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Hydrodynamic limit for a particle system with degenerate rates[3]

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Abstract. We study the hydrodynamic limit for some conservative particle systems with degenerate rates, namely with nearest neighbor exchange rates which vanish for certain configurations. These models belong to the class of *kinetically constrained lattice gases* (KCLG) which have been introduced and intensively studied in physical literature as simple models for the liquid/glass transition. Due to the degeneracy of rates there exist *blocked configurations* which do not evolve under the dynamics and in general the hyperplanes of configurations with a fixed number of particles can be decomposed into different irreducible sets. As a consequence, both the Entropy and Relative Entropy method cannot be straightforwardly applied to prove the hydrodynamic limit of KCLG. In particular, some care should be put when proving the One and Two-block Lemmas which guarantee local convergence to equilibrium. We show that, for initial profiles smooth enough and bounded away from zero and one, the macroscopic density profile for our KCLG evolves under the diffusive time scaling according to the *porous medium equation*. Then we prove the same result for more general profiles for a slightly perturbed dynamics obtained by adding jumps of the Symmetric Simple Exclusion. The role of the latter is to remove the degeneracy of rates and at the same time they are properly slowed down in order not to change the macroscopic behavior. The equilibrium fluctuations and the magnitude of the spectral gap for this perturbed model are also obtained.

Résumé. On étudie la limite hydrodynamique de systèmes de particules évoluant selon une dynamique de Kawasaki dont le taux de saut est nul si la configuration ne satisfait pas à des contraintes locales particulières. Ces modéles appartiennent à la classe de modéles avec contraintes cinétiques qui ont été introduits et etudiés en physique pour modéliser la transition liquide/verre. La présence de ces contraintes entraîne l'existence de *configurations gelées*. En particulier, les hyperplans de configurations avec un nombre fixé de particules peuvent etre decomposés en plusieurs composantes irréductibles. Par conséquent, les méthodes usuelles de l'Entropie et de l'Entropie Relative ne peuvent être utilisées pour établir la limite hydrodynamique. On démontre dans cet article que sous une échelle diffusive la densité évolue selon une *équation des milieux poreux* pour des profils de densité initiaux strictement positifs et réguliers, le même résultat est démontré pour des conditions initiales plus générales pour la dynamique superposée à une dynamique d'exclusion simple symmétrique sous-diffusive.

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Keywords: Hydrodynamic limit; Porous medium equation; Spectral gap; Degenerate rates.

1. Introduction

The purpose of this article is twofold. On the one hand we want to define a conservative interacting particle system whose macroscopic density profiles evolve according to (1.1), namely to obtain a microscopic derivation of the *porous medium equation*. On the other hand we wish to establish the hydrodynamic limit for *kinetically constrained lattice gases* (KCLG), a class of particle systems with exclusion constraints which have been introduced and analyzed in physical literature since the late 1980's (see for a review [6,8] and references therein). Indeed, as we shall explain, the standard methods cannot be straightforwardly applied to KCLG due to the degeneracy of nearest neighbor exchange rates.

The porous medium equation is a partial differential equation of the form

$$\begin{cases} \partial_t \rho(t, u) = \Delta \rho^m(t, u), \\ \rho(0, \cdot) = \rho_0(\cdot), \end{cases} \tag{1.1}$$

where $\Delta = \sum_{1 \leq j \leq d} \partial_{u_j}^2$ and $m \in \mathbb{N} \setminus \{1\}$. This can be rewritten in the divergence form as $\partial_t \rho(t, u) = \nabla(D(\rho(t, u)) \nabla(\rho(t, u)))$ with diffusion coefficient $D(\rho(t, u)) = m\rho^{m-1}(t, u)$. Note that $D(\rho)$ goes to zero as $\rho \to 0$, thus the equation looses its parabolic character.

One of the most important properties of (1.1) is that its solutions can be compactly supported at each fixed time or, in physical terms, that is has a finite speed of propagation. This is in strong contrast with the solutions of the classical heat equation. A nonnegative solution of the heat equation is always positive on its domain. A second observation is that the solutions of Eq. (1.1) can be continuous on the domain of definition without being smooth at the boundary. The existence of this kind of solutions is a direct consequence of the degeneracy of $D(\rho)$ as $\rho \to 0$. For a reference on the mathematical properties of the porous medium equation we refer to [9] and references therein. We also recall that such equation is relevant in different contexts in physical literature beyond the original motivation of describing the density of an ideal gas flowing isothermally through an homogeneous porous medium (corresponding to the choice $m = 2$).

A microscopic derivation of the porous medium equation has been already obtained in [2] by considering a model in which the occupation number is a continuous variable. Here we study instead models with discrete occupation variables: our microscopic dynamics are given by stochastic lattice gases with hard core exclusion, namely systems of interacting particles on the d-dimensional discrete torus $\mathbb{T}_N^d$ with the constraint that on each site there can be at most one particle. A configuration is therefore defined by giving for each site $x \in \mathbb{T}_N^d$ the occupation variable, $\eta(x) \in \{0, 1\}$, which stands for empty or occupied sites, respectively. Evolution is then given by a continuous time Markov process during which the jump of a particle from a site x to a nearest neighbor site y occurs at rate $c(x, y, \eta)$. The choice $c(x, y, \eta) = 1$ corresponds to the Symmetric Simple Exclusion process (SSEP) and, as is very well known, leads to the heat equation under diffusive re-scaling of time, namely $D(\rho) = 1$. In order to slow down the low density dynamics and obtain a diffusion coefficient which degenerates for $\rho \to 0$, we impose a local constraint (in addition to hard core exclusion) that must be satisfied to allow a particle jump. This constraint is imposed at the level of rates: for any nearest neighbor couple (x, y) we fix the exchange rate $c(x, y, \eta)$ to be zero if η does not satisfy a local constraint. The latter corresponds to requiring a minimal number of occupied sites in a proper neighborhood of (x, y). Since the typical number of particles in a given region is monotone with ρ, $D(\rho)$ will decrease as ρ is decreased. At the same time the rates are chosen in order to satisfy the detailed balance condition with respect to Bernoulli product measure at any density (as for SSEP), namely the constraints do not add further interactions beyond hard core exclusion.

The above properties of the rates imply that our models belong to the class of *kinetically constrained lattice gases* (KCLG) (see [1,6,8] and references therein), which are used in physical literature to model liquid/glass and more general jamming transitions. In this context, the constraints are devised to mimic the fact that the motion of a particle in a dense medium (e.g. a molecule in a low temperature liquid) can be inhibited by the geometrical constraints induced by the surrounding particles. Therefore the role of particles and vacancy is usually inverted with respect to our convention: in physical literature vacancies rather than particles are needed to facilitate motion, namely $c(x, y, \eta)$ is zero if there is not a minimal number of empty sites around (x, y).

According to the standard classification of physical literature, all the models we consider belong to the class of *noncooperative* KCLG. This means that the rates are such that it is possible to construct a proper finite group of particles, the *mobile cluster*, which has the following properties. There exists (at least one) allowed (i.e. with strictly

positive rates) sequence of nearest neighbor jumps which allows to shift the mobile cluster to any other position. Furthermore, this allowed path should be deterministic, i.e. independent on the value of the occupation variables on the remaining sites. Finally, the jump of any other particle to a neighboring site should be allowed when the mobile cluster is brought in a proper position in its vicinity. Therefore a configuration containing the mobile cluster can be connected to any other one with the same property by an allowed path. This, very loosely speaking, means that noncooperative KCLG should behave like a re-scaled SSEP with the mobile clusters playing the role of single particles for SSEP. Thus their diffusion coefficient is expected to degenerate to zero as power law with ρ for $\rho \to 0$, i.e. when the density of mobile clusters goes to zero (and the coefficient in the power law should depend on the size of the minimal mobile cluster).

Here we provide the first derivation of the hydrodynamic limit for a noncooperative KCLG. As we will explain, in our proof the noncooperative property will play a key role since we will use it to construct paths which allow to perform particle exchanges. The use of similar path arguments for KCLG had already been exploited in [1], were the scaling with the lattice size of the spectral gap and log Sobolev constant for noncooperative models in contact with particle reservoirs at the boundary were derived. A similar case in which the diffusion coefficient does not vanish has already been studied in [7]. Finally, we stress that all along our proofs of hydrodynamics (both for the Entropy and Relative Entropy method) we use an additional property of the rates: the fact that they are of gradient type. A natural development of our work would be to generalize the present results to noncooperative KCLG and/or to nongradient rates.

An outline of the paper follows. In Section 2 we introduce some notation, define our models and state the main results. The models we consider are either truly KCLG or perturbed models in which we add proper jumps of SSEP to remove the degeneracy of the exchange rates. In Section 3 the proof of the hydrodynamic limit via the Relative Entropy method for the KCLG is presented. The proof of the hydrodynamic limit for the perturbed models via the Entropy method is described in Section 4. The proof of a Replacement Lemma needed for the Entropy method is postponed to the Section 5. Finally, in Section 6, we study the spectral gap for both the unperturbed and perturbed dynamics.

2. Statement of results

The models we consider are continuous time Markov processes η_t with space state $\chi_N^d = \{0, 1\}^{\mathbb{T}_N^d}$, where $\mathbb{T}_N^d = \{0, 1, \ldots, N-1\}^d$ is the discrete d-dimensional torus. Let η denote a configuration in χ_N^d, x a site in $\mathbb{T}_N^d$ and $\eta(x) = 1$ if there is a particle at site x, otherwise $\eta(x) = 0$. The elementary moves which occur during evolution correspond to jump of particles among nearest neighbors, x and y, occurring at a rate $c(x, y, \eta)$ which depends both on the couple (x, y) and on the value of the configuration η in a finite neighborhood of x and y. Furthermore these rates are symmetric with respect of an $x-y$ exchange $c(x, y, \eta) = c(y, x, \eta)$ and are translation invariant. More precisely the dynamics is defined by means of an infinitesimal generator acting on local functions $f : \chi_N^d \to \mathbb{R}$ as

$$(\mathcal{L}_P f)(\eta) = \sum_{\substack{x,y \in \mathbb{T}_N^d \\ |x-y|=1}} c(x, y, \eta)\eta(x)\big(1 - \eta(y)\big)\big(f\big(\eta^{x,y}\big) - f(\eta)\big), \tag{2.1}$$

where $|x - y| = \sum_{1 \le i \le d} |x_i - y_i|$ is the sum norm in $\mathbb{R}^d$ and

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y. \end{cases} \tag{2.2}$$

In the sequel we consider the rates

$$c(x, x + e_j, \eta) = \eta(x - e_j) + \eta(x + 2e_j), \tag{2.3}$$

where $\{e_j, j = 1, \ldots, d\}$ denotes the canonical basis of $\mathbb{R}^d$ and we will prove all the theorems for this choice. This, as we will prove, leads in the hydrodynamic limit to the porous medium equation (1.1) for $m = 2$. Also, we can provide

for any other m a proper choice of the rates such that all proofs can be readily extended leading in the diffusive rescaling to the porous medium equation with the correspondent m. For instance in the case $m = 3$, the jump rates to be considered are

$$c(x, x + e_j, \eta) = \eta(x - e_j)\eta(x + 2e_j) + \eta(x - 2e_j)\eta(x - e_j) + \eta(x + 2e_j)\eta(x + 3e_j).$$

Note that both the above choices of the jump rates have the property of defining a *gradient system*, namely one for which the instantaneous current between the sites 0 and e_j:

$$W_{0,e_j}(\eta) = c(0, e_j, \eta)\{\eta(0)(1 - \eta(e_j)) - \eta(e_j)(1 - \eta(0))\}$$

can be rewritten as a function minus its translation. This property will be a key ingredient when deriving the hydrodynamic limit. Also, both the models are noncooperative in any dimension according to the definition given in the Introduction. Consider for example the rates (2.3) in one dimension. A possible choice for the mobile cluster is given by two particles at distance at most two. Let us describe the deterministic sequence of allowed moves (i.e. with strictly positive exchange rate) which we should perform to shift of one step to the right the mobile cluster when $\eta(x) = \eta(x + e_1) = 1$, i.e. to transform η into η' with $\eta'(x + e_1) = \eta'(x + 2e_1) = 1$, $\eta'(x) = \eta(x + 2e_1)$ and $\eta'(z) = \eta(z)$ for $z \notin (x, x + e_1, x + 2e_1)$. First we make the move $\eta \to \eta^{x+e_1,x+2e_1}$, which is allowed since $c(x + e_1, x + 2e_1, \eta) \geq \eta(x) = 1$. Then we perform the move $\eta^{x+e_1,x+2e_1} \to (\eta^{x+e_1,x+2e_1})^{x,x+e_1} = \eta'$ which is also allowed, since $c(x, x + e_1, \eta^{x+e_1,x+2e_1}) \geq \eta^{x+e_1,x+2e_1}(x + 2e_1) = \eta(x + e_1) = 1$. The case in which the particles are instead at distance two, $\eta(x) = \eta(x + 2e_1) = 1$, can be treated analogously. The second property which characterizes noncooperative models can also be readily checked: if we are given any two neighboring sites, $(y, y + e_1)$, the exchange of their occupation variables can be performed if we bring the mobile group of two particles in $(y - 2e_1, y - e_1)$ since $c(y, y + e_1, \eta) \geq \eta(y - e_1)$. It is then possible to verify that any two configurations η and η' with the same number of particles, $\sum \eta(x) = \sum \eta'(x)$, and both containing at least two particles at distance at most two can be connected one to another via a sequence of allowed jumps.

Let ν_α be the Bernoulli product measure in χ_N^d, with $\alpha \in (0, 1)$. Since $c(x, y, \eta) = c(y, x, \eta) \ \forall x, y \in \mathbb{T}_N^d$, the measures ν_α are reversible for this process $\forall \alpha$, as for SSEP. Due to the degeneracy of the rates other invariant measures arise naturally. For example in the one-dimensional setting any configuration η such that the distance between all the positions of two consecutive occupied sites is bigger than two has all the exchange rates equal to zero. Therefore it is a *blocked configuration* and a Dirac measure supported on it is an invariant measure for this process.

Let $\Sigma_{N,k}$ denote the hyperplane of configurations with k particles, namely

$$\Sigma_{N,k} = \left\{ \eta \in \chi_N^d : \sum_{x \in \mathbb{T}_N^d} \eta(x) = k \right\}, \tag{2.4}$$

which is invariant under the dynamics. We say that $\mathcal{O}$ is an irreducible component of $\Sigma_{N,k}$ if for every $\eta, \xi \in \mathcal{O}$ it is possible to go from η to ξ by a sequence of allowed jumps. For SSEP, the hyperplanes $\Sigma_{N,K}$ are irreducible components for any choice of k and N. In the presence of constraints, a more complicated decomposition in general arises. For example in $d = 1$ with the rates (2.3), the above observation on blocked configurations and on the noncooperative character of the model, leads to the following irreducible decomposition for the hyperplanes. If $k > N/3$, $\Sigma_{N,K}$ is irreducible. Instead, if $k \leq N/3$, $\Sigma_{N,k}$ is reducible and decomposable into the irreducible component which contains all configurations with at least one couple of particles at distance at most two plus many irreducible sets each containing a single blocked configuration. The irreducible decomposition in dimension $d > 1$ is more complicated. In this case the model is still noncooperative and a possible mobile cluster is given by a d-dimensional hypercube of particles of linear size 2. For any choice of the spatial dimension d, it is possible to identify a constant $C(d) < \infty$ such that the hyperplane $\Sigma_{N,k}$ is irreducible for $k > C(d)(N/3)^d$, while it is reducible in several components for $k \leq C(d)(N/3)^d$. In this case we have: (i) the irreducible component which contain configurations with at least one d-dimensional hypercube of particles of linear size two plus all configurations that can be connected to these; (ii) irreducible components which contain single (blocked) configuration; (iii) other irreducible components which contain neither blocked configurations nor any configuration belonging to (i). An example of irreducible set of the third kind for the rates (2.3) in $d = 2$ is for example the one that contains all configurations which have two particles at distance smaller or equal to two on a given line and are completely empty on the remaining lines.

In order to investigate the hydrodynamic limit, we need to settle some notation. Let $\mathbb{T}^d$ denote the d-dimensional torus. This will be the "macroscopic" space whose points are denoted with the letter u, v and correspond to $[uN]$, $[vN]$ in the discrete "microscopic" space T_N^d. For a given configuration η we define the *empirical measure* π^N as the positive measure on $\mathbb{T}^d$ which gives to each particle a mass $1/N^d$, namely

$$\pi^N(\eta, \mathrm{d}u) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x) \delta_{x/N}(\mathrm{d}u), \tag{2.5}$$

where δ_u denotes the Dirac measure at u. Then we consider the time evolution of this measure defined by

$$\pi_t^N(\mathrm{d}u) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_{tN^2}(x) \delta_{x/N}(\mathrm{d}u), \tag{2.6}$$

where as usual η_t is the process at time t which is generated by $\mathcal{L}_P$ when the configuration at time zero is η.

Fix now, an initial profile $\rho_0 : \mathbb{T}^d \to [0, 1]$ and denote by $(\mu^N)_{N \geq 1}$ a sequence of probability measures on χ_N^d.

Definition 1. *A sequence $(\mu^N)_{N \geq 1}$ is associated to ρ_0, if for every continuous function $H : \mathbb{T}^d \to \mathbb{R}$ and for every $\delta > 0$*

$$\lim_{N \to +\infty} \mu^N \left[\eta : \left| \frac{1}{N} \sum_{x \in \mathbb{T}_N^d} H\left(\frac{x}{N}\right) \eta(x) - \int_{\mathbb{T}^d} H(u) \rho_0(u) \, \mathrm{d}u \right| > \delta \right] = 0. \tag{2.7}$$

Note that the first term corresponds to the integral of H w.r.t. π^N, thus the above definition corresponds to asking that the sequence $\pi^N(\eta, \mathrm{d}u)$ converges in probability to $\rho_0(u) \, \mathrm{d}u$. Our goal consists in showing that, if at time $t = 0$ the empirical measures are associated to some initial profile ρ_0, at the macroscopic time t (i.e. the microscopic time tN^2) they are associated to a profile ρ_t which is the solution of the hydrodynamic equation (1.1). In other words we wish to prove that π_t^N converges in probability to $\rho(t, u) \, \mathrm{d}u$.

Fix $\varepsilon > 0$ and let $\rho_0 : \mathbb{T}^d \to [0, 1]$ be a profile of class $C^{2+\varepsilon}(\mathbb{T}^d)$. By Theorem A2.4.1 of [4], Eq. (1.1) admits a solution that we denote by $\rho(t, \cdot)$ which is of class $C^{1+\varepsilon, 2+\varepsilon}(\mathbb{R}_+ \times \mathbb{T}^d)$.

Here we also have to impose a bound condition on the initial profile, as the existence of a strictly positive constant δ_0 such that

$$\forall u \in \mathbb{T}^d \quad \delta_0 \leq \rho_0(u) \leq 1 - \delta_0. \tag{2.8}$$

Let $\nu_{\rho_0(\cdot)}^N$ be the product measure in χ_N^d such that:

$$\nu_{\rho_0(\cdot)}^N \{\eta, \eta(x) = 1\} = \rho_0(x/N).$$

For two measures μ and ν in χ_N^d denote by $H(\mu/\nu)$ the *relative entropy* of μ with respect to ν, defined by:

$$H(\mu/\nu) = \sup_f \left\{ \int f \, \mathrm{d}\mu - \log \int e^f \, \mathrm{d}\nu \right\}, \tag{2.9}$$

where the supreme is carried over all continuous functions.

Theorem 2.1. *Let $\rho_0 : \mathbb{T}^d \to [0, 1]$ be a initial profile of class $C^{2+\varepsilon}(\mathbb{T}^d)$ that satisfies condition (2.8). Let $(\mu^N)_{N \geq 1}$ be a sequence of probability measures on χ_N^d such that*:

$$H\left(\mu^N / \nu_{\rho_0(\cdot)}^N\right) = \mathrm{o}\left(N^d\right). \tag{2.10}$$

Then, for each $t \geq 0$

$$\pi_t^N(\mathrm{d}u) \underset{N \to +\infty}{\longrightarrow} \rho(t, u) \, \mathrm{d}u$$

in probability, where $\rho(t, u)$ is a smooth solution of Eq. (1.1).

The porous medium equation (1.1) presents an interesting behavior for profiles which vanish in some region of their domain. The previous method does not give us information in this situation since in order to apply it the initial profile has to satisfy condition (2.8). When these conditions are not verified, another possibility to derive the hydrodynamic limit is to use the Entropy method which is due to Guo, Papanicolau and Varadhan [3]. However the latter strongly relies on the ergodicity of the Markov process restricted to an hyperplane which, as we discussed above, does not hold for the process generated by $\mathcal{L}_P$. One way to overcome this problem is to perturb slightly the dynamics in such a way that the frozen states are destroyed, ergodicity is restored and the macroscopic hydrodynamic behavior still evolves according to (1.1). More precisely we consider a Markov process with generator given by

$$\mathcal{L}_\theta^N = \mathcal{L}_P + N^{\theta-2}\mathcal{L}_S, \tag{2.11}$$

where $0 < \theta < 2$, $\mathcal{L}_P$ was introduced in (2.1) and $\mathcal{L}_S$ is the generator of the SSEP, which acts on local functions $f : \chi_N^d \to \mathbb{R}$ as

$$(\mathcal{L}_S f)(\eta) = \sum_{\substack{x,y \in \mathbb{T}_N^d \\ |x-y|=1}} \frac{1}{2d}\eta(x)\big(1-\eta(y)\big)\big(f\big(\eta^{x,y}\big) - f(\eta)\big),$$

where $\eta^{x,y}$ was defined in (2.2). Since the Bernoulli product measures ν_α are invariant for both the processes generated by $\mathcal{L}_P$ and by $\mathcal{L}_S$, they are also invariant for the perturbed process generated by $\mathcal{L}_\theta^N$. Thanks to the restriction $0 < \theta < 2$, the generator of SSEP is slowed down and the Markov process generated by $\mathcal{L}_P$ and $\mathcal{L}_\theta^N$ will have the same hydrodynamic limit. Furthermore, since on any configuration each nearest neighbor exchange rate is strictly positive, the Markov process generated by $\mathcal{L}_\theta^N$ is ergodic on each hyperplane with fixed particle number and we can apply the Entropy method to derive its hydrodynamic limit.

For a probability measure μ on χ_N^d, denote by $\mathbb{P}_\mu^{\theta,N} = \mathbb{P}_\mu$ the probability measure on the space $D([0, T], \chi_N^d)$, induced by the Markov process with generator $\mathcal{L}_\theta^N$ speeded up by N^2 and with initial measure μ; and by $\mathbb{E}_\mu$ the expectation with respect to $\mathbb{P}_\mu$.

We start by introducing the definition of weak solutions of Eq. (1.1).

Definition 2. *Fix a bounded profile $\rho_0 : \mathbb{T}^d \to \mathbb{R}$. A bounded function $\rho : [0, T] \times \mathbb{T}^d \to \mathbb{R}$ is a* weak solution *of Eq.* (1.1) *if for every function $H : [0, T] \times \mathbb{T}^d \to \mathbb{R}$ of class $C^{1,2}([0, T] \times \mathbb{T}^d)$*

$$\int_0^T \mathrm{d}t \int_{\mathbb{T}^d} \mathrm{d}u \left\{ \rho(t, u)\partial_t H(t, u) + \big(\rho(t, u)\big)^2 \sum_{1 \leq i \leq d} \partial_{u_i}^2 H(t, u) \right\}$$

$$+ \int_{\mathbb{T}^d} \rho_0(u)H(0, u)\,\mathrm{d}u = \int_{\mathbb{T}^d} \rho(T, u)H(T, u)\,\mathrm{d}u. \tag{2.12}$$

The Entropy method requires the uniqueness of a weak solution of the hydrodynamic equation. This is a consequence of Theorem A2.4.4 of [4] together with the fact that there is no more than a particle per site.

Theorem 2.2. *Let $\rho_0 : \mathbb{T}^d \to [0, 1]$ and $(\mu^N)_{N \geq 1}$ be a sequence of probability measures on χ_N^d associated to the profile ρ_0. Then, for every $t \geq 0$, for every continuous function $H : \mathbb{T}^d \to \mathbb{R}$ and for every $\delta > 0$,*

$$\pi_t^N(\mathrm{d}u) \underset{N \to +\infty}{\longrightarrow} \rho(t, u)\,\mathrm{d}u$$

in probability, where $\rho(t, u)$ is the unique weak solution of Eq. (1.1).

Once we have established the Law of Large Numbers for the empirical measure for the process with generator $\mathcal{L}_\theta^N$, the next step is to obtain the *Central Limit Theorem* starting from the invariant measure ν_ρ.

For each $z > 0$ (resp. $z < 0$) define $h_z : \mathbb{T}^d \to \mathbb{R}$ by $h_z(u) = \sqrt{2}\cos(2\pi z \cdot u)$ (resp. $h_z(u) = \sqrt{2}\sin(2\pi z \cdot u)$) and let $h_0 = 1$. Here $\cdot$ denotes the inner product of $\mathbb{R}^d$. It is well known that the set $\{h_z, z \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{T}^d)$. In this space consider the operator $\Omega = 1 - \Delta$. A simple computation shows that $\Omega h_z = \gamma_z h_z$ where $\gamma_z = 1 + 4\pi^2 \|z\|^2$.

For a positive integer k, denote by $\mathcal{H}_k$ the space obtained as the completion of $C^\infty(\mathbb{T}^d)$ endowed with the inner product defined by $\langle f, g \rangle_k = \langle f, \Omega^k g \rangle$. Let $\mathcal{H}_{-k}$ denote the dual of $\mathcal{H}_k$ with respect to the inner product of $L^2(\mathbb{T}^d)$.

Recall the definition of the empirical measure in (2.6) where η_t denotes the Markov process with generator $\mathcal{L}_N^\theta$, with $0 < \theta < 2$. As we want to investigate the fluctuations of this measure, fix $\rho > 0$ and denote by $\mathcal{Y}^N$ the *density fluctuation field* acting on smooth functions H as

$$\mathcal{Y}_t^N(H) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}_N^d} H\left(\frac{x}{N}\right)(\eta_{tN^2}(x) - \rho). \tag{2.13}$$

Fix a positive integer $k \geq 3$ and denote by $D([0, T], \mathcal{H}_{-k})$ (resp. $C([0, T], \mathcal{H}_{-k})$) the space of $\mathcal{H}_{-k}$ functions, that are right continuous with left limits (resp. continuous), endowed with the uniform weak topology. Denote by $\mathcal{Q}_N$ the probability measure on $D([0, T], \mathcal{H}_{-k})$ induced by $\mathcal{Y}^N$ and ν_ρ; and by $\mathcal{Q}$ be the probability measure on $C([0, T], \mathcal{H}_{-k})$ corresponding to the stationary generalized Ornstein–Uhlenbeck process with mean 0 and covariance given by

$$E_{\mathcal{Q}}\big[Y_t(H)Y_s(G)\big] = \frac{\text{Var}(\nu_\rho, \eta(0))}{\sqrt{8\pi\rho(t - s)}} \int_{\mathbb{R}^d} \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}v\, \bar{H}(u)\bar{G}(v) \exp\left\{-\frac{(u - v)^2}{8(t - s)\rho}\right\}$$

for every $0 \leq s \leq t$ and H, G in $\mathcal{H}_k$. Here $\bar{H}$ and $\bar{G}$ are periodic functions equal to H, G in $\mathbb{T}^d$. It is not hard to show that the sequence of measures $(\mathcal{Q}_N)_N$ converges weakly to the probability measure $\mathcal{Q}$. The proof of this result is very close to the one presented for the Zero-Range process in [4] and for this reason we have omitted it. We note that, since the proof of the Boltzmann–Gibbs principle in [4] relies on the ergodicity of the Markov process, we can use the ergodic properties of the generator $\mathcal{L}_S$ to obtain the result. We also remark that if one considers the process η_t with generator $\mathcal{L}_P$, we obtain in the limit the same Ornstein–Uhlenbeck process as described above. We only stress that, in this case, we derive the Boltzmann–Gibbs principle in a similar way as we do in the proof of the One block estimate via the Relative Entropy method.

Finally, in Section 6 we investigate the spectral gap for the process in finite volume for $d = 1$. Our aim is to study the dependence on the size of the system of the spectral gap of the process. This, as is very well known, scales as $1/N^2$ for SSEP on all hyperplanes $\Sigma_{N,k}$, uniformly in k. For our models, due to presence of constraints, the uniformity in k is certainly lost, see Remark 6.2.

In order to illustrate our spectral gap results we need to introduce a few additional notation. Fix an integer N and denote by Λ_N the box of size N, $\Lambda_N = \{1, 2, \ldots, N\}$ and by χ_N the configuration space $\chi_N = \{0, 1\}^N$. In order to define the generator on χ_N we could use definition (2.1) with the sum restricted to $x, y \in \Lambda_N$. However, some care should be put when defining the jump rate for sites close to the boundary of Λ_N, since $c(x, y, \eta)$ as defined in (2.3) depend not only on the configuration on x and y but also on their neighboring sites which can be outside Λ_N. We denote by $\partial\Lambda_N$ the *boundary set* including all sites which do not belong to Λ_N and are nearest neighbor to at least one site in Λ_N, $\partial\Lambda_N = \{x \in \mathbb{Z} : x \notin \Lambda_N, \mathrm{d}(x, \Lambda_N) = 1\}$, where as usual the distance between a point x and the set Λ_N is the infimum of the distances between $y \in \Lambda_N$ and x. A possible way to define the finite volume generator is to imagine that the configuration in the boundary set is frozen to a reference configuration, σ, and to define the finite volume rates as $c^\sigma(x, y, \eta) = c(x, y, \eta \cdot \sigma)$ where c are the rates in (2.3) and $\eta \cdot \sigma \in \{0, 1\}^{|\Lambda_N| + |\partial\Lambda_N|}$ is the configuration equal to $\eta(x)$ on sites $x \in \Lambda_N$ and to $\sigma(x)$ on $x \in \partial\Lambda_N$. In the following we make the choice $\sigma(x) = 0$ in $x \in \partial\Lambda_N$ and we denote by $\mathcal{L}_{P,\Lambda_N}$ and $\mathcal{L}_{\theta,\Lambda_N}^N$ the Markov processes with this choice corresponding to (2.1) and (2.11). For sake of clarity, we explicitly write the second one

$$\big(\mathcal{L}_{\theta,\Lambda_N}^N f\big)(\eta) = \sum_{x \in \Lambda_N \setminus \{1, N-1\}} \big(c(x, x + 1, \eta) + N^{\theta-2}\big)\eta(x)\big(1 - \eta(x + 1)\big)\big(f\big(\eta^{x,x+1}\big) - f(\eta)\big)$$

$$+ \sum_{x \in \Lambda_N \setminus \{2, N\}} \big(c(x, x - 1, \eta) + N^{\theta-2}\big)\eta(x)\big(1 - \eta(x - 1)\big)\big(f\big(\eta^{x,x-1}\big) - f(\eta)\big)$$

$$+ \left(\eta(3) + N^{\theta-2}\right)\left(\eta(1) - \eta(2)\right)^2\left(f\left(\eta^{1,2}\right) - f(\eta)\right)$$

$$+ \left(\eta(N-2) + N^{\theta-2}\right)\left(\eta(N-1) - \eta(N)\right)^2\left(f\left(\eta^{N-1,N}\right) - f(\eta)\right), \qquad (2.14)$$

where $c(x, y, \eta)$ and $\eta^{x,y}$ were defined in (2.3) and (2.2), respectively.

Let, with a slight abuse of notation, $\Sigma_{N,k}$ denote again the hyperplanes with k particles, namely those in (2.4) but with the sum running over Λ_N. For each k, the Markov process generated by $\mathcal{L}^N_{\theta,\Lambda_N}$ is irreducible on $\Sigma_{N,k}$. The same holds for the process generated by $\mathcal{L}_{P,\Lambda_N}$ but only for $k > N/3$. This can be again proved by using the fact that the model is noncooperative with two particles at distance at most two being a mobile cluster. In both cases the unique invariant measure is the uniform measure, $\nu_{N,k}$.

For a generator $\mathcal{L}$ with invariant measure μ, denote by $\lambda_N(\mathcal{L})$ its spectral gap, defined by

$$\lambda_N(\mathcal{L}) = \inf_{f \in L^2(\mu)} \frac{\mathfrak{D}_{\mathcal{L}}(f, \mu)}{\mathrm{Var}(f, \mu)},$$

where $\mathfrak{D}_{\mathcal{L}}(f, \mu)$ denotes the Dirichlet form defined by

$$\mathfrak{D}_{\mathcal{L}}(f, \mu) = \int_{\chi^d_N} -f(\eta)\mathcal{L}f(\eta)\mu(\mathrm{d}\eta). \qquad (2.15)$$

In the following we will also use the shortened notation $\mathfrak{D}_P(f, \mu)$, $\mathfrak{D}_\theta(f, \mu)$ and $\mathfrak{D}_S(f, \mu)$ to denote the Dirichlet form with generator $\mathcal{L}_{P,\Lambda_N}$, $\mathcal{L}^N_{\theta,\Lambda_N}$ and $\mathcal{L}_{S,\Lambda_N}$, respectively. Let $\rho = k/N$. We obtain that:

Proposition 2.3. *Fix $k > N/3$. For the Markov process with generator $\mathcal{L}_{P,\Lambda_N}$, there exists a constant C that does not depend on N nor k such that*

$$\lambda_N\left(\mathcal{L}^N_{\theta,\Lambda_N}\right) \geq \lambda_N(\mathcal{L}_{P,\Lambda_N}) \geq C\frac{(\rho - 1/3)}{\rho N^2}.$$

Proposition 2.4. *Fix $k \leq N/3$. For the Markov process with generator $\mathcal{L}^N_{\theta,\Lambda_N}$, where $\theta = 1$, there exists a constant C that does not depend on N nor k such that*:

$$\lambda_N\left(\mathcal{L}^N_{\theta,\Lambda_N}\right) \geq C\frac{\rho^2}{N^2}.$$

3. The relative entropy method

In this section, we prove Theorem 2.1. Let $\rho_0 \colon \mathbb{T}^d \to [0, 1]$ be a profile of class $C^{2+\varepsilon}(\mathbb{T}^d)$ that satisfies the bound condition (2.8) and let $(\mu^N)_N$ be a sequence of probability measures on χ^d_N that satisfies (2.10). Fix a time $t \geq 0$. Denote by $S^{N,P}_t$ the semigroup associated to the generator $\mathcal{L}_P$ speeded up by N^2 and by μ^N_t the distribution of the process at time t. Denote by $\nu^N_{\rho(t,\cdot)}$ the product measure with slowly varying parameter associated to the profile $\rho(t, \cdot)$:

$$\nu^N_{\rho(t,\cdot)}\{\eta, \eta(x) = 1\} = \rho(t, x/N).$$

It is well known (see Chapter 6 of [4]) that in order to prove the Theorem 2.1 it is enough to show:

Theorem 3.1. *Let $\rho_0 \colon \mathbb{T}^d \to \mathbb{R}$ be an initial profile of class $C^{2+\varepsilon}(\mathbb{T}^d)$ that satisfies the bound condition (2.8) and $(\mu^N)_{N \geq 1}$ a sequence of probability measures in χ^d_N that satisfies the condition (2.10). Then, for every $t \geq 0$*

$$H\left(\mu^N_t / \nu^N_{\rho(t,\cdot)}\right) = \mathrm{o}\left(N^d\right),$$

where $\rho(t, u)$ is a smooth solution of (1.1).

Proof. Fix $\alpha \in (0, 1)$ and an invariant measure ν_α. Let

$$\psi_t^N = \frac{\mathrm{d}\nu_{\rho(t,\cdot)}^N}{\mathrm{d}\nu_\alpha}, \qquad f_N(t) = \frac{\mathrm{d}\mu_t^N}{\mathrm{d}\nu_\alpha}, \qquad H_N(t) = H\big(\mu_t^N / \nu_{\rho(t,\cdot)}^N\big).$$

Since the measures $\nu_{\rho(t,\cdot)}^N$ and ν_α are product, it is very simple to obtain an expression for ψ_t^N:

$$\psi_t^N(\eta) = \frac{1}{Z_t^N} \exp\Bigg\{ \sum_{x \in \mathbb{T}_N^d} \eta(x) \lambda(t, x/N) \Bigg\},$$

where

$$\lambda(t, u) = \log\left(\frac{\rho(t, u)(1 - \alpha)}{\alpha(1 - \rho(t, u))} \right),$$

and Z_t^N is a renormalizing constant.

In order to prove the result, we are going to show that

$$H_N(t) \le \mathrm{o}\big(N^d\big) + \frac{1}{\gamma} \int_0^t H_N(s)\,\mathrm{d}s$$

for some $\gamma > 0$, and apply Gronwall inequality to conclude.

There is a celebrated estimate for the entropy production due to Yau [10]:

$$\partial_t H_N(t) \le \int \Bigg\{ \frac{N^2 \mathcal{L}_P^* \psi_t^N(\eta)}{\psi_t^N(\eta)} - \partial_t \log \psi_t^N(\eta) \Bigg\} f_N(t)(\eta) \nu_\alpha(\mathrm{d}\eta), \tag{3.1}$$

where $\mathcal{L}_P^*$ is the adjoint operator of $\mathcal{L}_P$ in $L^2(\nu_\alpha)$.

Here and in the sequel, for a local function f we use the notation $\tilde{f}(\rho) = E_{\nu_\rho}[f(\eta)]$. By simple computations together with the One-block estimate in Lemma 3.2 we can rewrite $(\psi_t^N(\eta))^{-1}\{N^2 \mathcal{L}_P \psi_t^N(\eta) - \partial_t \psi_t^N(\eta)\}$ as

$$\sum_{x \in \mathbb{T}_N^d} \sum_{j=1}^{d} \partial_{u_j}^2 \lambda(t, x/N) \big\{ \tilde{h}\big(\eta^l(x)\big) - \tilde{h}\big(\rho(t, x/N)\big) - \tilde{h}'\big(\rho(t, x/N)\big)\big[\eta^l(x) - \rho(t, x/N)\big] \big\}$$

$$+ \sum_{x \in \mathbb{T}_N^d} \sum_{j=1}^{d} \big(\partial_{u_j} \lambda(t, x/N)\big)^2 \big\{ \tilde{g}\big(\eta^l(x)\big) - \tilde{g}\big(\rho(t, x/N)\big) - \tilde{g}'\big(\rho(t, x/N)\big)\big[\eta^l(x) - \rho(t, x/N)\big] \big\},$$

plus a term of order $\mathrm{o}(N^d)$, where

$$h_j(\eta) = \eta(0)\eta(e_j) + \eta(0)\eta(-e_j) - \eta(-e_j)\eta(e_j), \tag{3.2}$$

$$g_j(\eta) = \frac{1}{2} c(0, e_j, \eta)\big(\eta(0) - \eta(e_j)\big)^2 \tag{3.3}$$

and

$$\eta^l(x) = \frac{1}{(2l+1)^d} \sum_{|y-x| \le l} \eta(y). \tag{3.4}$$

Repeating standard arguments of the Relative Entropy method, the result follows. We refer the reader to Chapter 6 of [4] for details. $\qquad\qquad\square$

3.1. *One-block estimate*

The main difficulty in the derivation of the One-block estimate is the fact that, as already discussed, the process is not ergodic on hyperplanes with a fixed particle number. In order to overcome this problem, we separate the set of configurations into two sets: the irreducible component that contains all configuration with at least one d-dimensional hypercube of particles of linear size 2 (and all the configurations that can be connected to them) and the remaining configurations. In the first case the standard proof is easily adapted, while for the second case we will use as a key ingredient the fact that this set has small measure with respect to $\nu_{\rho(t,\cdot)}^N$.

Now we introduce some notation. Given a positive function f let

$$D_P(f,\mu) := \mathfrak{D}_P(\sqrt{f},\mu), \tag{3.5}$$

where $\mathfrak{D}_P(f,\mu)$ was defined in (2.15). $D_P(f,\mu)$ will also be called Dirichlet form associated to the generator $\mathcal{L}_P$ and a measure μ in χ_N^d. We denote by $f_t^{N,P}$ the Radon–Nikodym density of $\mu^{N,P}(t) = \frac{1}{t}\int_0^t \mu^N S_s^{N,P}\,ds$ with respect to ν_α. Finally, we let $\tau_x\eta$ be the configuration translated of x, i.e. $(\tau_x\eta)(y) = \eta(y+x)$ and we denote by $\tau_x f$ the translation of a function f, i.e. $(\tau_x f)(\eta) = f(\tau_x\eta)$. The following lemma holds.

Lemma 3.2 (One-block estimate). *Choose $\varepsilon > 0$. Then for every local function ψ there exists $\gamma_0 > 0$ s.t.: there exists $\ell_0 < \infty$ s.t. for any $\ell \geq \ell_0$ there exists $N_0(\ell)$ s.t. for any $N \geq N_0$ it holds*

$$\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} \tau_x V_{l,\psi}(\eta) f_t^{N,P}(\eta)\nu_\alpha(d\eta) \leq \frac{1}{\gamma N^d}\int_0^t H_N(s)\,ds + \varepsilon,$$

where

$$V_{l,\psi}(\eta) = \left|\frac{1}{(2l+1)^d}\sum_{|y|\leq l}\tau_y\psi(\eta) - \tilde{\psi}(\eta^l(0))\right| \tag{3.6}$$

and with $\eta^l(0)$ as defined in (3.4).

Proof. Fix $x \in \mathbb{T}_N^d$ and denote by $\mathcal{Q}_{x,l}$ the set of configurations in the box of center x and radius l containing at least one d-dimensional hypercube of linear size 2 which is completely filled:

$$\mathcal{Q}_{x,l} = \left\{\eta : \sum_{y\in C_l^x}\prod_{z\in Q_y}\eta(z) \geq 1\right\},$$

where $Q_y = \{z : z_i - y_i \in \{0,1\}\ \forall i \in \{1,\ldots,d\}\}$ and $C_l^x = \{y : |y-x| \leq l,\ Q_y \subset \mathbb{T}_N^d\}$. We denote by $\mathcal{E}$ the irreducible set which contains $\mathcal{Q}_{x,l}$ (and all configurations that can be connected via an allowed path to one in $\mathcal{E}$) and we split the integral that appears in the statement of the lemma into

$$\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x V_{l,\psi}(\eta)\mathbf{1}_{\{\mathcal{Q}_{x,l}\}}(\eta)f_t^{N,P}(\eta)\nu_\alpha(d\eta) \tag{3.7}$$

$$+\int \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\tau_x V_{l,\psi}(\eta)\mathbf{1}_{\{\mathcal{Q}_{x,l}^c\}}(\eta)f_t^{N,P}(\eta)\nu_\alpha(d\eta). \tag{3.8}$$

Note that the probability of the ergodic set $\mathcal{E}$ converges rapidly to one with l, indeed the following holds:

$$\nu_{\rho(s,\cdot)}^N(\mathcal{E}) \geq 1 - \left(1-\delta_0^{2^d}\right)^{l^d}, \tag{3.9}$$

where we used the fact that the initial profile is bounded away from zero, namely $\rho(s,u) \geq \delta_0\ \forall u \in \mathbb{T}^d$. Since $H(\mu^N/\nu_\alpha) = O(N^d)$ and the entropy decreases with time, it holds that $H(f_t^{N,P}) = O(N^d)$. This implies that the

Dirichlet form of $f_t^{N,P}$ is bounded above by CN^{d-2}. For the term (3.7), since we are restricting on an irreducible set, we can repeat the standard arguments of the One-block estimate and conclude:

$$\limsup_{l \to +\infty} \limsup_{N \to +\infty} \sup_{D_P(f,\nu_\alpha) \leq CN^{d-2}} \int \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V_{l,\psi}(\eta) 1_{\{Q_{x,l}\}}(\eta) f(\eta) \nu_\alpha(d\eta) = 0.$$

We now deal with the term (3.8) and, to keep notation simple, we drop the integral with respect to time. The entropy inequality allows to bound it above by

$$\frac{H(\mu_s^N / \nu_{\rho(s,u)}^N)}{\gamma N^d} + \frac{1}{\gamma N^d} \log \int \exp \left\{ \gamma \sum_{x \in \mathbb{T}_N^d} \tau_x V_{l,\psi}(\eta) 1_{\{Q_{x,l}^c\}}(\eta) \right\} \nu_{\rho(s,\cdot)}^N(d\eta)$$

for every $\gamma > 0$. The first term on last expression is $H_N(s)/\gamma N^d$. On the other hand, since $V_{l,\psi}$ is bounded and by Hölder inequality, the second term of last expression can be bounded by

$$\frac{1}{N^d \gamma (2l+1)^d} \sum_{x \in \mathbb{T}_N^d} \log \int \exp\{\gamma(2l+1)^d C 1_{\{Q_{x,l}^c\}}(\eta)\} \nu_{\rho(s,\cdot)}^N(d\eta)$$

$$= \frac{1}{N^d \gamma (2l+1)^d} \sum_{x \in \mathbb{T}_N^d} \log\left(\nu_{\rho(s,\cdot)}^N(Q_{x,l}^c)\left(\exp\{\gamma(2l+1)^d C\} - 1\right) + 1\right)$$

$$\leq \frac{1}{N^d \gamma (2l+1)^d} \left(\exp\{\gamma(2l+1)^d C\} - 1\right)\left(1 - \delta_0^{2^d}\right)^{l^d},$$

where the inequality was derived by using the upper bound on $\nu_{\rho(s,\cdot)}^N(Q_{x,l}^c)$ which follows from (3.9) and the inequality $\log(x+1) \leq x$. The desired result now follows provided that γ_0 is s.t. for $\gamma < \gamma_0$ it holds $C2^d \gamma + \log(1 - \delta_0^{2^d}) < 0$. $\square$

4. The entropy method

Now, we prove Theorem 2.2. The strategy of the proof is the same as given for the Zero-Range process in [4]. The main step is the derivation of the One-block and the Two-blocks estimate, which are presented in Section 5.

Fix a time $T > 0$. Let $\mathcal{M}_+$ be the space of finite positive measures on $\mathbb{T}^d$ endowed with the weak topology. Consider a sequence of probability measures $(Q_N)_{N \geq 1}$ on $D([0,T], \mathcal{M}_+)$ corresponding to the Markov process π_t^N as defined in (2.6), starting from μ^N.

First we prove that Q_N is a tight sequence. Then we prove the uniqueness of a limit point, by showing that the limit points of Q_N are concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure, equal to $\rho_0(u)\,du$ at the initial time and whose density is concentrated on weak solutions of the hydrodynamic equation (1.1). By the uniqueness of these solutions we conclude that π_t^N has a unique limit point, concentrated on the trajectory with density $\rho(t,u)$, where $\rho(t,u)$ is the weak solution of Eq. (1.1).

We divide the proof in several steps, to make the exposition clearer. Fix a smooth function $H : \mathbb{T}^d \to \mathbb{R}$. Recall the definition of the empirical measure in (2.6) and let

$$\langle \pi_t^N, H \rangle = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H\left(\frac{x}{N}\right) \eta_t(x),$$

where here (and in the following) with a slight abuse of notation η_t stands for $\eta_{N^2 t}$, namely the microscopic configuration at the macroscopic time t. By Lemma A1.5.1 of [4]

$$M_t^{N,H} = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t N^2 \mathcal{L}_\theta^N \langle \pi_s^N, H \rangle \, ds$$

is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(\eta_s, s \leq t)$, whose quadratic variation is given by

$$\int_0^t \frac{1}{N^{2d}} \sum_{x \in \mathbb{T}_N^d} \sum_{j=1}^d \left(\partial_{u_j}^N H\left(\frac{x}{N}\right) \right)^2 \tau_x g_j^N(\eta_s) \, ds, \tag{4.1}$$

where

$$\partial_{u_j}^N H\left(\frac{x}{N}\right) = N\left[H\left(\frac{x + e_j}{N}\right) - H\left(\frac{x}{N}\right) \right],$$

$g_j^N(\eta) = g_j(\eta) + (\eta(0) - \eta(e_j))^2 N^{\theta-2}$ and $g_j(\eta)$ as defined in (3.3).

By elementary computations we can rewrite the integral part of the martingale as

$$\int_0^t \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \sum_{j=1}^d \partial_{u_j}^2 H\left(\frac{x}{N}\right) \tau_x h_j^N(\eta_s) \, ds, \tag{4.2}$$

where

$$\partial_{u_j}^2 H\left(\frac{x}{N}\right) = N^2\left[H\left(\frac{x + e_j}{N}\right) + H\left(\frac{x - e_j}{N}\right) - 2H\left(\frac{x}{N}\right) \right]$$

and $h_j^N(\eta) = h_j(\eta) + N^{\theta-2}\eta(0)$ and $h_j(\eta)$ as defined in (3.2).

4.1. *Relative compactness*

Following the same arguments as for the Zero-Range in [4] it is easy to show that $(Q_N)_N$ is tight.

4.2. *Uniqueness of limit points*

At first we note that all limit points Q of $(Q_N)_N$ are concentrated on absolutely continuous measures with respect to the Lebesgue measure since there is at most one particle per site. The limit points are equal to $\rho_0(u) \, du$ at the initial time by the hypothesis of $(\mu^N)_N$ being associated to the profile ρ_0. But, in order to prove that the limit points are concentrated on weak solutions of Eq. (1.1), we need to write the integral part of the martingale (4.2) as a function of the empirical measure. This is the main difficulty in the proof of an hydrodynamical limit for a gradient system and we state it as a lemma:

Lemma 4.1 (Replacement lemma). *For every $\delta > 0$ and every local function ψ*

$$\limsup_{\varepsilon \to 0} \limsup_{N \to +\infty} \mathbb{P}_{\mu^N}\left[\int_0^T \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V_{\varepsilon N, \psi}(\eta_s) \, ds \geq \delta \right] = 0,$$

where $V_{l,\psi}$ was defined in (3.6).

We postpone the proof to the following section in order to make the exposition more clear. To keep notation simple, in the previous statement and hereafter we write εN for $[\varepsilon N]$, it's integer part.

Lemma 4.1 means that

$$\int_0^t \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \sum_{j=1}^d H\left(s, \frac{x}{N}\right) \{ \tau_x h_j(\eta_s) - \tilde{h}(\eta_s^{\varepsilon N}(x)) \} \, ds$$

vanishes in probability as $N \to +\infty$ and then as $\varepsilon \to 0$, for every continuous function H. This is the replacement that permits to close the integral part of the martingale in terms of the empirical measure. For details we refer the reader to Chapter 5 of [4].

5. Replacement lemma

This section is devoted to the proof of Lemma 4.1. Before introducing the proof we give some fundamental inequalities that are used in the sequel.

Fix a density α and an invariant measure ν_α, as reference. Let $(\mu^N)_N$ be a sequence of probability measures on χ_N^d and denote by S_t^N the semigroup associated to the generator $\mathcal{L}_\theta^N$. Denote by f_t^N the density of $\mu^N S_t^N$ with respect to ν_α. As the process is evolving on a torus we have that

$$H\left(\mu^N/\nu_\alpha\right) \le C(\alpha)N^d,$$

and since the entropy decreases with time we obtain that $H(\mu^N S_t^N/\nu_\alpha^N) \le C(\alpha)N^d$, see Proposition A1.9.2 of [4]. Denote by $(\mathcal{L}_\theta^N)^*$ the adjoint operator of $\mathcal{L}_\theta^N$ in $L^2(\nu_\alpha)$. Since the process is reversible with respect to ν_α, f_t^N satisfies the following equality:

$$\partial_t f_t^N = N^2\left(\mathcal{L}_\theta^N\right)^* f_t^N = N^2\mathcal{L}_\theta^N f_t^N$$

with initial condition $f_0^N = \frac{d\mu^N}{d\nu_\alpha^N}$.

For $f: \chi_N^d \to \mathbb{R}_+$, recall the definition of the Dirichlet form $D(f, \nu_\alpha) = \mathfrak{D}(\sqrt{f}, \nu_\alpha)$ and $\mathfrak{D}(f, \nu_\alpha)$ defined in (2.15). Since there exists a constant C, such that $H_N(f_0^N) \le CN^d$, it can be proved that for all $t > 0$ (see [4], Section 5.2), that

$$H_N\left(\frac{1}{t}\int_0^t f_s^N\, ds\right) \le CN^d, \qquad D_\theta\left(\frac{1}{t}\int_0^t f_s^N\, ds, \nu_\alpha\right) \le \frac{CN^{d-2}}{2t}. \tag{5.1}$$

The proof of the Replacement lemma relies on the well-known One-block and Two-blocks estimates. At first we reduce the dynamical problem, since the function depends on a trajectory, to a static one using the estimates obtained above. For that, we need to introduce some notation.

Let $\mu^N(T)$ be the Cesaro mean of $\mu^N S_t^N$, namely:

$$\mu^N(T) = \frac{1}{T}\int_0^T \mu^N S_t^N\, dt$$

and $\bar{f}_T^N$ the Radon–Nikodym density of $\mu^N(T)$ with respect to ν_α. Recall that in the beginning of this section we have obtained estimates for the entropy and the Dirichlet form of $\bar{f}_T^N$. As a consequence, to prove the Replacement lemma it is enough to show that

$$\limsup_{\varepsilon \to 0} \limsup_{N \to +\infty} \sup_{D_\theta(f, \nu_\alpha) \le CN^{d-2}} \int \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V_{\varepsilon N, \psi}(\eta) f(\eta)\nu_\alpha(d\eta) = 0, \tag{5.2}$$

for every $C < \infty$. This is a consequence of the following two results:

Lemma 5.1 (One-block estimate). *For every finite constant C,*

$$\limsup_{l \to +\infty} \limsup_{N \to +\infty} \sup_{D_\theta(f, \nu_\alpha) \le CN^{d-2}} \int \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V_{l, \psi}(\eta) f(\eta)\nu_\alpha(d\eta) = 0,$$

where $V_{l, \psi}$ was defined in (3.6).

Lemma 5.2 (Two-blocks estimate). *For every finite constant C,*

$$\limsup_{l \to +\infty} \limsup_{\varepsilon \to 0} \limsup_{N \to +\infty} \sup_{D_\theta(f, \nu_\alpha) \le CN^{d-2}} \sup_{|y| \le \varepsilon N} \int \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \left|\eta^l(x + y) - \eta^{\varepsilon N}(x)\right| f(\eta)\nu_\alpha(d\eta) = 0. \tag{5.3}$$

5.1. *One-block estimate*

Since in this case we are considering the perturbed process $\mathcal{L}_\theta^N$, we can use the One-block estimate for the SSEP speeded up by N^θ, which is enough to conclude the One-block estimate for the perturbed process, the idea to proceed is the following. By the definition of the Dirichlet form (2.15) and the generator $\mathcal{L}_\theta^N$:

$$D_\theta(f, v_\alpha) = D_P(f, v_\alpha) + N^{\theta-2} D_S(f, v_\alpha), \tag{5.4}$$

where D_P (resp. D_S) denotes the Dirichlet form as defined in (3.5). Since the Dirichlet form is always positive, and we are restricted to densities f for which $D_\theta(f, v_\alpha) \leq CN^{d-2}$, it holds that:

$$D_S(f, v_\alpha) \leq N^{2-\theta} D_\theta(f, v_\alpha) \leq CN^{d-\theta}.$$

Following the same arguments as for the Zero-Range in Section 5 of [4], it is not hard to show that the modified One-block estimate

$$\limsup_{l \to +\infty} \limsup_{N \to +\infty} \sup_{D_S(f,v_\alpha) \leq CN^{d-\theta}} \int \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V_{l,\psi}(\eta) f(\eta) v_\alpha(d\eta) = 0$$

holds for the SSEP. The main difference between the proofs, comes from the bounds on the Dirichlet forms. Having the bound $D_S(f, v_\alpha) \leq CN^{d-2}$ it provides the estimate $D_S^l(f_l, v_\alpha) \leq C/N^2$, where f_l is the conditional expectation of f with respect to σ-algebra generated by $\{\eta(x), |x| \leq l\}$, while the bound $D_S(f, v_\alpha) \leq CN^{d-\theta}$, provides the estimate $D_S^l(f_l, v_\alpha) \leq C/N^\theta$, which is enough to conclude the standard proof of the One-block estimate, since $\theta > 0$.

5.2. *Two-blocks estimate*

In order to show (5.3) it is enough to prove that

$$\limsup_{l \to +\infty} \limsup_{\varepsilon \to +\infty} \limsup_{N \to +\infty} \sup_{D_\theta(f,v_\alpha) \leq CN^{d-2}} \sup_{2l \leq |y| \leq \varepsilon N} \int \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \left| \eta^l(x) - \eta^l(x+y) \right| f(\eta) v_\alpha(d\eta) = 0.$$

We can write this integral as

$$\int \left| \eta^l(0) - \eta^l(y) \right| \bar{f}(\eta) v_\alpha(d\eta),$$

where $\bar{f}$ denotes the average of all space translations of f.

   In this case, we are able to separate the integral over configurations with small density of particles from the ones with high density. The first case is easily treated. For remaining, since we are taking configurations with high density of particles, we are able to construct a coupled process, in which each marginal evolves according to the SSEP and they are connected by jumps of $\mathcal{L}_P$. The main difference between the proof for the Zero-Range process in [4] comes from the estimate in the Dirichlet form of this process, which in this case we are able to show that is bounded by $C(d,l)\varepsilon^\theta$ and is enough to conclude. We divide the proof in several steps, in order to make the exposition clearer.

5.2.1. *Cut off of small densities*
At first we note that if we restrict last integral to the small density set

$$\Omega_{0,y} = \left\{ \eta : \sum_{|x| \leq l} \eta(x) \leq 2^{d+1} \right\} \cap \left\{ \eta : \sum_{|x-y| \leq l} \eta(x) \leq 2^{d+1} \right\},$$

it is bounded above by $\frac{C(d)}{(2l+1)^d}$, which vanishes as $l \to +\infty$. So, in fact we just have to consider the integral over the complementary set $\Omega_{0,y}^c$.

5.2.2. *Reduction to microscopic cubes*

Here we need to introduce some notation. Fix a positive integer l and a site x, denote by $\wedge_l(x)$ the box centered at x with radius l. Let $\chi^{2,l}$ denote the configuration space $\{0, 1\}^{\wedge_l(0)} \times \{0, 1\}^{\wedge_l(0)}$, $\xi = (\xi_1, \xi_2)$ the configurations of $\chi^{2,l}$ and by $\nu_\alpha^{2,l}$ the product measure ν_α restricted to $\chi^{2,l}$. Denote by $f_{y,l}$ the conditional expectation of f with respect to the σ-algebra generated by $\{\eta(z), z \in \wedge_l(0, y)\}$, where

$$\wedge_l(0, y) = \wedge_l(0) \cup \wedge_l(y).$$

Since, $\eta^l(0)$ and $\eta^l(y)$ depend on $\eta(x)$, for $x \in \wedge_l(0, y)$, we are able to replace $\bar{f}$ by $\bar{f}_{y,l}$ and rewrite our desired limit as

$$\limsup_{l \to +\infty} \limsup_{\varepsilon \to 0} \limsup_{N \to +\infty} \sup_{D_\theta(f, \nu_\alpha) \le CN^{d-2}} \sup_{2l < |y| \le 2\varepsilon N} \int_{\Omega_{0,y}^c} \left|\xi_1^l(0) - \xi_2^l(0)\right| \bar{f}_{y,l}(\xi) \nu_\alpha^{2,l}(d\xi) = 0.$$

5.2.3. *Estimates on the Dirichlet form*

At first we need to introduce some notation since we are working in the space $\Omega_{0,y}^c$. Recall from (2.15) the definition of the Dirichlet form of f. Since the Bernoulli product measures are homogeneous, we can define the Dirichlet form of f as:

$$D_\theta(f, \nu_\alpha) = \sum_{\substack{x,z \in \mathbb{T}_N^d \\ |x-z|=1}} I_{x,z}^\theta(f, \nu_\alpha),$$

where

$$I_{x,z}^\theta(f, \nu_\alpha) = I_{x,z}^P(f, \nu_\alpha) + N^{\theta-2} I_{x,z}^S(f, \nu_\alpha),$$

$$I_{x,z}^P(f, \nu_\alpha) = \int_{\chi_N^d} c(x, z, \eta) \left[\sqrt{f(\eta^{x,z})} - \sqrt{f(\eta)}\right]^2 \nu_\alpha(d\eta) \tag{5.5}$$

and

$$I_{x,z}^S(f) = \int_{\chi_N^d} \left[\sqrt{f(\eta^{x,z})} - \sqrt{f(\eta)}\right]^2 \nu_\alpha(d\eta).$$

By the definition of $\bar{f}$ and since the Dirichlet form is convex, this implies that $D_\theta(\bar{f}, \nu_\alpha) \le D_\theta(f, \nu_\alpha)$.

The main step in the proof, consists in obtaining an upper bound for the Dirichlet form of $\bar{f}_{y,l}$ from the upper bound of the Dirichlet form of $\bar{f}$, in such a way that we can use the ergodic properties of the Markov process. The idea is to obtain a limit density with Dirichlet form in $\wedge_l(0, y)$ equal to 0, and then by using the irreducibility we can decompose that density along the hyperplanes. As the boxes $\wedge_l(0)$ and $\wedge_l(y)$ have no communication, we cannot define the Dirichlet form in $\wedge_l(0, y)$ as the sum of the Dirichlet forms in $\wedge_l(0)$ and in $\wedge_l(y)$, we must add a term that connects both boxes. Define on positive densities $f : \chi^{2,l} \to \mathbb{R}_+$:

$$D_\theta^{2,l}(f, \nu_\alpha^{2,l}) = I_{0,0}^l(f, \nu_\alpha^{2,l}) + \sum_{\substack{x,z \in \wedge_l(0) \\ |x-z|=1}} I_{x,z}^{1,l}(f, \nu_\alpha^{2,l}) + \sum_{\substack{x,z \in \wedge_l(0) \\ |x-z|=1}} I_{x,z}^{2,l}(f, \nu_\alpha^{2,l}), \tag{5.6}$$

where

$$I_{x,z}^{1,l}(f, \nu_\alpha^{2,l}) = \int_{\chi^{2,l}} \left[\sqrt{f(\xi_1^{x,z}, \xi_2)} - \sqrt{f(\xi_1, \xi_2)}\right]^2 \nu_\alpha^{2,l}(d\xi_1, d\xi_2),$$

$$I_{x,z}^{2,l}(f, \nu_\alpha^{2,l}) = \int_{\chi^{2,l}} \left[\sqrt{f(\xi_1, \xi_2^{x,z})} - \sqrt{f(\xi_1, \xi_2)}\right]^2 \nu_\alpha^{2,l}(d\xi_1, d\xi_2).$$

We still have to define the term that connects both boxes, namely $I_{0,0}^l(f, \nu_\alpha^{2,l})$. Notice that $\Omega_{0,y}^c = \Omega_0^c \cup \Omega_y^c$, where

$$\Omega_0 = \left\{ \eta \in \{0,1\}^{\wedge_l(0)}: \sum_{x \in \wedge_l(0)} \eta(x) < 2^d + 1 \right\}$$

and

$$\Omega_y = \left\{ \eta \in \{0,1\}^{\wedge_l(y)}: \sum_{x \in \wedge_l(y)} \eta(x) < 2^d + 1 \right\}.$$

Then we define

$$I_{0,0}^l(f, \nu_\alpha^{2,l}) = \int_{\Omega_0^c} \left[\sqrt{f(\xi_1^{0,-}, \xi_2^{0,+})} - \sqrt{f(\xi)} \right]^2 \nu_\alpha^{2,l}(\mathrm{d}\xi_1, \mathrm{d}\xi_2)$$

$$+ \int_{\Omega_y^c} \left[\sqrt{f(\xi_1^{0,+}, \xi_2^{0,-})} - \sqrt{f(\xi)} \right]^2 \nu_\alpha^{2,l}(\mathrm{d}\xi_1, \mathrm{d}\xi_2),$$

where $\xi_i^{0,\pm} = \xi_i \pm \partial_0$ and ∂_0 is the configuration with one particle at site 0 and in the rest empty.

This Dirichlet form corresponds to a particle system on $\wedge_l(0) \times \wedge_l(0)$, where the marginal processes evolve as SSEP and where particles can jump from the origin of one marginal process to the origin of the other and vice-versa, according to the jumps of the generator $\mathcal{L}_P$. Using Schwarz inequality and the definition of $\bar{f}_{y,l}$, we have that:

$$I_{x,z}^{1,l}(\bar{f}_{y,l}, \nu_\alpha^{2,l}) \le I_{x,z}^S(\bar{f}, \nu_\alpha) \quad \text{and} \quad I_{x,z}^{2,l}(\bar{f}_{y,l}, \nu_\alpha^{2,l}) \le I_{x+y,z+y}^S(\bar{f}, \nu_\alpha).$$

Then, by equality (5.5) and the estimate on the Dirichlet form of f, it holds that $D_S(\bar{f}, \nu_\alpha) \le C N^{d-\theta}$ which together with $l/N < \varepsilon$, implies that:

$$\sum_{\substack{x,z \in \wedge_l(0) \\ |x-z|=1}} I_{x,z}^{1,l}(f, \nu_\alpha^{2,l}) + \sum_{\substack{x,z \in \wedge_l(0) \\ |x-z|=1}} I_{x,z}^{2,l}(f, \nu_\alpha^{2,l}) \le \frac{2(2l+1)^{d-1} 2l D_S(\bar{f}, \nu_\alpha)}{N^d}$$

$$\le C(d,l) N^{-\theta}. \tag{5.7}$$

In last expression $C(d,l)$ is a constant that depends on d and l. In what follows it may vary from line to line.

It remains to obtain an upper bound for $I_{0,0}^l(\bar{f}_{y,l})$. By using Schwarz inequality and the definition of $\bar{f}_{y,l}$, we get

$$I_{0,0}^l(\bar{f}_{y,l}, \nu_\alpha^{2,l}) \le E_{0,y}^{l,1}(\bar{f}, \nu_\alpha) + E_{0,y}^{l,2}(\bar{f}, \nu_\alpha), \tag{5.8}$$

where

$$E_{0,y}^{l,1}(\bar{f}, \nu_\alpha) := \int_{\Omega_0^c} \left[\sqrt{\bar{f}(\eta^{0,y})} - \sqrt{\bar{f}(\eta)} \right]^2 \nu_\alpha(\mathrm{d}\eta),$$

$$E_{0,y}^{l,2}(\bar{f}, \nu_\alpha) := \int_{\Omega_y^c} \left[\sqrt{\bar{f}(\eta^{0,y})} - \sqrt{\bar{f}(\eta)} \right]^2 \nu_\alpha(\mathrm{d}\eta).$$

Lemma 5.3. *Let f be a density such that $D_\theta(f, \nu_\alpha) \le C N^{d-2}$ and let $y \in \mathbb{T}_N^d : 2l < |y| \le 2\varepsilon N$. Then for $i = 1, 2$,*

$$E_{0,y}^{l,i}(f, \nu_\alpha) \le C(d,l) \varepsilon^\theta.$$

Proof. We give the proof in the case $i = 1$, the other case can be proved by a similar argument. The idea consists in expressing the exchange $\eta^{0,y}$ by means of allowed nearest neighbor exchanges $\eta^{x,x+1}$ of the generator $\mathcal{L}_\theta^N$. Since the

Fig. 1. Moving in the direction e_1.

Fig. 2. Moving in the direction e_2.

integral is restricted to Ω_0^c we have for certain $2^d + 1$ particles in $\wedge_l(0)$. We discuss for definiteness the path in the case $\eta(0) = 1$, $\eta(y) = 0$, the other possibility can be treated analogously. In order to bring the particle from 0 to y we first move 2^d of the particles in $\wedge_l(0)$ close to 0 by means of the jumps that corresponds to $\mathcal{L}_S$. Then we arrange them in order to form a d-dimensional hypercube of linear size 2, which is a mobile cluster. Now we can shift this cluster plus the particle originally in 0 in each of the d directions by using only the jumps in $\mathcal{L}_P$ and we bring them close to y. In Figs 1 and 2 we show as an example the path which allows to shift in the e_1 and e_2 direction the mobile 2×2 square of particles (black circles) plus the particle originally in 0 (grey circle).

Then we drop particle that was originally in 0 in site y and bring back the mobile cluster alone to $\wedge_l(0)$ again using moves in $\mathcal{L}_P$. Finally, using the jumps of $\mathcal{L}_S$, we put them in their initial positions.

Note that the above path is uniquely defined only after choosing the 2^d particles in the box $\wedge_l(0)$ which are used to form the mobile cluster and whose existence in guaranteed by the fact that the integral is restricted to Ω_0^c. This will bring an entropy factor which corresponds to the number of possible initial positions for the 2^d particles, which is bounded by l^{d2^d}.

Once for a given η the initial position of the 2^d particles which will be used to form the mobile cluster is fixed, namely $x_1, \ldots, x_{2^d}$, we let $\tau_{\gamma^{0,y}} \tau_{\gamma^{0,y}-1} \cdots \tau_1(\eta)$ be the sequence of nearest neighboring exchanges that represents the above described path. We can therefore rewrite, for any function g, the square difference $(g(\eta^{0,y}) - \bar{g}(\eta))^2$ as a telescopic sum:

$$\left[g\left(\eta^{0,y}\right) - g(\eta)\right]^2 = \left[\sum_{k=1}^{\gamma^{0,y}} g\left(\tau_k \prod_{i=1}^{k-1} \tau_i(\eta)\right) - g\left(\prod_{i=1}^{k-1} \tau_i(\eta)\right)\right]^2,$$

where $\gamma^{0,y}$ denotes the number of steps of the path which is less than $C(d)(l + \varepsilon N)$.

Now we separate the part of the path in which the moves are performed via jumps in $\mathcal{L}_S$ from the one that can be performed by using $\mathcal{L}_P$ and, by using the elementary inequality $(x + y)^2 \leq 2x^2 + 2y^2$, we bound last expression by

$$\left[\sum_{k=1}^{C(d)l} g\left(\tau_k \prod_{i=1}^{k-1} \tau_i(\eta)\right) - g\left(\prod_{i=1}^{k-1} \tau_i(\eta)\right)\right]^2 + \left[\sum_{k=C(d)l+1}^{C(d)N\varepsilon} g\left(\tau_k \prod_{i=1}^{k-1} \tau_i(\eta)\right) - g\left(\prod_{i=1}^{k-1} \tau_i(\eta)\right)\right]^2.$$

Note that in the previous expression we had to take into account the number of steps in the paths, which is order l for the terms involving moves of $\mathcal{L}_S$ and of order $N\varepsilon$ for those using $\mathcal{L}_P$, thanks to our choice of the path. Also note that the term on the left-hand side of last expression, refers to all the jumps inside the box of size l: when bringing the 2^d particles together and when putting them in their initial position.

We now apply the above inequality with the choice $g = \sqrt{f}$ and use again the Cauchy–Schwarz inequality on each single term and the fact that the rates of $\mathcal{L}_P$ (2.3) are bounded from below by 1 thanks to our choice of the path for all

the terms in the second telescopic sum. This leads to the bound

$$
E_{0,y}^{l,1}(f, \nu_\alpha) \leq C(d)l \sum_{\substack{a_1 \in \wedge_l(0) \\ \cdots \\ a_{2^d} \in \wedge_l(0)}} \int_{\Omega_0^c} 1_{\{a_1 = x_1, \ldots, a_{2^d} = x_{2^d}\}} \sum_{e_i} \left[\sqrt{\bar{f}(\eta^{e_i})} - \sqrt{\bar{f}(\eta)}\right]^2 \nu_\alpha(d\eta)
$$

$$
+ C(d)N\varepsilon \sum_{\substack{a_1 \in \wedge_l(0) \\ \cdots \\ a_{2^d} \in \wedge_l(0)}} \int_{\Omega_0^c} 1_{\{a_1 = x_1, \ldots, a_{2^d} = x_{2^d}\}} \sum_{\tilde{e}_i} c(\tilde{e}_i, \eta) \left[\sqrt{\bar{f}(\eta^{\tilde{e}_i})} - \sqrt{\bar{f}(\eta)}\right]^2 \nu_\alpha(d\eta),
$$

where $\{e_i\}_i$ denotes the bonds that we use inside $\wedge_l(0)$ when taking the 2^d particles whose initial positions are $x_1, \ldots, x_{2^d}$, close to the particle at the site 0, by the jumps of the exclusion process, while $\tilde{e}_i$ corresponds to the bonds used by the generator $\mathcal{L}_P$ when performing the rest of the path and $c(\tilde{e}_i, \eta)$ denotes the corresponding jump rate. Since there are l^{d2^d} chances for the initial positions $x_1, \ldots, x_{2^d}$, we can bound last expression by:

$$
(l+1)^{d2^d} C(d)l \sum_{x, x+1 \in \wedge_l(0)} I_{x,x+1}^S(\bar{f}, \nu_\alpha) + (l+1)^{d2^d} C(d)\varepsilon N \sum_{x, x+1 \in \wedge_l(0)} I_{x,x+1}^P(\bar{f}, \nu_\alpha).
$$

By the equality (5.5) and the bound on the Dirichlet form of f, it holds that:

$$
\forall x \in \mathbb{T}_N^d : \quad I_{x,x+1}^S(\bar{f}, \nu_\alpha) \leq \frac{C}{N^\theta}, \qquad I_{x,x+1}^P(\bar{f}, \nu_\alpha) \leq \frac{C}{N^2},
$$

which together with $l \leq \varepsilon N$, ends the proof.   □

By the definition of the Dirichlet form $D_\theta^{2,l}(f, \nu_\alpha^{2,l})$ in (5.6), together with (5.7) and the previous lemma, we can restrict to densities f that satisfy

$$
D_\theta^{2,l}(f, \nu_\alpha^{2,l}) \leq C(d,l)\varepsilon^\theta.
$$

So, to conclude the proof of the Two-blocks estimate it is enough to show that

$$
\limsup_{l \to +\infty} \limsup_{\varepsilon \to 0} \sup_{D_\theta^{2,l}(f, \nu_\alpha^{2,l}) \leq C(l)\varepsilon^\theta} \int_{\Omega_{0,y}^c} \left|\xi_1^l(0) - \xi_2^l(0)\right| f(\xi) \nu_\alpha^{2,l}(d\xi) = 0,
$$

where the supreme is carried over densities with respect to $\nu_\alpha^{2,l}$. Note that the parameter ε is appearing only on the bound of the Dirichlet form. The proof now follows the same lines as in the case of the Zero-Range process in [4], by starting to decompose the Dirichlet form along the hyperplanes and applying the equivalence of ensembles. For the detailed strategy see Section 5.5 of [4].

6. Spectral gap

In this section we analyze the magnitude of the spectral gap for the one-dimensional generators on finite boxes, $\mathcal{L}_{P, \Lambda_N}$ and $\mathcal{L}_{\theta, \Lambda_N}^N$, which have been defined in Section 2. Note that the results below on the scaling of the spectral gap with the lattice size have not been used in the previous sections to derive the hydrodynamic limit. This was possible thanks to the fact that we were considering a gradient choice of the rates. The following analysis of the spectral gap can be regarded as a first step towards the analysis of the non gradient version of our models, e.g. for the choice

$$
c'(x, x+e_j, \eta) = \begin{cases} 1 & \text{if } \eta(x-e_j) + \eta(x+2e_j) \geq 1, \\ 0 & \text{otherwise,} \end{cases}
$$

which is a nongradient version with the same kinetic constraints as our original choice (2.3), namely $c'(x, x + e_i, \eta) = 0$ if and only if $c(x, x + e_i, \eta) = 0$.

For simplicity in the following we drop Λ_N from our notation. The main ingredient of our proofs will be a comparison with the spectral gap $\lambda(\mathcal{L}_{LR})$ for the *long range exclusion process* and the use of path arguments. Let us start by recalling the definition of $\mathcal{L}_{LR}$ and the result in [5] for its spectral gap. The action of $\mathcal{L}_{LR}$ on local functions $f : \Sigma_{N,k} \to \mathbb{R}$ is given by

$$(\mathcal{L}_{LR}f)(\eta) = \sum_{x,y \in \Lambda_N} \frac{1}{N} \eta(x)\big(1 - \eta(y)\big)\big(f(\eta^{x,y}) - f(\eta)\big), \tag{6.1}$$

where $\eta^{x,y}$ as defined in (2.2).

Consider the Dirichlet form $\mathfrak{D}_{LR}$ associated to $\mathcal{L}_{LR}$, with respect to the uniform measure $\nu_{N,k}$, given explicitly by:

$$\mathfrak{D}_{LR}(f, \nu_{N,k}) = \frac{1}{N} \sum_{x,y \in \Lambda_N} \int_{\Sigma_{N,k}} \big(f(\eta^{x,y}) - f(\eta)\big)^2 \nu_{N,k}(\mathrm{d}\eta). \tag{6.2}$$

Quastel in [5], obtained that

$$\mathrm{Var}(f, \nu_{N,k}) \leq \mathfrak{D}_{LR}(f, \nu_{N,k}), \tag{6.3}$$

by computing precisely the eigenvalues of the generator $\mathcal{L}_{LR}$.

In the next section we will prove the following result:

Proposition 6.1. *Fix $k > N/3$. There exists a constant C not depending on N nor k such that*:

$$\mathfrak{D}_{LR}(f, \nu_{N,k}) \leq \frac{C\rho}{\rho - 1/3} N^2 \mathfrak{D}_P(f, \nu_{N,k}).$$

This, together with Eq. (6.3) leads immediately to

$$\mathrm{Var}(f, \nu_{N,k}) \leq \frac{C\rho}{\rho - 1/3} N^2 \mathfrak{D}_P(f, \nu_{N,k}). \tag{6.4}$$

Proof of Proposition 2.3. The second inequality is an immediate consequence of (6.4) and of the variational characterization of the spectral gap. The first inequality follows from the fact that for any function f it holds that $\mathfrak{D}_P(f, \nu_{N,k}) \leq \mathfrak{D}_\theta(f, \nu_{N,k})$ since the rates of our model are upper bounded by the unconstrained rates of SSEP. $\square$

The case $\rho = \frac{k}{N} \leq \frac{1}{3}$ is more demanding. Since the hyperplanes $\Sigma_{N,k}$ are no longer irreducible for $\mathcal{L}_P$, the corresponding spectral gap is zero. A natural issue is determining the magnitude of the spectral gap for $\mathcal{L}_P$ on the restricted irreducible set of configurations with k particles and at least one couple of particles at distance at most two. In this case the invariant measure is no longer the same one as for the long range jumps and the comparison with the latter process is no more useful. Instead, we will here consider for $k/N \leq 1/3$ the spectral gap for the modified generator $\mathcal{L}_\theta^N$ which, thanks to the addition of the exclusion part, is ergodic on the hyperplanes $\Sigma_{N,k}$ for any k and reversible with respect to $\nu_{N,k}$. In Section 6.2 we show that:

Proposition 6.2. *Fix $k \leq N/3$. Then for every $f \in L^2(\nu_{N,k})$ there exists a constant C not depending on N nor k such that*:

$$\mathfrak{D}_{LR}(f, \nu_{N,k}) \leq \frac{C}{\rho^2} N \mathfrak{D}_S(f, \nu_{N,k}) + \frac{C}{\rho} N^2 \mathfrak{D}_P(f, \nu_{N,k}).$$

As an immediate consequence for $\theta = 1$ we get

$$\mathrm{Var}(f, \nu_{N,k}) \leq \frac{C}{\rho^2} N^2 \mathfrak{D}_\theta(f, \nu_{N,k}). \tag{6.5}$$

Proof of Proposition 2.4. The results is an immediate consequence of (6.5) and the variational characterization of the spectral gap. □

6.1. *Proof of Proposition* 6.1

The idea is to express each exchange $\eta^{x,y}$ of $\mathcal{L}_{LR}$ by means of a sequence of allowed jumps of $\mathcal{L}_P$. The Dirichlet form associated to $\mathcal{L}_P$ is given explicitly by

$$\mathfrak{D}_P(f, \nu_{N,k}) = \frac{1}{2} \sum_{\substack{x,y \in \Lambda_N \\ |x-y|=1}} \int_{\Sigma_{N,k}} c(x, y, \eta)\big(f(\eta^{x,y}) - f(\eta)\big)^2 \nu_{N,k}(d\eta),$$

where $c(x, y, \eta)$ and $\eta^{x,y}$ were defined on (2.3) and (2.2), respectively. Consider for example the case $x < y$ and $\eta(x) = 1$, $\eta(y) = 0$. The restriction $\rho > \frac{1}{3}$ guarantees that there exists a couple of sites (a, b) both in Λ_N such that $\eta(a) = \eta(b) = 1$ and $|a - b| \leq 2$. Once the couple has been chosen we shift it close to sites $x - 2, x - 1$. We have thus reached a configuration with $\eta(x - 2) = \eta(x - 1) = \eta(x) = 1$, and we now shift the three particles to $(y - 3, y - 2, y - 1)$. The allowed path which performs the shift of one step to the right is the composition of the following three basic steps: $\eta \to \eta_1 = \eta^{x,x+1}$, $\eta_1 \to \eta_2 = \eta_1^{x-1,x}$ and $\eta_2 \to \eta_3 = \eta_2^{x-2,x-1}$. When after a proper number of one step shifts we reach the configuration η_n with $\eta(y - 3) = \eta(y - 2) = \eta(y - 1) = 1$, we can perform the exchange $\eta_n \to \eta_{n+1} = \eta_n^{y-1,y}$ which corresponds to dropping the particle at y. Then we make a similar backward path, this time shifting the two particles plus the vacancy until bringing the vacancy in x and finally we bring back the two particles to their original position (a, b).

As we did for Lemma 5.3, we have to take into account the entropic term corresponding to the possible initial positions of the couple of particles (a, b). Some care is required at this point, since using the rough bound N for this original position would lead to an additional factor N in the inequality among Dirichlet forms. Let us start by presenting a lower bound for the number of couples of particles at distance at most two.

Lower bound for the number of couples for $k > N/3$
Fix a configuration $\eta \in \Sigma_{N,k}$. Let $\mathcal{B}$ be the set of sites which are occupied and such that for each of them there is a particle at distance at most two, $\mathcal{B} = \{x : \eta(x) = 1, \eta(x+1) + \eta(x-1) + \eta(x+2) + \eta(x-2) \geq 1\}$. Let also $\mathcal{A}$ denote the remaining occupied sites $\mathcal{A} = \{x : \eta(x) = 1, x \in \Lambda_N \setminus \mathcal{B}\}$. The following holds:

$$k = |\mathcal{A}| + |\mathcal{B}|,$$
$$N \geq 3(|\mathcal{A}| - 1) + 1 + |\mathcal{B}|,$$

where for a set S, $|S|$ denotes the cardinality of the set. The minimum number of couples of particles at distance at most two and such that there are not common sites among different couples is $|\mathcal{B}|/2$, see e.g. Fig. 3 in which there are 8 neighboring particles but just 4 couples. This, together with the above established relations among $k, N, |\mathcal{B}|$, gives the following lower bound for the number of couples

$$\sum_{z \in \Lambda_N} \eta(z)\eta(z+1) + \eta(z)\eta(z+2) \geq |\mathcal{B}|/2 \geq \frac{3}{4}(k - N/3 - 2/3). \tag{6.6}$$

Introducing this term in the Dirichlet form of the long range exclusion (6.2) we bound it by:

$$\frac{2}{N|\mathcal{B}|} \sum_{x,y \in \Lambda_N} \sum_{\eta \in \Sigma_{N,k}} \sum_{z \in \Lambda_N} \sum_{l=1}^{2} \eta(z)\eta(z+l)\big(f(\eta^{x,y}) - f(\eta)\big)^2 \nu_{N,k}(\eta). \tag{6.7}$$

Fig. 3. Ergodic couples.

Let us now consider a term in the sum above

$$\eta(z)\eta(z+l)\big(f\big(\eta^{x,y}\big)-f(\eta)\big)^2$$

and let $x-(z+1)=n$ and $y-x=m$. By using the construction sketched above for a the possible path which connects η to $\eta^{x,y}$ via allowed elementary exchanges, it is possible to define a sequence η_i for $1\le i\le \gamma_z^{x,y}$ with $\gamma_z^{x,y}=5(m-1)+4(n-1)+2$ with the following properties: $\eta_1=\eta$, $\eta_{\gamma_z^{x,y}}=\eta^{x,y}$ and $\forall i$ there exists $x_i\in\Lambda_N$ such that $\eta_i=\eta_{i-1}^{x(i),x(i)+1}$ and $c(x(i),x(i)+1,\eta_{i-1})>0$, namely the exchanges are permitted for the generator $\mathcal{L}_P$. Therefore we can rewrite each $\eta(z)\eta(z+l)(f(\eta^{x,y})-f(\eta))^2$ as the telescopic sum:

$$\eta(z)\eta(z+l)\big(f\big(\eta^{x,y}\big)-f(\eta)\big)^2=\eta(z)\eta(z+l)\Bigg(\sum_{i=1}^{\gamma_z^{x,y}-1}f(\eta_n)-f(\eta_{n-1})\Bigg)^2.$$

By using this equality together with $c(x(i),x(i)+1,\eta_{i-1})>0$, the fact that the size of the path $\gamma_{x,y}^z$ is of O(N) and applying Cauchy–Schwarz inequality we can finally bound (6.7) from above by:

$$\frac{CN}{N|\mathcal{B}|}\sum_{x,y\in\Lambda_N}\sum_{\eta\in\Sigma_{N,k}}\sum_{z\in\Lambda_N}\sum_{l=1}^{2}\eta(z)\eta(z+l)\sum_{\tilde e_i}c(i,i+1,\eta)\big(f\big(\eta^{\tilde e_i}\big)-f(\eta)\big)^2\nu_{N,k}(\eta),$$

where $\tilde e_i=\{i,i+1\}$ denotes one bond that we have used when performing the path that takes the particle from x to y using the couple at the sites z and $z+1$ and C is a constant independent on N and k which may vary from line to line. Estimating over all x and $y\in\Lambda_N$, last expression is bounded above by

$$\frac{CN^3}{N|\mathcal{B}|}\sum_{\eta\in\Sigma_{N,k}}\sum_{z\in\Lambda_N}\eta(z)\sum_{x}c(x,x+1,\eta)\big(f\big(\eta^{x,x+1}\big)-f(\eta)\big)^2\nu_{N,k}(\eta).$$

By the estimate (6.6) and since the number of particles is k, last expression is bounded above by

$$\frac{C\rho}{\rho-1/3}N^2\mathfrak{D}_P(f,\nu_{N,k}).$$

6.2. *Proof of Proposition 6.2*

Now, fix an integer k such that $\rho=\frac{k}{N}\le\frac{1}{3}$. In this case, we are going to show that

$$\mathfrak{D}_{LR}(f,\nu_{N,k})\le\frac{C}{\rho^2}N\mathfrak{D}_S(f,\nu_{N,k})+\frac{C}{\rho}N^2\mathfrak{D}_P(f,\nu_{N,k}),\tag{6.8}$$

which is enough to conclude. As before, the idea consists in expressing a path from x to y using the admissible jumps of $\mathcal{L}_\theta^N$.

For this choice of k we are no more guaranteed that there exist two particles at distance at most two, which was a key ingredient to construct the path in the high density regime. The idea will be to make use of the simple exclusion jumps to construct such mobile clusters and then proceed as before in a similar way as we did in the proof of Lemma 5.3.

Fix a distance j and denote by $\mathcal{B}_j$ the set of particles at distance at most j, $\mathcal{B}_j=\{x:\eta(x)=1,\exists l\in(-j,-1)\cup(1,j)$ s.t. $\eta(x+l)=1\}$ and by $\mathcal{A}_j$ the remaining particles. Then, the following holds:

$$k=|\mathcal{A}_j|+|\mathcal{B}_j|,$$
$$N\ge(j+1)\big(|\mathcal{A}_j|-1\big)+1+|\mathcal{B}_j|.$$

This inequality is obtained in the same manner as before considering that now the minimum distance that the $\mathcal{A}_j$-particles have to be is $j+1$.

As before, the minimum number of couples that one can have is $|\mathcal{B}_j|/2$. By simple computations we obtain the lower bound

$$\sum_{z \in \Lambda_N} \sum_{l=1}^{j} \eta(z)\eta(z+l) \geq \frac{|\mathcal{B}_j|}{2} \geq \frac{j+1}{2j}\left(k - \frac{N}{j+1} - \frac{j}{j+1}\right). \tag{6.9}$$

Introducing this inequality in the Dirichlet form of the long range exclusion (6.2), we bound it from above by:

$$\frac{C}{N|\mathcal{B}_j|} \sum_{x,y \in \Lambda_N} \sum_{\eta \in \Sigma_{N,k}} \sum_{z \in \Lambda_N} \sum_{l=1}^{j} \eta(z)\eta(z+l)\big(f(\eta^{x,y}) - f(\eta)\big)^2 \nu_{N,k}(\eta).$$

For each configuration η and each choice x, y, z and l, we can now construct a path which first bring together the two particles in z and $z+l$ by using the jumps of the simple exclusion and then uses this mobile cluster to perform the exchange of occupation variables in x and y, as in the proof of Lemma 5.3. Here $\mathcal{L}_S$, also denotes the generator of the Symmetric Simple Exclusion process restricted to the box Λ_N, given on local functions by

$$(\mathcal{L}_S f)(\eta) = \sum_{\substack{x,y \in \Lambda_N \\ |x-y|=1}} \frac{1}{2}\eta(x)\big(1 - \eta(y)\big)\big(f(\eta^{x,y}) - f(\eta)\big).$$

Since the jumps of the exclusion are used just to put the neighboring particles at a distance equal to two, the size of the path for this process is $O(j)$. With this purpose, write $f(\eta^{x,y}) - f(\eta)$ as a telescopic sum, use the elementary inequality $(x+y)^2 \leq 2x^2 + 2y^2$ and then the Cauchy–Schwarz inequality to bound last expression by:

$$\frac{Cj}{N|\mathcal{B}_j|} \sum_{x,y \in \Lambda_N} \sum_{\eta \in \Sigma_{N,k}} \sum_{z \in \Lambda_N} \sum_{l=1}^{j} \eta(z)\eta(z+l) \sum_{e_i}\big(f(\eta^{e_i}) - f(\eta)\big)^2 \nu_{N,k}(\eta) \tag{6.10}$$

$$+ \frac{CN}{N|\mathcal{B}_j|} \sum_{x,y \in \Lambda_N} \sum_{\eta \in \Sigma_{N,k}} \sum_{z \in \Lambda_N} \sum_{l=1}^{j} \eta(z)\eta(z+l) \sum_{\tilde{e}_i} c(i,i+1,\eta)\big(f(\eta^{\tilde{e}_i}) - f(\eta)\big)^2 \nu_{N,k}(\eta), \tag{6.11}$$

where e_i denotes the bonds that we have used when bringing the neighboring particles at a distance equal to two, while $\tilde{e}_i = \{i, i+1\}$ denotes the bonds we have used when performing the remaining part of the path that takes the particle from x to y. Note, that there is a factor j multiplying the first expression which comes from the size of the path for the jumps of the exclusion, while for the other process the size of the path is of $O(N)$.

First we deal with the jumps that concerns $\mathcal{L}_P$. As before, we bound (6.11) from above by

$$\frac{CjN^3}{N|\mathcal{B}_j|} \sum_{\eta \in \Sigma_{N,k}} \sum_{z \in \Lambda_N} \eta(z) \sum_x c(x,x+1,\eta)\big(f(\eta^{x,x+1}) - f(\eta)\big)^2 \nu_{N,k}(\eta),$$

and since $\eta \in \Sigma_{N,k}$, we obtain the bound:

$$\frac{Cj\rho}{\rho - 1/j} N^2 \mathfrak{D}_P(f, \nu_{N,k}).$$

Now, we bound (6.10), by

$$\frac{CjN^2}{N|\mathcal{B}_j|} \sum_{\eta \in \Sigma_{N,k}} \sum_{z \in \Lambda_N} \sum_{l=1}^{j} \eta(z) \sum_x \big(f(\eta^{x,x+1}) - f(\eta)\big)^2 \nu_{N,k}(\eta).$$

Since the number of particles is k, we can bound last expression by:

$$\frac{Cj^2\rho}{(\rho - 1/j)} N \mathfrak{D}_S(f, \nu_{N,k}).$$

Reorganizing these facts together we obtain that:

$$\mathfrak{D}_{LR}(f, \nu_{N,k}) \leq \frac{Cj^2\rho}{(\rho - 1/j)} N \mathfrak{D}_S(f, \nu_{N,k}) + \frac{Cj\rho}{\rho - 1/j} N^2 \mathfrak{D}_P(f, \nu_{N,k}).$$

Optimizing over j, (6.8) follows.

Remark 6.1. *For sake of simplicity we have presented the spectral gap results only in the one-dimensional setting. By a proper modification of the path arguments and an accurate estimate on the minimal number of mobile clusters it is possible to obtain for $d > 1$ an analogous result as the one in Proposition 2.3 if the density $\rho = k/N$ is such that $k > C(d)(N/3)^d$.*

Remark 6.2. *Taking for instance the density fluctuation field as defined in (2.13) and the reference measure the Bernoulli product measure ν_ρ, by simple computations we obtain that*

$$\text{Var}\big(\mathcal{Y}_t^N(H), \nu_\rho\big) = \rho(1 - \rho)\|H\|_2^2,$$

while the Dirichlet form corresponding to $\mathcal{L}_P$ equals to:

$$\mathfrak{D}_P\big(\mathcal{Y}_t^N(H), \nu_\rho\big) = \frac{1}{N^2}\rho^2(1 - \rho)\|H'\|_2^2.$$

So, if we consider H, such that $\|H\|_2^2 = \|H'\|_2^2$, then:

$$\text{Var}\big(\mathcal{Y}_t^N(H), \nu_\rho\big) = \frac{N^2}{\rho}\mathfrak{D}_P\big(\mathcal{Y}_t^N(H), \nu_\rho\big),$$

which implies that the spectral gap $\lambda_N(\mathcal{L}_{P,\Lambda_N}) \leq \frac{\rho}{N^2}$. This is in agreement with the bound that we have obtained in (6.8), *when considering the spectral gap with respect to the uniform measure.*

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