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LARGE DEVIATIONS FOR RANDOM WALK IN A SPACE–TIME PRODUCT ENVIRONMENT[1]

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We consider random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}^d$ in a space–time product environment $\omega \in \Omega$. We take the point of view of the particle and focus on the environment Markov chain $(T_{n,X_n}\omega)_{n\geq 0}$ where T denotes the shift on Ω. Conditioned on the particle having asymptotic mean velocity equal to any given ξ, we show that the empirical process of the environment Markov chain converges to a stationary process μ_ξ^∞ under the averaged measure. When $d \geq 3$ and ξ is sufficiently close to the typical velocity, we prove that averaged and quenched large deviations are equivalent and when conditioned on the particle having asymptotic mean velocity ξ, the empirical process of the environment Markov chain converges to μ_ξ^∞ under the quenched measure as well. In this case, we show that μ_ξ^∞ is a stationary Markov process whose kernel is obtained from the original kernel by a Doob h-transform.

1. Introduction. Random walk in a random environment (RWRE) is one of the standard models in the study of random media. It aims to capture the essence of the motion of a particle in a disordered medium. Mathematically, it is a discrete time Markov chain on $\mathbb{Z}^d$ with random transition probabilities which are sampled from a given joint distribution and kept fixed throughout the walk. See Sznitman [11] or Zeitouni [13] for a survey.

Instead, if we assume that the transition probabilities at distinct states are i.i.d. and are freshly sampled at each time step, we get what is known as random walk in a space–time product environment. Here is the precise formulation: For each $n \in \mathbb{Z}$ and $x, y \in \mathbb{Z}^d$, we write $\pi_{n,n+1}(x, x+y)$ to denote the random probability of the particle being at position $x+y$ at time $n+1$ given it is at position x at time n. We refer to the random probability vector $\omega_{n,x} := (\pi_{n,n+1}(x, x+y))_{y\in\mathbb{Z}^d}$ as the environment at position x at time n and assume the environment $\omega := (\omega_{n,x})_{n\in\mathbb{Z},x\in\mathbb{Z}^d}$ to be an i.i.d. collection. The environments form a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ where $\mathcal{B}$ is the Borel σ-algebra on Ω and $\mathbb{P}$ is a product measure. For every $n \in \mathbb{Z}$, we write $\mathcal{B}_n^+$ and $\mathcal{B}_n^-$ to mean the σ-algebras generated by $(\omega_{m,x} : x \in \mathbb{Z}^d, m \geq n)$ and $(\omega_{m,x} : x \in \mathbb{Z}^d, m \leq n)$, respectively.

Given $\omega \in \Omega$, the Markov chain $(X_n)_{n\geq k}$ starting at position $x \in \mathbb{Z}^d$ at time $k \in \mathbb{Z}$ induces a measure $P_{k,x}^\omega$ on the space of paths, called the "quenched" measure.

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The semi-direct product $P_{k,x} := \mathbb{P} \times P^{\omega}_{k,x}$ is referred to as the "averaged" measure. Expectations with respect to $\mathbb{P}$, $P^{\omega}_{k,x}$ and $P_{k,x}$ are denoted by $\mathbb{E}$, $E^{\omega}_{k,x}$ and $E_{k,x}$, respectively. For convenience, we sometimes write Z_{n+1} to mean $X_{n+1} - X_n$.

We note that if $(X_n)_{n\geq 0}$ is random walk on $\mathbb{Z}^d$ in a space–time product environment, then $(n, X_n)_{n\geq 0}$ can be viewed as RWRE on $\mathbb{Z}^{d+1}$.

We set $U := \{z \in \mathbb{Z}^d : |z| = 1\}$. In order to provide short proofs, we assume that the walk is nearest-neighbor and the environment is uniformly elliptic, that is, $\mathbb{P}(\pi_{0,1}(0, z) > 0) = 0$ unless $z \in U$, and there exists a constant $c > 0$ such that $\mathbb{P}(\pi_{0,1}(0, z) \geq c) = 1$ for $z \in U$.

We define the shifts $(T_{m,y})_{m\in\mathbb{Z}, y\in\mathbb{Z}^d}$ on Ω by $(T_{m,y}\omega)_{n,x} = \omega_{n+m,x+y}$. Given a random path $(X_n)_{n\geq 0}$, we consider $(T_{n,X_n}\omega)_{n\geq 0}$ which is a Markov process with state space Ω. It is referred to as the "environment Markov chain" and its transition kernel is given by $\overline{\pi}(\omega, T_{1,z}\omega) := \pi_{0,1}(0, z)$ for every $z \in U$. What it does is simply observe the environment from the point of view of the particle. This is a standard approach in the study of random media. See, for example, De Masi et al. [2], Kipnis and Varadhan [5], Kozlov [6], Olla [7] or Papanicolaou and Varadhan [8].

By a generalization of a technique first given in Kozlov [6], Rassoul-Agha [9] shows that the environment Markov chain has a unique invariant measure that is absolutely continuous relative to $\mathbb{P}$ on every $\mathcal{B}^+_n$.

In Section 2, we focus on the averaged measure. The marginal of $P_{o,o}$ on paths is classical random walk with transition vector $(q(z))_{z\in U}$ given by $q(z) = \mathbb{E}[\pi_{0,1}(0, z)]$ for every $z \in U$. Therefore, the law of large numbers (LLN) for the mean velocity of the particle under $P_{o,o}$ is valid and the limiting velocity is $\xi_o := \sum_{z\in U} q(z)z$. The averaged large deviation principle (LDP) for the mean velocity of the particle is simply Cramér's theorem (see, e.g., Dembo and Zeitouni [3]) and the rate function I_a is the convex conjugate of the logarithm of the moment generating function ϕ, given by

$$\phi(\theta) = \sum_{z\in U} q(z)e^{\langle\theta,z\rangle}. \tag{1.1}$$

We set $\mathcal{D} := \{\xi \in \mathbb{R}^d : I_a(\xi) < \infty\}$. Given $\xi \in \mathrm{int}(\mathcal{D})$, we consider the event defined by the particle having asymptotic mean velocity ξ. If $\xi \neq \xi_o$, this is a rare event and the exponential rate of decay of its $P_{o,o}$-probability is given by $I_a(\xi) > 0$. Conditioned on this event, we expect the environment Markov chain to behave differently. Indeed, we show that the empirical process of the environment Markov chain under this conditioning converges to a stationary process uniquely determined by ξ. Here is how we accomplish this: We first give a definition.

DEFINITION 1. For every $\xi \in \mathrm{int}(\mathcal{D})$, we define a measure $\overline{\mu}^{\infty}_{\xi}$ on $\Omega \times U^{\mathbb{N}}$ in the following way: There exists a unique $\theta \in \mathbb{R}^d$ satisfying $\xi = \nabla \log \phi(\theta)$. For every N, M and $K \in \mathbb{N}$, we take any bounded function $f : \Omega \times U^{\mathbb{N}} \to \mathbb{R}$ such that $f(\cdot,(z_i)_{i\geq 1})$ is independent of $(z_i)_{i>K}$ and $\mathcal{B}^+_{-N}\cap\mathcal{B}^-_M$-measurable for each $(z_i)_{i\geq 1}$. Then

$$\begin{aligned}\int f\,d\overline{\mu}^\infty_\xi\\ := E_{o,o}\bigl[e^{\langle\theta,X_{N+M+K+1}\rangle-(N+M+K+1)\log\phi(\theta)}f(T_{N,X_N}\omega,(Z_{N+i})_{i\geq 1})\bigr]\end{aligned}\tag{1.2}$$

where X_i and $Z_i = X_i - X_{i-1}$ are as defined earlier.

In Proposition 1, we show that $\overline{\mu}^\infty_\xi$ is well defined. In Proposition 2, we show that $\overline{\mu}^\infty_\xi$ induces a stationary process μ^∞_ξ with values in Ω. Let us define the events that we use in the statement of the main theorem of Section 2.

DEFINITION 2. For every $\xi\in\operatorname{int}(\mathcal{D})$, N, M and $K\in\mathbb{N}$, f as in Definition 1, $\varepsilon>0$ and $n\in\mathbb{N}$, we define the event

$$A^\varepsilon_{\xi,n}(f) := \left\{\left|\frac{1}{n}\sum_{j=0}^{n-1} f(T_{j,X_j}\omega,(Z_{j+i})_{i\geq 1}) - \int f\,d\overline{\mu}^\infty_\xi\right| > \varepsilon\right\}.\tag{1.3}$$

Given $\delta>0$, we define the event

$$D^\delta_{\xi,n} := \left\{\left|\frac{X_n}{n}-\xi\right|\leq\delta\right\}.\tag{1.4}$$

Finally, we prove the following theorem.

THEOREM 1. *For every $\xi\in\operatorname{int}(\mathcal{D})$, N, M and $K\in\mathbb{N}$, f as in Definition 1 and $\varepsilon>0$, there exists $\delta_o>0$ such that for every $\delta>0$ with $\delta<\delta_o$*

$$\limsup_{n\to\infty}\frac{1}{n}\log P_{o,o}(A^\varepsilon_{\xi,n}(f)|D^\delta_{\xi,n})<0$$

where the events $A^\varepsilon_{\xi,n}(f)$ and $D^\delta_{\xi,n}$ are defined in (1.3) and (1.4), respectively.

In Section 3, we focus on the quenched measure. Varadhan [12] proves the quenched LDP for the mean velocity of the particle for the related model of RWRE. In our case, even though we can think of $(n,X_n)_{n\geq 0}$ as RWRE on $\mathbb{Z}^{d+1}$, Varadhan's result is not directly applicable since he assumes that the environment is uniformly elliptic, which we do not have in the "time" direction. However, one expects that a modification of his argument should work. Instead of taking this route, we develop an alternative technique and prove the following theorem.

THEOREM 2. *If $d\geq 3$, then there exists $\eta>0$ such that the quenched LDP for the mean velocity of the particle holds in the η-neighborhood of ξ_o and the rate function is identically equal to the rate function I_a of the averaged LDP in this neighborhood.*

REMARK 1. Theorem 2 is similar in flavor to the results of Flury [4], Song and Zhou [10] and Zygouras [14] on the related model of random walk with a random potential. For multidimensional random walk in a product environment, Varadhan [12] proves that the averaged LDP also holds and the corresponding rate function has the same zero set with the quenched rate function. However, it is not known whether the two rate functions agree on a neighborhood of the LLN velocity. The case of RWRE on $\mathbb{Z}$ is closely studied by Comets, Gantert and Zeitouni [1] who in particular show that there is no neighborhood of the LLN velocity on which the two rate functions agree.

Once again, conditioned on the particle having asymptotic mean velocity ξ, we can ask what the empirical process of the environment Markov chain converges to, but this time under the quenched measure. Whenever the quenched LDP for the mean velocity holds in a neighborhood of ξ with rate $I_a(\xi)$ at ξ—in particular when $d \geq 3$ and $|\xi - \xi_o| < \eta$—one expects that the answer should be again μ_ξ^∞. The following theorem is a result to this effect.

THEOREM 3. *For every $\xi \in \operatorname{int}(\mathcal{D})$, if the quenched LDP for the mean velocity holds in a neighborhood of ξ with rate $I_a(\xi)$ at ξ, then for every N, M and $K \in \mathbb{N}$, f as in Definition 1 and $\varepsilon > 0$, there exists $\delta_o > 0$ such that for every $\delta > 0$ with $\delta < \delta_o$, $\mathbb{P}$-a.s.*

$$\limsup_{n\to\infty} \frac{1}{n} \log P_{o,o}^{\omega}(A_{\xi,n}^{\varepsilon}(f) | D_{\xi,n}^{\delta}) < 0,$$

where the events $A_{\xi,n}^{\varepsilon}(f)$ and $D_{\xi,n}^{\delta}$ are defined in (1.3) and (1.4), respectively.

Finally, in Section 4, we reveal the structure of μ_ξ^∞.

THEOREM 4. *For $d \geq 3$ and $|\xi - \xi_o| < \eta$ with η as in Theorem 2, we let $\theta \in \mathbb{R}^d$ be the unique solution of $\xi = \nabla \log \phi(\theta)$. There exists a $\mathcal{B}_o^+$-measurable function $u^\theta > 0$ which satisfies $\int u^\theta \, d\mathbb{P} = 1$ and $\mathbb{P}$-a.s.*

$$u^\theta(\omega) = \sum_{z \in U} \overline{\pi}(\omega, T_{1,z}\omega) e^{\langle \theta, z \rangle - \log \phi(\theta)} u^\theta(T_{1,z}\omega).$$

We define a transformed kernel $\overline{\pi}^\theta$ on Ω by

$$\overline{\pi}^\theta(\omega, T_{1,z}\omega) = \overline{\pi}(\omega, T_{1,z}\omega) \frac{u^\theta(T_{1,z}\omega)}{u^\theta(\omega)} e^{\langle \theta, z \rangle - \log \phi(\theta)}.$$

μ_ξ^∞ is the unique stationary Markov process with transition kernel $\overline{\pi}^\theta$ and whose marginal μ_ξ^1 is absolutely continuous relative to $\mathbb{P}$ on every $\mathcal{B}_n^+$.

In other words, conditioned on the particle having asymptotic mean velocity ξ, the environment Markov chain chooses to switch from kernel $\overline{\pi}$ to kernel $\overline{\pi}^\theta$. The most economical tilt in terms of large deviations is given by a Doob h-transform.

2. Conditioning under the averaged measure. For any $n \in \mathbb{N}$ and $\theta \in \mathbb{R}^d$, since the environment is i.i.d.,

$$E_{o,o}\big[e^{\langle \theta, X_n\rangle}\big] = \phi(\theta)^n \tag{2.1}$$

with the notation in (1.1). By Cramér's theorem, the LDP for the mean velocity of the particle holds under $P_{o,o}$ with the rate function I_a given by

$$I_a(\xi) = \sup_{\theta'}\{\langle \theta', \xi\rangle - \log\phi(\theta')\} = \langle \theta, \xi\rangle - \log\phi(\theta), \tag{2.2}$$

where θ is the unique solution of $\xi = \nabla \log\phi(\theta)$. Due to our nearest-neighbor and ellipticity assumptions,

$$\mathcal{D} = \{\xi \in \mathbb{R}^d : I_a(\xi) < \infty\} = \{(\xi^1, \dots, \xi^d) \in \mathbb{R}^d : |\xi^1| + \cdots + |\xi^d| \le 1\}.$$

PROPOSITION 1. *For every $\xi \in \mathrm{int}(\mathcal{D})$, the measure $\overline{\mu}_\xi^\infty$ on $\Omega \times U^{\mathbb{N}}$, given in Definition 1, is well defined.*

PROOF. For every N, M and $K \in \mathbb{N}$, we consider any f as in Definition 1. We set $L := N + M + K + 1$. Since $f(\cdot, (z_i)_{i\ge1})$ is independent of $(z_i)_{i>K}$ and $\mathcal{B}_{-N}^+ \cap \mathcal{B}_M^-$-measurable for each $(z_i)_{i\ge1}$, we see that for every N', M' and $K' \in \mathbb{N}$ with $N \le N'$, $M \le M'$ and $K \le K'$, $f(\cdot, (z_i)_{i\ge1})$ is independent of $(z_i)_{i>K'}$ and $\mathcal{B}_{-N'}^+ \cap \mathcal{B}_{M'}^-$-measurable for each $(z_i)_{i\ge1}$ as well. So, we need to show that (1.2) does not change if we replace N by $N+1$, M by $M+1$, or K by $K+1$.

Let us start with the argument for N. We observe that

$$\begin{aligned}
&E_{o,o}\big[e^{\langle\theta, X_{L+1}\rangle - (L+1)\log\phi(\theta)} f(T_{N+1, X_{N+1}}\omega, (Z_{N+1+i})_{i\ge1})\big]\\
&\quad= \sum_x \mathbb{E}\big(E_{o,o}^\omega\big[e^{\langle\theta, X_1\rangle - \log\phi(\theta)}, X_1 = x\big] \qquad (2.3)\\
&\qquad\qquad \times E_{1,x}^\omega\big[e^{\langle\theta, X_{L+1}-X_1\rangle - L\log\phi(\theta)} f(T_{N+1,X_{N+1}}\omega, (Z_{N+1+i})_{i\ge1})\big]\big)\\
&\quad= \sum_x E_{o,o}\big[e^{\langle\theta, X_1\rangle - \log\phi(\theta)}, X_1 = x\big] \qquad (2.4)\\
&\qquad\qquad \times E_{1,x}\big[e^{\langle\theta, X_{L+1}-X_1\rangle - L\log\phi(\theta)} f(T_{N+1,X_{N+1}}\omega, (Z_{N+1+i})_{i\ge1})\big]\\
&\quad= \sum_x E_{o,o}\big[e^{\langle\theta, X_1\rangle - \log\phi(\theta)}, X_1 = x\big] \qquad (2.5)\\
&\qquad\qquad \times E_{o,o}\big[e^{\langle\theta, X_L\rangle - L\log\phi(\theta)} f(T_{N,X_N}\omega, (Z_{N+i})_{i\ge1})\big]\\
&\quad= E_{o,o}\big[e^{\langle\theta, X_L\rangle - L\log\phi(\theta)} f(T_{N,X_N}\omega, (Z_{N+i})_{i\ge1})\big]
\end{aligned}$$

holds. We note that each term of the sum in (2.3) is the $\mathbb{P}$-expectation of two random variables; the first one is $\mathcal{B}_0^-$-measurable and the second one is $\mathcal{B}_1^+$-measurable. We make use of this independence to obtain (2.4). We also note that we use the stationarity of $\mathbb{P}$ under shifts to obtain (2.5) from (2.4). Hence, (1.2) does not change if we replace N by $N+1$.

Similarly, if we replace M by $M+1$ in (1.2),

$$
\begin{aligned}
&E_{o,o}\big[e^{\langle\theta,X_{L+1}\rangle-(L+1)\log\phi(\theta)}f(T_{N,X_N}\omega,(Z_{N+i})_{i\geq1})\big]\\
&\quad=\sum_x \mathbb{E}\big(E^{\omega}_{o,o}\big[e^{\langle\theta,X_L\rangle-L\log\phi(\theta)}f(T_{N,X_N}\omega,(Z_{N+i})_{i\geq1}),X_L=x\big]\\
&\qquad\qquad\qquad\qquad\times E^{\omega}_{L,x}\big[e^{\langle\theta,X_{L+1}-X_L\rangle-\log\phi(\theta)}\big]\big)\\
&\quad=\sum_x E_{o,o}\big[e^{\langle\theta,X_L\rangle-L\log\phi(\theta)}f(T_{N,X_N}\omega,(Z_{N+i})_{i\geq1}),X_L=x\big]\\
&\qquad\quad\times E_{L,x}\big[e^{\langle\theta,X_{L+1}-X_L\rangle-\log\phi(\theta)}\big]\\
&\quad=E_{o,o}\big[e^{\langle\theta,X_L\rangle-L\log\phi(\theta)}f(T_{N,X_N}\omega,(Z_{N+i})_{i\geq1})\big]
\end{aligned}
\tag{2.6}
$$

where, once again, we use the fact that each term of the sum in (2.6) is the $\mathbb{P}$-expectation of two random variables; the first one is $\mathcal{B}^-_{L-1}$-measurable and the second one is $\mathcal{B}^+_L$-measurable.

We finally note that the argument for K is the same as the one for M. $\square$

PROPOSITION 2. *$\overline{\mu}^{\infty}_{\xi}$ induces a stationary process μ^{∞}_{ξ} with values in Ω.*

PROOF. We define $\bar{S}:\Omega\times U^{\mathbb{N}}\to\Omega\times U^{\mathbb{N}}$ by

$$\bar{S}:(\omega,(z_i)_{i\geq1})\mapsto(T_{1,z_1}\omega,(z_i)_{i\geq2})$$

and the projection map $\Psi:\Omega\times U^{\mathbb{N}}\to\Omega$ by $\Psi:(\omega,(z_i)_{i\geq1})\mapsto\omega$. Let us show that $\overline{\mu}^{\infty}_{\xi}$ is invariant under $\bar{S}$:

For every N,M and $K\in\mathbb{N}$, f as in Definition 1 and $(z_i)_{i\geq1}$, we see that $f\circ\bar{S}(\omega,(z_i)_{i\geq1})=f(T_{1,z_1}\omega,(z_i)_{i\geq2})$ is $\mathcal{B}^+_{-(N-1)}\cap\mathcal{B}^-_{M+1}$-measurable and independent of $(z_i)_{i>K+1}$. By definition,

$$
\begin{aligned}
&\int f\circ\bar{S}\,d\overline{\mu}^{\infty}_{\xi}\\
&\quad=E_{o,o}\big[e^{\langle\theta,X_{N+M+K+2}\rangle-(N+M+K+2)\log\phi(\theta)}\\
&\qquad\qquad\times f\circ\bar{S}\big(T_{N-1,X_{N-1}}\omega,\big(Z_{(N-1)+i}\big)_{i\geq1}\big)\big]\\
&\quad=E_{o,o}\big[e^{\langle\theta,X_{N+M+K+2}\rangle-(N+M+K+2)\log\phi(\theta)}f(T_{N,X_N}\omega,(Z_{N+i})_{i\geq1})\big]\\
&\quad=\int f\,d\overline{\mu}^{\infty}_{\xi}.
\end{aligned}
$$

Therefore, under $\overline{\mu}^{\infty}_{\xi}$, $(\Psi\circ\bar{S}^k(\cdot))_{k\geq0}$ extends to a stationary process with values in Ω, whose distribution we denote by μ^{∞}_{ξ}. $\square$

PROOF OF THEOREM 1. Under the hypotheses of the theorem, we take any $\varepsilon > 0$, $\delta > 0$ and recall (1.3) and (1.4). We define the event

$$A^{\varepsilon,+}_{\xi,n}(f) := \left\{ \frac{1}{n} \sum_{j=0}^{n-1} f(T_{j,X_j}\omega, (Z_{j+i})_{i\geq 1}) - \int f \, d\overline{\mu}^{\infty}_{\xi} > \varepsilon \right\}. \tag{2.7}$$

We similarly define $A^{\varepsilon,-}_{\xi,n}(f)$ and have $A^{\varepsilon}_{\xi,n}(f) = A^{\varepsilon,+}_{\xi,n}(f) \cup A^{\varepsilon,-}_{\xi,n}(f)$. So, it suffices to prove Theorem 1 for only, say, $A^{\varepsilon,+}_{\xi,n}(f)$.

To simplify the notation, we write

$$F_j := f(T_{j,X_j}\omega, (Z_{j+i})_{i\geq 1}) - \int f \, d\overline{\mu}^{\infty}_{\xi}. \tag{2.8}$$

Then (2.7) becomes

$$A^{\varepsilon,+}_{\xi,n}(f) = \left\{ \frac{1}{n} \sum_{j=0}^{n-1} F_j > \varepsilon \right\}.$$

Since $\xi \in \operatorname{int}(\mathcal{D})$, the unique solution θ of $\xi = \nabla \log \phi(\theta)$ satisfies

$$I_a(\xi) = \langle \theta, \xi \rangle - \log \phi(\theta).$$

By a standard change of measure argument and the averaged LDP, we see that for any $z > 0$,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log P_{o,o}(A^{\varepsilon,+}_{\xi,n}(f) | D^{\delta}_{\xi,n}) \\
&\quad \leq \limsup_{n\to\infty} \frac{1}{n} \log P_{o,o}(A^{\varepsilon,+}_{\xi,n}(f), D^{\delta}_{\xi,n}) - \liminf_{n\to\infty} \frac{1}{n} \log P_{o,o}(D^{\delta}_{\xi,n}) \\
&\quad \leq \limsup_{n\to\infty} \frac{1}{n} \log E_{o,o}\left[e^{\langle \theta, X_n \rangle}, A^{\varepsilon,+}_{\xi,n}(f), \left| \frac{X_n}{n} - \xi \right| \leq \delta \right] \\
&\qquad - \langle \theta, \xi \rangle + I_a(\xi) + |\theta|\delta \\
&\quad \leq \limsup_{n\to\infty} \frac{1}{n} \log E_{o,o}\left[e^{\langle \theta, X_n \rangle - n \log \phi(\theta)}, A^{\varepsilon,+}_{\xi,n}(f) \right] + |\theta|\delta \\
&\quad \leq \limsup_{n\to\infty} \frac{1}{n} \log E_{o,o}\left[e^{\langle \theta, X_n \rangle - n \log \phi(\theta) + z \sum_{j=0}^{n-1} F_j} \right] - z\varepsilon + |\theta|\delta,
\end{aligned} \tag{2.9}$$

where the last line is obtained by Chebyshev's inequality. We set $L := N + M + K + 1$ and note that

$$\begin{aligned}
&E_{o,o}\left[e^{\langle \theta, X_n \rangle - n \log \phi(\theta) + z \sum_{j=0}^{n-1} F_j} \right] \\
&\quad \leq \prod_{i=0}^{L-1} E_{o,o}\left[e^{\langle \theta, X_n \rangle - n \log \phi(\theta) + Lz(F_i + F_{L+i} + F_{2L+i} + \cdots)} \right]^{1/L}
\end{aligned} \tag{2.10}$$

holds by an application of Hölder's inequality under $e^{\langle\theta,X_n\rangle-n\log\phi(\theta)}\,dP_{o,o}$.

For any $i\in\{0,\dots,L-1\}$, we let $k=k(i)$ be the largest integer such that $kL+i<n$. Then for $n\geq 2L$,

$$\begin{aligned}
&E_{o,o}\big[e^{\langle\theta,X_n\rangle-n\log\phi(\theta)+Lz(F_i+\cdots+F_{(k-1)L+i}+F_{kL+i})}\big]\\
&\quad=\sum_x \mathbb{E}\big(E^{\omega}_{o,o}\big[e^{\langle\theta,X_{kL+i-N}\rangle-(kL+i-N)\log\phi(\theta)+Lz(F_i+\cdots+F_{(k-1)L+i})},\\
&\qquad\qquad\qquad\qquad\qquad\qquad X_{kL+i-N}=x\big]\\
&\qquad\quad\times E^{\omega}_{kL+i-N,x}\big[e^{\langle\theta,X_n-X_{kL+i-N}\rangle-(n-(kL+i-N))\log\phi(\theta)+Lz(F_{kL+i})}\big]\big).
\end{aligned}$$

Each term of the above sum is the $\mathbb{P}$-expectation of the product of two random variables and these variables are independent since for every $(z_i)_{i\geq1}$, $f(\cdot,(z_i)_{i\geq1})$ is $\mathcal{B}^+_{-N}\cap\mathcal{B}^-_M$-measurable and independent of $(z_i)_{i>K}$. Using this and the fact that $\mathbb{P}$ is invariant under shifts, we write

$$\begin{aligned}
&E_{o,o}\big[e^{\langle\theta,X_n\rangle-n\log\phi(\theta)+Lz(F_i+\cdots+F_{(k-1)L+i}+F_{kL+i})}\big]\\
&\quad=\sum_x E_{o,o}\big[e^{\langle\theta,X_{kL+i-N}\rangle-(kL+i-N)\log\phi(\theta)+Lz(F_i+\cdots+F_{(k-1)L+i})}, X_{kL+i-N}=x\big]\\
&\qquad\quad\times E_{o,o}\big[e^{\langle\theta,X_{n-(kL+i-N)}\rangle-(n-(kL+i-N))\log\phi(\theta)+LzF_N}\big]\\
&\quad=E_{o,o}\big[e^{\langle\theta,X_{kL+i-N}\rangle-(kL+i-N)\log\phi(\theta)+Lz(F_i+\cdots+F_{(k-1)L+i})}\big]\\
&\qquad\times E_{o,o}\big[e^{\langle\theta,X_{n-(kL+i-N)}\rangle-(n-(kL+i-N))\log\phi(\theta)+LzF_N}\big].
\end{aligned}$$

Iterating this, we get

$$\begin{aligned}
&E_{o,o}\big[e^{\langle\theta,X_n\rangle-n\log\phi(\theta)+Lz(F_i+\cdots+F_{(k-1)L+i}+F_{kL+i})}\big]\\
&\quad=E_{o,o}\big[e^{\langle\theta,X_{L+i-N}\rangle-(L+i-N)\log\phi(\theta)+LzF_i}\big]\\
&\qquad\times E_{o,o}\big[e^{\langle\theta,X_L\rangle-L\log\phi(\theta)+LzF_N}\big]^{k-1}\\
&\qquad\times E_{o,o}\big[e^{\langle\theta,X_{n-(kL+i-N)}\rangle-(n-(kL+i-N))\log\phi(\theta)+LzF_N}\big].
\end{aligned}$$

Since $n-(kL+i-N)\leq L+N<\infty$ and f is bounded, the first and the last terms of the above product are bounded and

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\log E_{o,o}\big[e^{\langle\theta,X_n\rangle-n\log\phi(\theta)+Lz(F_i+\cdots+F_{(k-1)L+i}+F_{kL+i})}\big]\\
&\quad=\frac{1}{L}\log E_{o,o}\big[e^{\langle\theta,X_L\rangle-L\log\phi(\theta)+LzF_N}\big].
\end{aligned}$$

Recalling (2.10), we now know that

$$\limsup_{n\to\infty}\frac{1}{n}\log E_{o,o}\big[e^{\langle\theta,X_n\rangle-n\log\phi(\theta)+z\sum_{j=0}^{n-1}F_j}\big]$$

$$\leq \frac{1}{L} \log E_{o,o}\big[e^{\langle \theta, X_L\rangle - L\log\phi(\theta) + LzF_N}\big]$$
$$=: \zeta(z).$$

Because of (2.9), to conclude the proof, it suffices to show that $\zeta(z) = o(z)$. But $\zeta(0) = 0$ and we recall (2.8) to see that

$$\zeta'(0) = E_{o,o}\big[e^{\langle \theta, X_L\rangle - L\log\phi(\theta)} F_N\big] = 0$$

precisely by Definition 1. Hence, we are done. □

3. Conditioning under the quenched measure. In this section, we obtain the function u^θ mentioned in the statement of Theorem 4, derive some of its properties and define the transformed kernel $\overline{\pi}^\theta$ also mentioned in Theorem 4. Finally, having built the necessary machinery, we prove Theorems 2 and 3.

3.1. *The main estimate.* In the rest of the article, the following family of functions play a central role:

DEFINITION 3. For every $\theta \in \mathbb{R}^d$, $x \in \mathbb{Z}^d$ and $n, N \in \mathbb{Z}$ with $n < N$, we define

$$u_N^\theta(\omega, n, x) := \frac{E_{n,x}^\omega[e^{\langle \theta, X_N - X_n\rangle}]}{\phi(\theta)^{N-n}} \tag{3.1}$$

where ϕ is given in (1.1).

The main estimate that enables us to obtain u^θ and establish the equivalence of quenched and averaged large deviations is given as

LEMMA 1. *If $d \geq 3$, then there exists $\bar{\eta} > 0$ such that for every $\theta \in \mathbb{R}^d$ with $|\theta| < \bar{\eta}$, $x \in \mathbb{Z}^d$ and $n \in \mathbb{Z}$, we have*

$$\sup_{N>n} \|u_N^\theta(\cdot, n, x)\|_{L^2(\mathbb{P})} < \infty.$$

PROOF. It suffices to prove the lemma for $n = 0$ and $x = 0$.

$$\begin{aligned} G_N(\theta) := \|u_N^\theta(\cdot, 0, 0)\|_{L^2(\mathbb{P})}^2 &= \frac{\mathbb{E}(E_{o,o}^\omega[e^{\langle \theta, X_N\rangle}]^2)}{\phi(\theta)^{2N}} \\ &= \sum_{\substack{x_o=0, x_1, \ldots, x_N \\ y_o=0, y_1, \ldots, y_N}} \prod_{i=0}^{N-1} \mathbb{E}(\pi_{i,i+1}(x_i, x_{i+1})\pi_{i,i+1}(y_i, y_{i+1})) \\ &\qquad \times \frac{e^{\langle \theta, x_{i+1} - x_i\rangle}}{\phi(\theta)} \frac{e^{\langle \theta, y_{i+1} - y_i\rangle}}{\phi(\theta)} \end{aligned} \tag{3.2}$$

$$= \sum_{\substack{x_o=0,x_1,\ldots,x_N\\ y_o=0,y_1,\ldots,y_N}} \prod_{i=0}^{N-1} \frac{\mathbb{E}(\pi_{i,i+1}(x_i,x_{i+1})\pi_{i,i+1}(y_i,y_{i+1}))}{q(x_{i+1}-x_i)q(y_{i+1}-y_i)} \times q^\theta(x_{i+1}-x_i)q^\theta(y_{i+1}-y_i),$$

where $q^\theta(z) := q(z)\frac{e^{\langle\theta,z\rangle}}{\phi(\theta)}$ for every $z \in U$. For every $x \in \mathbb{Z}^d$, we let $\hat{P}_x^\theta$ be the probability measure on paths starting at x and induced by $(q^\theta(z))_{z\in U}$. We write $\hat{E}_x^\theta$ to denote expectation with respect to $\hat{P}_x^\theta$.

We note that $\mathbb{E}(\pi_{i,i+1}(x_i,x_{i+1})\pi_{i,i+1}(y_i,y_{i+1})) = q(x_{i+1}-x_i)q(y_{i+1}-y_i)$ unless $x_i = y_i$. For every $x, y \in U$, let us set

$$V(x,y) := \log\left(\frac{\mathbb{E}(\pi_{0,1}(0,x)\pi_{0,1}(0,y))}{q(x)q(y)}\right).$$

By uniform ellipticity, V is bounded by some constant $\bar{V}$. With this notation,

$$G_N(\theta) = \hat{E}_o^\theta\hat{E}_o^\theta\bigl[e^{\sum_{i=0}^{N-1}\delta_{X_i=Y_i}V(X_{i+1}-X_i,Y_{i+1}-Y_i)}\bigr].$$

We let $\tau := \inf\{k \geq 0 : X_k = Y_k\}$, $\tau^+ := \inf\{k > 0 : X_k = Y_k\}$ and decompose $G_N(\theta)$ with respect to the first steps X_1 and Y_1:

$$\begin{aligned}
G_N(\theta) &= \sum_{x,y} q^\theta(x)q^\theta(y)e^{V(x,y)}\sum_{k=0}^{N-2}\hat{P}_x^\theta\hat{P}_y^\theta(\tau=k)G_{N-k-1}(\theta)\\
&\quad+\sum_{x,y} q^\theta(x)q^\theta(y)e^{V(x,y)}\hat{P}_x^\theta\hat{P}_y^\theta(\tau\geq N-1)\\
&=\sum_{k=0}^{N-2}\left(\sum_{x,y} q^\theta(x)q^\theta(y)e^{V(x,y)}\hat{P}_x^\theta\hat{P}_y^\theta(\tau=k)\right)G_{N-k-1}(\theta)\\
&\quad+\sum_{x,y} q^\theta(x)q^\theta(y)e^{V(x,y)}\hat{P}_x^\theta\hat{P}_y^\theta(\tau\geq N-1).
\end{aligned}$$

We simplify the last expression by defining

$$\begin{aligned}
B_k(\theta) &:= \sum_{x,y} q^\theta(x)q^\theta(y)e^{V(x,y)}\hat{P}_x^\theta\hat{P}_y^\theta(\tau=k),\\
C_N(\theta) &:= \sum_{x,y} q^\theta(x)q^\theta(y)e^{V(x,y)}\hat{P}_x^\theta\hat{P}_y^\theta(\tau\geq N-1)
\end{aligned}$$

and obtain the following equation:

$$G_N(\theta) = \sum_{k=0}^{N-2} B_k(\theta)G_{N-k-1}(\theta) + C_N(\theta). \tag{3.3}$$

Now, we use the dimension. For every x, y such that $x \neq y$, under the product measure $\hat{P}_x^\theta \hat{P}_y^\theta$, $(X_i - Y_i)_{i\geq 0}$ is a symmetric random walk and since $d \geq 3$, it has positive probability of never hitting the origin. Therefore,

$$\lim_{N\to\infty} C_N(\theta) = \inf_N C_N(\theta) = \sum_{x,y} q^\theta(x) q^\theta(y) e^{V(x,y)} \hat{P}_x^\theta \hat{P}_y^\theta(\tau = \infty) > 0.$$

By (3.2), we know that $G_M(0) = 1$ for every M. Plugging it in (3.3), we get

$$1 = \sum_{k=0}^{N-2} B_k(0) + C_N(0).$$

Taking $N \to \infty$ gives us

$$\sum_{k=0}^{\infty} B_k(0) < 1. \tag{3.4}$$

We would like to show that

$$B(\theta) := \sum_{k=0}^{\infty} B_k(\theta)$$

is continuous in θ at 0. Since $\theta \mapsto B_k(\theta)$ is continuous for each k, it suffices to argue that the tail of this sum is small, uniformly in θ in a neighborhood of 0. Indeed,

$$\begin{aligned} \sum_{k=N}^{\infty} B_k(\theta) &\leq e^{\bar{V}} \sum_{x,y} q^\theta(x) q^\theta(y) \hat{P}_x^\theta \hat{P}_y^\theta(N \leq \tau < \infty) \\ &= e^{\bar{V}} \hat{P}_o^\theta \hat{P}_o^\theta(N+1 \leq \tau^+ < \infty) \\ &\leq e^{\bar{V}} \sum_{k=N+1}^{\infty} \hat{P}_o^\theta \hat{P}_o^\theta(X_k = Y_k). \end{aligned} \tag{3.5}$$

Since $d \geq 3$ and the covariance of $X_1 - Y_1$ under $\hat{P}_o^\theta \hat{P}_o^\theta$ is a nonsingular matrix whose entries are continuous in θ, the local CLT implies that the sum in (3.5) is the tail of a series which converges uniformly in θ in a neighborhood of 0.

Now that we know $\theta \mapsto B(\theta)$ is continuous at 0, we recall (3.4) and see that there exists $\bar{\eta} > 0$ such that for every $\theta \in \mathbb{R}^d$ with $|\theta| < \bar{\eta}$, $B(\theta) < 1$. Letting $C(\theta) := \sup_M C_M(\theta)$, we turn to (3.3) and conclude that

$$\sup_{M\leq N} G_M(\theta) \leq \frac{C(\theta)}{1 - B(\theta)} < \infty.$$

Taking $N \to \infty$ gives the desired result. $\square$

3.2. *Obtaining the function u^θ.* From now on, we consider $d \geq 3$ and θ as in Lemma 1. For every $x \in \mathbb{Z}^d$ and $n, N \in \mathbb{Z}$ with $n < N$, we recall (3.1) and observe that $\mathbb{P}$-a.s.

$$\begin{aligned} u_N^\theta(\omega, n, x) &= \frac{E_{n,x}^\omega[e^{\langle\theta, X_N - X_n\rangle}]}{\phi(\theta)^{N-n}} \\ &= \sum_y \pi_{n,n+1}(x, y) e^{\langle\theta, y-x\rangle} \frac{E_{n+1,y}^\omega[e^{\langle\theta, X_N - X_{n+1}\rangle}]}{\phi(\theta)^{N-n}} \\ &= \sum_y \pi_{n,n+1}(x, y) e^{\langle\theta, y-x\rangle - \log\phi(\theta)} u_N^\theta(\omega, n+1, y). \end{aligned}$$

$(u_N^\theta(\cdot, n, x))_{N>n}$ is a nonnegative martingale and $\mathbb{P}$-a.s. converges to a limit $u^\theta(\cdot, n, x)$ which satisfies

$$u^\theta(\omega, n, x) = \sum_y \pi_{n,n+1}(x, y) e^{\langle\theta, y-x\rangle - \log\phi(\theta)} u^\theta(\omega, n+1, y). \tag{3.6}$$

By Lemma 1, $(u_N^\theta(\cdot, n, x))_{N>n}$ is uniformly bounded in $L^2(\mathbb{P})$ and, therefore, the convergence takes place also in $L^2(\mathbb{P})$.

3.3. *Some properties of u^θ.* For every $x \in \mathbb{Z}^d$ and $n, N \in \mathbb{Z}$ with $n < N$, we know by (2.1) that $\|u_N^\theta(\cdot, n, x)\|_{L^1(\mathbb{P})} = 1$. Since $(u_N^\theta(\cdot, n, x))_{N>n}$ converges to $u^\theta(\cdot, n, x)$ in $L^2(\mathbb{P})$, we immediately see that $\|u^\theta(\cdot, n, x)\|_{L^1(\mathbb{P})} = 1$ and $u^\theta(\cdot, n, x) \in L^2(\mathbb{P})$.

Next, we observe that $\mathbb{P}$-a.s.

$$\begin{aligned} u_N^\theta(T_{n,x}\omega, 0, 0) &= \frac{E_{o,o}^{T_{n,x}\omega}[e^{\langle\theta, X_N - X_o\rangle}]}{\phi(\theta)^N} \\ &= \frac{E_{n,x}^\omega[e^{\langle\theta, X_{N+n} - X_n\rangle}]}{\phi(\theta)^N} \\ &= u_{N+n}^\theta(\omega, n, x). \end{aligned}$$

Taking $N \to \infty$, we see that $\mathbb{P}$-a.s.

$$u^\theta(T_{n,x}\omega, 0, 0) = u^\theta(\omega, n, x). \tag{3.7}$$

We abbreviate the notation by setting

$$u^\theta(\omega) := u^\theta(\omega, 0, 0). \tag{3.8}$$

Since $u_N^\theta(\cdot, 0, 0)$ is $\mathscr{B}_0^+$-measurable, it follows that u^θ is $\mathscr{B}_0^+$-measurable as well.

Using (3.7) and (3.8), we put (3.6) in the following form: $\mathbb{P}$-a.s.

$$u^\theta(\omega) = \sum_{z\in U} \overline{\pi}(\omega, T_{1,z}\omega) e^{\langle\theta,z\rangle - \log\phi(\theta)} u^\theta(T_{1,z}\omega). \tag{3.9}$$

Finally, let us prove that $u^\theta > 0$ holds $\mathbb{P}$-a.s. We already know that $u^\theta \geq 0$ holds $\mathbb{P}$-a.s. Clearly, (3.9) implies that $\{\omega : u^\theta(\omega) = 0\}$ is invariant under $T_{1,z}$ for every $z \in U$. Since the product environment $\mathbb{P}$ is ergodic under shifts, $\mathbb{P}(u^\theta(\omega) = 0)$ is either 0 or 1. But we know that $\|u^\theta(\cdot, n, x)\|_{L^1(\mathbb{P})} = 1$ and, therefore, we conclude that $\mathbb{P}(u^\theta(\omega) = 0) = 0$.

Now, we are ready to define a new transition kernel $\overline{\pi}^\theta$ on Ω by a Doob h-transform: For every $z \in U$, $\mathbb{P}$-a.s.

$$\overline{\pi}^\theta(\omega, T_{1,z}\omega) := \overline{\pi}(\omega, T_{1,z}\omega) \frac{u^\theta(T_{1,z}\omega)}{u^\theta(\omega)} e^{\langle\theta,z\rangle - \log\phi(\theta)}. \tag{3.10}$$

$\overline{\pi}^\theta$ induces a probability measure $P_{k,x}^{\theta,\omega}$ on particle paths starting at position x at time k and we write $E_{k,x}^{\theta,\omega}$ to denote expectation under this measure.

3.4. *Proofs of Theorems 2 and 3.*

PROOF OF THEOREM 2. For $d \geq 3$ and $\bar{\eta}$ as in Lemma 1, we recall (3.1) and observe that if $|\theta| < \bar{\eta}$ then

$$\lim_{n\to\infty} \frac{1}{n} \log E_{o,o}^\omega[e^{\langle\theta,X_n\rangle}] = \log\phi(\theta) + \lim_{n\to\infty} \frac{1}{n} \log u_n^\theta(\omega, 0, 0) = \log\phi(\theta) \tag{3.11}$$

because $\lim_{n\to\infty} u_n^\theta(\omega) = u^\theta(\omega) > 0$ holds $\mathbb{P}$-a.s. Since $\log\phi$ is strictly convex and $\xi_o = \nabla\log\phi(0)$,

$$\{\nabla\log\phi(\theta) : |\theta| < \bar{\eta}\}$$

is an open set containing the LLN velocity ξ_o. Hence, there exists $\eta > 0$ such that for every $\xi \in \mathcal{D}$ with $|\xi - \xi_o| < \eta$ there is a unique θ satisfying $|\theta| < \bar{\eta}$ and $\xi = \nabla\log\phi(\theta)$. Because $\theta \mapsto \log\phi(\theta)$ is analytic, (3.11) and the Gärtner–Ellis theorem (Dembo and Zeitouni [3], page 44) immediately imply the desired result. $\square$

PROOF OF THEOREM 3. Under the conditions of the theorem, we recall the proof of Theorem 1 and see that for the unique θ with $\xi = \nabla\log\phi(\theta)$

$$\limsup_{n\to\infty} \frac{1}{n} \log E_{o,o}[e^{\langle\theta,X_n\rangle - n\log\phi(\theta)}, A_{\xi,n}^\varepsilon(f)] =: \gamma < 0.$$

Fixing $\alpha > 0$, for every $n \in \mathbb{N}$, we define the events

$$B_n' := \{\omega : E_{o,o}^\omega[e^{\langle\theta,X_n\rangle - n\log\phi(\theta)}, A_{\xi,n}^\varepsilon(f)] > e^{n(\gamma+\alpha)}\}$$

on Ω. Then

$$\begin{aligned}\mathbb{P}(B_n') &\le \int_{B_n'} E_{o,o}^{\omega}\big[e^{\langle\theta, X_n\rangle - n\log\phi(\theta)}, A_{\xi,n}^{\varepsilon}(f)\big]e^{-n(\gamma+\alpha)}\,d\mathbb{P}\\ &\le E_{o,o}\big[e^{\langle\theta, X_n\rangle - n\log\phi(\theta)}, A_{\xi,n}^{\varepsilon}(f)\big]e^{-n(\gamma+\alpha)}.\end{aligned}$$

Therefore, $\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}(B_n') \le -\alpha$, and in particular $\sum_{n=1}^{\infty}\mathbb{P}(B_n') < \infty$. By the Borel–Cantelli lemma, $\mathbb{P}(B_n' \text{ i.o.}) = 0$. In other words, $\mathbb{P}$-a.s.

$$E_{o,o}^{\omega}\big[e^{\langle\theta, X_n\rangle - n\log\phi(\theta)}, A_{\xi,n}^{\varepsilon}(f)\big] \le e^{n(\gamma+\alpha)}$$

for sufficiently large n. Thus,

$$\limsup_{n\to\infty}\frac{1}{n}\log E_{o,o}^{\omega}\big[e^{\langle\theta, X_n\rangle - n\log\phi(\theta)}, A_{\xi,n}^{\varepsilon}(f)\big] \le \gamma + \alpha.$$

Since $\alpha > 0$ is arbitrary, we actually see that for $\mathbb{P}$-a.e. ω

$$\limsup_{n\to\infty}\frac{1}{n}\log E_{o,o}^{\omega}\big[e^{\langle\theta, X_n\rangle - n\log\phi(\theta)}, A_{\xi,n}^{\varepsilon}(f)\big] \le \gamma.$$

Let us now finish the proof of the theorem:

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log P_{o,o}^{\omega}(A_{\xi,n}^{\varepsilon}(f)|D_{\xi,n}^{\delta})\\ &\quad\le \limsup_{n\to\infty}\frac{1}{n}\log P_{o,o}^{\omega}(A_{\xi,n}^{\varepsilon}(f), D_{\xi,n}^{\delta}) - \liminf_{n\to\infty}\frac{1}{n}\log P_{o,o}^{\omega}(D_{\xi,n}^{\delta})\\ &\quad\le \limsup_{n\to\infty}\frac{1}{n}\log E_{o,o}^{\omega}\left[e^{\langle\theta, X_n\rangle}, A_{\xi,n}^{\varepsilon}(f), \left|\frac{X_n}{n} - \xi\right| \le \delta\right]\\ &\qquad - \langle\theta,\xi\rangle + I_a(\xi) + |\theta|\delta\\ &\quad\le \limsup_{n\to\infty}\frac{1}{n}\log E_{o,o}^{\omega}\big[e^{\langle\theta, X_n\rangle - n\log\phi(\theta)}, A_{\xi,n}^{\varepsilon}(f)\big] + |\theta|\delta\\ &\quad\le \gamma + |\theta|\delta\\ &\quad< 0\end{aligned}$$

when $\delta > 0$ is sufficiently small. In the above estimate, we use the fact that the quenched LDP holds in a neighborhood of ξ with rate

$$I_a(\xi) = \langle\theta,\xi\rangle - \log\phi(\theta)$$

at ξ, which is true by hypothesis. $\square$

4. Identifying μ_ξ^∞ as a stationary Markov process. For $d \geq 3$ and $|\xi - \xi_o| < \eta$ with η as in Theorem 2, we let $\theta \in \mathbb{R}^d$ be the unique solution of $\xi = \nabla \log \phi(\theta)$. We can put $\overline{\mu}_\xi^\infty$ in a nicer form. For every N, M and $K \in \mathbb{N}$ and any f as in Definition 1, setting $L := N + M + K + 1$, we have

$$
\begin{aligned}
&\int f\, d\overline{\mu}_\xi^\infty \\
&\quad = E_{o,o}\bigl[e^{\langle\theta, X_L\rangle - L\log\phi(\theta)} f(T_{N,X_N}\omega, (Z_{N+i})_{i\geq 1})\bigr] \\
&\quad = \sum_x \mathbb{E}\bigl(E_{o,o}^\omega\bigl[e^{\langle\theta, X_L\rangle - L\log\phi(\theta)} f(T_{N,X_N}\omega, (Z_{N+i})_{i\geq 1}), X_L = x\bigr]\bigr) \\
&\qquad\qquad \times \mathbb{E}(u^\theta(T_{L,x}\omega)) \\
&\quad = \sum_x \mathbb{E}\bigl(E_{o,o}^\omega\bigl[e^{\langle\theta, X_L\rangle - L\log\phi(\theta)} u^\theta(T_{L,x}\omega) f(T_{N,X_N}\omega, (Z_{N+i})_{i\geq 1}), X_L = x\bigr]\bigr) \\
&\quad = \mathbb{E}\biggl(u^\theta(\omega) E_{o,o}^\omega\biggl[e^{\langle\theta, X_L\rangle - L\log\phi(\theta)} \frac{u^\theta(T_{L,X_L}\omega)}{u^\theta(\omega)} f(T_{N,X_N}\omega, (Z_{N+i})_{i\geq 1})\biggr]\biggr) \\
&\quad = \mathbb{E}(u^\theta(\omega) E_{o,o}^{\theta,\omega}[f(T_{N,X_N}\omega, (Z_{N+i})_{i\geq 1})]), \qquad (4.1)
\end{aligned}
$$

where we use the facts that $\mathbb{E}(u^\theta(T_{L,x}\cdot)) = 1$ and $u^\theta(T_{L,x}\cdot)$ is $\mathcal{B}_L^+$-measurable.

We note that (4.1) is independent of M and K. This immediately tells us that the marginal μ_ξ^1 of μ_ξ^∞ is absolutely continuous relative to $\mathbb{P}$ on every $\mathcal{B}_{-N}^+$. Here is how: We fix $N \in \mathbb{N}$. For any $M \in \mathbb{N}$ and any bounded $\mathcal{B}_{-N}^+ \cap \mathcal{B}_M^-$-measurable $h : \Omega \to \mathbb{R}$, we have

$$
\begin{aligned}
\int h\, d\mu_\xi^1 &= \mathbb{E}(u^\theta(\omega) E_{o,o}^{\theta,\omega}[h(T_{N,X_N}\omega)]) \\
&\leq \|u^\theta\|_{L^2(\mathbb{P})} \|E_{o,o}^{\theta,\omega}[h(T_{N,X_N}\omega)]\|_{L^2(\mathbb{P})} \\
&\leq \|u^\theta\|_{L^2(\mathbb{P})} \biggl\| \sum_{|x|\leq N} |h(T_{N,x}\omega)| \biggr\|_{L^2(\mathbb{P})} \\
&\leq (2N+1)^d \|u^\theta\|_{L^2(\mathbb{P})} \|h\|_{L^2(\mathbb{P})}.
\end{aligned}
$$

Since such functions are dense in $L^2(\Omega, \mathcal{B}_{-N}^+, \mathbb{P})$, it follows by the Riesz representation theorem that

$$
\frac{d\mu_\xi^1}{d\mathbb{P}}\bigg|_{\mathcal{B}_{-N}^+} \in L^2(\mathbb{P}). \qquad (4.2)
$$

PROOF OF THEOREM 4. We have obtained u^θ in (3.8) and defined π^θ in (3.10). For every $N, K \in \mathbb{N}$, we take any bounded $f : \Omega^{K+1} \to \mathbb{R}$ and $g : \Omega^{\mathbb{N}} \to \mathbb{R}$

such that

$$f(\omega, T_{1,z_1}\omega, \dots, T_{K,z_1+\cdots+z_K}\omega) g(T_{K,z_1+\cdots+z_K}\omega, T_{K+1,z_1+\cdots+z_{K+1}}\omega, \dots)$$

is $\mathcal{B}_{-N}^{+}$-measurable for any $(z_i)_{i\geq 1}$. Then

$$\begin{aligned}
&\int f(\omega_1, \dots, \omega_{K+1}) g(\omega_{K+1}, \omega_{K+2}, \dots)\, d\mu_\xi^\infty \\
&\quad = \int f(\omega, T_{1,z_1}\omega, \dots, T_{K,z_1+\cdots+z_K}\omega) \\
&\qquad\quad \times g(T_{K,z_1+\cdots+z_K}\omega, T_{K+1,z_1+\cdots+z_{K+1}}\omega, \dots)\, d\overline{\mu}_\xi^\infty \\
&\quad = \mathbb{E}(u^\theta(\omega) E_{o,o}^{\theta,\omega}[f(T_{N,X_N}\omega, \dots, T_{N+K,X_{N+K}}\omega) g(T_{N+K,X_{N+K}}\omega, \dots)]) \\
&\quad = \mathbb{E}(u^\theta(\omega) E_{o,o}^{\theta,\omega}[f(T_{N,X_N}\omega, \dots, T_{N+K,X_{N+K}}\omega) \\
&\qquad\qquad\qquad \times E_{N+K,X_{N+K}}^{\theta,\omega}[g(T_{N+K,X_{N+K}}\omega, \dots)]]) \\
&\quad = \int f(\omega_1, \dots, \omega_{K+1}) E_{o,o}^{\theta,\omega_{K+1}}[g(\omega_{K+1}, T_{1,X_1}\omega_{K+1}, \dots)]\, d\mu_\xi^\infty
\end{aligned}$$

where we use (4.1) and the Markov property. This proves that μ_ξ^∞ is indeed a Markov process with state space Ω and transition kernel $\overline{\pi}^\theta$.

We already know that μ_ξ^∞ is a stationary process. Hence, its marginal μ_ξ^1 is an invariant measure for $\overline{\pi}^\theta$. Since μ_ξ^1 is absolutely continuous relative to $\mathbb{P}$ on every $\mathcal{B}_{-N}^{+}$ [by (4.2)], it follows that μ_ξ^1 is the unique invariant measure for $\overline{\pi}^\theta$ with that absolute continuity property (see Rassoul-Agha [9]). $\square$

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