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The triangle and the open triangle

Gady Kozma

Department of Mathematics, Weizmann Institute of Science, POB 26, Rehovot 76100, Israel. E-mail: gady.kozma@weizmann.ac.il

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Abstract. We show that for percolation on any transitive graph, the triangle condition implies the open triangle condition.

Résumé. Nous montrons que dans le cas de la percolation sur un graphe transitif la "condition du triangle" est équivalente à celle du "triangle ouvert".

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1. Introduction

Let G be a vertex-transitive[1] connected graph, and let p be some number in $[0,1]$. We say that p-percolation on G satisfies the triangle condition if for some $v \in G$

$$\sum_{x,y\in G} \mathbb{P}(v \leftrightarrow x)\mathbb{P}(x \leftrightarrow y)\mathbb{P}(y \leftrightarrow v) < \infty, \tag{1}$$

where $x \leftrightarrow y$ implies that there exists an open path between x and y. Here and below we abuse notations by denoting "v is a vertex of G" by $v \in G$. Of course, by transitivity, the sum is in fact independent of v. This note is far too short to explain the importance of the triangle condition. Suffices to say that it the triangle condition holds at the *critical* p, then many exponents take their *mean-field* values. See [1,2,11,12] for corollaries of the triangle condition. On the other hand, the triangle condition holds in many interesting cases, see [7,9] for the graphs $\mathbb{Z}^d$ with d sufficiently large, and [10,13] for various other transitive graphs. See also the related [14]. See [5] or [3] for a general introduction to percolation.

In many applications the triangle condition (1) is not so convenient to use. One instead uses the *open* triangle condition, which states that

$$\lim_{w\to\infty} \sum_{x,y\in G} \mathbb{P}(v \leftrightarrow x)\mathbb{P}(x \leftrightarrow y)\mathbb{P}(y \leftrightarrow w) = 0,$$

where here and below the limit means that $d(v,w) \to \infty$ where $d(v,w)$ is the graph (or shortest path) distance. Clearly, the open triangle condition implies the (closed) triangle condition (recall that if y and y' are neighbors in the graph then $\mathbb{P}(x \leftrightarrow y) \geq c\mathbb{P}(x \leftrightarrow y')$ for some constant c independent of x, y and y'). The contents of Lemma 2.1 of Barsky and Aizenman [2] is the reverse implication. The proof in [2] is specific to the graph $\mathbb{Z}^d$ as it uses the Fourier

[1] A vertex-transitive graph, and any other notion not specifically defined, may be found in Wikipedia.

transform of the function $f(x) = \mathbb{P}(\vec{0} \leftrightarrow x)$. The purpose of this note is to generalize this to any transitive graph, namely

Theorem. *Let G be a vertex-transitive graph and let $p \in [0,1]$. Assume G satisfies the triangle condition at p. Then G satisfies the open triangle condition at p.*

This result is not particularly important. For example, in [13] the author simply circumvents the problem by working directly with the open triangle condition. The advantage of making the triangle condition "the" marker for mean-field behavior is mostly aesthetic. The real reason for the existance of this note is to demonstrate an application of operator theory, specifically of spectral theory, to percolation. Operator theory has enchanced the research of random walk significantly ([8] is a personal favorite), and one might hope that by analogy it would do the same for percolation, but this has yet to happen. I aim to remedy this situation, even if by very little.

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2. The proof

Before starting the proof proper, let us make a short heuristic argument. Define the infinite matrix

$$B(v,w) = \mathbb{P}(v \leftrightarrow w), \tag{2}$$

where in the notation we assume that $v \leftrightarrow v$ always so $B(v,v) = 1$. By [1] B, considered as an (unbounded) operator on $l^2(G)$ is a positive operator. Hence the same holds for

$$Q(v,w) = \sum_{x,y} B(v,x)B(x,y)B(y,w) \tag{3}$$

which is just B^3 (as an infinite matrix or as an unbounded operator). It is possible to take the square root of any positive operator, so denote $S = \sqrt{Q}$. We get

$$Q(v,w) = \langle Q\mathbf{1}_v, \mathbf{1}_w\rangle = \langle S\mathbf{1}_v, S\mathbf{1}_w\rangle,$$

where $\mathbf{1}_v$ is the element of $l^2(G)$ defined by

$$\mathbf{1}_v(x) = \begin{cases} 1, & v = x, \\ 0, & v \neq x. \end{cases}$$

Hence the triangle condition $Q(v,v) < \infty$ implies that $\|Sv\| < \infty$. But S is invariant to the automorphisms of G (as a root of Q which is invariant to them) so $S\mathbf{1}_w$ is a map of $S\mathbf{1}_v$ under an automorphism φ taking v to w. But any vector in l^2 is almost orthogonal to sufficiently far away "translations" (namely, the automorphisms of G), so $\langle S\mathbf{1}_v, S\mathbf{1}_w\rangle \to 0$ as the graph distance of v and w goes to ∞, as required.

Why is this even a heuristic and not a full proof? Because of the benign looking expression $\langle Q\mathbf{1}_v, \mathbf{1}_w\rangle$ which is in fact meaningless. Q is an unbounded operator and hence it cannot be applied to any vector in $l^2(G)$, and there is nothing guaranteeing that $\mathbf{1}_v$ will be in its domain. For example, in a sufficiently spread-out lattice in $\mathbb{R}^d$ one has that $\mathbb{P}(x \leftrightarrow y) \approx |x-y|^{2-d}$ [6] which gives with a simple calculation that the triangle condition holds whenever $d > 6$ while $Q\mathbf{1}_v \in l^2$ only when $d > 12$.

The proof below circumvents this problem by decomposing B into a sum of positive bounded operators using specific properties of B. Somebody more versed in the theory of unbounded operators might have constructed a more direct proof.

We start the proof proper with

Definition. *Let φ be an automorphism of the graph G. We define the isometry $\Phi = \Phi_\varphi$ of $l^2(G)$ corresponding to φ by*

$$\big(\Phi(f)\big)(v) = f\big(\varphi^{-1}(v)\big). \tag{4}$$

It is easy to check that $\Phi\mathbf{1}_v = \mathbf{1}_{\varphi(v)}$ and that the support of Φf is φ(the support of f).

Lemma. *Let $f \in l^2(G)$, let $v \in G$ and let $\delta > 0$. Then there exists an $R = R(f, \delta, v)$ such that for any w such that $d(v, w) > R$ and any automorphism φ of G taking v to w one has*

$$\left|\langle \Phi_\varphi f, f\rangle\right| < \delta. \tag{5}$$

This lemma is standard and easy, but let us prove it nonetheless.

Proof of the Lemma. Let $A \subset G$ be some finite set of vertices such that

$$\sqrt{\sum_{v\notin A}\left|f(v)\right|^2} < \frac{1}{3\|f\|}\delta.$$

Write now

$$f = f_{\text{loc}} + f_{\text{glob}}, \quad \text{where } f_{\text{loc}} = f \cdot \mathbf{1}_A.$$

By the definition of A, $\|f_{\text{glob}}\| < \frac{1}{3\|f\|}\delta$, and so by Cauchy–Schwarz,

$$\left|\langle \Phi f, f\rangle\right| \le \left|\langle \Phi f_{\text{loc}}, f_{\text{loc}}\rangle\right| + 2\|f_{\text{glob}}\| \cdot \|f_{\text{loc}}\| + \|f_{\text{glob}}\|^2 < \left|\langle \Phi f_{\text{loc}}, f_{\text{loc}}\rangle\right| + \delta. \tag{6}$$

Define now

$$R = 2\max_{x\in A} d(v, x).$$

To see (5), let w and φ be as above. We get, for any $x \in A$,

$$d\big(\varphi(x), v\big) \ge d(v, w) - d\big(\varphi(x), w\big).$$

Now, $d(\varphi(x), w) = d(\varphi(x), \varphi(v)) = d(x, v) \le \frac{1}{2}R$ because φ is an automorphism of G. Hence we get

$$d\big(\varphi(x), v\big) > R - \frac{1}{2}R$$

implying that $\varphi(x) \notin A$ as it is too far. In other words, $A \cap \varphi(A) = \varnothing$ which implies that $\langle \Phi_\varphi f_{\text{loc}}, f_{\text{loc}}\rangle = 0$. With (6), the lemma is proved. □

Proof of the Theorem. We will not keep p in the notations as it does not change throughout the proof. For every $n \in \mathbb{N}$ and every $v, w \in G$, let $B_n(v, w)$ be defined by

$$B_n(v, w) = \mathbb{P}\big(v \leftrightarrow w, \left|\mathcal{C}(v)\right| = n\big),$$

where $\mathcal{C}(v)$ is the cluster of v i.e. the set of vertices connected to v by open paths, and $|\mathcal{C}(v)|$ is the number of vertices in $\mathcal{C}(v)$. Clearly $B_n(v, w) \ge 0$ and

$$B(v, w) = \sum_{n=1}^{\infty} B_n(v, w), \tag{7}$$

where B is as above (2). Therefore we may write

$$\begin{aligned} Q(v, w) &\overset{(3)}{=} \sum_{x,y} B(v, x)B(x, y)B(y, w) \overset{(7)}{=} \sum_{x,y} B(v, x)\left(\sum_{n=1}^{\infty} B_n(x, y)\right)B(y, w) \\ &= \sum_{n=1}^{\infty}\sum_{x,y} B(v, x)B_n(x, y)B(y, w), \end{aligned} \tag{8}$$

where the change of order of summation in the last equality is justified since all terms are positive. Now, the vector

$$B\mathbf{1}_w = \big(B(y,w)\big)_{y\in G}$$

is in $l^2(G)$ because

$$\sum_y B(y,w)^2 \le \sum_{y,x} B(w,y)B(y,x)B(x,w) < \infty.$$

Further, each B_n, considered as an operator on $l^2(G)$ is bounded, because the sum of the (absolute values of the) entries in each row and each column is finite. From this we conclude that $B_n B\mathbf{1}_w \in l^2(G)$ and we may present the sum in (8) in an l^2 notation as

$$Q(v,w) = \sum_{n=1}^{\infty} \langle B_n B\mathbf{1}_v, B\mathbf{1}_w\rangle. \tag{9}$$

Next we employ the argument of Aizenman and Newman [1] to show that B_n is a positive operator. This means that $B_n(v,w) = B_n(w,v)$ (which is obvious) and that $\langle B_n f, f\rangle \ge 0$ for any (real-valued) $f \in l^2$. It is enough to verify this for f with finite support. But in this case we can write

$$\begin{aligned}\langle B_n f, f\rangle &= \sum_{v,w} f(v)f(w)\mathbb{P}\big(v\leftrightarrow w, \big|\mathcal{C}(v)\big| = n\big) \overset{(*)}{=} \mathbb{E}\bigg(\sum_{v,w} f(v)f(w)\mathbf{1}_{\{v\leftrightarrow w, |\mathcal{C}(v)|=n\}}\bigg)\\ &= \mathbb{E}\bigg(\sum_{\mathcal{C}\,\text{s.t.}\,|\mathcal{C}|=n}\ \sum_{v,w\in\mathcal{C}} f(v)f(w)\bigg) = \mathbb{E}\bigg(\sum_{\mathcal{C}\,\text{s.t.}\,|\mathcal{C}|=n}\bigg(\sum_{v\in\mathcal{C}} f(v)\bigg)^2\bigg) \ge 0,\end{aligned}$$

where $(*)$ is where we used the fact that f has finite support to justify taking the expectation out of the sum. The notation $\mathbf{1}_E$ here is for the indicator of the event E. Thus B_n is positive.

We now apply the spectral theorem for *bounded* positive operators to take the square root of B_n. See [4], Lemma 6.3.5 for the specific case of taking the root of a positive operator and Chapter 7 for general spectral theory. Denote $S_n = \sqrt{B_n}$. This implies, of course, that $S_n^2 = B_n$ but also that S_n is positive and that it commutes with any operator Φ that commutes with B_n.

Returning to (9) we now write

$$Q(v,w) = \sum_{n=1}^{\infty}\big\langle S_n^2 B\mathbf{1}_v, B\mathbf{1}_w\big\rangle = \sum_{n=1}^{\infty}\langle S_n B\mathbf{1}_v, S_n B\mathbf{1}_w\rangle. \tag{10}$$

The fact that $Q(v,v) < \infty$ therefore implies that

$$\sum_{n=1}^{\infty} \|S_n B\mathbf{1}_v\|^2 < \infty. \tag{11}$$

Our only use of the triangle condition.

We now use the lemma, and we use it with

$$f_{\text{lemma}} = S_n B\mathbf{1}_v, \qquad v_{\text{lemma}} = v.$$

We get for every n,

$$\lim_{R\to\infty}\max_{\{\varphi: d(v,\varphi(v))>R\}} \big|\langle \Phi_\varphi S_n B\mathbf{1}_v, S_n B\mathbf{1}_v\rangle\big| = 0. \tag{12}$$

Some standard abstract nonsense shows that the invariance of B_n to automorphisms of the graph i.e. the fact that $B_n(x, y) = B_n(\varphi(x), \varphi(y))$ implies that $B_n\Phi = \Phi B_n$. Hence also $S_n\Phi = \Phi S_n$ so

$$\langle \Phi S_n B\mathbf{1}_v, S_n B\mathbf{1}_v\rangle = \langle S_n B\Phi\mathbf{1}_v, S_n B\mathbf{1}_v\rangle = \langle S_n B\mathbf{1}_{\varphi(v)}, S_n B\mathbf{1}_v\rangle.$$

This allows to rewrite (12) as

$$\lim_{w\to\infty}\langle S_n B\mathbf{1}_w, S_n B\mathbf{1}_v\rangle = 0.$$

This gives, using dominated convergence, that

$$\lim_{w\to\infty}\sum_{n=1}^{\infty}\langle S_n B\mathbf{1}_v, S_n B\mathbf{1}_w\rangle = 0.$$

We can use dominated convergence since

$$\sum_{n=1}^{\infty}\bigl|\langle S_n B\mathbf{1}_v, S_n B\mathbf{1}_w\rangle\bigr| \le \sum_{n=1}^{\infty}\|S_n B\mathbf{1}_v\|\cdot\|S_n B\mathbf{1}_w\| = \sum_{n=1}^{\infty}\|S_n B\mathbf{1}_v\|^2 \overset{(11)}{<} \infty.$$

Since the sum is the same as $Q(v, w)$ (recall (10)), the theorem is proved. □

Closing remark. *Comparing the proof here to that of Barsky and Aizenman* [2], *it seems as if there is something missing in their argument. This is not true. Justifying the change of order of summation in* [2] *is completely standard – for example, by examining Cesàro sums – and does not deserve any special remark.*

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