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CONTINUITY PROPERTIES AND INFINITE DIVISIBILITY OF STATIONARY DISTRIBUTIONS OF SOME GENERALIZED ORNSTEIN–UHLENBECK PROCESSES

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Properties of the law μ of the integral $\int_0^\infty c^{-N_{t-}}\, dY_t$ are studied, where $c > 1$ and $\{(N_t, Y_t), t \geq 0\}$ is a bivariate Lévy process such that $\{N_t\}$ and $\{Y_t\}$ are Poisson processes with parameters a and b, respectively. This is the stationary distribution of some generalized Ornstein–Uhlenbeck process. The law μ is parametrized by c, q and r, where $p = 1 - q - r$, q, and r are the normalized Lévy measure of $\{(N_t, Y_t)\}$ at the points $(1, 0)$, $(0, 1)$ and $(1, 1)$, respectively. It is shown that, under the condition that $p > 0$ and $q > 0$, $\mu_{c,q,r}$ is infinitely divisible if and only if $r \leq pq$. The infinite divisibility of the symmetrization of μ is also characterized. The law μ is either continuous-singular or absolutely continuous, unless $r = 1$. It is shown that if c is in the set of Pisot–Vijayaraghavan numbers, which includes all integers bigger than 1, then μ is continuous-singular under the condition $q > 0$. On the other hand, for Lebesgue almost every $c > 1$, there are positive constants C_1 and C_2 such that μ is absolutely continuous whenever $q \geq C_1 p \geq C_2 r$. For any $c > 1$ there is a positive constant C_3 such that μ is continuous-singular whenever $q > 0$ and $\max\{q, r\} \leq C_3 p$. Here, if $\{N_t\}$ and $\{Y_t\}$ are independent, then $r = 0$ and $q = b/(a + b)$.

1. Introduction. A generalized Ornstein–Uhlenbeck process $\{V_t, t \geq 0\}$ with initial condition V_0 is defined as

$$V_t = e^{-\xi_t}\left(V_0 + \int_0^t e^{\xi_{s-}}\, d\eta_s\right),$$

where $\{(\xi_t, \eta_t), t \geq 0\}$ is a bivariate Lévy process, independent of V_0. See Carmona, Petit and Yor [3, 4] for basic properties. Such processes arise in a variety of situations such as risk theory (e.g., Paulsen [18]), option pricing (e.g., Yor [26]) or financial time series (e.g., Klüppelberg, Lindner and Maller [13]), to name just a few. They also constitute a natural continuous time analogue of random recurrence equations, as studied by de Haan and Karandikar [11]. Lindner and Maller [16] have shown that a generalized Ornstein–Uhlenbeck process admits a strictly stationary solution which is not degenerate to a constant process with a suitable V_0 if

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and only if

$$\int_0^{\infty-} e^{-\xi_{s-}}\, dL_s := \lim_{t\to\infty} \int_0^t e^{-\xi_{s-}}\, dL_s \tag{1.1}$$

exists and is finite almost surely and not degenerate to a constant random variable. The distribution of (1.1) then gives the unique stationary distribution. Here, $\{(\xi_t, L_t), t \geq 0\}$ is another bivariate Lévy process, defined in terms of $\{(\xi_t, \eta_t)\}$ by

$$L_t = \eta_t + \sum_{0<s\leq t} \bigl(e^{-(\xi_s - \xi_{s-})} - 1\bigr)(\eta_s - \eta_{s-}) - t a^{1,2}_{\xi,\eta},$$

where $a^{1,2}_{\xi,\eta}$ denotes the $(1,2)$-element in the Gaussian covariance matrix of the Lévy–Khintchine triplet of $\{(\xi_t, \eta_t)\}$. Conversely, $\{(\xi_t, \eta_t)\}$ can be reconstructed from $\{(\xi_t, L_t)\}$ by

$$\eta_t = L_t + \sum_{0<s\leq t} (e^{\xi_s - \xi_{s-}} - 1)(L_s - L_{s-}) + t a^{1,2}_{\xi,L}.$$

Note that, if $\{\xi_t\}$ and $\{\eta_t\}$ are independent, then $L_t = \eta_t$ for all t. The convergence of integral (1.1) was characterized by Erickson and Maller [6] and generalized by Kondo, Maejima and Sato [14] to the case when $\{(\xi_t, L_t)\}$ is an $\mathbb{R} \times \mathbb{R}^d$ valued Lévy process with $d \in \mathbb{N}$.

Suppose now that $\{(\xi_t, L_t)\}$ is a bivariate Lévy process such that (1.1) converges almost surely and is finite, and denote by

$$\mu := \mathcal{L}\left(\int_0^{\infty-} e^{-\xi_{s-}}\, dL_s\right)$$

the distribution of the integral. If $\xi_t = t$ is deterministic, then it is well known that μ is self-decomposable, hence is infinitely divisible as well as absolutely continuous (if not degenerate to a Dirac measure, which happens only if $\{L_t\}$ is also deterministic). Other cases where μ is self-decomposable include the case where $\{\xi_t\}$ is stochastic, but spectrally negative (cf. Bertoin, Lindner and Maller [1]). On the other hand, as remarked by Samorodnitsky, μ is not infinitely divisible if, for example, $\xi_t = N_t + \alpha t$ with a Poisson process $\{N_t, t \geq 0\}$ and a positive drift $\alpha > 0$ and $L_t = t$ (cf. Klüppelberg, Lindner and Maller [13], page 408). Continuity properties of μ for general $\{(\xi_t, L_t)\}$ were studied by Bertoin, Lindner and Maller [1], who showed that μ cannot have atoms unless μ is a Dirac measure, with this degenerate case also being characterized. Gjessing and Paulsen [8] derived the distribution of μ in a variety of situations; however, in all cases considered the distribution turned out to be absolutely continuous.

With these results in mind, it is natural to ask, first, whether μ will always be absolutely continuous for general $\{(\xi_t, L_t)\}$, unless μ degenerates to a Dirac measure and, second, what is the condition for μ to be infinitely divisible. The present article will give the negative answer to the first question, showing many cases of μ being continuous-singular and, to the second question, provide a necessary and sufficient condition in a restricted class. Namely we will restrict our attention to the case $(\xi_t, L_t) = ((\log c)N_t, Y_t)$, where c is a constant greater than 1 and $\{N_t\}$ and $\{Y_t\}$ are Poisson processes with parameters a and b, respectively, with $\{(N_t, Y_t)\}$ being a bivariate Lévy process. Thus we will study in detail

$$\mu = \mathcal{L}\biggl(\int_0^{\infty-} e^{-(\log c)N_{s-}}\, dY_s\biggr) = \mathcal{L}\biggl(\int_0^{\infty-} c^{-N_{s-}}\, dY_s\biggr). \tag{1.2}$$

The integral here is an improper Stieltjes integral pathwise. From the strong law of large numbers, we see that the integral exists and is finite. Even in this class the problems of infinite divisibility and continuity properties turn out to have rich substance. Let T be the first jump time of $\{N_t\}$. Then

$$\int_0^{\infty-} c^{-N_{s-}}\, dY_s = Y_T + \int_T^{\infty-} c^{-N_{s-}}\, dY_s = Y_T + c^{-1}\int_0^{\infty-} c^{-N'_{s-}}\, dY'_s,$$

where $\{(N'_t, Y'_t)\}$ is an independent copy of $\{(N_t, Y_t)\}$. Hence, letting $\rho = \mathcal{L}(Y_T)$, we obtain

$$\widehat{\mu}(z) = \widehat{\rho}(z)\widehat{\mu}(c^{-1}z), \qquad z \in \mathbb{R}, \tag{1.3}$$

where $\widehat{\mu}(z)$ and $\widehat{\rho}(z)$ denote the characteristic functions of μ and ρ. It follows that

$$\widehat{\mu}(z) = \widehat{\mu}(c^{-k}z)\prod_{n=0}^{k-1}\widehat{\rho}(c^{-n}z), \qquad k \in \mathbb{N},$$

and hence

$$\widehat{\mu}(z) = \prod_{n=0}^{\infty}\widehat{\rho}(c^{-n}z). \tag{1.4}$$

In general, if a distribution μ satisfies (1.3) with some distribution ρ, then μ is called c^{-1}-*decomposable*. Our study of the law μ is based on this c^{-1}-decomposability. The expression (1.4) shows that the law ρ controls μ. The properties of c^{-1}-decomposable distributions are studied by Wolfe [25], Bunge [2], Watanabe [23] and others. In particular, it is known that any nondegenerate c^{-1}-decomposable distribution is either continuous-singular or absolutely continuous (Wolfe [25]). A distribution μ is self-decomposable if and only if μ is c^{-1}-decomposable for all $c > 1$; in this case μ and ρ are infinitely divisible. In general if a distribution μ satisfies (1.3) with ρ being infinitely divisible, then μ is called c^{-1}-*semi-self-decomposable*. We note that, when $c = e$ and $\{N_t\}$ and $\{Y_t\}$ are independent, Kondo, Maejima and Sato [14] recognizes that μ is e^{-1}-decomposable and either continuous-singular or absolutely continuous.

The Lévy process $\{(N_t, Y_t)\}$ is a bivariate compound Poisson process with Lévy measure concentrated on the three points $(1, 0)$, $(0, 1)$ and $(1, 1)$ and the amounts of the measure of these points are denoted by u, v and w. Letting

$p = u/(u+v+w)$, $q = v/(u+v+w)$ and $r = w/(u+v+w)$ be the normalized Lévy measure on these three points, we will see that μ is determined by c, q and r and ρ is determined by q and r, and hence denote $\mu = \mu_{c,q,r}$ and $\rho = \rho_{q,r}$. We call r the *dependence parameter* of $\{(N_t, Y_t)\}$, since $r = 0$ is equivalent to the independence of $\{N_t\}$ and $\{Y_t\}$ and $r = 1$ means $N_t = Y_t$ for all t. If $r = 0$, then $\rho = \mathcal{L}(Y_T)$ is a geometric distribution, which is infinitely divisible, and hence μ is also infinitely divisible. But, if $r > 0$, the situation is more complicated. In Section 2 we will give a complete description of the condition of infinite divisibility of $\mu_{c,q,r}$ and $\rho_{q,r}$ in terms of their parameters. It will turn out that infinite divisibility of $\mu_{c,q,r}$ does not depend on c. It is shown in Niedbalska-Rajba [17] that there exists a c^{-1}-decomposable infinitely divisible distribution μ that satisfies (1.3) with a noninfinitely-divisible ρ. But, in our case, it will turn out that $\mu_{c,q,r}$ is infinitely divisible if and only if $\rho_{q,r}$ is so. Further, under the condition that $0 < q < 1$ and $p > 0$, it will turn out that $\rho_{q,r}$ is infinitely divisible if and only if the dependence parameter is so small that $r \le pq$. We also address the problem of infinite divisibility of the symmetrizations μ^{sym} and ρ^{sym} of μ and ρ. Infinite divisibility of a distribution implies that of its symmetrization, but there is a noninfinitely-divisible distribution whose symmetrization is infinitely divisible, which is pointed out in pages 81–82 in Gnedenko and Kolmogorov [9]. A complete description of infinite divisibility of μ^{sym} and ρ^{sym} will be given, which provides new examples of this phenomenon in [9]. In the proof of noninfinite-divisibility, we use three methods: (1) Katti's condition for distributions on nonnegative integers; (2) Lévy–Khintchine type representation of characteristic functions with signed measures in place of Lévy measures; (3) representation of the Laplace transforms of infinitely divisible distributions on $[0,\infty)$ in the form $e^{-\varphi(\theta)}$ with $\varphi'(\theta)$ being completely monotone.

Section 3 is devoted to the study of continuous-singularity and absolute continuity of $\mu_{c,q,r}$. If $q = 0$, then it will be shown that

$$\widehat{\mu_{c,0,r}}(z) = \prod_{n=0}^{\infty} [(1-r) + re^{ic^{-n}z}], \qquad z \in \mathbb{R},$$

so that μ is an *infinite Bernoulli convolution* (usage of this word is not fixed; here we follow Watanabe [23]). The question of singularity and absolute continuity of infinite Bernoulli convolutions has been investigated by many authors but, even if $r = 1/2$, characterization of all $c > 1$ for which the distribution is absolutely continuous is an open problem. See Peres, Schlag and Solomyak [19], Peres and Solomyak [20], Watanabe [23] and the references therein. We shall exclude the case $q = 0$ from our consideration, but we will show that the notions and techniques developed in the study of infinite Bernoulli convolutions and b-decomposable measures are effectively applied. Here, unlike in the study of infinite divisibility, the parameter c plays a crucial role. If c has an algebraic property of being a Pisot–Vijayaraghavan (P.V.) number, then we will show that $\mu_{c,q,r}$

is continuous-singular under the condition that $q > 0$. For example, all integers greater than one and some irrationals such as $(1+\sqrt{5})/2$ are P.V. numbers. On the other hand, if c is the reciprocal of a Peres–Solomyak (P.S.) number, then it will be shown that there are positive constants C_1 and C_2 such that $\mu_{c,q,r}$ is absolutely continuous with bounded continuous density whenever $q \geq C_1 p \geq C_2 r$. It is known that Lebesgue almost all reals in $(1,\infty)$ are reciprocals of P.S. numbers. In general, under the condition $0 < q < 1$, we can estimate $\dim(\mu_{c,q,r})$, the Hausdorff dimension of $\mu_{c,q,r}$ defined as the infimum of the Hausdorff dimensions of E over all Borel sets E satisfying $\mu_{c,q,r}(E) = 1$ (in some papers, including [23], this is called upper Hausdorff dimension and denoted by $\dim^*$). Using a powerful theorem of Watanabe [23] for any c^{-1}-decomposable distribution satisfying (1.3) with a discrete distribution ρ, we see that $\dim(\mu_{c,q,r}) \leq H(\rho_{q,r})/\log c$, where $H(\rho_{q,r})$ is the entropy of $\rho_{q,r}$. It follows that $\mu_{c,q,r}$ is continuous-singular, if $H(\rho_{q,r})/\log c < 1$. Thus, for any $c > 1$, there is a positive constant C_3 such that $\mu_{c,q,r}$ is continuous-singular whenever $q > 0$ and $\max\{q,r\} \leq C_3 p$.

In Section 3 we also study, in the case where $\mu_{c,q,r}$ is infinitely divisible, continuity properties of the convolution power $(\mu_{c,q,r})^{t*}$ of $\mu_{c,q,r}$, that is, the distribution at time t of the Lévy process associated with $\mu_{c,q,r}$. It is shown that if c is a P.V. number, then $(\mu_{c,q,r})^{t*}$ is continuous-singular for all $t > 0$, while, if c is the reciprocal of a P.S. number, then there is $t_1 = t_1(c,q,r) \in (0,\infty)$ such that $(\mu_{c,q,r})^{t*}$ is continuous-singular for all $t \in (0,t_1)$ and absolutely continuous for all $t \in (t_1,\infty)$. Thus the present paper provides a new class of Lévy processes with a remarkable time evolution in distribution. See Section 27 in Sato [21] and Watanabe's survey [24] for such time evolution. We emphasize that here the distribution $\mu_{c,q,r}$ arises naturally as the stationary distribution of a generalized Ornstein–Uhlenbeck process.

The case of $\{N_t\}$ and $\{Y_t\}$ being independent (i.e., $r = 0$) is of special interest. The properties of $\mu_{c,q,0}$ are included in the results of Section 3 mentioned above. As explicit examples, $\mu_{e,q,0}$ with $c = e$ is continuous-singular if $q \leq 1 - \log 2 \approx 0.30685$; $\mu_{c,1/2,0}$ with $q = 1/2$ is continuous-singular if $c > 4$. We can prove more results for $\mu_{c,q,0}$ than for general $\mu_{c,q,r}$, since the Lévy measure of $\mu_{c,q,0}$ is increasing with respect to q. Thus, for any $c > 1$, there exists q_1 with $0 < q_1 \leq 1$ such that $\mu_{c,q,0}$ is continuous-singular for all $q \in (0,q_1)$ and absolutely continuous for all $q \in (q_1,1)$. It will be shown that $q_1 = 1$ for any P.V. number $c > 1$ and that $q_1 < 1$ whenever c is the reciprocal of a P.S. number, so that $q_1 < 1$ for Lebesgue almost all $c > 1$.

Throughout the paper, the set of all positive integers will be denoted by $\mathbb{N} = \{1,2,3,\ldots\}$, while we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The set of integers is denoted by $\mathbb{Z}$. The Dirac measure at a point x will be denoted by δ_x. For general definitions and properties regarding Lévy processes and infinitely divisible distributions, we refer to Sato [21].

2. Necessary and sufficient conditions for infinite divisibility. Suppose that $\{(N_t, Y_t), t \geq 0\}$ is a bivariate Lévy process such that $\{N_t\}$ is a Poisson process with parameter $a > 0$ and $\{Y_t\}$ is a Poisson process with parameter $b > 0$. It then follows easily that $\{(N_t, Y_t)\}$ has no Gaussian part, no drift, and a Lévy measure $\nu_{(N,Y)}$ concentrated on the set $\{(1,0), (0,1), (1,1)\}$, consisting of three points (e.g., [21], Proposition 11.10). Denote

$$u := \nu_{(N,Y)}(\{(1,0)\}), \qquad v := \nu_{(N,Y)}(\{(0,1)\}), \qquad w := \nu_{(N,Y)}(\{(1,1)\}).$$

Then $u, v, w \geq 0$, $u + w = a$ and $v + w = b$. Let

$$p := \frac{u}{u+v+w}, \qquad q := \frac{v}{u+v+w}, \qquad r := \frac{w}{u+v+w},$$

so that $p, q, r \in [0,1]$, $p + q + r = 1$, $p + r > 0$ and $q + r > 0$. These give the normalized Lévy measure on the three points. The two processes $\{N_t\}$ and $\{Y_t\}$ are independent if and only if $r = 0$. If $r = 1$, then $N_t = Y_t$ for all t with probability one. So we call r the *dependence parameter* of $\{(N_t, Y_t)\}$. The law μ in (1.2) depends on c, u, v and w. But it will turn out (Proposition 2.1) that μ depends only on c, q and r. Thus, for $c > 1$ denote

$$\mu_{c,q,r} := \mathcal{L}\left(\int_0^{\infty-} c^{-N_{s-}}\, dY_s\right). \tag{2.1}$$

If $r = 1$, then

$$\int_0^{\infty-} c^{-N_{s-}}\, dY_s = \int_0^{\infty-} c^{-N_{s-}}\, dN_s = \sum_{j=0}^{\infty} c^{-j} = \frac{c}{c-1},$$

which is degenerate to a constant. So, from now on, we assume that $p + q > 0$ in addition to the above-mentioned conditions $p + r > 0$ and $q + r > 0$. That is, $p, q, r < 1$. In this section we are interested in whether $\mu_{c,q,r}$ is infinitely divisible or not. It is also of interest whether the symmetrization $(\mu_{c,q,r})^{\mathrm{sym}}$ of $\mu_{c,q,r}$ is infinitely divisible or not. Recall that the symmetrization μ^{sym} of a distribution μ is defined to be the distribution with characteristic function $|\widehat{\mu}(z)|^2$. Infinite divisibility of μ implies that of μ^{sym}, but the converse is not true, as is mentioned in the Introduction.

We define $\rho_{q,r}$ in the following way: If $q > 0$, denote by σ_q a geometric distribution with parameter $1 - q$, that is, $\sigma_q(\{k\}) = (1-q)q^k$ for $k = 0, 1, \ldots$ and denote

$$\rho = \rho_{q,r} := (1 + r/q)\sigma_q - (r/q)\delta_0, \tag{2.2}$$

so that $\rho_{q,r}$ is a probability distribution concentrated on $\mathbb{N}_0$ with

$$\rho_{q,r}(\{0\}) = (1 + r/q)(1-q) - (r/q) = p; \tag{2.3}$$

if $q = 0$, let $\rho_{0,r}$ be a Bernoulli distribution with parameter $r \in (0,1)$, that is,

$$\rho_{0,r}(\{1\}) = 1 - \rho_{0,r}(\{0\}) = r. \tag{2.4}$$

PROPOSITION 2.1. *We have*

$$\widehat{\mu}_{c,q,r}(z) = \widehat{\rho}_{q,r}(z)\widehat{\mu}_{c,q,r}(c^{-1}z), \qquad z \in \mathbb{R}. \tag{2.5}$$

In particular, $\mu_{c,q,r}$ is c^{-1}-decomposable and determined by c, q and r.

PROOF. As is explained in the Introduction, we have only to show that $\mathcal{L}(Y_T) = \rho_{q,r}$, where T is the time of the first jump of $\{N_t\}$, that is, the time of the first jump of $\{(N_t, Y_t)\}$ with size in $\{(1,0), (1,1)\}$. Let S_i be the size of the ith jump of $\{(N_t, Y_t)\}$. Then we have for $k \geq 1$

$$Y_T = k \iff [S_1 = \cdots = S_{k-1} = (0,1), S_k = (1,1)]$$
$$\text{or}$$
$$[S_1 = \cdots = S_k = (0,1), S_{k+1} = (1,0)],$$

as well as

$$Y_T = 0 \iff S_1 = (1,0).$$

Since

$$P[S_i = (1,0)] = p, \qquad P[S_i = (0,1)] = q, \qquad P[S_i = (1,1)] = r,$$

it follows that $P(Y_T = 0) = p$ and, for $k \geq 1$, $P(Y_T = k) = q^{k-1}r + q^k p$. From this it follows easily that $\mathcal{L}(Y_T) = \rho_{q,r}$ for $q > 0$, while it is a Bernoulli distribution with parameter r for $q = 0$. $\square$

We can now formulate criteria when $\rho_{q,r}$ and $\mu_{c,q,r}$ and their symmetrizations are infinitely divisible. As is seen in (1.4), infinite divisibility of $\rho_{q,r}$ implies that of $\mu_{c,q,r}$. Similarly, infinite divisibility of $(\rho_{q,r})^{\mathrm{sym}}$ implies that of $(\mu_{c,q,r})^{\mathrm{sym}}$. The converse of these two implications is by no means clear, as we know Niedbalska-Rajba's example mentioned in the Introduction. However the following theorem will say that the converse is true for $\mu_{c,q,r}$ and $\rho_{q,r}$ and for $(\mu_{c,q,r})^{\mathrm{sym}}$ and $(\rho_{q,r})^{\mathrm{sym}}$. Thus infinite divisibility of $\mu_{c,q,r}$ does not depend on c. Another remarkable consequence is that $(\mu_{c,q,r})^{\mathrm{sym}}$ can be infinitely divisible without $\mu_{c,q,r}$ being infinitely divisible and that $(\rho_{q,r})^{\mathrm{sym}}$ can be infinitely divisible without $\rho_{q,r}$ being infinitely divisible.

THEOREM 2.2. *Assume that the parameters p, q, r satisfy $p, q, r < 1$. Let $c > 1$. For $\mu_{c,q,r}$ defined in (2.1) and $\rho_{q,r}$ in (2.2) and (2.4), the following hold true*:

(a) *If $p = 0$, then $\rho_{q,r}$ and $\mu_{c,q,r}$ are infinitely divisible.*

(b) *If $p > 0$ and $q > 0$, then the following conditions are equivalent*:

  (i) *$\mu_{c,q,r}$ is infinitely divisible.*

  (ii) *$\rho_{q,r}$ is infinitely divisible.*

(iii) $r \le pq$.

(c) *If $p > 0$, $q > 0$ and $r > pq$, then the following conditions are equivalent*:

(i) $(\mu_{c,q,r})^{\mathrm{sym}}$ *is infinitely divisible.*

(ii) $(\rho_{q,r})^{\mathrm{sym}}$ *is infinitely divisible.*

(iii) $p \le qr$.

(d) *If $q = 0$, then none of $\rho_{q,r}$, $\mu_{c,q,r}$, $(\rho_{q,r})^{\mathrm{sym}}$ and $(\mu_{c,q,r})^{\mathrm{sym}}$ is infinitely divisible.*

In the proof, we will first settle the question of infinite divisibility of $\rho_{q,r}$ and $(\rho_{q,r})^{\mathrm{sym}}$.

LEMMA 2.3. *Assume $q > 0$ and let $\rho = \rho_{q,r}$. Then the following hold true*:

(a) *If $r \le pq$, or if $p = 0$, then ρ is infinitely divisible.*

(b) *If $r > pq$ and $p > 0$, then ρ is not infinitely divisible.*

(c) *Assume that $r > pq$ and $p > 0$. Then ρ^{sym} is infinitely divisible if and only if*

$$p \le qr. \tag{2.6}$$

We remark that if $0 \le \alpha \le 1$, then $(1-\alpha)\sigma_q + \alpha\delta_0$ is infinitely divisible, since convex combinations of two geometric distributions are infinitely divisible (see pages 379–380 in Steutel and van Harn [22]), and the Dirac measure δ_0 is a limit of geometric distributions. Assertions (a) and (b) show to what extent this fact can be generalized to negative α.

PROOF OF LEMMA 2.3. Since $\widehat{\sigma}_q(z) = (1-q)/(1-qe^{iz})$, we have

$$\widehat{\rho}(z) = \frac{p + re^{iz}}{1 - qe^{iz}}, \qquad z \in \mathbb{R}. \tag{2.7}$$

(a) If $p = 0$, then $\rho(\{0\}) = 0$ and $\rho(\{k\}) = (1-q)q^{k-1}$ for $k = 1, 2, \dots$ and thus ρ is a geometric distribution translated by 1, hence infinitely divisible. So assume that $r \le pq$. Then $p > 0$, recalling that $p + r > 0$. Since $p = (1-q)/(1 + r/p)$, it follows from (2.7) that

$$\widehat{\rho}(z) = \exp\left[\log(1-q) - \log\left(1 + \frac{r}{p}\right) + \log\left(1 + \frac{r}{p}e^{iz}\right) - \log(1 - qe^{iz})\right].$$

Hence

$$\widehat{\rho}(z) = \exp\left[\sum_{k=1}^{\infty} (e^{ikz} - 1)\frac{q^k}{k}\left(1 - \left(-\frac{r}{pq}\right)^k\right)\right]. \tag{2.8}$$

Recall that $r/(pq) \le 1$. It follows that ρ is infinitely divisible with Lévy measure $\nu_\rho(\{k\}) = k^{-1}q^k(1 - (-r/(pq))^k)$, $k = 1, 2, \dots$, and drift 0.

(b) Now assume that $r > pq$ and $p > 0$. By Katti's criterion ([12] or Corollary 51.2 of [21]), a distribution $\sum_{n=0}^{\infty} p_n \delta_n$ with $p_0 > 0$ is infinitely divisible if and only if there are $q_n \geq 0$, $n = 1, 2, \ldots$, such that

$$np_n = \sum_{k=1}^{n} kq_k p_{n-k}, \qquad n = 1, 2, \ldots.$$

In fact, the equations above determine q_n, $n = 1, 2, \ldots$, successively in a unique way; infinite divisibility of $\sum_{n=0}^{\infty} p_n \delta_n$ is equivalent to nonnegativity of all q_n. Now let $p_n = \rho(\{n\})$. The first two equations are $p_1 = q_1 p_0$ and $2p_2 = q_1 p_1 + 2q_2 p_0$. Hence $q_1 = p_1/p_0 > 0$, but

$$q_2 = \frac{2p_2 - q_1 p_1}{2p_0} = \frac{(1 + r/q)(1 - q)q^2}{2p^2}[1 - q - (r/q)(1 + q)] < 0,$$

since $r > pq$. This shows that ρ is not infinitely divisible.

(c) Assume again that $r > pq$ and $p > 0$. From (2.7) it can be seen that $\widehat{\rho}$ will have a real zero if and only if $p = r$. In that case, $|\widehat{\rho}|^2$ will also have a real zero, and hence ρ^{sym} cannot be infinite divisible in agreement with the fact that (2.6) is violated for $p = r$. So in the following we assume that $p \neq r$. From (2.7) we have

$$\log(|\widehat{\rho}(z)|^2) = \log(p^2 + 2pr\cos z + r^2) - \log(1 - 2q\cos z + q^2).$$

Write

$$A = \frac{2pr}{p^2 + r^2}, \qquad B = \frac{2q}{1 + q^2}, \qquad C = \frac{p^2 + r^2}{1 + q^2}.$$

Then $0 < A < 1$, $0 < B < 1$, and $C > 0$ (recall that $0 < q < 1$ and $p \neq r$) and we obtain

$$\begin{aligned}
\log(|\widehat{\rho}(z)|^2) &= \log C + \log(1 + A\cos z) - \log(1 - B\cos z) \\
&= \log C - \sum_{k=1}^{\infty} k^{-1}(-A)^k \cos^k z + \sum_{k=1}^{\infty} k^{-1} B^k \cos^k z \\
&= \log C + \sum_{k=1}^{\infty} k^{-1} 2^{-k} \big(-(-A)^k + B^k\big) \sum_{l=0}^{k} \binom{k}{l} \cos(k - 2l)z,
\end{aligned}$$

since $\cos^k z = 2^{-k} \sum_{l=0}^{k} \binom{k}{l} \cos(k - 2l)z$. Letting $z = 0$, we represent $\log C$ by A and B and get

$$\log(|\widehat{\rho}(z)|^2) = \sum_{k=1}^{\infty} D_k \sum_{l=0}^{k} \binom{k}{l} \big(\cos(k - 2l)z - 1\big),$$

where

$$D_k = k^{-1} 2^{-k} \big(-(-A)^k + B^k\big). \tag{2.9}$$

Then we get, with $\lfloor (k-1)/2 \rfloor$ denoting the largest integer not exceeding $(k-1)/2$,

$$\begin{aligned}\log(|\widehat{\rho}(z)|^2) &= 2\sum_{k=1}^{\infty} D_k \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{l} \bigl(\cos(k-2l)z - 1\bigr) \\ &= 2\sum_{k=1}^{\infty} D_k \sum_{m} \binom{k}{(k-m)/2} (\cos mz - 1),\end{aligned}$$

where m runs over $k, k-2, \dots, 3, 1$, if k is odd ≥ 1 and over $k, k-2, \dots, 4, 2$, if k is even ≥ 2. Since $\sum_{k=1}^{\infty} 2^k |D_k| \leq \sum_{k=1}^{\infty} k^{-1}(A^k + B^k) < \infty$, we can change the order of summation and obtain

$$\log(|\widehat{\rho}(z)|^2) = 2\sum_{m=1}^{\infty} E_m (\cos mz - 1) \tag{2.10}$$

with

$$E_m = \sum_{h=0}^{\infty} D_{m+2h} \binom{m+2h}{h}. \tag{2.11}$$

This means that

$$\log(|\widehat{\rho}(z)|^2) = \int_{\mathbb{R}} \bigl(e^{ixz} - 1 - ixz 1_{(-1,1)}(x)\bigr)\nu(dx), \tag{2.12}$$

where ν is the symmetric signed measure

$$\nu = \sum_{m=1}^{\infty} E_m (\delta_m + \delta_{-m}). \tag{2.13}$$

Let $F = r/(pq)$. Then $F > 1$. A simple calculation then shows that $A \leq B$ if and only if $F - 1 \leq q^2(F^2 - F)$, which is equivalent to $1 \leq q^2 F$, that is, (2.6). Now, if (2.6) holds, then $A \leq B$ and hence $D_k \geq 0$ for all k, which implies $E_m \geq 0$ for all m and ρ^{sym} is infinitely divisible with the Lévy–Khintchine representation (2.12) with (2.13). If (2.6) does not hold, then $A > B$, $D_k < 0$ for all even k, and $E_m < 0$ for all even m, which implies, by (2.12) and (2.13), that ρ^{sym} is not infinitely divisible (see Exercise 12.3 of [21]). $\square$

PROOF OF THEOREM 2.2. Write $\mu = \mu_{c,q,r}$ and $\rho = \rho_{q,r}$. (a) Suppose $p = 0$. Then ρ is infinitely divisible by Lemma 2.3, and hence so is μ by (2.5).

(b) Suppose that $p, q > 0$. Under these conditions, the equivalence of (ii) and (iii) follows from Lemma 2.3. Further, (ii) implies (i) by (2.5), so that it remains to show that (i) implies (iii). For that, suppose that $r > pq$, and in order to show that μ is not infinitely divisible, we will distinguish three cases: $p = r$, $p > r$ and $p < r$. The first case is easy, but in the second and third cases we have to use rather involved arguments resorting to different conditions that guarantee noninfinite-divisibility.

Case 1: Suppose that $p = r$. Then $\widehat{\rho}$ will have a real zero as argued in the proof of Lemma 2.3(c). By (2.5), $\widehat{\mu}$ will also have a real zero, so that μ cannot be infinitely divisible.

Case 2: Suppose that $p > r$. Then $\widehat{\rho}$ can be expressed as in (2.8) with the same derivation. Together with (2.5) and (1.4) this implies

$$\widehat{\mu}(z) = \exp\left[\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(e^{imc^{-n}z} - 1)\frac{1}{m}(q^m - (-r/p)^m)\right], \qquad z \in \mathbb{R}. \tag{2.14}$$

Absolute convergence of this double series follows from $c > 1$, $q < 1$ and $r/p < 1$. Define the real numbers a_m, $m \in \mathbb{N}$, and the signed measure ν by

$$a_m := \frac{1}{m}(q^m - (-r/p)^m) \quad \text{and} \quad \nu := \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_m \delta_{c^{-n}m}.$$

It follows that $\widehat{\mu}$ in (2.14) has the same form as the Lévy–Khintchine representation with the signed measure ν in place of a Lévy measure, so that infinite divisibility of μ is equivalent to the signed measure ν having negative part 0; see Exercise 12.3 in [21]. Thus, to show that μ is not infinitely divisible, we will show that there is a point x such that $\nu(\{x\}) < 0$. Since $r/p > q$, it follows that $a_m < 0$ if m is even and that $a_m > 0$ if m is odd. If c^k is irrational for all $k \in \mathbb{N}$, then the points $c^{-n}m$ with $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$ are distinct, which implies $\nu(\{c^{-n}m\}) < 0$ for all even m and μ is not infinitely divisible.

Suppose that c^k is rational for some $k \in \mathbb{N}$. Let k_0 be the smallest such k and write $c^{k_0} = \alpha/\beta$ with $\alpha, \beta \in \mathbb{N}$ such that α and β have no common divisor. Let f be the largest $t \in \mathbb{N}_0$ such that 2^t divides β. Let m be even. Denote

$$G_m := \{(n', m') \in \mathbb{N}_0 \times \mathbb{N} : c^{-n'}m' = m,\ m' \text{ odd}\},$$
$$H_m := \{(n', m') \in \mathbb{N}_0 \times \mathbb{N} : c^{-n'}m' = m,\ m' \text{ even}\}.$$

Then

$$\nu(\{m\}) = \sum_{(n',m') \in G_m \cup H_m} a_{m'} \le a_m + \sum_{(n',m') \in G_m} a_{m'}. \tag{2.15}$$

We claim that the set G_m contains at most one element. To show this, let $(n', m') \in G_m$. Then $n' \neq 0$ and $c^{n'} = m'/m$ and thus $n' = lk_0$ for some $l \in \mathbb{N}$. Then $m'/m = (\alpha/\beta)^l$ and hence β^l divides m and m/β^l is odd. Thus $m = 2^{lf}m''$ with some odd integer m''. It follows that lf is determined by m. Hence l is determined by m and c. Hence n' is determined by m and c, which shows that G_m contains at most one element. It also follows that $f \ge 1$ whenever $G_m \neq \varnothing$ for some even m. If there is some even m such that $G_m = \varnothing$, then $\nu(\{m\}) < 0$ by (2.15), and we are done. So suppose from now on that $G_m \neq \varnothing$ for every even m. Let $m_j = 2^{jf}$ for $j = 1, 2, \ldots$. The argument above shows that the unique element (n'_j, m'_j) in G_{m_j}

is given by $n'_j = jk_0$ and $m'_j = c^{jk_0} m_j$. Noting that $0 < q < r/p < 1$ and $c > 1$, choose j so large that $q^{m_j} \le 2^{-1}(r/p)^{m_j}$ and $m'_j = m_j c^{jk_0} > 2m_j$. Then

$$a_{m_j} = \frac{1}{m_j}(q^{m_j} - (r/p)^{m_j}) \le -\frac{1}{2m_j}(r/p)^{m_j}$$

and

$$a_{m'_j} = \frac{1}{m'_j}(q^{m'_j} + (r/p)^{m'_j}) \le \frac{3}{2m'_j}(r/p)^{m'_j} < \frac{3}{4m_j}(r/p)^{2m_j}.$$

Thus

$$\nu(\{m_j\}) \le a_{m_j} + a_{m'_j} \le \frac{1}{2m_j}(r/p)^{m_j}\bigl(-1 + (3/2)(r/p)^{m_j}\bigr) < 0$$

for large enough j, showing that μ is not infinitely divisible under the conditions of Case 2.

Case 3: Suppose that $p < r$ and, by way of contradiction, assume that μ is infinitely divisible. Denote by $L_\mu(\theta) = \int_{\mathbb{R}} e^{-\theta x}\mu(dx)$, $\theta \ge 0$, the Laplace transform of μ. Then $L_\mu(\theta) = e^{-\varphi(\theta)}$ where φ has a completely monotone derivative $\psi(\theta)$ on $(0,\infty)$, that is, $(-1)^n\psi^{(n)}(\theta) \ge 0$ on $(0,\infty)$ for $n = 0, 1, \ldots$ (see Feller [7], page 450). By (2.5) and (2.7) we have

$$\varphi(\theta) = -\log L_\mu(\theta) = -\sum_{n=0}^{\infty} \log \frac{p + r f_n(\theta)}{1 - q f_n(\theta)}, \tag{2.16}$$

where $f_0(\theta) = e^{-\theta}$ and $f_n(\theta) = \exp(-c^{-n}\theta) = f_0(c^{-n}\theta)$, $n = 1, 2, \ldots$. Convergence of the summation in (2.16) is easily established. Since $\psi = \frac{d}{d\theta}\varphi$ is completely monotone, so is $\theta \mapsto c^{-1}\psi(c^{-1}\theta) = \frac{d}{d\theta}(\varphi(c^{-1}\theta))$. Consider the function

$$\xi(\theta) = \frac{p}{p + re^{-\theta}}, \qquad \theta \in (0,\infty).$$

Then $1/(1 - qf_0(\theta)) - \xi(\theta)$ is the difference of two completely monotone functions, because

$$\frac{d}{d\theta}\bigl(\varphi(\theta) - \varphi(c^{-1}\theta)\bigr) = \frac{d}{d\theta}\left(-\log \frac{p + rf_0(\theta)}{1 - qf_0(\theta)}\right) = \frac{1}{1 - qf_0(\theta)} - \xi(\theta).$$

Since $1/(1 - qf_0(\theta)) = \sum_{k=0}^{\infty} q^k e^{-k\theta}$ is completely monotone, $\xi(\theta)$ is itself the difference of two completely monotone functions. Applying Bernstein's theorem, there must exist a signed measure σ on $[0,\infty)$ such that $\int_{[0,\infty)} e^{-\theta x}|\sigma|(dx) < \infty$ and $\xi(\theta) = \int_{[0,\infty)} e^{-\theta x}\sigma(dx)$ for all $\theta \in (0,\infty)$. However, introducing the signed measure $\tau := \sum_{k=0}^{\infty}(-r/p)^k\delta_k$, we have

$$\xi(\theta) = \sum_{k=0}^{\infty}\left(-\frac{r}{p}e^{-\theta}\right)^k = \int_{[0,\infty)} e^{-\theta x}\tau(dx),$$

if $\theta > \theta_0 := \log(r/p)$. Thus $e^{-\theta_0 x}\sigma(dx)$ and $e^{-\theta_0 x}\tau(dx)$ have a common Laplace transform $\xi(\theta_0+\theta)$, $\theta > 0$. Now from the uniqueness theorem in Laplace transform theory (page 430 of Feller [7]) combined with the Hahn–Jordan decomposition of signed measures, it follows that $e^{-\theta_0 x}\sigma(dx) = e^{-\theta_0 x}\tau(dx)$, that is, $\sigma = \tau$. But $\int_{[0,\infty)} e^{-\theta x}|\tau|(dx) = \infty$ for $0 < \theta \le \theta_0$, contradicting the property of σ. This finishes the proof of (b).

(c) Suppose that $p, q > 0$ and that $r > pq$. The equivalence of (ii) and (iii) then follows from Lemma 2.3, and (ii) implies (i) by (2.5), so that it remains to show that (i) implies (iii). If $p = r$, then $|\widehat{\rho}|^2$ and hence $|\widehat{\mu}|^2$ have real zeros as shown in the proof of Lemma 2.3(c) and μ^{sym} is not infinitely divisible. Hence we can assume that $p \ne r$. With A, B, D_k and E_m as in the proof of Lemma 2.3(c), it follows from $|\widehat{\mu}(z)|^2 = \prod_{n=0}^{\infty} |\widehat{\rho}(c^{-n}z)|^2$ and (2.10) that

$$\log(|\widehat{\mu}(z)|^2) = 2\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} E_m\big(\cos(mc^{-n}z) - 1\big). \tag{2.17}$$

Since

$$\begin{aligned}
&2\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} |E_m||\cos(mc^{-n}z) - 1| \\
&\quad = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} |D_k| \sum_{l=0}^{k} \binom{k}{l} \big|\cos\big((k-2l)c^{-n}z\big) - 1\big| \\
&\quad \le \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} |D_k| 2^k (kc^{-n}z)^2 \le z^2 \sum_{n=0}^{\infty} c^{-2n} \sum_{k=1}^{\infty} k(A^k + B^k) < \infty,
\end{aligned}$$

we can consider the right-hand side of (2.17) as an integral with respect to a signed measure. Thus

$$\log(|\widehat{\mu}(z)|^2) = \int_{\mathbb{R}} \big(e^{ixz} - 1 - ixz1_{(-1,1)}(x)\big)\widetilde{\nu}(dx), \tag{2.18}$$

where $\widetilde{\nu}$ is the symmetric signed measure

$$\widetilde{\nu} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} E_m(\delta_{mc^{-n}} + \delta_{-mc^{-n}}). \tag{2.19}$$

Now suppose that $p > qr$. As observed in the proof of Lemma 2.3(c), this is equivalent to $A > B$. In order to show that μ is not infinitely divisible, we use Exercise 12.3 of [21] again. We need to show that $\widetilde{\nu}$ has a nontrivial negative part. Recall that $E_m > 0$ for all odd m and $E_m < 0$ for all even m. If c^k is irrational for all $k \in \mathbb{N}$, then $\widetilde{\nu}(\{m\}) = E_m < 0$ for even m. Hence, suppose that c^k is rational for some $k \in \mathbb{N}$. We first estimate E_m. Since $\binom{m+2h}{h} \le 2^{m+2h}$, it follows from (2.9) and (2.11) that

$$|E_m| \le \sum_{h=0}^{\infty} \frac{1}{m+2h} 2A^{m+2h} \le \frac{2A^m}{m(1-A^2)}, \qquad m \in \mathbb{N}. \tag{2.20}$$

Choose $\gamma \in (0,1)$ such that $A/\gamma < 1$, and choose $\alpha \in \mathbb{N}$ such that $(\alpha+1/2)/(\alpha+1) \geq \gamma$. By Stirling's formula, there exists a constant $d_1 > 0$ such that for every $m \in \mathbb{N}$,

$$\begin{aligned}\binom{m+2\alpha m}{\alpha m} &\geq d_1 \left(\frac{m+2\alpha m}{(m+\alpha m)\alpha m}\right)^{1/2} \frac{(m+2\alpha m)^{m+2\alpha m}}{(m+\alpha m)^{m+\alpha m}(\alpha m)^{\alpha m}} \\ &\geq \frac{d_1}{(\alpha m)^{1/2}} (2\gamma)^{m+2\alpha m}.\end{aligned}$$

Since $D_k < 0$ for every even k, we conclude

$$\begin{aligned}|E_m| &\geq |D_{m+2\alpha m}| \binom{m+2\alpha m}{\alpha m} \\ &\geq \frac{d_1}{(\alpha m)^{1/2}(m+2\alpha m)} (A\gamma)^{m+2\alpha m} \big(1-(B/A)^{m+2\alpha m}\big)\end{aligned} \tag{2.21}$$

for every even $m \geq 2$. For even m define G_m, H_m, k_0 and f as in the proof of (b)—Case 2. If $G_m = \varnothing$ for some even m, then $\widetilde{\nu}(\{m\}) \leq E_m < 0$ similarly to (2.15). So suppose that $G_m \neq \varnothing$ for all even $m \geq 2$. As seen in the proof of (b), this implies that G_m consists of a single element and that $f \geq 1$. Let $m = m_j = 2^{jf}$ with $j \in \mathbb{N}$, then the unique element (n'_j, m'_j) in G_{m_j} satisfies $m'_j/m_j = c^{jk_0}$. Recall that m'_j is odd by the definition of G_m. For large j, we then have $m'_j/2 > m_j + 2\alpha m_j$, and from (2.20) and (2.21) it follows that there exists some constant $d_2 > 0$ such that

$$\frac{E_{m'_j}}{|E_{m_j}|} \leq d_2 \sqrt{m'_j} (A/\gamma)^{m'_j/2} \to 0 \qquad \text{as } j \to \infty,$$

so that $\nu(\{m_j\}) \leq E_{m_j} + E_{m'_j} < E_{m_j}/2 < 0$ for large j, finishing the proof of (c).

(d) Suppose $q = 0$. By (2.4), $\rho = \rho_{0,r}$ is Bernoulli distributed with parameter r. Further, $\mu = \mu_{c,0,r}$ is the distribution of $\sum_{n=0}^{\infty} c^{-n} U_n$, where $\{U_n, n \in \mathbb{N}\}$ is an i.i.d. sequence with distribution ρ. The support of μ is then a subset of $[0, c/(c-1)]$. It follows that also ρ^{sym} and μ^{sym} have bounded support. Moreover none of them is degenerate to a Dirac measure. Hence they are not infinitely divisible. $\square$

EXAMPLE 2.4. (a) Let $p = q > 0$. Then $\rho_{q,r}$, $\mu_{c,q,r}$, $(\rho_{q,r})^{\mathrm{sym}}$ and $(\mu_{c,q,r})^{\mathrm{sym}}$ will all be infinitely divisible if $r \in [0, 3-2\sqrt{2}]$, and none of them is infinitely divisible if $r > 3-2\sqrt{2} \approx 0.17157$. Recall that r is the dependence parameter.

(b) Let $2p = q > 0$. Then $\rho_{q,r}$ and $\mu_{c,q,r}$ will be infinitely divisible for $r \in [0, (13-3\sqrt{17})/4]$ and fail to be infinitely divisible for $r > (13-3\sqrt{17})/4 \approx 0.15767$. On the other hand, $(\rho_{q,r})^{\mathrm{sym}}$ and $(\mu_{c,q,r})^{\mathrm{sym}}$ are infinitely divisible if and only if $r \in [0, (13-3\sqrt{17})/4] \cup [1/2, 1)$.

3. Continuous-singularity and absolute continuity. We continue to study the distribution

$$\mu_{c,q,r} = \mathcal{L}\left(\int_0^{\infty-} c^{-N_{s-}}\, dY_s\right)$$

defined by a process $\{(N_t, Y_t), t \geq 0\}$ and a constant $c > 1$ in Section 2. The parameters p, q and r with $p + q + r = 1$ are assumed to satisfy $p, q, r < 1$ and $p, q, r \geq 0$ throughout this section (see the first paragraph of Section 2). In this section continuity properties of $\mu_{c,q,r}$ are considered. Since $\mu_{c,q,r}$ is c^{-1}-decomposable and nondegenerate, it is either continuous-singular or absolutely continuous, as Wolfe's theorem [25] says. So our problem is to specify the continuous-singular case and the absolutely continuous case. To get complete criteria for the two cases is a difficult problem, far from being achieved.

We use two classes of numbers, namely Pisot–Vijayaraghavan (P.V.) numbers (sometimes called Pisot numbers) and Peres–Solomyak (P.S.) numbers. A number $c > 1$ is called a *P.V. number* if there exists a polynomial $F(x)$ with integer coefficients with leading coefficient 1 such that c is a simple root of $F(x)$ and all other roots have a modulus of less than 1. Every positive integer greater than 1 is a P.V. number, but also $(1 + \sqrt{5})/2$ and the unique real root of $x^3 - x - 1 = 0$ are nontrivial examples. There exist countably infinitely many P.V. numbers which are not integers. See Peres, Schlag and Solomyak [19] for related information. On the other hand, following Watanabe [23], we call c^{-1} a *P.S. number* if $c > 1$ and if there are $p_0 \in (1/2, 1)$ and $k \in \mathbb{N}$ such that the kth power of the characteristic function of the distribution of $\sum_{n=0}^{\infty} c^{-n} U_n$, where $\{U_n\}$ is Bernoulli i.i.d. with $P[U_n = 0] = 1 - P[U_n = 1] = p_0$, is integrable. Watanabe [23] points out that the paper [20] of Peres and Solomyak contains the proof that the set of P.S. numbers in the interval (0, 1) has Lebesgue measure 1. However, according to [23], an explicit example of a P.S. number is not known so far. As follows from the results of [23], the set of P.V. numbers and the set of reciprocals of P.S. numbers are disjoint.

THEOREM 3.1. *Assume that c is a P.V. number and that $q > 0$. Then $\mu_{c,q,r}$ is continuous-singular.*

Recall that the assumption $q > 0$ merely excludes the case of infinite Bernoulli convolutions.

PROOF OF THEOREM 3.1. Write $\mu = \mu_{c,q,r}$. The following proof of continuous-singularity of μ is based on an idea of Erdős [5]. It is enough to show that it is not absolutely continuous. Thus, by virtue of the Riemann–Lebesgue theorem, it is enough to find a sequence $z_k \to \infty$ such that

$$\limsup_{k\to\infty} |\widehat{\mu}(z_k)| > 0.$$

By the definition of a P.V. number, there is a polynomial $F(x) = x^N + a_{N-1}x^{N-1} + \cdots + a_1 x + a_0$ such that $a_{N-1}, \ldots, a_0 \in \mathbb{Z}$, $F(c) = 0$, and the totality $\{\alpha_1, \ldots, \alpha_N\}$ of roots of $F(x)$ satisfies $\alpha_1 = c$ and $|\alpha_j| < 1$ for $2 \le j \le N$. Choose $z_k = 2\pi c^k$. Now we divide the proof into three cases: (Case 1) $p > 0$ and $r \le pq$; (Case 2) $p = 0$; (Case 3) $p > 0$ and $r > pq$. Recall that $q > 0$ is always assumed.

Case 1: As in the proofs of Theorem 2.2(b)—Case 2, we have

$$\widehat{\mu}(z) = \exp\left[\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(e^{imc^{-n}z} - 1)a_m\right], \qquad z \in \mathbb{R}, \tag{3.1}$$

with

$$a_m = m^{-1}q^m\bigl(1 - (-r/(pq))^m\bigr) \ge 0. \tag{3.2}$$

The double series above is absolutely convergent. We then have

$$|\widehat{\mu}(z)| = \exp\left[-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\bigl(1 - \cos(mc^{-n}z)\bigr)a_m\right]. \tag{3.3}$$

Thus

$$|\widehat{\mu}(z_k)| = \exp\left[-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(1 - \cos(2\pi mc^{k-n}))a_m\right] = \exp\left[-\sum_{m=1}^{\infty}(S_m + R_m)a_m\right]$$

with

$$S_m = \sum_{n=0}^{k}(1 - \cos(2\pi mc^{k-n})), \qquad R_m = \sum_{n=k+1}^{\infty}(1 - \cos(2\pi mc^{k-n})).$$

Now

$$S_m = \sum_{n=0}^{k}(1 - \cos(2\pi mc^n)) = \sum_{n=0}^{k}\left(1 - \cos\left(2\pi m\sum_{j=2}^{N}{\alpha_j}^n\right)\right),$$

since $c^n = \sum_{j=1}^N {\alpha_j}^n - \sum_{j=2}^N {\alpha_j}^n$ and $\sum_{j=1}^N {\alpha_j}^n$ is an integer. The latter is a consequence of the symmetric function theorem in algebra (e.g., Lang [15], Section IV.6), implying that $\sum_{j=1}^N {\alpha_j}^n$, as a symmetric function of $\alpha_1, \ldots, \alpha_N$, can be expressed as a polynomial with integer coefficients in the elementary symmetric functions, which are integer valued themselves since F has integer coefficients with leading coefficient 1. Choose $0 < \delta < 1$ such that $|\alpha_j| < \delta$ for $j = 2, \ldots, N$. Then, with some constants C_1, C_2, C_3,

$$\begin{aligned} S_m &\le C_1\sum_{n=0}^{k}\left(m\sum_{j=2}^{N}{\alpha_j}^n\right)^2 \le C_2 m^2\sum_{n=0}^{k}\sum_{j=2}^{N}|\alpha_j|^{2n} \\ &\le C_3 m^2\sum_{n=0}^{k}\delta^{2n} \le C_3 m^2/(1 - \delta^2). \end{aligned}$$

Further, we have

$$R_m \le C_1 \sum_{n=1}^{\infty} (mc^{-n})^2 = C_1 m^2/(c^2-1).$$

Hence, it follows that

$$|\widehat{\mu}(z_k)| \ge \exp\left[-\sum_{m=1}^{\infty} a_m m^2 \left(\frac{C_3}{1-\delta^2} + \frac{C_1}{c^2-1}\right)\right].$$

This shows that $\limsup_{k\to\infty} |\widehat{\mu}(z_k)| > 0$, since $\sum_{m=1}^{\infty} a_m m^2 < \infty$.

Case 2: We have

$$|\widehat{\rho}_{q,r}(z)| = \exp\left[\sum_{m=1}^{\infty} (\cos mz - 1)\frac{q^m}{m}\right]$$

by the remark at the beginning of the proof of Lemma 2.3(a). Hence the situation is the same as in Case 1.

Case 3: Recall the proof of Lemma 2.3(c). We have

$$|\widehat{\rho}_{q,r}(z)|^2 = \exp\left[2\sum_{m=1}^{\infty} E_m(\cos mz - 1)\right]$$

with E_m of (2.11). Hence

$$|\widehat{\mu}(z)|^2 = \prod_{n=0}^{\infty} |\widehat{\rho}_{q,r}(c^{-n}z)|^2 \ge \exp\left[-2\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} E_m^+(1-\cos(mc^{-n}z))\right],$$

where $E_m^+ = \max\{E_m, 0\}$. We have $\sum_{m=1}^{\infty} E_m^+ m^2 < \infty$, since

$$\begin{aligned}\sum_{m=1}^{\infty} E_m^+ m^2 &\le \sum_{m=1}^{\infty}\sum_{h=0}^{\infty} m^2 |D_{m+2h}| \binom{m+2h}{h} \\ &= \sum_{m=1}^{\infty} \sum_{k-m \text{ even } \ge 0} m^2 |D_k| \binom{k}{(k-m)/2} \\ &\le \sum_{k=1}^{\infty} k^2 |D_k| \sum_{l=0}^{\lfloor (k-1)/2\rfloor} \binom{k}{l} \le \sum_{k=1}^{\infty} \frac{k}{2}(A^k + B^k) < \infty,\end{aligned}$$

noting that $\sum_{l=0}^{\lfloor (k-1)/2\rfloor} \binom{k}{l} \le 2^{k-1}$ and $|D_k| \le k^{-1}2^{-k}(A^k + B^k)$ with $0 < A < 1$ and $0 < B < 1$. Hence we obtain $\limsup_{k\to\infty} |\widehat{\mu}(z_k)|^2 > 0$ exactly in the same way as in Case 1. $\square$

THEOREM 3.2. *Assume that c^{-1} is a P.S. number. Then there exists $\varepsilon = \varepsilon(c) \in (0,1)$ such that $\mu_{c,q,r}$ is absolutely continuous with bounded continuous density whenever $p > 0$, $r \le pq$ and $q \ge 1-\varepsilon$, or whenever $p = 0$ and $q \ge 1-\varepsilon$. In particular, there exist constants $C_1 = C_1(c) > 0$ and $C_2 = C_2(c) > 0$ such that $\mu_{c,q,r}$ is absolutely continuous with bounded continuous density whenever $q \ge C_1 p \ge C_2 r$.*

Recall that Lebesgue almost all $c \in (1,\infty)$ are the reciprocals of P.S. numbers.

PROOF OF THEOREM 3.2. Let $\mu = \mu_{c,q,r}$. Let $p_0 \in (1/2, 1)$ and $k \in \mathbb{N}$ as in the definition of a P.S. number. The following proof was suggested by an argument of Watanabe [23]. Let $K := k|\log(2p_0 - 1)|/2$, which is positive. Then (2.4) of [23] tells us that

$$\int_{-\infty}^{\infty} \exp\left\{\alpha \sum_{n=0}^{\infty} \left(\cos(c^{-n}u) - 1\right)\right\} du < \infty \qquad \text{whenever } \alpha \ge K. \tag{3.4}$$

Under the condition that $p > 0$, $q > 0$ and $r \le pq$, we have (3.3) with a_m of (3.2). Let $\alpha_0 = \sum_{m=1}^{\infty} a_m$. Then it follows from Jensen's inequality that

$$\begin{aligned}
\int_{-\infty}^{\infty} |\widehat{\mu}(z)|\,dz &= 2\int_0^{\infty} \exp\left[\frac{1}{\alpha_0}\sum_{m=1}^{\infty} a_m\left(\alpha_0 \sum_{n=0}^{\infty}\left(\cos(mc^{-n}z) - 1\right)\right)\right] dz \\
&\le 2\int_0^{\infty} \left[\frac{1}{\alpha_0}\sum_{m=1}^{\infty} a_m \exp\left(\alpha_0 \sum_{n=0}^{\infty}\left(\cos(mc^{-n}z) - 1\right)\right)\right] dz \\
&= \frac{2}{\alpha_0}\left(\sum_{m=1}^{\infty}\frac{a_m}{m}\right)\int_0^{\infty} \exp\left(\alpha_0 \sum_{n=0}^{\infty}\left(\cos(c^{-n}u) - 1\right)\right) du.
\end{aligned}$$

The last integral is finite whenever $\alpha_0 \ge K$ by (3.4). We have $a_m \ge m^{-1}q^m$ for m odd, and it follows that α_0 tends to ∞ as $q \uparrow 1$. Thus there is $\varepsilon = \varepsilon(c)$ such that $\alpha_0 \ge K$ for all $q \ge 1-\varepsilon$. Hence μ has bounded continuous density whenever $p > 0$, $r \le pq$ and $q \ge 1-\varepsilon$. The case when $p = 0$ and $q \ge 1-\varepsilon$ follows similarly, with $a_m = m^{-1}q^m$ in the above calculations.

To see the second half of the theorem, suppose that $q \ge C_1 p \ge C_2 r$ with $C_1, C_2 > 0$. Then $p > 0$ since $p + r > 0$, and hence $q > 0$. Thus

$$q = \left(1 + \frac{p}{q} + \frac{r}{q}\right)^{-1} \ge \left(1 + \frac{p}{q} + \frac{C_1 p}{C_2 q}\right)^{-1} \ge \left(1 + \frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}.$$

Hence, $q \ge 1 - \varepsilon(c)$ if C_1 and C_2 are large enough. We also have

$$\frac{r}{pq} \le \frac{C_1}{C_2}\left(1 + \frac{1}{C_1} + \frac{1}{C_2}\right).$$

Hence $r/(pq) \le 1$ if C_1 is fixed and C_2 is large. Thus there are C_1 and C_2 such that $q \ge 1 - \varepsilon(c)$, $p > 0$ and $r/(pq) \le 1$ whenever $q \ge C_1 p \ge C_2 r$. $\square$

Now we use the *entropy* $H(\rho)$ of a discrete probability measure ρ on $\mathbb{R}$. Here discrete means that ρ is concentrated on a countable set. We define

$$H(\rho) := -\sum_{a \in C} \rho(\{a\}) \log \rho(\{a\}),$$

where C is the carrier (the set of points with positive mass) of ρ.

THEOREM 3.3. *Assume that* $q > 0$. *Then*

$$H(\rho_{q,r}) = (q+r)\left(\log \frac{1}{1-q} + \frac{1}{1-q} \log \frac{1}{q} - \log \frac{q+r}{q}\right) + p \log \frac{1}{p}, \tag{3.5}$$

where $p \log(1/p)$ *is understood to be zero for* $p = 0$. *The following are true*:

(a) *The Hausdorff dimension of* $\mu_{c,q,r}$ *is estimated as*

$$\dim(\mu_{c,q,r}) \le \frac{H(\rho_{q,r})}{\log c}. \tag{3.6}$$

(b) *For each* $c > 1$, *there exists a constant* $C_3 = C_3(c) > 0$ *such that* $\mu_{c,q,r}$ *is continuous-singular whenever* $\max\{q, r\} \le C_3 p$.

(c) *Fix* q *and* r. *Then there exists a constant* $C_4 = C_4(q, r) > 0$ *such that* $\mu_{c,q,r}$ *is continuous-singular whenever* $c \ge C_4$.

The estimate (3.6) is meaningful only when $H(\rho_{q,r})/\log c < 1$, as the Hausdorff dimension of any measure on the line is less than or equal to 1. In this case (3.6) not only tells the continuous-singularity of $\mu_{c,q,r}$, but also gives finer information on a set of full measure for $\mu_{c,q,r}$.

PROOF OF THEOREM 3.3. Recall that $\rho_{q,r}$ is defined by (2.2). The geometric distribution σ_q has entropy

$$H(\sigma_q) = -\log(1-q) - \frac{q}{1-q} \log q \tag{3.7}$$

and $H(\rho_{q,r})$ is readily calculated as

$$H(\rho_{q,r}) = (1 + r/q)[H(\sigma_q) + (1-q)\log(1-q) - q \log(1 + r/q)] - p \log p,$$

which shows (3.5).

(a) Applying the remarkable Theorem 2.2 of Watanabe [23] on c^{-1}-decomposable distributions, we obtain (3.6).

(b) Notice that, since μ is continuous-singular or absolutely continuous, it must be continuous-singular if its Hausdorff dimension is less than 1. Suppose that $\max\{q, r\} \le C_3 p$ with $C_3 > 0$. Then $p > 0$ and hence $p = (1 + q/p + r/p)^{-1} \ge$

$(1+2C_3)^{-1}$, which tends to 1 as $C_3 \to 0$. Hence $q \to 0$ and $r \to 0$ as $C_3 \to 0$. Thus

$$H(\rho_{q,r}) = (q+r)\left(\log\frac{1}{1-q} + \frac{q}{1-q}\log\frac{1}{q} + \log\frac{1}{q+r}\right) + p\log\frac{1}{p} \to 0$$

and hence

$$\sup\{H(\rho_{q,r}) : \max\{q,r\} \le C_3 p\} \to 0, \qquad C_3 \to 0.$$

This shows (b).

(c) For given q, r, take any $C_4 > \exp H(\rho_{q,r})$. Then the assertion follows from the estimate (3.6). □

It is known that the distributions of some Lévy processes have time evolution, that is, change their qualitative properties as time passes (see Chapters 5 and 10 of Sato [21] and Watanabe [24]). It was Watanabe [23] who showed that the distributions of some semi-self-decomposable processes have time evolution in continuous-singularity and absolute continuity. It is of interest that the Lévy process $\{Z_t, t \ge 0\}$ determined by the distribution $\mu_{c,q,r}$ when it is infinitely divisible gives an explicit example of time evolution of this sort as the following theorem shows. Note that $\mathcal{L}(Z_t) = (\mu_{c,q,r})^{t*}$ for $t \ge 0$.

THEOREM 3.4. *Assume that either $p > 0$, $q > 0$ and $r \le pq$ or $p = 0$. Write $\rho = \rho_{q,r}$ and $\mu = \mu_{c,q,r}$. Then the following are true*:

(a) *There are $t_1 = t_1(c,q,r)$ and $t_2 = t_2(c,q,r)$ with $0 < t_1 \le t_2 \le \infty$ such that μ^{t*} is continuous-singular for all $t \in (0, t_1)$, absolutely continuous without bounded continuous density for all $t \in (t_1, t_2)$ if $t_1 < t_2$, and absolutely continuous with bounded continuous density for all $t \in (t_2, \infty)$ if $t_2 < \infty$.*

(b) *If c is a P.V. number, then $t_1 = \infty$, that is, μ^{t*} is continuous-singular for all $t > 0$.*

(c) *If c^{-1} is a P.S. number, then $t_2 < \infty$. Thus $t_2 < \infty$ for Lebesgue almost all $c > 1$.*

(d) *The entropy $H(\rho^{t*})$ is a finite, continuous, strictly increasing function of $t \in [0, \infty)$, vanishing at $t = 0$, and*

$$\dim(\mu^{t*}) \le \frac{H(\rho^{t*})}{\log c} \qquad \textit{for all } t \ge 0. \tag{3.8}$$

(e) *Fix q, r and $t > 0$. Then μ^{t*} is continuous-singular for all sufficiently large $c > 1$.*

LEMMA 3.5. *If ρ is a distribution on $\mathbb{Z}$ with finite absolute moment of order $1 + \varepsilon$ for some $\varepsilon > 0$, then its entropy $H(\rho)$ is finite.*

PROOF. Let $\rho = \sum_{m=-\infty}^{\infty} p_m \delta_m$. Then $\sum_{m=-\infty}^{\infty} |m|^{1+\varepsilon} p_m < \infty$. Hence there is a constant $C > 0$ such that $p_m \le C|m|^{-1-\varepsilon}$. The function $f(x) = x \log(1/x)$ is increasing for $0 \le x \le e^{-1}$. Hence,

$$H(\rho) \le \sum_{|m| \le m_0} p_m \log(1/p_m) + \sum_{|m| > m_0} C|m|^{-1-\varepsilon} \log((C|m|^{-1-\varepsilon})^{-1}) < \infty$$

with an appropriate choice of m_0. $\square$

PROOF OF THEOREM 3.4. Observe that under our assumption μ is infinitely divisible (see Theorem 2.2), so that μ^{t*} is definable for all $t \ge 0$. If $p > 0$ and $r \le pq$, it follows from (2.8) that ρ is a compound Poisson distribution, concentrated on $\mathbb{N}_0$, with finite second moment, since $\sum_{m=1}^{\infty} m^2 a_m < \infty$ for $a_m = \nu_\rho(\{m\})$. If $p = 0$, then ρ is a geometric distribution shifted by 1 (see the proof of Lemma 2.3). In both cases, $H(\rho) < \infty$ by Lemma 3.5. The property that μ is c^{-1}-decomposable is preserved to convolution powers, since (1.3) implies

$$\widehat{\mu^{t*}}(z) = \widehat{\rho^{t*}}(z)\widehat{\mu^{t*}}(c^{-1}z)$$

for any $t \ge 0$. Thus we have

$$\widehat{\mu^{t*}}(z) = \prod_{n=0}^{\infty} \widehat{\rho^{t*}}(c^{-n}z) = \exp\left(it\gamma_p^0 z + t\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(e^{imc^{-n}z} - 1)a_m\right), \tag{3.9}$$

where $\gamma_p^0 = 0$ for $p > 0$ and $\gamma_p^0 = \sum_{n=0}^{\infty} c^{-n} = c/(c-1)$ for $p = 0$. If $p > 0$, $q > 0$ and $r = 0$, then ρ is geometric. If $p = 0$, then the result is the same as in the case of ρ being a geometric distribution, since shifts do not change entropy, Hausdorff dimension and continuity properties. Hence, from now on, we exclude the case $p = 0$. Let us prove (d) first. We obtain the properties of $H(\rho^t)$ from Proposition 5.1 of Watanabe [23] or Exercise 29.24 of Sato [21]. Applying Theorem 2.2 of [23] to the law μ^{t*}, we get (3.8).

(a) It follows from (d) that μ^{t*} is continuous-singular for all sufficiently small $t > 0$. Let t_1 be the supremum of $t > 0$ for which μ^{t*} is continuous-singular. Then $0 < t_1 \le \infty$. Noting that if $t < t'$, then μ^{t*} is a convolution factor of $\mu^{t'*}$, use Lemma 27.1 of [21]. Thus μ^{t*} is continuous-singular for all $t \in (0, t_1)$ and absolutely continuous for all $t \in (t_1, \infty)$ if $t_1 < \infty$. Let t_2 be the infimum of $t > t_1$ for which μ^{t*} has bounded continuous density. Use the fact that a distribution is absolutely continuous with bounded continuous density if a convolution factor of it has this property. Then we obtain the assertion (a).

(b) The proof is the same as that of Theorem 3.1 with a_m replaced by ta_m.

(c) Let c^{-1} be a P.S. number. Then μ^{t*} is absolutely continuous with bounded continuous density for all sufficiently large t, which is shown in the same way as Theorem 3.2, or we can apply Theorem 2.1 of Watanabe [23].

(e) Obvious consequence of (d). $\square$

In the rest of this section we consider the case where $\{N_t\}$ and $\{Y_t\}$ are independent, that is, the case where $r = 0$. The following theorem is largely a repetition of Theorems 3.1–3.3 in this case but, since the Lévy measure of $\mu_{c,q,0}$ is increasing with respect to q, we obtain stronger statements.

THEOREM 3.6. *Under the condition that* $0 < q < 1$ *and* $r = 0$ (*i.e.*, $\{N_t\}$ *and* $\{Y_t\}$ *are independent*), *the following are true*:

(a) *For any* $c > 1$ *there are constants* $q_1 = q_1(c)$ *and* $q_2 = q_2(c)$ *satisfying* $0 < q_1 \le q_2 \le 1$ *with the following properties*: $\mu_{c,q,0}$ *is continuous-singular for all* $q \in (0, q_1)$, *absolutely continuous without bounded continuous density for all* $q \in (q_1, q_2)$ *if* $q_1 < q_2$, *and absolutely continuous with bounded continuous density for all* $q \in (q_2, 1)$ *if* $q_2 < 1$.

(b) *If* c *is a P.V. number, then* $q_1 = 1$, *that is*, $\mu_{c,q,0}$ *is continuous-singular for all* q.

(c) *If* c^{-1} *is a P.S. number, then* $q_2 < 1$. *Hence* $q_2 < 1$ *for Lebesgue almost all* $c > 1$.

(d) *The Hausdorff dimension of* $\mu_{c,q,0}$ *is estimated as*

$$\dim(\mu_{c,q,0}) \le \frac{H(\rho_{q,0})}{\log c}, \tag{3.10}$$

where

$$H(\rho_{q,0}) = -\log(1-q) - \frac{q}{1-q}\log q, \tag{3.11}$$

which is a finite, *continuous*, *strictly increasing function of* $q \in (0, 1)$ *and tends to* 0 *as* $q \downarrow 0$.

(e) *Fix* $c > 1$. *If* q *is so small that* $H(\rho_{q,0}) < \log c$, *then* $\mu_{c,q,0}$ *is continuous-singular. In particular, if*

$$0 < q < 1 - (\log 2/\log c), \tag{3.12}$$

then $\mu_{c,q,0}$ *is continuous-singular.*

(f) *Fix* $q > 0$. *If* c *is so large that* $c > \exp H(\rho_{q,0})$, *then* $\mu_{c,q,0}$ *is continuous-singular.*

PROOF. Let us begin with the proof of (d). The estimate (3.10) follows from (3.6) of Theorem 3.3. The expression (3.11) is exactly (3.7), since $\rho_{q,0} = \sigma_q$.

(e) and (f) These come from (d), as a distribution with Hausdorff dimension < 1 cannot be absolutely continuous. We get the sufficient condition (3.12), since

$$H(\rho_{q,0}) = \frac{1}{1-q}\left((1-q)\log\frac{1}{1-q} + q\log\frac{1}{q}\right) \le \frac{1}{1-q}\log 2$$

by strict concavity of the function $\log x$.

(a) Recall that $\mu_{c,q,0}$ has Lévy measure

$$\nu_{c,q,0} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{q^m}{m} \delta_{c^{-n}m}.$$

Hence, if $q < q'$, then $\mu_{c,q,0}$ is a convolution factor of $\mu_{c,q',0}$. Now the proof is obtained by the same argument as in the proof of (a) of Theorem 3.4.

(b) Consequence of Theorem 3.1.

(c) This follows from Theorem 3.2. $\square$

EXAMPLE 3.7. (a) In the case $c = e$, $\mu_{e,q,0}$ is continuous-singular if

$$q \le 1 - \log 2 \approx 0.30685.$$

This follows from (3.12) in Theorem 3.6.

(b) In the case $q = 1/2$, $\mu_{c,1/2,0}$ is continuous-singular if $c > 4$ since $H(\rho_{1/2,0}) = 2\log 2$, as (f) of Theorem 3.6 says.

In the independent case ($0 < q < 1$ and $r = 0$), the assumption in Theorem 3.4 is satisfied. So the assertions on time evolution of $\mu_{c,q,0}$ hold true as in that theorem. It is of interest to estimate $H(\rho_{q,0}{}^{t*})$ appearing in the right-hand side of (3.8).

PROPOSITION 3.8. *If* $0 < q < 1$, *then*

$$H(\rho_{q,0}{}^{t*}) \le t\left[\frac{1}{p}\left(1 + 2\log\frac{1}{p}\right) + \frac{q}{p}\log\frac{1}{t}\right] \qquad \text{for } 0 < t \le 1, \tag{3.13}$$

where $p = 1 - q$. *The right-hand side of* (3.13) *is a strictly increasing function of* $t \in (0, 1]$ *which tends to* 0 *as* $t \downarrow 0$.

PROOF. Write $\rho = \rho_{q,0}$. Since ρ equals geometric distribution σ_q with parameter p, the distribution ρ^{t*}, $t > 0$, is a negative binomial distribution with parameters t and p, that is,

$$\rho^{t*}(\{k\}) = \binom{-t}{k} p^t (-q)^k, \qquad k \in \mathbb{N}_0.$$

To estimate $H(\rho^{t*})$ from above, observe that $tp^t q^k/k \le \rho^{t*}(\{k\}) \le tq^k$ for $0 < t \le 1$ and $k \in \mathbb{N}$ so that

$$\begin{aligned} H(\rho^{t*}) &= -\sum_{k=0}^{\infty} \rho^{t*}(\{k\}) \log \rho^{t*}(\{k\}) \\ &\le -(\log p^t) + \sum_{k=1}^{\infty} tq^k \bigl(\log k - \log(tp^t) - k\log q\bigr) \\ &\le t\left[\log\frac{1}{p} + \frac{1}{p}\log\frac{1}{p} - \frac{q}{p}\log(tp^t) - \frac{q}{p^2}\log q\right], \end{aligned}$$

where we used $\sum_{k=1}^{\infty} kq^k = q/p^2$ and

$$\sum_{k=1}^{\infty} q^k \log k \leq \sum_{k=1}^{\infty} q^k \sum_{n=1}^{k} \frac{1}{n} = \frac{1}{p} \log \frac{1}{p},$$

compare Gradshteyn and Ryzhik [10], Formula 1.513.6. Recalling that $p^t \geq p$ since $t \leq 1$ and this can be further estimated to

$$H(\rho^{t*}) \leq t \left[\frac{2}{p} \log \frac{1}{p} + \frac{q}{p} \log \frac{1}{t} + \frac{q}{p^2} \log \frac{1}{q} \right].$$

Together with $(q/p)\log(1/q) = (q/p)\log(1 + p/q) \leq 1$, this gives (3.13). $\square$

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