{10}L^{-a/2} + C_{23}L^{-a/8}.$$

Now, for $m \ge 2$, let

$$\begin{aligned} a_m := 2\bigl[&C_{12}(\epsilon)\exp\bigl(-C_{13}(\epsilon)(m-1)L\bigr) + aLh(m-1) \\ &+ C_{12}(\epsilon)\exp(-C_{13}(\epsilon)mL) + \phi_0(\mathcal{I}_0)h(m)\bigr]. \end{aligned}$$

Inequalities (85) and (86) entail assumptions (3), (4), (5), (6) of Lemma 35. Now observe that, for L large enough, $\sum_{n=1}^{+\infty} a_n \exp(tn) < +\infty$ for $t > 0$ small enough. As L goes to infinity, $\sum_{m=1}^{+\infty} a_m$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m\ge1}$. Part (a) then follows from Lemma 34. $\square$

PROOF OF PART (b). We reuse exactly the same definitions as in the proof of part (b) of Lemma 36, except that we look for a different sequence $(a_n)_{n\ge1}$ such that assumptions (1)–(6) of Lemma 35 are satisfied. Assume that $n \ge 2$. By the strong Markov property and Lemma 26, we have for any $1 \le k \le n-1$, on $\{D < +\infty\}$ a.s.

$$\begin{aligned} &\mathbb{Q}^{\epsilon}_w\bigl(T'(nL) - T'(kL) \ge (n-k)L/\alpha_1|\mathcal{F}'_{T'((k-1)L)}\bigr) \\ &\quad \le C_{12}(\epsilon)\exp\bigl(-C_{13}(\epsilon)(n-k)L\bigr). \end{aligned}$$

As in Lemma 36, we have that, on $\{D < +\infty\}$ a.s.

$$\mathbb{Q}^{\epsilon}_w\bigl[\Delta^n_k > 2^{-(n-k)}p, T'(nL) - T'(kL) \le (n-k)L/\alpha_1|\mathcal{F}'_{T'((k-1)L)}\bigr] \le aLh(n-k).$$

We deduce that, for $n \ge 2$, and $1 \le k \le n-1$, on $\{D < +\infty\}$, a.s.

$$\mathbb{Q}^{\epsilon}_w\bigl(A^n_k|\mathcal{F}'_{T'((k-1)L)}\bigr) \le C_{12}(\epsilon)\exp\bigl(-C_{13}(\epsilon)(n-k)L\bigr) + aLh(n-k). \tag{89}$$

Similarly, using Lemma 32, which is possible since $\phi_{r-L}(w) \le p$, we have that

$$\mathbb{Q}^{\epsilon}_w\bigl(T'(nL) \ge nL/\alpha_1, D < +\infty\bigr) \le C_{25}(\epsilon)L\exp(-C_{26}(\epsilon)nL).$$

As in the proof of Lemma 36, we have that

$$\mathbb{Q}^{\epsilon}_w[\psi'_0(T'(nL)) > 2^{-n}p, T(nL) \le nL/\alpha_1] \le C_{27}Lh(n).$$

We deduce that

$$\mathbb{Q}_w^\epsilon(A_0^n) \le C_{25}(\epsilon) L \exp(-C_{26}(\epsilon) n L) + C_{27} L h(n). \tag{90}$$

Now, for $n \ge 2$, as in Lemma 36 a.s.

$$\mathbb{Q}_w^\epsilon\big(B_n' | \mathcal{F}'_{T'((n-1)L)}\big) \le C_{23} L^{-a/8} \tag{91}$$

and

$$\mathbb{Q}_w^\epsilon(B_1) \le C_{28} L^{-aM/(16(M+1))}. \tag{92}$$

Now we see that, by inequalities (91) and (92), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1 := C_{23} L^{-a/8} + C_{28} L^{-aM/(16(M+1))}.$$

Now, for $m \ge 2$, let

$$\begin{aligned} a_m := 2\big[&C_{12}(\epsilon) \exp\big(-C_{13}(\epsilon)(m-1)L\big) + aLh(m-1) \\ &+ C_{25}(\epsilon) L \exp(-C_{26}(\epsilon) n L) + C_{27} L h(m)\big]. \end{aligned}$$

Inequalities (89) and (90) entail assumptions (3), (4), (5), (6) of Lemma 35. Now observe that, for L large enough, $\sum_{n=1}^{+\infty} a_n \exp(tn) < +\infty$ for $t > 0$ small enough. As L goes to infinity, $\sum_{m=1}^{+\infty} a_m$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m \ge 1}$. Part (b) then follows from Lemma 34. $\square$

PROOF OF PART (c). We use exactly the same definitions as in the proof of Lemma 36(c), except that we look for a different sequence $(a_n)_{n \ge 1}$ such that assumptions (1)–(6) of Lemma 35 are satisfied.

Assume that $n \ge 2$. Exactly as in the proof of part (a) of the present lemma, we can prove that, for $n \ge 2$, and $1 \le k \le n-1$, a.s.

$$\mathbb{Q}_{\mathcal{I}_0}^\epsilon\big(A_k^n | \mathcal{F}_{T((k-1)L)}\big) \le C_{12}(\epsilon) \exp\big(-C_{13}(\epsilon)(n-k)L\big) + aLh(n-k). \tag{93}$$

As in the proof of Lemma 36(c),

$$\mathbb{Q}_{a\delta_0}^\epsilon(A_0^n) \le \phi_0(a\delta_0) h(n). \tag{94}$$

Similarly, a.s.

$$\mathbb{Q}_{a\delta_0}^\epsilon\big(B_n' | \mathcal{F}_{T((n-1)L)}\big) \le C_{23} L^{-a/8} \tag{95}$$

and

$$\mathbb{Q}_{a\delta_0}^\epsilon(B_1) \le \phi_0(a\delta_0) h(1) + C_{10} L^{-a/2} + C_{23} L^{-a/8}. \tag{96}$$

Now we see that, by inequalities (95) and (96), (1) and (2) of Lemma 35 are satisfied if we let

$$a_1 := \phi_0(a\delta_0) h(1) + C_{10} L^{-a/2} + C_{23} L^{-a/8}.$$

Now, for $m \geq 2$, let

$$a_m := 2\big[C_{12}(\epsilon)\exp\big(-C_{13}(\epsilon)(m-1)L\big) + aLh(m-1) + \phi_0(a\delta_0)h(m)\big].$$

Inequalities (93) and (94) entail assumptions (3), (4), (5), (6) of Lemma 35. Now observe that, for L large enough, $\sum_{n=1}^{+\infty} a_n \exp(tn) < +\infty$ for $t > 0$ small enough. As L goes to infinity, $\sum_{m=1}^{+\infty} a_m$ goes to zero, as can be checked by studying each term in the definition of $(a_m)_{m\geq 1}$. Part (c) then follows from Lemma 34. $\square$

LEMMA 38. *Let $(Y_i)_{i\geq 1}$ be a sequence of random variables on a probability space $(O, \mathcal{H}, \mathbb{T})$, and $(\mathcal{H}_i)_{i\geq 0}$ an nondecreasing sequence of sub-σ-algebras of $\mathcal{H}$ such that $\mathcal{H}_0 = \{\varnothing, O\}$. Assume that the following properties hold*:

- *for all $i \geq 1$, Y_i is measurable with respect to $\mathcal{H}_i$*;
- *there exists an integer $q \geq 1$ and a constant $0 < c_1(q) < +\infty$ such that a.s. $\mathbb{E}_{\mathbb{T}}(Y_i^{2q}|\mathcal{H}_{i-1}) \leq c_1(q)$.*

Then there exists a constant $0 < c_2(q) < +\infty$, depending only on q and $c_1(q)$, such that for all $t \geq 0$ and $n \geq 1$,

$$\mathbb{T}\left(\sup_{k\geq n} k^{-1}\left|Y_1 + \cdots + Y_k - \sum_{i=1}^{k} \mathbb{E}_{\mathbb{T}}(Y_i|\mathcal{H}_{i-1})\right| \geq t\right) \leq c_2(q)n^{-q}t^{-2q}.$$

PROOF. Observe that $\mathbb{E}_{\mathbb{T}}(Y_i|\mathcal{H}_{i-1})$ exists and is finite for all i since $\mathbb{E}_{\mathbb{T}}(Y_i^{2q}|\mathcal{H}_{i-1}) < +\infty$. Now let $Z_i := Y_i - \mathbb{E}_{\mathbb{T}}(Y_i|\mathcal{H}_{i-1})$. Observe that, with our assumptions, $\mathbb{E}_{\mathbb{T}}(Z_i|\mathcal{H}_{i-1}) = 0$ a.s. Moreover, thanks, e.g., to Jensen's inequality, $\mathbb{E}_{\mathbb{T}}(Z_i^{2q}|\mathcal{H}_{i-1}) \leq c_3(q)$, where $c_3(q)$ depends only on q and $c_1(q)$.

We now prove by induction on ℓ that, for all $\ell \in [\![0, q]\!]$ there exists a constant $0 < c_4(\ell) < +\infty$, depending only on ℓ, q and $c_1(q)$, such that, for all $n \geq 1$,

$$\mathbb{E}_{\mathbb{T}}\big((Z_1 + \cdots + Z_n)^{2\ell}\big) \leq c_4(\ell)n^{\ell}. \tag{97}$$

For $\ell = 0$, the result is trivially true for all $n \geq 1$. Now consider $\ell \in [\![0, q-1]\!]$, assume that the result holds for ℓ, and let us prove that it holds for $\ell + 1$. For all $n \geq 1$,

$$\mathbb{E}_{\mathbb{T}}\big((Z_1 + \cdots + Z_{n+1})^{2\ell+2}\big) = \sum_{k=0}^{2\ell+2} \binom{2\ell+2}{k} \mathbb{E}_{\mathbb{T}}\big((Z_1 + \cdots + Z_n)^{2\ell+2-k} Z_{n+1}^k\big).$$

With our assumptions, $\mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell+1}Z_{n+1}) = 0$. Now, by Jensen's inequality, $\mathbb{E}_{\mathbb{T}}(Z_{n+1}^2|\mathcal{H}_n) \leq c_3(q+1)^{1/(q+1)}$ a.s. By our induction hypothesis, we see that $\mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell}) \leq c_4(\ell)n^{\ell}$, with $c_4(\ell)$ depending only on q, ℓ, and $c_1(q)$. As a consequence, $\mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell}Z_{n+1}^2) \leq c_4(\ell)c_3(q)^{1/(q+1)}n^q$. On the other hand, by Jensen's inequality, for $k \geq 3$, $\mathbb{E}_{\mathbb{T}}|(Z_1 + \cdots + Z_n)^{2\ell+2-k}| \leq \mathbb{E}_{\mathbb{T}}((Z_1 + \cdots + Z_n)^{2\ell})^{(2\ell+2-k)/2\ell} \leq (c_4(\ell)n^{\ell})^{(2\ell+2-k)/2\ell}$. Similarly, $\mathbb{E}_{\mathbb{T}}(|Z_{n+1}^k||$

$\mathcal{H}_n)\le c_3(q)^{k/2q}$ a.s., so that $|\mathbb{E}_{\mathbb{T}}((Z_1+\cdots+Z_n)^{2\ell+2-k}Z_{n+1}^k)|\le c_3(q)^{k/2q)}\times (c_4(\ell)n^\ell)^{(2\ell+2-k)/2\ell}$. Putting these estimates together, we obtain that the difference

$$\mathbb{E}_{\mathbb{T}}\big((Z_1+\cdots+Z_{n+1})^{2\ell+2}\big)-\mathbb{E}_{\mathbb{T}}\big((Z_1+\cdots+Z_n)^{2\ell+2}\big)$$

is bounded above by

$$\binom{2\ell+2}{2}c_4(\ell)c_3(q)^{1/q}n^\ell+\sum_{k=3}^{2\ell}\binom{2\ell+2}{2\ell+2-k}c_3(q)^{k/2q}(c_4(\ell)n^\ell)^{(2\ell+2-k)/2\ell}.$$

Since the are only terms of order n^ℓ or less in the right-hand side of the above inequality, summing, we deduce that $\mathbb{E}_{\mathbb{T}}((Z_1+\cdots+Z_n)^{2\ell+2})\le c_4(\ell+1)n^{\ell+1}$ for all $n\ge 1$, with a constant $c_4(\ell+1)$ depending only on ℓ, q, and $c_1(q)$, so the induction step from q to $q+1$ is complete.

Now observe that the sequence $(M_k)_{k\ge 0}$ defined by $M_0:=0$ and $M_k:=k^{-1}(Z_1+\cdots+Z_k)$ is a martingale with respect to $(\mathcal{H}_k)_{k\ge 0}$. As a consequence, using the maximal inequality for martingales and inequality (97), we see that, for all integers $n\ge 1$ and $\ell\ge 0$,

$$\mathbb{T}\Big(\sup_{2^\ell n\le k\le 2^{\ell+1}n}|M_k|\ge t\Big)\le c_4(q)(2^{\ell+1}n)^{-q}t^{-2q}.$$

By the union bound,

$$\mathbb{T}\left(\sup_{k\ge n}k^{-1}\left|Y_1+\cdots+Y_k-\sum_{i=1}^k\mathbb{E}_{\mathbb{T}}(Y_i|\mathcal{H}_{i-1})\right|\ge t\right)$$

is bounded above by

$$\sum_{\ell=0}^{+\infty}\mathbb{T}\Big(\sup_{2^\ell n\le k\le 2^{\ell+1}n}|M_k|\ge t\Big)$$

and so by

$$\sum_{\ell=0}^{+\infty}c_4(q)(2^{\ell+1}n)^{-q}t^{-2q}.$$

The conclusion follows. □

LEMMA 39. *Let $(Y_i)_{i\ge 1}$ be a sequence of nonnegative integer-valued random variables on a probability space $(O,\mathcal{H},\mathbb{T})$, and $(\mathcal{H}_i)_{i\ge 0}$ an nondecreasing sequence of sub-σ-algebras of $\mathcal{H}$ such that $\mathcal{H}_0=\{\varnothing,O\}$. Assume that the following properties hold*:

- *for all $i\ge 1$, Y_i is measurable with respect to $\mathcal{H}_i$*;

- *there exists* $0 < c_1, c_2 < +\infty$ *such that for all* $i \geq 1$ *and* $k \geq 0$, $\mathbb{T}(Y_i \geq t|\mathcal{H}_{i-1}) \leq c_1 \exp(-c_2 k)$.

Then there exists c_3 *depending only on* c_1, c_2 *such that, for all* $t > c_3$, *there exist* $0 < c_5, c_6 < +\infty$ *such that, for all* $1 \leq n \leq m$, $\mathbb{T}(Y_1 + \cdots + Y_n \geq mt) \leq c_5 \exp(-c_6 m)$.

PROOF. For $0 < \lambda < c_2$, one has a.s.

$$\begin{aligned}\mathbb{E}_{\mathbb{T}}(\exp(\lambda Y_i)|\mathcal{H}_{i-1}) &\leq 1 + \sum_{k=1}^{+\infty}(e^{\lambda k} - e^{\lambda(k-1)})\mathbb{T}(Y_i \geq k|\mathcal{H}_{i-1})\\ &\leq 1 + c_1(1 - e^{-\lambda})\frac{e^{\lambda - c_2}}{1 - e^{\lambda - c_2}}.\end{aligned}$$

Letting $j(\lambda) := c_1(1 - e^{-\lambda})\frac{e^{\lambda - c_2}}{1 - e^{\lambda - c_2}}$, we deduce that

$$\mathbb{E}_{\mathbb{T}}\bigl(\exp\bigl(\lambda(Y_1 + \cdots + Y_m)\bigr)\bigr) \leq \bigl(1 + j(\lambda)\bigr)^m.$$

Then, by Markov's inequality,

$$\mathbb{T}(Y_1 + \cdots + Y_m \geq mt) \leq \exp(-m\lambda t)\mathbb{E}_{\mathbb{T}}\bigl(\exp\bigl(\lambda(Y_1 + \cdots + Y_m)\bigr)\bigr),$$

so that

$$\mathbb{T}(Y_1 + \cdots + Y_m \geq mt) \leq \exp\bigl[-m\bigl(\lambda t + \log\bigl(1 + j(\lambda)\bigr)\bigr)\bigr]. \tag{98}$$

As λ goes to zero, we see that $j(\lambda) = c_3\lambda + o(\lambda)$, with $c_3 := \frac{e^{-c_2}}{1 - e^{-c_2}}$. Choosing λ small enough in (98) yields the result when $n = m$. For $n \leq m$, observe that by assumption $Y_1 + \cdots + Y_n \leq Y_1 + \cdots + Y_m$. $\square$

LEMMA 40. *For* $L \geq L_0$, *there exists* $0 < C_{32}, C_{33} < +\infty$ *such that, for all* $\epsilon \in [0, \epsilon_0]$, *and all* $k \geq 1$,

(a) $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(r_{S_k} > kC_{33} + u, K > k) \leq C_{32}k^2u^{-4}$;
(b) $\mathbb{Q}^{\epsilon}_{a\delta_0}(r_{S_k} > kC_{33} + u, U = +\infty, K > k) \leq C_{32}k^2u^{-4}$.

PROOF. Fix $L \geq L_0$. Observe that, for any $k \geq 1$, on $\{K > k\}$,

$$r_{S_k} = r_0 + (r_{S_1} - r_0) + \sum_{j=1}^{k-1}(r_{S_{j+1}} - r_{D_j} + r_{D_j} - r_{S_j})\mathbf{1}(K \geq j). \tag{99}$$

Observe that, for $w = X_{r_{S_j}}$ with $1 \leq j \leq K$, denoting $w = (F, r, A)$, the three conditions $w \in \mathbb{L}_1$, $\phi_{r-L}(w) \leq p$, and $m_{r-L^{1/4},r}(w) \geq aL^{1/4}/2$ are satisfied. As a consequence, by Lemma 27 and the strong Markov property, for all $1 \leq j \leq k-1$, and all $t > 0$, a.s. $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(r_{D_j} - r_{S_j} > t, K \geq j|\mathcal{F}_{S_j}) \leq C_{18}(t^{-M'} + L\exp(-C_{19}t))$.

Now letting, for $j\geq 1$, $Y_j:=(r_{D_j}-r_{S_j})\mathbf{1}(K\geq j)$, and $\mathcal{H}_j:=\mathcal{F}_{S_{j+1}}$, we see that the assumptions of Lemma 38 are satisfied with $q=2$, since $M'\geq a+8$.

Thanks to the above observation on $w=X_{r_{S_j}}$, and to the fact that, on $\{K\geq j\}$, $r_{S_{j+1}}-r_{D_j}=LJ_{r_{D_j}}$, we see that, by Lemma 36(b) and the strong Markov property, for all $1\leq j\leq k-1$, and all $t>0$, a.s. $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(r_{S_{j+1}}-r_{D_j}>t, K\geq j|\mathcal{F}_{S_j})\leq C_{29}(\lfloor L^{-1}t\rfloor)^{3-M'}$. Similarly, thanks to Lemma 36(a), one also has that, for all $t>0$, a.s. $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(r_{S_1}-r_0>t, K\geq j|\mathcal{F}_{S_j})\leq C_{29}(\lfloor L^{-1}t\rfloor)^{3-M'}$.

Now letting $Y_1:=r_{S_1}-r_0$, and, for $j\geq 2$, $Y_j:=(r_{S_j}-r_{D_{j-1}})\mathbf{1}(K\geq j)$, and $\mathcal{H}_j:=\mathcal{F}_{S_j}$, we see that the assumptions of Lemma 38 are again satisfied with $q=2$. Applying Lemma 38, we deduce the existence of two constants c,d not depending on ϵ such that for all $k\geq 1$ and $u>0$,

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\left(\sum_{j=1}^{k-1}(r_{D_j}-r_{S_j})\mathbf{1}(K\geq j)>kc+u, K>k\right)\leq dk^2u^{-4}$$

and

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\left(r_{S_1}-r_0+\sum_{j=1}^{k-1}(r_{S_{j+1}}-r_{D_j})\mathbf{1}(K\geq j)>kc+u, K>k\right)\leq dk^2u^{-4}.$$

Part (a) of the lemma then follows from the two above inequalities, (99) and the union bound.

To prove part (b), we note that, for all $k\geq 1$, on $\{K>k, U=+\infty\}$,

$$\begin{aligned}r_{S_k}&=r_0+(r_{S_1}-r_0)\mathbf{1}(U=+\infty)\\&\quad+\sum_{j=1}^{k-1}(r_{S_{j+1}}-r_{D_j}+r_{D_j}-r_{S_j})\mathbf{1}(K\geq j).\end{aligned}\tag{100}$$

We can use the same argument as in the proof of part (a) to deal with $\sum_{j=1}^{k-1}(r_{D_j}-r_{S_j})\mathbf{1}(K\geq j)$ and $\sum_{j=1}^{k-1}(r_{S_{j+1}}-r_{D_j})\mathbf{1}(K\geq j)$. To deal with the remaining term $(r_{S_1}-r_0)\mathbf{1}(U=+\infty)$, observe that $r_{S_1}-r_0=LJ_0$, and apply Lemma 36(c). □

LEMMA 41. *For all $\epsilon\in(0,\epsilon_0]$, and $L\geq L_1(\epsilon)$, there exist $0<C_{34}(\epsilon)$, $C_{35}(\epsilon)$, $C_{36}(\epsilon)<+\infty$ such that, for all $k\leq m$,*

(a) $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(r_{S_k}>mC_{34}(\epsilon), K>k)\leq C_{35}(\epsilon)\exp(-C_{36}(\epsilon)m)$;
(b) $\mathbb{Q}^{\epsilon}_{a\delta_0}(r_{S_k}>mC_{34}(\epsilon), U=+\infty, K>k)\leq C_{35}(\epsilon)\exp(-C_{36}(\epsilon)m)$.

PROOF. Adapt the proof of Lemma 40, using Lemma 39 instead of Lemma 38, and Lemma 37 instead of Lemma 36. □

PROPOSITION 17. *For all $L\geq L_0$, there exists $0<C_{37}<+\infty$ not depending on ϵ such that, for all $\epsilon\in[0,\epsilon_0]$,*

(a) $\mathbb{E}^{\epsilon}_{\mathcal{I}_0}(\kappa^2) \le C_{37}, \mathbb{E}^{\epsilon}_{\mathcal{I}_0}((r_\kappa)^2) \le C_{37}$;
(b) $\mathbb{E}^{\epsilon}_{a\delta_0}(\kappa^2 | U = +\infty) \le C_{37}, \mathbb{E}^{\epsilon}_{a\delta_0}((r_\kappa)^2 | U = +\infty) \le C_{37}$.

PROOF. Observe that, for any integer $\ell \ge 1$,

$$\{\kappa > t\} \subset \{K > \ell\} \cup \bigcup_{k=1}^{\ell} \{K = k, S_k > t\},$$

whence

$$\{\kappa > t\} \subset \{K > \ell\} \cup \bigcup_{k=1}^{\ell} \{K = k, r_{S_k} \ge \lfloor \alpha_1 t \rfloor\} \cup \left\{\bigcup_{s \ge t} r_s < \lfloor \alpha_1 s \rfloor\right\}. \tag{101}$$

By the union bound,

$$\begin{aligned} \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(\kappa > t) \le \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(K > \ell) + \sum_{k=1}^{\ell} \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(r_{S_k} \ge \lfloor \alpha_1 t \rfloor, K = k) \\ + \mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\left(\bigcup_{s \ge t} r_s < \lfloor \alpha_1 s \rfloor\right). \end{aligned} \tag{102}$$

Now remember δ_3 defined in Corollary 5 and let $\ell := -4\log((1-\delta_3)^{-1}\lceil t \rceil)$. By (25), $\phi_{r-L}(\mathcal{I}_0) \le p$ so that $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(D < +\infty) \le 1 - \delta_3$. Moreover, for all $j \ge 1$, on $K \ge j$, $\phi_{r-L}(X_{r_{S_j}})$, so that, by the strong Markov property, we have a.s. $\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(D < +\infty | \mathcal{F}_{S_j}) \le 1 - \delta_3$. We deduce that

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(K > \ell) \le (1 - \delta_3)^{\ell} \le t^{-4}. \tag{103}$$

Now observe that, for large enough t (not depending on ϵ), $\lfloor \alpha_1 t \rfloor \ge \ell C_{33} + \alpha_1 t/2$. Using Lemma 40(a), we deduce that, for all $1 \le k \le \ell$,

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}(r_{S_k} > \lfloor \alpha_1 t \rfloor, K > k) \le C_{32} k^2 (\alpha_1 t/2)^{-4}. \tag{104}$$

Finally, by Lemma 20,

$$\mathbb{Q}^{\epsilon}_{\mathcal{I}_0}\left[\bigcup_{s \ge t} r_s < \lfloor \alpha_1 s \rfloor\right] \le C_5 t^{-M'}. \tag{105}$$

Plugging (103), (104) and (105) into (102), we deduce the conclusion of part (a) regarding $\mathbb{E}^{\epsilon}_{\mathcal{I}_0}(\kappa^2)$. The conclusion for $\mathbb{E}^{\epsilon}_{\mathcal{I}_0}((r_\kappa)^2)$ follows by an application of Lemma 2.

As for part (b), observe that the estimate in (103) is still valid when $\mathcal{I}_0$ is replaced by $a\delta_0$. On the other hand, the estimate obtained in (104) follows from Lemma 40(b). Then, by definition, the event $U = +\infty$ rules out the event $\bigcup_{s \ge t} r_s < \lfloor \alpha_1 s \rfloor$. Part (b) is then proved exactly as part (a), noting that, $Q_{a\delta_0}(U = +\infty) \ge 1 - \delta_2$. $\square$

PROPOSITION 18. *For all* $\epsilon \in (0, \epsilon_0]$, *and* $L \geq L_1(\epsilon)$, *there exists* $0 < C_{38}(\epsilon)$, $C_{39}(\epsilon) < +\infty$ *such that*:

(a) $\mathbb{E}^{\epsilon}_{\mathcal{I}_0}(\exp(-C_{38}(\epsilon)\kappa)) \leq C_{39}(\epsilon), \mathbb{E}^{\epsilon}_{\mathcal{I}_0}(\exp(-C_{38}(\epsilon)r_\kappa) \leq C_{39}(\epsilon)$;
(b) $\mathbb{E}^{\epsilon}_{a\delta_0}(\exp(-C_{38}(\epsilon)\kappa|U = +\infty) \leq C_{39}(\epsilon), \mathbb{E}^{\epsilon}_{a\delta_0}(\exp(-C_{38}(\epsilon)r_\kappa|U = +\infty) \leq C_{39}(\epsilon)$.

PROOF. The proof is very similar to the proof of Proposition 17, but this time, we use $\ell := \lfloor (1/2)C_{34}(\epsilon)^{-1}\alpha_1 t\rfloor$, so that the right-hand side of (103) now decays exponentially as $t \to +\infty$.

We then use Lemma 41 instead of Lemma 40, noting that, for large enough t, $\lfloor \alpha_1 t\rfloor \geq \ell C_{34}(\epsilon)$. Finally, we use Lemma 22 instead of Lemma 20, and the conclusion follows as in the proof of Proposition 17. □

It remains to note that Proposition 12 is a mere rephrasing of Propositions 17 and 18 above.

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