*The Annals of Probability*
2009, Vol. 37, No. 4, 1587–1604
DOI: 10.1214/08-AOP444


# LOG-SOBOLEV INEQUALITIES: DIFFERENT ROLES OF RIC AND HESS

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Let $P_t$ be the diffusion semigroup generated by $L := \Delta + \nabla V$ on a complete connected Riemannian manifold with $\mathrm{Ric} \geq -(\sigma^2 \rho_o^2 + c)$ for some constants $\sigma, c > 0$ and $\rho_o$ the Riemannian distance to a fixed point. It is shown that $P_t$ is hypercontractive, or the log-Sobolev inequality holds for the associated Dirichlet form, provided $-\mathrm{Hess}_V \geq \delta$ holds outside of a compact set for some constant $\delta > (1+\sqrt{2})\sigma\sqrt{d-1}$. This indicates, at least in finite dimensions, that Ric and $-\mathrm{Hess}_V$ play quite different roles for the log-Sobolev inequality to hold. The supercontractivity and the ultracontractivity are also studied.

**1. Introduction.** Let $M$ be a $d$-dimensional completed connected noncompact Riemannian manifold and $V \in C^2(M)$ such that

$$Z := \int_M e^{V(x)}\,dx < \infty, \tag{1.1}$$

where $dx$ is the volume measure on $M$. Let $\mu(dx) = Z^{-1}e^{V(x)}\,dx$. Under (1.1) it is easy to see that $H_0^{2,1}(\mu) = W^{2,1}(\mu)$, where $H_0^{2,1}(\mu)$ is the completion of $C_0^1(M)$ under the Sobolev norm $\|f\|_{2,1} := \mu(f^2 + |\nabla f|^2)^{1/2}$, and $W^{2,1}(\mu)$ is the completion of the class $\{f \in C^1(M) : f + |\nabla f| \in L^2(\mu)\}$ under $\|\cdot\|_{2,1}$. Then the $L$-diffusion process is nonexplosive and its semigroup $P_t$ is uniquely determined. Moreover, $P_t$ is symmetric in $L^2(\mu)$ so that $\mu$ is $P_t$-invariant. It is well known by the Bakry–Emery criterion (see [4]) that

$$\mathrm{Ric} - \mathrm{Hess}_V \geq K \tag{1.2}$$

for some constant $K > 0$ implies the Gross log-Sobolev inequality [14],

$$\mu(f^2 \log f^2) := \int_M f^2 \log f^2\, d\mu \leq C\mu(|\nabla f|^2), \qquad \mu(f^2) = 1,\ f \in C^1(M) \tag{1.3}$$

Received October 2007; revised August 2008.

[1]Supported in part by WIMCS, Creative Research Group Fund of the National Natural Science Foundation of China (No. 10721091) and the 973-Project.

*AMS 2000 subject classifications.* 60J60, 58G32.

*Key words and phrases.* Log-Sobolev inequality, Ricci curvature, Riemannian manifold, diffusion semigroup.

for $C = 2/K$. This result was extended by Chen and the author [9] to the situation that $\mathrm{Ric} - \mathrm{Hess}_V$ is uniformly positive outside a compact set. In the case that $\mathrm{Ric} - \mathrm{Hess}_V$ is bounded below, sufficient concentration conditions of $\mu$ for (1.3) to hold are presented in [1, 19, 20]. Obviously, in a condition on $\mathrm{Ric} - \mathrm{Hess}_V$ the Ricci curvature and $-\mathrm{Hess}_V$ play the same role.

What can we do when $\mathrm{Ric} - \mathrm{Hess}_V$ is unbounded below? It seems very hard to confirm the log-Sobolev inequality with the unbounded below condition of $\mathrm{Ric} - \mathrm{Hess}_V$. Therefore, in this paper we try to clarify the roles of Ric and $-\mathrm{Hess}_V$ in the study of the log-Sobolev inequality. Let us first recall the gradient estimate of $P_t$, which is a key point in the above references to prove the log-Sobolev inequality.

Let $x_t$ be the $L$-diffusion process starting at $x$, and let $v \in T_xM$. Due to Bismut [6] and Elworthy–Li [11], under a reasonable lower bound condition of $\mathrm{Ric} - \mathrm{Hess}_V$, one has

$$\langle \nabla P_t f, v\rangle = \mathbb{E}\langle \nabla f(x_t), v_t\rangle, \qquad t > 0, f \in C_b^1(M),$$

where $v_t \in T_{x_t}M$ solves the equation

$$D_t v_t := /\!/_{t\to 0}^{-1} \frac{d}{dt} /\!/_{t\to 0} v_t = -(\mathrm{Ric} - \mathrm{Hess}_V)^{\#}(v_t)$$

for $/\!/_{t\to 0} : T_{x_t}M \to T_xM$ the associated stochastic parallel displacement, and $(\mathrm{Ric} - \mathrm{Hess}_V)^{\#}(v_t) \in T_{x_t}M$ with

$$\langle (\mathrm{Ric} - \mathrm{Hess}_V)^{\#}(v_t), X\rangle := (\mathrm{Ric} - \mathrm{Hess}_V)(v_t, X), \qquad X \in T_{x_t}M.$$

Thus, for the gradient of $P_t$, which is a short distance behavior of the diffusion process, a condition on $\mathrm{Ric} - \mathrm{Hess}_V$ appears naturally.

On the other hand, however, Ric and $-\mathrm{Hess}_V$ play very different roles for long distance behaviors. For instance, Let $\rho_o$ be the Riemannian distance function to a fixed point $o \in M$. If $\mathrm{Ric} \ge -k$ and $-\mathrm{Hess}_V \ge \delta$ for some $k \ge 0$, $\delta \in \mathbb{R}$, the Laplacian comparison theorem implies

$$L\rho_o \le \sqrt{k(d-1)}\coth\big[\sqrt{k/(d-1)}\,\rho_o\big] - \delta\rho_o.$$

Therefore, for large $\rho_o$, the Ric lower bound leads to a bounded term while that of $-\mathrm{Hess}_V$ provides a linear term. The same phenomena appears in the formula on distance of coupling by parallel displacement (cf. [3], (2.3), (2.4)), which implies the above Bismut–Elworthy–Li formula by letting the initial distance tend to zero (cf. [15]). Here, $k \ge 0$ is essential for our framework, since the manifold has to be compact, if Ric is bounded below by a positive constant.

Since the log-Sobolev inequality is always available on bounded regular domains, it is more likely a long-distance property of the diffusion process. So, Ric and $-\mathrm{Hess}_V$ should take different roles in the study of the log-Sobolev inequality. Indeed, it has been observed by the author [20] that (1.3) holds for some

$C > 0$, provided Ric is bounded below and $-\operatorname{Hess}_V$ is uniformly positive outside a compact set. This indicates that for the log-Sobolev inequality, the positivity of $-\operatorname{Hess}_V$ is a dominative condition, which allows the Ricci curvature to be bounded below by an arbitrary negative constant, and hence, allows $\operatorname{Ric} - \operatorname{Hess}_V$ to be globally negative on $M$.

The first aim of this paper is to search for the weakest possibility of curvature lower bound for the log-Sobolev inequality to hold under the condition

$$-\operatorname{Hess}_V \geq \delta \qquad \text{outside a compact set} \tag{1.4}$$

for some constant $\delta > 0$. This condition is reasonable as the log-Sobolev inequality implies $\mu(e^{\lambda\rho_o^2}) < \infty$ for some $\lambda > 0$ (see, e.g., [2, 17]).

According to the following Theorem 1.1 and Example 1.1, we conclude that under (1.4) the optimal curvature lower bound condition for (1.3) to hold is

$$\inf_M \{\operatorname{Ric} + \sigma^2 \rho_o^2\} > -\infty \tag{1.5}$$

for some constant $\sigma > 0$, such that $\delta > (1 + \sqrt{2})\sigma\sqrt{d-1}$. More precisely, let $\theta_0 > 0$ be the smallest positive constant, such that for any connected complete noncompact Riemannian manifold $M$ and $V \in C^2(M)$, such that $Z := \int_M e^{V(x)}\,dx < \infty$, the conditions (1.4) and (1.5) with $\delta > \sigma\theta_0\sqrt{d-1}$, implies (1.3) for some $C > 0$. Due to Theorem 1.1 and Example 1.1 below, we conclude that

$$\theta_0 \in [1, 1 + \sqrt{2}].$$

The exact value of $\theta_0$ is however unknown.

THEOREM 1.1. *Assume that* (1.4) *and* (1.5) *hold for some constants* $c, \delta, \sigma > 0$ *with* $\delta > (1 + \sqrt{2})\sigma\sqrt{d-1}$. *Then* (1.3) *holds for some* $C > 0$.

EXAMPLE 1.1. Let $M = \mathbb{R}^2$ be equipped with the rotationally symmetric metric

$$ds^2 = dr^2 + \{re^{kr^2}\}^2\,d\theta^2,$$

under the polar coordinates $(r, \theta) \in [0, \infty) \times \mathbb{S}^1$ at 0, where $k > 0$ is a constant, then (see, e.g., [13])

$$\operatorname{Ric} = -\frac{(d^2/dr^2)(re^{kr^2})}{re^{kr^2}} = -4k - 4k^2r^2.$$

Thus, (1.5) holds for $\sigma = 2k$. Next, take $V = -k\rho_o^2 - \lambda(\rho_o^2 + 1)^{1/2}$ for some $\lambda > 0$. By the Hessian comparison theorem and the negativity of the sectional curvature, we obtain (1.4) for $\delta = 2k$. Since $d = 2$ and

$$e^{V(x)}\,dx = re^{-\lambda(1+r^2)^{1/2}}\,dr\,d\theta, \tag{1.6}$$

one has $Z < \infty$ and $\delta = 2k = \sigma\sqrt{d-1}$. But the log-Sobolev inequality is not valid since by Herbst's inequality it implies $\mu(e^{r\rho_o^2}) < \infty$ for some $r > 0$, which is, however, not the case due to (1.6). Since in this example one has $\delta > \sigma\theta\sqrt{d-1}$ for any $\theta < 1$, according to the definition of $\theta_0$, we conclude that $\theta_0 \geq 1$.

Following the line of [19, 20], the key point in the proof of Theorem 1.1 will be a proper Harnack inequality of type

$$(P_t f(x))^\alpha \leq C_\alpha(t, x, y) P_t f^\alpha(y), \qquad t > 0, x, y \in M,$$

for any nonnegative $f \in C_b(M)$, where $\alpha > 1$ is a constant and $C_\alpha \in C((0,\infty), M^2)$ is a positive function. Such an inequality was established in [19] for $\mathrm{Ric} - \mathrm{Hess}_V$ bounded below and extended in [3] to a more general situation with Ric satisfying (1.5).

The Harnack inequality presented in [3] contains a leading term $\exp[\rho(x, y)^4]$, which is, however, too large to be integrability w.r.t. $\mu \times \mu$ under our conditions. So, to prove Theorem 1.1, we shall present a sharper Harnack inequality in Section 3 by refining the coupling method introduced in [3] (see Proposition 3.1 below). This inequality, together with the concentration of $\mu$ ensured by (1.4) and (1.5), will imply the hypercontractivity of $P_t$. To establish this new Harnack inequality, some necessary preparations are presented in Section 2.

Finally, in the same spirit of Theorem 1.1, the supercontractivity and ultracontractivity of $P_t$ are studied in Section 4 under explicit conditions on Ric and $-\mathrm{Hess}_V$.

## 2. Preparations.

We first study the concentration of $\mu$ by using (1.4) and (1.5), for which we need to estimate $L\rho_o$ from above according to [5] and references within.

LEMMA 2.1. *If* (1.4) *and* (1.5) *hold, then there exists a constant* $C_1 > 0$ *such that*

$$L\rho_o^2 \leq C_1(1+\rho_o) - 2\big(\delta - \sigma\sqrt{d-1}\big)\rho_o^2 \tag{2.1}$$

*holds outside* cut$(o)$, *the cut-locus of* $o$. *If moreover* $\delta > \sigma\sqrt{d-1}$ *then* $Z < \infty$ *and* $\mu(e^{\lambda\rho_o^2}) < \infty$ *for all* $\lambda < \frac{1}{2}(\delta - \sigma\sqrt{d-1})$.

PROOF. By (1.5) we have $\mathrm{Ric} \geq -(c + \sigma^2\rho_o^2)$ for some constant $c > 0$. By the Laplacian comparison theorem this implies that

$$\Delta\rho_o \leq \sqrt{(c+\sigma^2\rho_o^2)(d-1)}\,\coth\big[\sqrt{(c+\sigma^2\rho_o^2)/(d-1)}\;\rho_o\big]$$

holds outside cut$(o)$. Thus, outside cut$(o)$ one has

$$\begin{aligned}\Delta\rho_o^2 &\leq 2\rho_o\sqrt{(c+\sigma^2\rho_o^2)(d-1)}\,\coth\big[\sqrt{(c+\sigma^2\rho_o^2)/(d-1)}\;\rho_o\big] + 2\\ &\leq 2d + 2\rho_o\sqrt{(c+\sigma^2\rho_o^2)(d-1)},\end{aligned} \tag{2.2}$$

where the second inequality follows from the fact that

$$r\cosh r \le (1+r)\sinh r, \qquad r \ge 0.$$

On the other hand, for $x \notin \mathrm{cut}(o)$ and $U$ the unit tangent vector along the unique minimal geodesic $\ell$ form $o$ to $x$, by (1.4) there exists a constant $c_1 > 0$ independent of $x$ such that

$$\langle \nabla V, \nabla \rho_o\rangle(x) = \langle \nabla V, U\rangle(o) + \int_0^{\rho_o(x)} \mathrm{Hess}_V(U,U)(\ell_s)\,ds \le c_1 - \delta\rho_o(x).$$

Combining this with (2.2) we prove (2.1).

Finally, let $\delta > \sigma\sqrt{d-1}$ and $0 < \lambda < \frac{1}{2}(\delta - \sigma\sqrt{d-1})$. By (2.1) we have

$$\begin{aligned} Le^{\lambda\rho_o^2} &\le \lambda e^{\lambda\rho_o^2}\big(C_1(1+\rho_o) - 2\big(\delta - \sigma\sqrt{d-1}\big)\rho_o^2 + 4\lambda\rho_o^2\big) \\ &\le c_2 - c_3\rho_o^2 e^{\lambda\rho_o^2} \end{aligned}$$

for some constants $c_2, c_3 > 0$. By [5], Proposition 3.2, this implies $Z < \infty$ and

$$\int_M \rho_o^2 e^{\lambda\rho_o^2}\,d\mu \le \frac{c_2}{c_3} < \infty. \qquad \square$$

LEMMA 2.2. *Let $x_t$ be the $L$-diffusion process with $x_0 = x \in M$. If (1.4) and (1.5) hold with $\delta > \sigma\sqrt{d-1}$, then for any $\delta_0 \in (\sigma\sqrt{d-1}, \delta)$ there exists a constant $C_2 > 0$ such that*

$$\begin{aligned} &\mathbb{E}\exp\bigg[\frac{(\delta_0 - \sigma\sqrt{d-1})^2}{4}\int_0^T \rho_o(x_t)^2\,dt\bigg] \\ &\quad \le \exp\bigg[C_2 T + \frac{1}{4}\big(\delta_0 - \sigma\sqrt{d-1}\big)\rho_o(x)^2\bigg], \qquad T > 0, x \in M. \end{aligned}$$

PROOF. By Lemma 2.1, we have

$$L\rho_o^2 \le C - 2\big(\delta_0 - \sigma\sqrt{d-1}\big)\rho_o^2$$

outside $\mathrm{cut}(o)$ for some constant $C > 0$. Then the Itô formula for $\rho_o(x_t)$ due to Kendall [16] implies that

$$d\rho_o^2(x_t) \le 2\sqrt{2}\rho_o(x_t)\,db_t + \big[C - 2\big(\delta_0 - \sigma\sqrt{d-1}\big)\rho_o^2(x_t)\big]\,dt \tag{2.3}$$

holds for some Brownian motion $b_t$ on $\mathbb{R}$. This implies that the $L$-diffusion process is nonexplosive so that

$$T_n := \inf\{t \ge 0 : \rho_o(x_t) \ge n\} \to \infty$$

as $n \to \infty$. Indeed, (2.3) implies that

$$n\mathbb{P}(T_n \le t) \le \mathbb{E}\rho_o(x_{t\wedge T_n})^2 \le \rho_o(x)^2 + Ct, \qquad n \ge 1, t > 0.$$

Hence, $\mathbb{P}(T_n \le t) \to 0$ as $n \to \infty$ for any $t > 0$. This implies $\lim_{n\to\infty} T_n = \infty$ a.s.

For any $\lambda > 0$ and $n \ge 1$, it follows from (2.3) that

$$\begin{aligned}
&\mathbb{E}\exp\biggl[2\lambda\bigl(\delta_0 - \sigma\sqrt{d-1}\bigr)\int_0^{T\wedge T_n} \rho_o^2(x_t)\,dt\biggr] \\
&\quad\le e^{\lambda\rho_o^2(x)+C\lambda T}\,\mathbb{E}\exp\biggl[2\sqrt{2}\lambda\int_0^{T\wedge T_n} \rho_o(x_t)\,db_t\biggr] \\
&\quad\le e^{\lambda\rho_o^2(x)+C\lambda T}\biggl(\mathbb{E}\exp\biggl[16\lambda^2\int_0^{T\wedge T_n} \rho_o^2(x_t)\,dt\biggr]\biggr)^{1/2},
\end{aligned}$$

where in the last step we have used the inequality

$$\mathbb{E}e^{M_t} \le \bigl(\mathbb{E}e^{2\langle M\rangle_t}\bigr)^{1/2}$$

for $M_t = 2\sqrt{2}\lambda\int_0^{t\wedge T_n} \rho_o(X_s)\,db_s$. This follows immediately from the Schwartz inequality and the fact that $\exp[2M_t - 2\langle M\rangle_t]$ is a martingale. Thus, taking

$$\lambda = \tfrac{1}{8}\bigl(\delta_0 - \sigma\sqrt{d-1}\bigr),$$

we obtain

$$\begin{aligned}
&\mathbb{E}\exp\biggl[\frac{1}{4}\bigl(\delta_0 - \sigma\sqrt{d-1}\bigr)^2\int_0^{T\wedge T_n} \rho_o^2(x_t)\,dt\biggr] \\
&\quad\le \exp\biggl[\frac{1}{4}\bigl(\delta_0 - \sigma\sqrt{d-1}\bigr)\rho_o^2(x) + C_2 T\biggr]
\end{aligned}$$

for some $C_2 > 0$. Then the proof is completed by letting $n \to \infty$. □

Finally, we recall the coupling argument introduced in [3] for establishing the Harnack inequality of $P_t$.

Let $T > 0$ and $x \ne y \in M$ be fixed. Then the $L$-diffusion process starting from $x$ can be constructed by solving the following Itô stochastic differential equation:

$$d_I x_t = \sqrt{2}\Phi_t\,dB_t + \nabla V(x_t)\,dt, \qquad x_0 = x,$$

where $d_I$ is the Itô differential on manifolds introduced in [12] (see also [3]), $B_t$ is the $d$-dimensional Brownian motion, and $\Phi_t$ is the horizontal lift of $x_t$ onto the orthonormal frame bundle $O(M)$.

To construct another diffusion process $y_t$ starting from $y$ such that $x_T = y_T$, as in [3], we add an additional drift term to the equation (as explained in [3], Section 3, we may and do assume that the cut-locus of $M$ is empty)

$$d_I y_t = \sqrt{2}P_{x_t,y_t}\Phi_t\,dB_t + \nabla V(y_t)\,dt + \xi_t U(x_t, y_t)1_{\{t<\tau\}}\,dt, \qquad y_0 = y,$$

where $P_{x_t,y_t}$ is the parallel transformation along the unique minimal geodesic $\ell$ from $x_t$ to $y_t$, $U(x_t, y_t)$ is the unit tangent vector of $\ell$ at $y_t$, $\xi_t \geq 0$ is a smooth function of $x_t$ to be determined, and

$$\tau := \inf\{t \geq 0 : x_t = y_t\}$$

is the coupling time. Since all terms involved in the equation are regular enough, there exists a unique solution $y_t$. Furthermore, since the additional term containing $1_{\{t<\tau\}}$ vanishes from the coupling time on, one has $x_t = y_t$ for $t \geq \tau$ due to the uniqueness of solutions.

LEMMA 2.3. *Assume that* (1.4) *and* (1.5) *hold with* $\delta \geq 2\sigma\sqrt{d-1}$. *Then there exists a constant* $C_3 > 0$ *independent of* $x$, $y$ *and* $T$ *such that* $x_T = y_T$ *holds for* $\xi_t := C_3 + 2\sigma\sqrt{d-1}\rho_o(x_t) + \frac{\rho(x,y)}{T}$.

PROOF. According to Section 2 in [3], we have

$$\begin{aligned} d\rho(x_t, y_t) = {} & \{I(x_t, y_t) + \langle \nabla V, \nabla \rho(\cdot, y_t)\rangle(x_t) \\ & + \langle \nabla V, \nabla \rho(x_t, \cdot)\rangle(y_t) - \xi_t\}\,dt, \qquad t < \tau, \end{aligned} \tag{2.4}$$

where

$$I_Z(x_t, y_t) = \sum_{i=1}^{d-1} \int_0^{\rho(x_t,y_t)} \big(|\nabla_U J_i|^2 - \langle R(U, J_i)U, J_i\rangle\big)(\ell_s)\,ds$$

for $R$ the Riemann curvature tensor, $U$ the unit tangent vector of the minimal geodesic $\ell : [0, \rho(x_t, y_t)] \to M$ from $x_t$ to $y_t$, and $\{J_i\}_{i=1}^{d-1}$ the Jacobi fields along $\ell$, which, together with $U$, consist of an orthonormal basis of the tangent space at $x_t$ and $y_t$ and satisfy

$$J_i(y_t) = P_{x_t,y_t} J_i(x_t), \qquad i = 1, \ldots, d-1.$$

By (1.5) we take a constant $c \geq 0$ such that $\mathrm{Ric} \geq -(c + \sigma^2\rho_o^2)$. Letting

$$K(x_t, y_t) = \sup_{\ell([0,\rho(x_t,y_t)])} \{c + \sigma^2\rho_o^2\},$$

we obtain from Wang [21], Theorem 2.14 (see also [7, 8]), that

$$I(x_t, y_t) \leq 2\sqrt{K(x_t, y_t)(d-1)}\tanh\left[\frac{\rho(x_t, y_t)}{2}\sqrt{K(x_t, y_t)/(d-1)}\right]. \tag{2.5}$$

Moreover, by (1.4) there exist two constants $r_0, r_1 > 0$ such that $-\mathrm{Hess}_V \geq \delta$ outside $B(o, r_0)$ but $\leq r_1$ on $B(o, r_0)$, where $B(o, r_0)$ is the closed geodesic ball at $o$ with radius $r_0$. Since the length of $\ell$ contained in $B(o, r_0)$ is less than $2r_0$, we conclude that

$$\begin{aligned}
&\langle \nabla V, \nabla \rho(\cdot, y_t)\rangle(x_t) + \langle \nabla V, \nabla \rho(x_t, \cdot)\rangle(y_t) \\
&\quad = \int_0^{\rho(x_t, y_t)} \mathrm{Hess}_V(U, U)(\ell_s)\, ds \le 2r_0 r_1 - \big(\rho(x_t, y_t) - 2r_0\big)^+ \delta \\
&\quad \le c_1 - \delta\rho(x_t, y_t)
\end{aligned}$$

for some constant $c_1 > 0$. Combining this with (2.4), (2.5) and

$$\xi_t = C_3 + 2\sigma\sqrt{d-1}\, \rho_o(x_t) + \frac{\rho(x, y)}{T},$$

we arrive at

$$\begin{aligned}
d\rho(x_t, y_t) \le \bigg\{&2\sqrt{K(x_t, y_t)(d-1)} + c_1 - \delta\rho(x_t, y_t) \\
&- C_3 - 2\sigma\sqrt{d-1}\rho_o(x_t) - \frac{\rho(x, y)}{T}\bigg\}\, dt
\end{aligned}$$

for $t < \tau$. Noting that

$$\begin{aligned}
\sqrt{K(x_t, y_t)} &\le \big(c + \sigma^2[\rho_o(x_t) + \rho(x_t, y_t)]^2\big)^{1/2} \\
&\le \sqrt{c} + \sigma[\rho_o(x_t) + \rho(x_t, y_t)],
\end{aligned}$$

and $\delta \ge 2\sigma\sqrt{d-1}$, one has

$$2\sqrt{K(x_t, y_t)(d-1)} - \delta\rho(x_t, y_t) - 2\sigma\sqrt{d-1}\rho_o(x_t) \le 2\sqrt{c(d-1)}.$$

Thus, when $C_3 \ge c_1 + 2\sqrt{c(d-1)}$ we have

$$d\rho(x_t, y_t) \le -\frac{\rho(x, y)}{T}\, dt, \qquad t < \tau,$$

so that

$$0 = \rho(x_\tau, y_\tau) \le \rho(x, y) - \int_0^\tau \frac{\rho(x, y)}{T}\, dt = \frac{T - \tau}{T}\rho(x, y),$$

which implies that $\tau \le T$ and hence, $x_T = y_T$. $\square$

**3. Harnack inequality and proof of Theorem 1.1.** We first prove the following Harnack inequality using results in Section 2.

PROPOSITION 3.1. *Assume that* (1.4) *and* (1.5) *hold with* $\delta > (1 + \sqrt{2})\sigma \times \sqrt{d-1}$. *Then there exist* $C > 0$ *and* $\alpha > 1$ *such that*

$$(P_T f(y))^\alpha \le (P_T f^\alpha(x)) \exp\bigg[\frac{C}{T}\rho(x, y)^2 + C\big(T + \rho_o(x)^2\big)\bigg] \tag{3.1}$$

*holds for all* $x, y \in M$, $T > 0$ *and nonnegative* $f \in C_b(M)$.

PROOF. According to Lemma 2.3, we take

$$\xi_t = C_3 + 2\sigma\sqrt{d-1}\rho_o(x_t) + \frac{\rho(x,y)}{T},$$

such that $\tau \le T$ and $x_T = y_T$. Obviously, $y_t$ solves the equation

$$d_I y_t = \sqrt{2}\tilde{\Phi}_t\, d\tilde{B}_t + \nabla V(y_t)\, dt$$

for $\tilde{\Phi}_t := P_{x_t,y_t}\Phi_t$ being the horizontal lift of $y_t$, and $\tilde{B}_t$ solving the equation

$$d\tilde{B}_t = dB_t + \frac{1}{\sqrt{2}}\tilde{\Phi}_t^{-1}\xi_t U(x_t, y_t)1_{\{t<\tau\}}\, dt.$$

By the Girsanov theorem and the fact that $\tau \le T$, the process $\{\tilde{B}_t : t \in [0, T]\}$ is a $d$-dimensional Brownian motion under the probability measure $R\mathbb{P}$ for

$$R := \exp\biggl[-\frac{1}{\sqrt{2}}\int_0^\tau \langle P_{x_t,y_t}\Phi_t\, dB_t, \xi_t U(x_t, y_t)\rangle - \frac{1}{4}\int_0^\tau \xi_t^2\, dt\biggr].$$

Thus, under this probability measure $\{y_t : t \in [0, T]\}$ is generated by $L$. In particular, $P_T f(y) = \mathbb{E}[f(y_T)R]$. Combining this with the Hölder inequality and noting that $x_T = y_T$, we obtain

$$\begin{aligned} P_T f(y) &= \mathbb{E}[f(y_T)R] = \mathbb{E}[f(x_T)R] \\ &\le (P_T f^\alpha(x))^{1/\alpha}\bigl(\mathbb{E}R^{\alpha/(\alpha-1)}\bigr)^{(\alpha-1)/\alpha}. \end{aligned}$$

That is,

$$(P_T f(y))^\alpha \le (P_T f^\alpha(x))\bigl(\mathbb{E}R^{\alpha/(\alpha-1)}\bigr)^{\alpha-1}. \tag{3.2}$$

Since for any continuous exponential integrable martingale $M_t$ and any $\beta, p > 1$, the process $\exp[\beta p M_t - \frac{p^2\beta^2}{2}\langle M\rangle_t]$ is a martingale, by the Hölder inequality one has

$$\begin{aligned} \mathbb{E}e^{\beta M_t - (\beta/2)\langle M\rangle_t} &= \mathbb{E}\bigl[e^{\beta M_t - (\beta^2 p/2)\langle M\rangle_t} \cdot e^{(\beta(\beta p-1)/2)\langle M\rangle_t}\bigr] \\ &\le \mathbb{E}\bigl(e^{(\beta p(\beta p-1)/(2(p-1)))\langle M\rangle_t}\bigr)^{(p-1)/p}. \end{aligned} \tag{3.3}$$

By taking $\beta = \alpha/(\alpha-1)$ we obtain

$$\begin{aligned} &\bigl(\mathbb{E}R^{\alpha/(\alpha-1)}\bigr)^{\alpha-1} \\ &\quad\le \biggl\{\mathbb{E}\exp\biggl[\frac{p\alpha(p\alpha-\alpha+1)}{8(p-1)(\alpha-1)^2}\int_0^T \xi_t^2\, dt\biggr]\biggr\}^{(\alpha-1)(p-1)/p}, \qquad p > 1. \end{aligned} \tag{3.4}$$

Since $\delta > (1+\sqrt{2})\sigma\sqrt{d-1}$, we may take $\delta_0 \in ((1+\sqrt{2})\sigma\sqrt{d-1}, \delta)$, small $\varepsilon' > 0$ and large $C_4 > 0$, independent of $T$, $x$ and $y$, such that

$$\begin{aligned} \xi_t^2 &= \biggl(C_3 + 2\sigma\sqrt{d-1}\rho_o(x_t) + \frac{\rho(x,y)}{T}\biggr)^2 \\ &\le (1-\varepsilon')\biggl[C_4 + \frac{C_4\rho(x,y)^2}{T^2} + 2\bigl(\delta_0 - \sigma\sqrt{d-1}\bigr)^2\rho_o(x_t)^2\biggr] \end{aligned}$$

holds. Moreover, since

$$\lim_{p\downarrow 1}\lim_{\alpha\uparrow\infty}\frac{p\alpha(p\alpha-\alpha+1)}{8(p-1)(\alpha-1)^2}=\frac{1}{8}, \tag{3.5}$$

there exist $p, \alpha > 1$ such that

$$\begin{aligned}\frac{p\alpha(p\alpha-\alpha+1)}{8(p-1)(\alpha-1)^2}&\int_0^T \xi_t^2\,dt\\ &\le C_4T+\frac{C_4\rho(x,y)^2}{T}+\frac{(\delta_0-\sigma\sqrt{d-1})^2}{4}\int_0^T \rho_o(x_t)^2\,dt.\end{aligned}$$

Combining this with (3.4) and Lemma 2.2, we obtain

$$\left(\mathbb{E}R^{\alpha/(\alpha-1)}\right)^{\alpha-1}\le \exp\left[C_5T+\frac{C_5\rho(x,y)}{T}+C_5\rho_o(x)^2\right],\qquad T>0, x\in M,$$

for some constant $C_5 > 0$. This completes the proof by (3.2). □

PROOF OF THEOREM 1.1. By Proposition 3.1, let $\alpha > 1$ and $C > 0$ such that (3.1) holds. Since $\delta > \sigma\sqrt{d-1}$, we may take $T > 0$ such that

$$\frac{C}{T}\le \varepsilon := \frac{1}{8}\left(\delta-\sigma\sqrt{d-1}\right).$$

Then for any nonnegative $f \in C_b(M)$ with $\mu(f^\alpha) = 1$, since $\mu$ is $P_T$-invariant, it follows from (3.1) that

$$\begin{aligned}1&=\int_M P_Tf^\alpha(x)\mu(dx)\ge (P_Tf(y))^\alpha\int_M e^{-\varepsilon\rho(x,y)^2-C(1+\rho_o(x)^2)}\mu(dx)\\ &\ge (P_Tf(y))^\alpha\int_{\{\rho_o\le 1\}} e^{-\varepsilon(1+\rho_o(y))^2-2C}\mu(dx)\\ &\ge \varepsilon'(P_Tf(y))^\alpha\exp[-2\varepsilon\rho_o(y)^2],\qquad y\in M,\end{aligned}$$

for some constant $\varepsilon' > 0$. Thus,

$$\int_M (P_Tf(y))^{2\alpha}\mu(dy)\le \frac{1}{\varepsilon'}\int_M e^{4\varepsilon\rho_o(y)^2}\mu(dy)<\infty,$$

according to Lemma 2.1. This implies that

$$\|P_T\|_{L^\alpha(\mu)\to L^{2\alpha}(\mu)}<\infty.$$

Therefore, the log-Sobolev inequality (1.3) holds for some constant $C > 0$, due to the uniformly positively improving property of $P_t$ (see [20], proof of Theorem 1.1, and [1]). □

**4. Supercontractivity and ultracontractivity.** Recall that $P_t$ is called supercontractive if $\|P_t\|_{2\to 4} < \infty$ for all $t > 0$ while ultracontractive if $\|P_t\|_{2\to\infty} < \infty$ for all $t > 0$ (see [10]). In the present framework these two properties are stronger than the hypercontractivity: $\|P_t\|_{2\to 4} \le 1$ for some $t > 0$, which is equivalent to (1.3) due to Gross [14].

PROPOSITION 4.1. *Under* (1.4) *and* (1.5), $P_t$ *is supercontractive if and only if* $\mu(\exp[\lambda\rho_o^2]) < \infty$ *for all* $\lambda > 0$, *while it is ultracontractive if and only if* $\|P_t \exp[\lambda\rho_o^2]\|_\infty < \infty$ *for all* $t, \lambda > 0$.

PROOF. The proof is similar to that of [18], Theorem 2.3. Let $f \in L^2(\mu)$ with $\mu(f^2) = 1$. By (3.1) for $\alpha = 2$ and noting that $\mu$ is $P_t$-invariant, we obtain

$$\begin{aligned}1 &\ge (P_T f(y))^2 \int_M \exp\left[-\frac{C}{T}\rho(x,y)^2 - C\big(T + \rho_o(x)^2\big)\right]\mu(dx)\\ &\ge (P_T f(y))^2 \exp\left[-\frac{2C}{T}\big(\rho_o(y)^2 + 1\big) - C(T+1)\right]\mu(B(o,1)).\end{aligned}$$

Hence, for any $T > 0$ there exists a constant $\lambda_T > 0$ such that

$$|P_T f| \le \exp[\lambda_T(1 + \rho_o^2)], \qquad T > 0, \mu(f^2) = 1. \tag{4.1}$$

(1) If $\mu(e^{\lambda\rho_o^2}) < \infty$ for any $\lambda > 0$, (4.1) yields that

$$\|P_T\|_{2\to 4}^4 \le \mu\big(e^{4\lambda_T(1+\rho_o^2)}\big) < \infty, \qquad T > 0.$$

Conversely, if $P_t$ is supercontractive then the super log-Sobolev inequality (cf. [10])

$$\mu(f^2 \log f^2) \le r\mu(|\nabla f|^2) + \beta(r), \qquad r > 0, \mu(f^2) = 1,$$

holds for some $\beta : (0,\infty) \to (0,\infty)$. By [2] (see also [17, 18]), this inequality implies $\mu(e^{\lambda\rho_o^2}) < \infty$ for all $\lambda > 0$.

(2) By (4.1) and the semigroup property,

$$\|P_T\|_{2\to\infty} \le \big\|P_{T/2} e^{\lambda_{T/2}(1+\rho_o^2)}\big\|_\infty < \infty, \qquad T > 0,$$

provided $\|P_t e^{\lambda\rho_o^2}\|_\infty < \infty$ for any $t, \lambda > 0$. Conversely, since the ultracontractivity is stronger than the supercontractivity, it implies that $e^{\lambda\rho_o^2} \in L^2(\mu)$ for any $\lambda > 0$ as explained above. Therefore,

$$\|P_t e^{\lambda\rho_o^2}\|_\infty \le \|P_t\|_{2\to\infty} \|e^{\lambda\rho_o^2}\|_2 < \infty, \qquad \lambda > 0.$$

Then the proof is completed. □

To derive explicit conditions for the supercontractivity and ultracontractivity, we consider the following stronger version of (1.4):

$$-\mathrm{Hess}_V \geq \Phi \circ \rho_o \qquad \text{holds outside a compact subset of } M \tag{4.2}$$

for a positive increasing function $\Phi$ with $\Phi(r) \uparrow \infty$ as $r \uparrow \infty$. We then aim to search for reasonable conditions on positive increasing function $\Psi$ such that

$$\mathrm{Ric} \geq -\Psi \circ \rho_o \tag{4.3}$$

implies the supercontractivity and/or ultracontractivity.

THEOREM 4.2. *If* (4.3) *and* (4.2) *hold for some increasing positive functions* $\Phi$ *and* $\Psi$ *such that*

$$\lim_{r\to\infty} \Phi(r) = \lim_{r\to\infty} \frac{(\int_0^r \Phi(s)\,ds)^2}{\Phi(r)} = \infty, \tag{4.4}$$

$$\begin{aligned} &\sqrt{\Psi(r+t)(d-1)} \\ &\quad \leq \theta \int_0^r \Phi(s)\,ds + \frac{1}{2}\int_0^{t/2} \Phi(s)\,ds + C, \qquad r,t \geq 0, \end{aligned} \tag{4.5}$$

*for some constants* $\theta \in (0, 1/(1+\sqrt{2}))$ *and* $C > 0$. *Then* $P_t$ *is supercontractive. Furthermore*, *if*

$$\int_1^\infty \frac{ds}{\sqrt{s}\int_0^{\sqrt{r}} \Phi(u)\,du} < \infty, \tag{4.6}$$

*then* $P_t$ *is ultracontractive. More precisely, for*

$$\Gamma_1(r) := \frac{1}{\sqrt{r}}\int_0^{\sqrt{r}} \Phi(s)\,ds, \qquad \Gamma_2(r) := \int_r^\infty \frac{ds}{\sqrt{s}\int_0^{\sqrt{s}} \Phi(u)\,du}, \qquad r > 0,$$

(4.6) *implies*

$$\|P_t\|_{2\to\infty} \leq \exp\left[c + \frac{c}{t}\big(1 + \Gamma_1^{-1}(c/t) + \Gamma_2^{-1}(t/c)\big)\right] < \infty, \qquad t > 0, \tag{4.7}$$

*for some constant* $c > 0$ *and*

$$\Gamma_1^{-1}(s) := \inf\{t \geq 0 : \Gamma_1(t) \geq s\}, \qquad s \geq 0.$$

PROOF. (a) Replacing $c+\rho_o^2$ by $\Psi\circ\rho_o$ and noting that $\mathrm{Hess}_V \le -\Phi\circ\rho_o$ for large $\rho_o$, the proof of Lemma 2.1 implies

$$L\rho_o^2 \le c_1(1+\rho_o) - 2\rho_o\left(\int_0^{\rho_o}\Phi(s)\,ds - \sqrt{\Psi\circ\rho_o(d-1)}\right) \tag{4.8}$$

for some constant $c_1>0$. Combining this with (4.5) and noting that $\frac{1}{\rho_o}\times\int_0^{\rho_o}\Phi(s)\,ds\to\infty$ as $\rho_o\to\infty$, we conclude that for any $\lambda>0$,

$$\begin{aligned} Le^{\lambda\rho_o^2} &\le C - \frac{2\lambda\rho_o\sqrt{2}}{1+\sqrt{2}}e^{\lambda\rho_o^2}\int_0^{\rho_o}\Phi(s)\,ds + 4\lambda^2\rho_o^2e^{\lambda\rho_o^2} \\ &\le C + C(\lambda) - \lambda\rho_o e^{\lambda\rho_o^2}\int_0^{\rho_o}\Phi(s)\,ds, \end{aligned}\tag{4.9}$$

where $C>0$ is a universal constant and

$$\begin{aligned} C(\lambda) &:= \sup_{r>0} re^{\lambda r^2}\left\{4\lambda^2 r - \frac{\lambda}{(1+\sqrt{2})^2}\int_0^r\Phi(s)\,ds\right\} \\ &= \sup_{r^2\le\Gamma_1^{-1}(4(1+\sqrt{2})^2\lambda)} re^{\lambda r^2}\left\{4\lambda^2 r - \frac{\lambda}{(1+\sqrt{2})^2}\int_0^r\Phi(s)\,ds\right\} \\ &\le 4\lambda^2\Gamma_1^{-1}(4(1+\sqrt{2})^2\lambda)\exp[\lambda\Gamma_1^{-1}(4(1+\sqrt{2})^2\lambda)] \\ &\le \exp[4\lambda + 2\lambda\Gamma_1^{-1}(4(1+\sqrt{2})^2\lambda)] < \infty. \end{aligned}\tag{4.10}$$

Therefore, (1.1) holds and

$$\mu(e^{\lambda\rho_o^2}) < \infty, \qquad \lambda > 0. \tag{4.11}$$

(b) By (4.5), (4.8) and Kendall's Itô formula [16] as in the proof of Lemma 2.2, we have

$$d\rho_o^2(x_t) \le 2\sqrt{2}\rho_o(x_t)\,db_t + \left(C_1 - \frac{2\sqrt{2}\rho_o(x_t)(1+\varepsilon)}{1+\sqrt{2}}\int_0^{\rho_o(x_t)}\Phi(s)\,ds\right)dt$$

for some constants $\varepsilon, C_1>0$, where $x_t$ and $b_t$ are in the proof of Lemma 2.2. Let

$$\varphi(r) = \int_0^r \frac{ds}{\sqrt{s}}\int_0^{\sqrt{s}}\Phi(u)\,du, \qquad r\ge 0. \tag{4.12}$$

We arrive at

$$\begin{aligned} d\varphi\circ\rho_o^2(x_t) &\le 2\sqrt{2}\rho_o(x_t)\varphi'\circ\rho_o^2(x_t)\,db_t + 4\rho_o^2(x_t)\varphi''\circ\rho_o^2(x_t)\,dt \\ &\quad + \varphi'\circ\rho_o^2(x_t)\left(C_1 - \frac{2\sqrt{2}\rho_o(x_t)(1+\varepsilon)}{1+\sqrt{2}}\int_0^{\rho_o(x_t)}\Phi(s)\,ds\right)dt. \end{aligned}$$

From (4.4) we see that

$$\frac{\rho_o \varphi'' \circ \rho_o^2}{\varphi' \circ \rho_o^2 \int_0^{\rho_o} \Phi(s)\,ds} \le \frac{\Phi \circ \rho_o}{2(\int_0^{\rho_o} \Phi(s)\,ds)^2},$$

which goes to zero as $\rho_o \to \infty$. Then there exists a constant $C_2 > C_1$ such that

$$\begin{aligned} d\varphi \circ \rho_o^2(x_t) &\le 2\sqrt{2}\biggl(\int_0^{\rho_o(x_t)} \Phi(s)\,ds\biggr)\,db_t \\ &\qquad + C_2\,dt - \frac{2\sqrt{2}}{1+\sqrt{2}}\biggl(\int_0^{\rho_o(x_t)} \Phi(s)\,ds\biggr)^2\,dt. \end{aligned}$$

This implies that for any $\lambda > 0$,

$$\begin{aligned} &\mathbb{E}\exp\biggl[\frac{2\sqrt{2}\,\lambda}{1+\sqrt{2}} \int_0^T \biggl(\int_0^{\rho_o(x_t)} \Phi(s)\,ds\biggr)^2 dt\biggr] \\ &\quad \le e^{C_2\lambda T + \lambda\varphi\circ\rho_o^2(x)}\mathbb{E}\exp\biggl[2\sqrt{2}\lambda \int_0^T \biggl(\int_0^{\rho_o(x_t)} \Phi(s)\,ds\biggr)\,db_t\biggr] \\ &\quad \le e^{C_2\lambda T + \lambda\varphi\circ\rho_o^2(x)}\biggl(\mathbb{E}\exp\biggl[16\lambda^2 \int_0^T \biggl(\int_0^{\rho_o(x_t)} \Phi(s)\,ds\biggr)^2 dt\biggr]\biggr)^{1/2}. \end{aligned}$$

Taking

$$\lambda = \frac{\sqrt{2}}{8(1+\sqrt{2})},$$

we arrive at

$$\begin{aligned} &\mathbb{E}\exp\biggl[\frac{1}{2(1+\sqrt{2})^2} \int_0^T \biggl(\int_0^{\rho_o(x_t)} \Phi(s)\,ds\biggr)^2 dt\biggr] \\ &\quad \le e^{2C_2T + \varphi\circ\rho_o^2(x)\sqrt{2}/8(1+\sqrt{2})}. \end{aligned} \tag{4.13}$$

(c) Let $\gamma : [0, \rho(x_t, y_t)] \to M$ be the minimal geodesic from $x_t$ to $y_t$, and $U$ its tangent unit vector. By (4.2), there exists a constant $C_3 > 0$ such that

$$\begin{aligned} &\langle \nabla V, \nabla\rho(\cdot, y_t)\rangle(x_t) + \langle \nabla V, \nabla\rho(x_t, \cdot)\rangle(y_t) \\ &\quad = \int_0^{\rho(x_t,y_t)} \mathrm{Hess}_V(U_s, U_s)\,ds \le C_3 - \int_0^{\rho(x_t,y_t)/2} \Phi(s)\,ds. \end{aligned} \tag{4.14}$$

To understand the last inequality, we assume, for instance, that $\rho_o(x_t) \ge \rho_o(y_t)$ so that by the triangle inequality,

$$\rho_o(\gamma_s) \ge \rho_o(x_t) - s \ge \rho(x_t, y_t)/2 - s, \qquad s \in [0, \rho(x_t, y_t)/2].$$

For the coupling constructed in Section 3, one concludes from (4.14) and the proof of Lemma 2.3 that

$$
\begin{aligned}
d\rho(x_t, y_t) \le \Big\{ & 2\sqrt{K(x_t, y_t)(d-1)} + C_4 \\
& - \int_0^{\rho(x_t,y_t)/2} \Phi(s)\,ds - \xi_t \Big\}\,dt, \qquad t < \tau,
\end{aligned}
\tag{4.15}
$$

holds for some constant $C_4 > 0$, where

$$K(x_t, y_t) := \sup_{\ell([0,\rho(x_t,y_t)])} \Psi \circ \rho_o \le \Psi\big(\rho_o(x_t) + \rho(x_t, y_t)\big),$$

and $\ell$ is the minimal geodesic from $x_t$ to $y_t$. Combining (4.5) and (4.15), we obtain

$$d\rho(x_t, y_t) \le \Big\{ C_4 + 2\theta \int_0^{\rho_o(x_t)} \Phi(s)\,ds - \xi_t \Big\}\,dt, \qquad t < \tau.$$

So, taking

$$\xi_t = C_4 + 2\theta \int_0^{\rho_o(x_t)} \Phi(s)\,ds + \frac{\rho(x, y)}{T},$$

we arrive at

$$d\rho(x_t, y_t) \le -\frac{\rho(x, y)}{T}\,dt, \qquad t < \tau.$$

This implies $\tau \le T$, and hence $x_T = y_T$ a.s.

Combining (4.5) with (3.4) and (3.5) we conclude that for the present choice of $\xi_t$ there exist $\alpha, p, C_5 > 1$ such that

$$
\begin{aligned}
\big(\mathbb{E} R^{\alpha/(\alpha-1)}\big)^{p/(p-1)} \le \mathbb{E} \exp\Big[ & \frac{1}{2(1+\sqrt{2})^2} \int_0^T \Big( \int_0^{\rho(x_t)} \Phi(s)\,ds \Big)^2 dt \\
& + C_5 T + \frac{C_5}{T} \rho(x, y)^2 \Big].
\end{aligned}
$$

Combining this with (4.13) and (3.2) we obtain

$$(P_T f(y))^\alpha \le (P_T f^\alpha(x)) \exp\Big[ CT + \frac{C}{T}\rho(x, y)^2 + C\varphi \circ \rho^2(x) \Big] \tag{4.16}$$

holds for some $\alpha, C > 1$, any positive $f \in C_b(M)$ and all $x, y \in M, T > 0$.

(d) For any positive $f \in C_b(M)$ with $\mu(f^\alpha) = 1$, (4.16) implies that

$$(P_T f(y))^\alpha \int_{B(o,1)} \exp\Big[ -CT - \frac{C}{T}\rho(x, y)^2 - C\varphi^2(x) \Big] \mu(dx) \le 1.$$

Therefore, there exists a constant $C' > 0$ such that

$$(P_T f(y))^\alpha \le \exp\Big[ C'(1+T) + \frac{C'}{T}\rho(y)^2 \Big], \qquad y \in M, T > 0. \tag{4.17}$$

Combining this with (4.11) we obtain

$$\|P_T\|_{\alpha\to p\alpha} < \infty, \qquad T > 0, p > 1.$$

This is equivalent to the supercontactivity by the Riesz–Thorin interpolation theorem and $\|P_t\|_{1\to 1} = 1$. Thus, the first assertion holds.

(e) To prove (4.7), it suffices to consider $t \in (0, 1]$ since $\|P_t\|_{2\to\infty}$ is decreasing in $t > 0$. So, below we assume that $T \le 1$. By (4.17) and the fact that $(P_{2T} f)^\alpha \le P_T(P_T f)^\alpha$, we have

$$\|P_{2T}\|_{\alpha\to\infty} \le \|P_T e^{2C'\rho_o^2/T}\|_\infty e^{C'(1+T)}, \qquad T > 0. \tag{4.18}$$

Therefore, by the Riesz–Thorin interpolation theorem and $\|P_t\|_{1\to 1} = 1$, for the ultracontractivity it suffices to show that

$$\|P_T e^{\lambda\rho_o^2}\|_\infty < \infty, \qquad \lambda, T > 0. \tag{4.19}$$

Since $\Phi$ is increasing, it is easy to check that

$$\eta(r) := \sqrt{r}\int_0^{\sqrt{r}} \Phi(s)\,ds, \qquad r \ge 0,$$

is convex, and so is $s \mapsto s\eta(\frac{\log s}{\lambda})$ for $\lambda > 0$. Thus, it follows from (4.9) and the Jensen inequality that

$$h_{\lambda,x}(t) := \mathbb{E}e^{\lambda\rho_o^2(x_t)} < \infty, \qquad x_0 = x \in M, \lambda, t > 0,$$

and

$$\frac{d^+}{dt} h_{\lambda,x}(t) \le C + C(\lambda) - \lambda h_{\lambda,x}(t)\eta(\lambda^{-1}\log h_{\lambda,x}(t)), \qquad t > 0.$$

This implies (4.19), provided (4.6) holds. This can be done by considering the following two situations:

(1) Since $h_{\lambda,x}(t)$ is decreasing provided $\lambda h_{\lambda,x}(t)\eta(\lambda^{-1}\log h_{\lambda,x}(t)) > C + C(\lambda)$, if

$$\lambda h_{\lambda,x}(0)\eta(\lambda^{-1}\log h_{\lambda,x}(0)) \le 2C + 2C(\lambda),$$

then

$$h_{\lambda,x}(t) \le \sup\{r \ge 1 : \lambda r\eta(\lambda^{-1}\log r) \le 2C + 2C(\lambda)\} \le \frac{1}{\lambda}\big(2C + 2C(\lambda)\big) + C''$$

for some constant $C'' > 0$.

(2) If $\lambda h_{\lambda,x}(0)\eta(\lambda^{-1}\log h_{\lambda,x}(0)) > 2C + 2C(\lambda)$, then $h_{\lambda,x}(t)$ is decreasing in $t$ up to

$$t_\lambda := \inf\{t \ge 0 : \lambda h_{\lambda,x}(t)\eta(\lambda^{-1}\log h_{\lambda,x}(t)) \le 2C + 2C(\lambda)\}.$$

Indeed,

$$\frac{d^+}{dt} h_{\lambda,x}(t) \le -\frac{\lambda}{2} h_{\lambda,x}(t)\eta(\lambda^{-1}\log h_{\lambda,x}(t)), \qquad t \le t_\lambda.$$

Thus,

$$\int_{h_{\lambda,x}(T\wedge t_\lambda)}^{\infty} \frac{dr}{r\eta(\lambda^{-1}\log r)} \ge \frac{\lambda}{2}(T \wedge t_\lambda).$$

This is equivalent to

$$\Gamma_2\big(\lambda^{-1}\log h_{\lambda,x}(T\wedge t_\lambda)\big) \ge \tfrac{1}{2}(T\wedge t_\lambda).$$

Hence,

$$h_{\lambda,x}(T\wedge t_\lambda) \le \exp\big[\lambda\Gamma_2^{-1}\big(\tfrac{1}{2}(T\wedge t_\lambda)\big)\big].$$

Since it is reduced to case (1) if $T > t_\lambda$ by regarding $t_\lambda$ as the initial time, in conclusion we have

$$\sup_{x\in M} h_{\lambda,x}(T) \le \max\left\{\exp[\lambda\Gamma_2^{-1}(T/2)],\ C'' + \frac{1}{\lambda}\big(2C + 2C(\lambda)\big)\right\}.$$

Therefore, (4.7) follows from (4.18), (4.10) with $\lambda = 2C'/T$, and the Riesz interpolation theorem. $\square$

Finally, we note that a simple example for conditions in Theorem 4.2 to hold is

$$\Phi(s) = s^{\alpha-1}, \qquad \Psi(s) = \varepsilon s^{2\alpha}$$

for $\alpha > 1$ and small enough $\varepsilon > 0$. In this case $P_t$ is ultracontractive with

$$\|P_t\|_{2\to\infty} \le \exp\big[c\big(1 + t^{-(\alpha+1)/(\alpha-1)}\big)\big], \qquad t > 0,$$

for some $c > 0$.

**Acknowledgments.** The author would like to thank the referees for their careful reading and valuable comments on an earlier version of the paper.

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