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On the regularity of stochastic currents, fractional Brownian motion and applications to a turbulence model

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Abstract. We study the pathwise regularity of the map

$$\varphi \mapsto I(\varphi) = \int_0^T \langle \varphi(X_t), \mathrm{d}X_t \rangle,$$

where φ is a vector function on $\mathbb{R}^d$ belonging to some Banach space V, X is a stochastic process and the integral is some version of a stochastic integral defined via regularization. A continuous version of this map, seen as a random element of the topological dual of V will be called *stochastic current*. We give sufficient conditions for the current to live in some Sobolev space of distributions and we provide elements to conjecture that those are also necessary. Next we verify the sufficient conditions when the process X is a d-dimensional fractional Brownian motion (fBm); we identify regularity in Sobolev spaces for fBm with Hurst index $H \in (1/4, 1)$. Next we provide some results about general Sobolev regularity of currents when W is a standard Wiener process. Finally we discuss applications to a model of random vortex filaments in turbulent fluids.

Résumé. Nous étudions la régularité trajectorielle de l'opérateur

$$\varphi \mapsto I(\varphi) = \int_0^T \langle \varphi(X_t), \mathrm{d}X_t \rangle,$$

où φ est une fonction vectorielle à valeurs dans $\mathbb{R}^d$ appartenant à un certain espace de Banach V, X est un processus stochastique et l'intégrale est une certaine version d'une intégrale stochastique définie via régularisation. Une version continue d'un tel opérateur, interprétée comme une variable aléatoire à valeurs dans le dual topologique de V sera appelée *courant stochastique*. Nous donnons des conditions suffisantes pour que le courant se situe dans un certain espace de Sobolev de distributions. De plus nous donnons des arguments qui permettent de conjecturer que ces conditions sont aussi nécessaires. Successivement nous vérifions la validité de ces conditions lorsque le processus X est un mouvement brownien fractionnaire (mbf) d-dimensionnel; en particulier, nous identifions la régularité de Sobolev pour un mbf d'indice de Hurst $H \in (1/4, 1)$. Par suite, nous fournissons quelques résultats sur la régularité générale de Sobolev de courants relative à un mouvement brownien standard. Enfin nous discutons une application à un modèle de filaments de vorticité dans un fluide turbulent.

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1. Introduction

We consider stochastic integrals, loosely speaking of the form

$$I(\varphi) = \int_0^T \langle \varphi(X_t), \mathrm{d}X_t \rangle, \tag{1}$$

where (X_t) is a Wiener process or a fractional Brownian motion with Hurst parameter H in a certain range. We are interested in the pathwise continuity properties with respect to φ: we would like to establish that the random generalized field $\varphi \mapsto I(\varphi)$ has a version that is a.s. continuous in φ in certain topologies. In the language of geometric measure theory, such a property means that the stochastic integral defines pathwise a *current*, with the regularity specified by the topologies that we have found.

This problem is motivated by the study of fluidodynamical models. In [4], the energy of a random vortex filament has an expression involving a stochastic double integral related to Wiener process in the form

$$\int_{[0,T]^2} f(X_s - X_t) \,\mathrm{d}X_s \,\mathrm{d}X_t, \tag{2}$$

where $f(x) = K_\alpha(x)$ and where $K_\alpha(x)$ is the kernel of the pseudo-differential operator $(1-\Delta)^{-\alpha}$ (precise definitions will be given in Section 2.3). f is therefore a continuous singular function at zero.

The difficulty there comes from the appearance of anticipating integrands and from the singularity of f at zero. [4] gives sense to this integral in some Stratonovich sense. Moreover that paper explores the connection with self-intersection local time considered for instance by Le Gall in [10].

The work [17] considers a similar double integral in the case of fractional Brownian motion with Hurst index $H > \frac{1}{2}$ using Malliavin–Skorokhod anticipating calculus.

A natural approach is to interpret the previous double integral as a symmetric (or eventually) forward integral in the framework of stochastic calculus via regularization, see [21] for a survey. We recall that when X is a semimartingale, the forward (respectively symmetric) integral $\int_0^t \varphi(X)\,\mathrm{d}^- X$ (resp. $\int_0^t \varphi(X)\,\mathrm{d}^\circ X$) coincides with the corresponding Itô (resp. Stratonovich) integral. The double stochastic integral considered by [4] coincides in fact with the symmetric integral introduced here. So (2) can be interpreted as

$$\int_{[0,T]^2} f(X_s - X_t) \,\mathrm{d}^\circ X_s \,\mathrm{d}^\circ X_t. \tag{3}$$

In this paper, X will be a fractional Brownian motion with Hurst index $H > 1/4$ but a complete study of the existence of integrals (3) will be not yet performed here because of heavy technicalities. We will only essentially consider their regularized versions. Now, those double integrals are naturally in correspondence with currents related to I defined in (1). The investigation of those currents is strictly related to "pathwise stochastic calculus" in the spirit of *rough paths* theory by Lyons and coauthors [15,16]. Here we aim at exploring the "pathwise character" of stochastic integrals via regularization. A first step in this direction was done in [13], where the authors showed that forward integrals of the type $\int_0^T \varphi(X)\,\mathrm{d}^- X$, when X is a one-dimensional semimartingale or a fractional Brownian motion with Hurst index $H > \frac{1}{2}$, can be regarded as a.s. uniform approximations of their regularization $I_\varepsilon^-(\varphi)$ (see Section 3), instead of the usual convergence in probability.

This analysis of currents related to stochastic integrals was started in [9] using an approach based on spectral analysis. The approach presented here is not based on the Fourier transform and contains new general ideas with respect to [9]. Informally speaking, it is based on the formula

$$\int_0^T \langle \varphi(X_t), \mathrm{d}X_t \rangle = \int_{\mathbb{R}^d} \left\langle (1-\Delta)^\alpha \varphi(x), \int_0^T K_\alpha(x - X_t)\,\mathrm{d}X_t \right\rangle \mathrm{d}x. \tag{4}$$

This formula decouples φ and X and replaces the problem of the pathwise dependence of $I(\varphi)$ on the infinite dimensional parameter φ with the problem of the pathwise dependence of $\int_0^T K_\alpha(x - X_t)\,\mathrm{d}X_t$ on the finite dimensional parameter $x \in \mathbb{R}^d$. Another form of decoupling is also one of the ingredients of the Fourier approach of [9] but the novelty here is that we can take better advantages from the properties of the underlying process (like, for example, the existence of a density). Moreover formula (4) produces at least two new results.

First, we can treat in an essentially optimal way the case of fractional Brownian motion, making use of its Gaussian properties. For $H > 1/2$ results in this direction can be extracted from the estimates proved in [17] again by spectral analysis. However, with the present approach we may treat the case $H \in (1/4, 1/2)$ as well.

Second, in the case of the Brownian motion, we may work with functions φ in the Sobolev spaces of Banach type H_p^α, with $p > 1$, instead of only the Hilbert topologies H_2^α considered in [9], with the great advantage that it is sufficient to ask less differentiability on φ (any $\alpha > 1$ suffices), at the price of a larger p (depending on α and the space dimension). In this way we may cover, for instance, the class $\varphi \in C^{1,\varepsilon}$ treated in [16], see also [14]; the approach here is entirely different and does not rely on rough paths, see Remark 39.

Finally, we apply these ideas to random vortex filaments. In the case of the fractional Brownian motion we prove new results about the finiteness of the kinetic energy of the filaments (such a property is expected to be linked to the regularity of the pathwise current). In the case of the Brownian motion optimal conditions for a finite energy were already proved in [4,5], while a sufficient condition when $H > 1/2$ has been found in [17]. The results of the present work provide new regularity properties of the random filaments, especially for the parameter range $H \in (1/4, 1/2)$.

2. Generalities

2.1. *Stochastic currents*

Let (X_t) be a stochastic process such that $X_0 = 0$ a.s. on a probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$. Let $T > 0$. Let V be a Banach space of vector fields $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ and $\mathcal{D} \subset V$ be a dense subset. Assume that a stochastic integral

$$I(\varphi) = \int_0^T \langle \varphi(X_t), \mathrm{d}X_t \rangle$$

is well defined, in a suitable sense (Itô, etc.), for every $\varphi \in \mathcal{D}$. Our first aim is to define it for every $\varphi \in V$. In addition, we would like to prove that it has a pathwise redefinition according to the following definition.

Definition 1. *The family of r.v. $\{I(\varphi)\}_{\varphi\in\mathcal{D}}$ has a* pathwise redefinition *on V if there exists a measurable mapping $\xi : \Omega \to V'$ such that for every $\varphi \in \mathcal{D}$*

$$I(\varphi)(\omega) = \big(\xi(\omega)\big)(\varphi) \quad \textit{for } P\textit{-a.e. } \omega \in \Omega. \tag{5}$$

Then, if we succeed in our objective:

- for every $\varphi \in V$ we consider the r.v. $\omega \mapsto (\xi(\omega))(\varphi)$ as the definition of the stochastic integral $I(\varphi)$ (now extended to the class $\varphi \in V$), and
- for P-a.e. $\omega \in \Omega$, we consider the linear continuous mapping $\varphi \mapsto (\xi(\omega))(\varphi)$ as a pathwise redefinition of the stochastic integral on V.

Formally, the candidate for ξ is the expression

$$\xi(x) = \int_0^T \delta(x - X_t)\,\mathrm{d}X_t,$$

where δ is here the d-dimensional Dirac measure. Indeed, always formally,

$$\xi(\varphi) = \int_{\mathbb{R}^d} \langle \xi(x), \varphi(x) \rangle \,\mathrm{d}x = \int_0^T \left\langle \left(\int_{\mathbb{R}^d} \delta(x - X_t)\varphi(x)\,\mathrm{d}x \right), \mathrm{d}X_t \right\rangle = \int_0^T \langle \varphi(X_t), \mathrm{d}X_t \rangle = I(\varphi).$$

We remark that this viewpoint is inspired by the theory of currents; with other methods (spectral ones) it was developed in [9].

2.2. *Decoupling by duality*

As we said in the introduction, our approach is based on a proper rigorous version of formula (4). One way to interpret it is by the following duality argument, which we describe only at a formal level.

Let W be another Banach space and $\Lambda : V \to W$ be an isomorphism. Proceeding formally as above we have

$$\int_0^T \langle \varphi(X_t), \mathrm{d}X_t \rangle_{\mathbb{R}^d} = \langle \varphi, \xi \rangle_{V,V'} = \langle \Lambda^{-1}\Lambda\varphi, \xi \rangle_{V,V'} = \langle \Lambda\varphi, (\Lambda^{-1})^*\xi \rangle_{W,W'}$$

(notice that $\Lambda^{-1} : W \to V$, $(\Lambda^{-1})^* : V' \to W'$). Our aim essentially amounts to proving that $(\Lambda^{-1})^*\xi : \Omega \to W'$ is a well-defined random variable.

This reformulation becomes useful if the spaces W, W' are easier to handle than V and V', and the operator $(\Lambda^{-1})^*$ has a kernel $K(x, y)$ as an operator in function spaces:

$$\big((\Lambda^{-1})^* f\big)(x) = \int K(x, y) f(y)\,\mathrm{d}y.$$

In such a case, formally

$$\begin{aligned} \big((\Lambda^{-1})^*\xi\big)(x) &= \int K(x, y)\left(\int_0^T \delta(y - X_t)\,\mathrm{d}X_t\right)\mathrm{d}y \\ &= \int_0^T \left(\int K(x, y)\delta(y - X_t)\,\mathrm{d}y\right)\mathrm{d}X_t \\ &= \int_0^T K(x, X_t)\,\mathrm{d}X_t. \end{aligned}$$

Below we make a rigorous version of this representation by choosing $V = H_p^\alpha(\mathbb{R}^d)$, $W = L^p(\mathbb{R}^d)$, $\Lambda = (1 - \Delta)^{\alpha/2}$, $K(x, y) = K_{\alpha/2}(x - y)$ (notations are given in the next section).

2.3. *Rigorous setting*

Denote by $S(\mathbb{R}^d)$ the space of rapidly decreasing infinitely differentiable *vector fields* $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, by $S'(\mathbb{R}^d)$ its dual (the space of tempered distributional fields) and by $\mathcal{F}$ the Fourier transform

$$(\mathcal{F}\varphi)(\ell) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle x, \ell \rangle} \varphi(x)\,\mathrm{d}x, \quad \ell \in \mathbb{R}^d$$

which is an isomorphism in both $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$. Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. For every $s \in \mathbb{R}$, let $\Lambda_s : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ be the pseudo-differential operator defined as

$$\Lambda_s \varphi = \mathcal{F}^{-1}\big(1 + |\ell|^2\big)^{s/2} \mathcal{F}\varphi.$$

We shall also denote it by $(1 - \Delta)^{s/2}$. In this paper $\mathcal{D}$ of Section 2.1 will always indicate $\mathcal{S}(\mathbb{R}^d)$.

Let $H_p^s(\mathbb{R}^d)$, with $p > 1$ and $s \in \mathbb{R}$, be the Sobolev space of vector fields $\varphi \in S'(\mathbb{R}^d)$ such that

$$\|\varphi\|_{H_p^s}^p := \int_{\mathbb{R}^d} \big|(1 - \Delta)^{s/2}\varphi(x)\big|^p\,\mathrm{d}x < \infty;$$

see [22], Section 2.3.3, where the definition chosen here for brevity is given as a characterization. From the very definitions of Λ_s and $H_p^s(\mathbb{R}^d)$, the operator Λ_s is an isomorphism from $H_p^s(\mathbb{R}^d)$ onto $L^p(\mathbb{R}^d)$ (the Lebesgue space of p-integrable *vector fields*).

Another fact often used in the paper is that the dual space $(H^s_p(\mathbb{R}^d))'$ is $H^{-s}_{p'}(\mathbb{R}^d)$:

$$\big(H^s_p(\mathbb{R}^d)\big)' = H^{-s}_{p'}(\mathbb{R}^d), \quad \frac{1}{p} + \frac{1}{p'} = 1;$$

see [22], Section 2.6.1. Moreover, being Λ_s an isomorphism from $H^s_p(\mathbb{R}^d)$ onto $L^p(\mathbb{R}^d)$, its dual operator $\Lambda_s^\star$ is an isomorphism from $L^{p'}(\mathbb{R}^d)$ onto $H^{-s}_{p'}(\mathbb{R}^d)$.

It is known that negative fractional powers of a positive self-adjoint operator A in a Hilbert space H, such that $-A$ generates the semigroup $T(t)$, have the representation

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\,\mathrm{d}t,$$

where Γ is the standard Gamma function; see [18], formula (6.9). Taking $A = (1-\Delta)$ in the Hilbert space $H = L^2(\mathbb{R}^d)$, we have

$$\big(T(t)\varphi\big)(x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4t)-t} \varphi(y)\,\mathrm{d}y$$

and thus, for $\alpha > 0$,

$$\begin{aligned}\big((1-\Delta)^{-\alpha}\varphi\big)(x) &= \frac{(4\pi t)^{-d/2}}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4t)-t}\varphi(y)\,\mathrm{d}y\,\mathrm{d}t \\ &= \int_{\mathbb{R}^d} \left[\frac{1}{\Gamma(\alpha)(4\pi)^{d/2}} \int_0^\infty t^{\alpha-1-d/2} \mathrm{e}^{-|x-y|^2/(4t)-t}\,\mathrm{d}t\right] \varphi(y)\,\mathrm{d}y.\end{aligned}$$

In fact this formula can be proved more elementarily from the definition of $(1-\Delta)^{-\alpha}\varphi$ and the formula

$$\lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathrm{e}^{-\lambda t}\,\mathrm{d}t,$$

then taking $\lambda = 1 + |\ell|^2$ and the Fourier transform of the Gaussian density. This fact implies that the operator $(1-\Delta)^{-\alpha}$, which originally is an isomorphism between $H_2^{-2\alpha}(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, considered by restriction as a bounded linear operator in $L^2(\mathbb{R}^d)$, has a kernel $K_\alpha(\cdot)$,

$$\big((1-\Delta)^{-\alpha}\varphi\big)(x) = \int_{\mathbb{R}^d} K_\alpha(x-y)\varphi(y)\,\mathrm{d}y \tag{6}$$

given by

$$K_\alpha(x) = \frac{1}{\Gamma(\alpha)(4\pi)^{d/2}} \int_0^\infty t^{\alpha-d/2} \mathrm{e}^{-|x|^2/(4t)-t} \frac{\mathrm{d}t}{t}.$$

The following estimates are not optimized as far as the exponential decay is concerned; we just state a version sufficient for our purposes. The proofs of the following two lemmas are in the Appendix.

Lemma 2. *There exist positive constants $c_{\alpha,d}$, $C_{\alpha,d}$ such that*:

(1) *For $0 < \alpha < \frac{d}{2}$, we have*

$$K_\alpha(x) = |x|^{2\alpha-d}\rho(x), \tag{7}$$

where $c_{\alpha,d}\mathrm{e}^{-2|x|^2} \le \rho(x) \le C_{\alpha,d}\mathrm{e}^{-|x|/8}$;

(2) *For $\alpha > \frac{d}{2}$, we have*

$$c_{\alpha,d}\mathrm{e}^{-|x|^2/4} \le K_\alpha(x) \le C_{\alpha,d}\mathrm{e}^{-|x|/8} \tag{8}$$

for two positive constants $c'_{\alpha,d}$, $C'_{\alpha,d}$. Moreover

$$K_\alpha(x) = K_\alpha(0) - \rho'(x)|x|^{-d+2\alpha} \ge 0 \tag{9}$$

with $0 < K_\alpha(0) < \infty$ and where $c_{\alpha,d}\mathrm{e}^{-2|x|^2} \le \rho'(x) \le C_{\alpha,d}\mathrm{e}^{-|x|/8}$;

(3) *Finally, when $\alpha = d/2$ we have*

$$K_\alpha(x) \le C_{\alpha,d} \log|x|\mathrm{e}^{-a_\alpha|x|}, \tag{10}$$

where a_α is another positive constant.

Remark 3. *In particular ρ and ρ' are bounded.*

In the applications we will need also some control on $\Delta K_\alpha(x)$ which is provided by the next lemma.

Lemma 4. *It holds that $-\Delta K_\alpha(x) = K_\alpha(x) - K_{\alpha-1}(x)$. Then, when $\alpha < d/2 + 1$,*

$$|-\Delta K_\alpha(x)| \le \rho''(x)|x|^{2\alpha-d-2}, \tag{11}$$

where ρ'' is positive, bounded above, locally bounded away from zero below and depends on α.

2.4. *Regularity of stochastic currents*

With these notations and preliminaries in mind, we may state a first rigorous variant of formula (4). Given a continuous stochastic process $(X_t)_{t\ge 0}$ on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$, given $\varepsilon > 0$, let $(D_\varepsilon X_t)_{t\ge 0}$ be any one of the following discrete derivatives:

$$\frac{X_{t+\varepsilon} - X_t}{\varepsilon}, \qquad \frac{X_t - X_{t-\varepsilon}}{\varepsilon}, \qquad \frac{X_{t+\varepsilon} - X_{t-\varepsilon}}{2\varepsilon},$$

where we understand that $X_{t-\varepsilon} = 0$ for $t < \varepsilon$. The following integral

$$I_\varepsilon(\varphi) = \int_0^T \langle \varphi(X_t), D_\varepsilon X_t \rangle \,\mathrm{d}t$$

is well defined P-a.s. as a classical Riemann integral, at least for every continuous vector field φ.

Lemma 5. *Given $\alpha, \varepsilon > 0$, with probability one the function $t \mapsto K_{\alpha/2}(x - X_t)D_\varepsilon X_t$ is integrable for a.e. $x \in \mathbb{R}^d$, the function*

$$\eta_\varepsilon(x) := \int_0^T K_{\alpha/2}(x - X_t) D_\varepsilon X_t \,\mathrm{d}t$$

is in $L^1(\mathbb{R}^d)$ and for any $\varphi \in S(\mathbb{R}^d)$ we have

$$\int_0^T \langle \varphi(X_t), D_\varepsilon X_t \rangle \,\mathrm{d}t = \int_{\mathbb{R}^d} \left\langle (1-\Delta)^{\alpha/2}\varphi(x), \int_0^T K_{\alpha/2}(x - X_t) D_\varepsilon X_t \,\mathrm{d}t \right\rangle \mathrm{d}x. \tag{12}$$

Proof. Notice that $(1-\Delta)^{\alpha/2}\varphi \in S(\mathbb{R}^d)$ and, by (6),

$$\varphi = (1-\Delta)^{-\alpha/2}(1-\Delta)^{\alpha/2}\varphi = \int_{\mathbb{R}^d} K_{\alpha/2}(\cdot - x)\big[(1-\Delta)^{\alpha/2}\varphi\big](x)\,\mathrm{d}x.$$

Thus

$$\int_0^T \big\langle \varphi(X_t), D_\varepsilon X_t\big\rangle \,\mathrm{d}t = \int_0^T \left\langle \int_{\mathbb{R}^d} K_{\alpha/2}(X_t - x)\big[(1-\Delta)^{\alpha/2}\varphi\big](x)\,\mathrm{d}x, D_\varepsilon X_t \right\rangle \mathrm{d}t.$$

Denote by $\widehat{K}_{\alpha/2}(x)$ the function equal to $K_{\alpha/2}(x)$ for $x \neq 0$, infinite for $x = 0$. Suppose for a moment that

$$P\left(\int_0^T \int_{\mathbb{R}^d} \widehat{K}_{\alpha/2}(X_t - x)\,\mathrm{d}x\,\mathrm{d}t < \infty\right) = 1. \tag{13}$$

Then the integrability properties stated in the lemma will hold. Since

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \big|\big\langle \big[(1-\Delta)^{\alpha/2}\varphi\big](x), D_\varepsilon X_t \big\rangle\big| < \infty \quad \text{a.s.},$$

using the Fubini theorem we will get (12) [notice that $K_{\alpha/2}(X_t - x) = K_{\alpha/2}(x - X_t)$].

Thus we have only to prove (13). Since $K_{\alpha/2}$ is positive, we may apply again the Fubini theorem and analyze $\int_0^T (\int_{\mathbb{R}^d} \widehat{K}_{\alpha/2}(X_t - x)\,\mathrm{d}x)\,\mathrm{d}t$. But we have, for every $y \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \widehat{K}_{\alpha/2}(y - x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \widehat{K}_{\alpha/2}(x)\,\mathrm{d}x.$$

This quantity is finite for every $\alpha > 0$, from the estimates of Lemma 2. The proof is now complete. □

Below we need a criterion to decide when η_ε (defined in the previous lemma) belongs to $L^2(\mathbb{R}^d)$. It is thus useful to introduce the following condition which ensures the existence of the representation given in Lemma 8.

Condition 6. *$\int_0^T\int_0^T \widehat{K}_\alpha(X_t - X_s)\,\mathrm{d}t\,\mathrm{d}s < \infty$ P-a.s.*

Remark 7. *$\int_0^T\int_0^T \widehat{K}_\alpha(X_t - X_s)\,\mathrm{d}t\,\mathrm{d}s < \infty$, implies that the function $(t,s) \mapsto X_t - X_s$ is different from zero except possibly on a zero measure set of $[0,T]^2$, and that the well-defined function $(t,s) \mapsto \widehat{K}_\alpha(X_t - X_s)$ is Lebesgue integrable on $[0,T]^2$. From now on the notation $\widehat{K}_\alpha$ will simply be replaced by K_α.*

Lemma 8. *Under Condition 6 we have the following double integral representation for the norm of η_ε defined in Lemma 5:*

$$\|\eta_\varepsilon\|^2_{L^2(\mathbb{R}^d)} = \int_0^T \int_0^T K_\alpha(X_t - X_s)\langle D_\varepsilon X_t, D_\varepsilon X_s\rangle\,\mathrm{d}t\,\mathrm{d}s.$$

Proof. We have

$$\begin{aligned}
\|\eta_\varepsilon\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \big\langle \eta_\varepsilon(x), \eta_\varepsilon(x)\big\rangle\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \left\langle \int_0^T K_{\alpha/2}(x - X_t) D_\varepsilon X_t\,\mathrm{d}t, \int_0^T K_{\alpha/2}(x - X_s) D_\varepsilon X_s\,\mathrm{d}s \right\rangle \mathrm{d}x \\
&= \int_0^T \int_0^T \int_{\mathbb{R}^d} K_{\alpha/2}(x - X_t) K_{\alpha/2}(x - X_s) \langle D_\varepsilon X_t, D_\varepsilon X_s\rangle\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s
\end{aligned}$$

if we can apply the Fubini theorem. Then it is sufficient to use the property

$$\int_{\mathbb{R}^d} K_{\alpha/2}(x-y)K_{\alpha/2}(x-z)\,\mathrm{d}x = K_\alpha(y-z).$$

Since the process X is continuous, $(t,s)\mapsto D_\varepsilon X_s D_\varepsilon X_t$ is a continuous two-parameter process which on $[0,T]^2$ is a.s. bounded. Then a sufficient condition to apply the Fubini theorem is

$$\int_0^T\int_0^T\int_{\mathbb{R}^d} K_{\alpha/2}(x-X_t)K_{\alpha/2}(x-X_s)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s<\infty.$$

Since the integrand is positive, it is equal to

$$\int_0^T\int_0^T\left(\int_{\mathbb{R}^d} K_{\alpha/2}(x-X_t)K_{\alpha/2}(x-X_s)\,\mathrm{d}x\right)\mathrm{d}t\,\mathrm{d}s=\int_0^T\int_0^T K_\alpha(X_t-X_s)\,\mathrm{d}t\,\mathrm{d}s.$$

Invoking Condition 6 we can conclude the proof. □

The double integral representation of the norm of η_ε will play a major role in the following, so we introduce the notation

$$Z_{\alpha,\varepsilon}:=\int_0^T\int_0^T K_\alpha(X_t-X_s)\langle D_\varepsilon X_t, D_\varepsilon X_s\rangle\,\mathrm{d}t\,\mathrm{d}s.$$

Lemma 9. *Assume Condition 6 holds for any $\alpha\ge\overline{\alpha}$. Then the function $\alpha\mapsto Z_{\alpha,\varepsilon}\ge 0$ is decreasing for $\alpha\ge\overline{\alpha}$.*

Proof. Denote $\eta_{\alpha,\varepsilon}(x)=\int_0^T K_{\alpha/2}(x-X_t)D_\varepsilon X_t\,\mathrm{d}t$ making explicit the dependence on α. It is not difficult to prove that, if $\overline{\alpha}\le\alpha\le\beta$, we have $\eta_{\beta,\varepsilon}=(1-\Delta)^{(\alpha-\beta)/2}\eta_{\alpha,\varepsilon}$. Then $Z_{\beta,\varepsilon}=\|\eta_{\beta,\varepsilon}\|^2_{L^2(\mathbb{R}^d)}=\|(1-\Delta)^{(\alpha-\beta)/2}\eta_{\alpha,\varepsilon}\|^2_{L^2(\mathbb{R}^d)}\le\|\eta_{\alpha,\varepsilon}\|^2_{L^2(\mathbb{R}^d)}=Z_{\alpha,\varepsilon}$, since being $\alpha-\beta\le 0$, the operator $(1-\Delta)^{(\alpha-\beta)/2}$ has a norm bounded by one. □

We will assume below the following condition on the convergence of the regularized integrals:

Condition 10. *For every $\varphi\in S(\mathbb{R}^d)$, $I_\varepsilon(\varphi)$ converges in probability to some r.v., denoted by $I(\varphi)$.*

Under Condition 10, the mapping $\varphi\mapsto I(\varphi)$ is *a priori* defined only on $S(\mathbb{R}^d)$ with values in the set $L^0(\Omega)$ of random variables. Its extension to $\varphi\in H_2^\alpha(\mathbb{R}^d)$ is a result of the next theorem.

Theorem 11. *Assume Conditions 6 and 10, and the a priori bound*

$$\sup_{\varepsilon\in(0,1)} E\left[\int_0^T\int_0^T K_\alpha(X_t-X_s)\langle D_\varepsilon X_t, D_\varepsilon X_s\rangle\,\mathrm{d}t\,\mathrm{d}s\right]<\infty. \tag{14}$$

Then:

(i) *The mappings $\varphi\in S(\mathbb{R}^d)\mapsto I_\varepsilon(\varphi), I(\varphi)\in L^0(\Omega)$ take values in $L^2(\Omega)$ and extend (uniquely) to linear continuous mappings from $H_2^\alpha(\mathbb{R}^d)$ to $L^2(\Omega)$. Moreover, for every $\varphi\in H_2^\alpha(\mathbb{R}^d)$, $I_\varepsilon(\varphi)\to I(\varphi)$ in probability and in $L^{2-\delta}(\Omega)$ for every $\delta>0$.*

(ii) *In addition, there exist random elements $\xi_\varepsilon,\xi:\Omega\to H_2^{-\alpha}(\mathbb{R}^d)$ [in fact belonging to $L^2(\Omega;H_2^{-\alpha}(\mathbb{R}^d))$] that constitute pathwise redefinitions of I_ε and I on $V=H_2^\alpha(\mathbb{R}^d)$, in the sense of Definition 1. Moreover, $\xi_\varepsilon\to\xi$ weakly in $L^2(\Omega;H_2^{-\alpha}(\mathbb{R}^d))$: this means that for all $\Psi\in L^2(\Omega,H_2^\alpha(\mathbb{R}^d))$. $E(\langle\xi_\varepsilon,\Psi\rangle)\to E(\langle\xi,\Psi\rangle)$.*

Remark 12. *If the convergence in probability in Condition 10 is replaced by a convergence in $L^2(\Omega)$, then we can take $\delta=0$ in statement* (i) *of Theorem 11.*

Proof of Theorem 11. *Step 1* (mean square results). By the assumptions and the previous lemma we have $\sup_{\varepsilon\in(0,1)} E[\|\eta_\varepsilon\|^2_{L^2(\mathbb{R}^d)}] < \infty$. From (12), for $\varphi \in S(\mathbb{R}^d)$ we have

$$I_\varepsilon(\varphi) = \langle (1-\Delta)^{\alpha/2}\varphi, \eta_\varepsilon \rangle_{L^2(\mathbb{R}^d)}.$$

Therefore, always for $\varphi \in S(\mathbb{R}^d)$,

$$\big|I_\varepsilon(\varphi)\big| \le \|\eta_\varepsilon\|_{L^2(\mathbb{R}^d)} \big\|(1-\Delta)^{\alpha/2}\varphi\big\|_{L^2(\mathbb{R}^d)} \le C_\alpha \|\eta_\varepsilon\|_{L^2(\mathbb{R}^d)} \|\varphi\|_{H_2^\alpha(\mathbb{R}^d)},$$

$$E\big[\big|I_\varepsilon(\varphi)\big|^2\big] \le C_\alpha \|\varphi\|^2_{H_2^\alpha(\mathbb{R}^d)} E\big[\|\eta_\varepsilon\|^2_{L^2(\mathbb{R}^d)}\big] \le C'_\alpha \|\varphi\|^2_{H_2^\alpha(\mathbb{R}^d)}.$$

Immediately we have $I_\varepsilon(\varphi) \in L^2(\Omega)$ for every $\varphi \in S(\mathbb{R}^d)$, and the mapping $\varphi \mapsto I_\varepsilon(\varphi)$ extends (uniquely by density) to a linear continuous mapping from $H_2^\alpha(\mathbb{R}^d)$ to $L^2(\Omega)$.

Given $\varphi \in S(\mathbb{R}^d)$, since $I_\varepsilon(\varphi) \to I(\varphi)$ in probability, uniform integrability arguments and $E[|I_\varepsilon(\varphi)|^2] \le C'_\alpha \|\varphi\|^2_{H_2^\alpha(\mathbb{R}^d)}$, yield $I_\varepsilon(\varphi) \to I(\varphi)$ in $L^{2-\delta}(\Omega)$ for every $\delta > 0$; moreover, it is not difficult to deduce $I(\varphi) \in L^2(\Omega)$ and $E[|I(\varphi)|^2] \le C'_\alpha \|\varphi\|^2_{H_2^\alpha(\mathbb{R}^d)}$. As before, this implies that the mapping $\varphi \mapsto I(\varphi)$ extends uniquely to a linear continuous mapping from $H_2^\alpha(\mathbb{R}^d)$ to $L^2(\Omega)$. Now, with these extensions, it is not difficult to show that $I_\varepsilon(\varphi) \to I(\varphi)$ in $L^{2-\delta}(\Omega)$ for every $\delta > 0$ also for every $\varphi \in H_2^\alpha(\mathbb{R}^d)$.

Step 2 (pathwise results). We still have to construct ξ_ε and ξ. Recalling that $\eta_\varepsilon \in L^2(\Omega; L^2(\mathbb{R}^d))$, ξ_ε is simply defined as $[(1-\Delta)^{\alpha/2}]^\star \eta_\varepsilon$, element of $L^2(\Omega; H_2^{-\alpha}(\mathbb{R}^d))$, where $A^\star$ denotes the dual of an operator A.

To this end, recall that $(1-\Delta)^{\alpha/2}$ is an isomorphism between $H_2^\alpha(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, and thus the dual operator $[(1-\Delta)^{\alpha/2}]^\star$ is an isomorphism between the dual spaces $L^2(\mathbb{R}^d)$ and $H_2^{-\alpha}(\mathbb{R}^d)$ (we identify $L^2(\mathbb{R}^d)$ with its dual).

The family $\{\eta_\varepsilon\}$ is bounded in $L^2(\Omega; L^2(\mathbb{R}^d))$, hence there exist a sequence η_{ε_n} weakly convergent to some η in $L^2(\Omega; L^2(\mathbb{R}^d))$: $E\langle \eta_{\varepsilon_n}, Y\rangle_{L^2(\mathbb{R}^d)} \to E\langle \eta, Y\rangle_{L^2(\mathbb{R}^d)}$ for every $Y \in L^2(\Omega; L^2(\mathbb{R}^d))$. We set

$$\xi := \big[(1-\Delta)^{\alpha/2}\big]^\star \eta, \qquad \xi_\varepsilon := \big[(1-\Delta)^{\alpha/2}\big]^\star \eta_\varepsilon \tag{15}$$

random element of $H_2^{-\alpha}(\mathbb{R}^d)$. We shall see that this definition does not depend on the sequence ε_n. We have to prove that for every $\varphi \in S(\mathbb{R}^d)$, $I(\varphi)(\omega) = (\xi(\omega))(\varphi)$ for P-a.s. $\omega \in \Omega$. Equivalently we have to prove that for every $\varphi \in S(\mathbb{R}^d)$

$$I(\varphi)(\omega) = \langle \eta(\omega), (1-\Delta)^{\alpha/2}\varphi \rangle_{L^2(\mathbb{R}^d)} \quad \text{for } P\text{-a.s. } \omega \in \Omega. \tag{16}$$

We already know that $I_\varepsilon(\varphi) = \langle (1-\Delta)^{\alpha/2}\varphi, \eta_\varepsilon\rangle_{L^2(\mathbb{R}^d)}$ for every $\varepsilon > 0$. Choose Y above of the form $Y = F(1-\Delta)^{\alpha/2}\varphi$ with generic $F \in L^2(\Omega)$. Given $\varphi \in S(\mathbb{R}^d)$, we know that

$$E\big[F\langle \eta_{\varepsilon_n}, (1-\Delta)^{\alpha/2}\varphi\rangle_{L^2(\mathbb{R}^d)}\big] \to E\big[F\langle \eta, (1-\Delta)^{\alpha/2}\varphi\rangle_{L^2(\mathbb{R}^d)}\big]$$

for every $F \in L^2(\Omega)$. Hence

$$E\big[F I_\varepsilon(\varphi)\big] \to E\big[F\langle \eta, (1-\Delta)^{\alpha/2}\varphi\rangle_{L^2(\mathbb{R}^d)}\big]$$

but we also know that $I_\varepsilon(\varphi) \to I(\varphi)$ in $L^{2-\delta}(\Omega)$ for every $\delta > 0$. We get

$$E\big[F I(\varphi)\big] = E\big[F\langle \eta, (1-\Delta)^{\alpha/2}\varphi\rangle_{L^2(\mathbb{R}^d)}\big]$$

at least for every bounded random variable F, hence (16) holds true. This also implies that the definition of ξ does not depend on the sequence ε_n.

Step 3 (weak convergence). A consequence of (15) is that

$$\|\xi_\varepsilon\|^2_{H_2^{-\alpha}(\mathbb{R}^d)} = \|\eta_\varepsilon\|^2_{L^2(\mathbb{R}^d)}.$$

Using Lemma 8 and Assumption 14, it follows that $\|\xi_\varepsilon\|^2_{L^2(\Omega, H_2^{-\alpha}(\mathbb{R}^d))}$ is bounded in $\varepsilon > 0$. We remark that the linear space $\mathcal{C}$ of simple random elements of the type $\sum_{i=1}^n \psi_i Z_i$, where $Z_i \in L^\infty(\Omega)$ and $\psi_i \in H_2^\alpha(\mathbb{R}^d)$ is dense in the Hilbert space $\mathcal{H} := L^2(\Omega, H_2^{-\alpha}(\mathbb{R}^d))$. We consider the linear forms $T^\varepsilon, T : \mathcal{H} \to \mathbb{R}$ defined by $T^\varepsilon(\Psi) = E(\langle \xi^\varepsilon, \Psi\rangle)$, $T(\Psi) = E(\langle \xi, \Psi\rangle)$. Since ξ_ε are pathwise redefinitions of I, then point (i) implies that, for every $\Psi \in \mathcal{C}$, $T^\varepsilon(\Psi) \to T(\Psi)$ when $\varepsilon \to 0$. Finally by the Banach–Steinhaus theorem we obtain $T^\varepsilon(\Psi) \to T(\Psi)$ for any $\Psi \in \mathcal{H}$ and so ξ_ε weakly converges to ξ. □

To state a possible converse of Theorem 11, it is useful to introduce a weaker version of Definition 1.

Definition 13. *Let $A \in \mathcal{F}$ be such that $P(A) > 0$. We say that $\{I(\varphi)\}_{\varphi\in\mathcal{D}}$ has a* pathwise redefinition on V restricted to A *if there exists a measurable mapping $\xi : A \to V'$ such that for every $\varphi \in S(\mathbb{R}^d)$ Eq. (5) holds true for P-a.e. $\omega \in A$.*

Theorem 14. *Assume Conditions 6 and 10, and the a priori bound*

$$E\left[\int_0^T\int_0^T K_\alpha(X_t - X_s)\langle D_\varepsilon X_t, D_\varepsilon X_s\rangle \,\mathrm{d}t\,\mathrm{d}s\right] < \infty$$

for every $\varepsilon > 0$. If there exists $A \in \mathcal{F}$ with $P(A) > 0$ such that $\{I(\varphi)\}_{\varphi\in H^\alpha(\mathbb{R}^d)}$ has a pathwise redefinition on $H_2^\alpha(\mathbb{R}^d)$ restricted to A, then

$$\limsup_{\varepsilon\to 0}\int_0^T\int_0^T K_\alpha(X_t - X_s)\langle D_\varepsilon X_t, D_\varepsilon X_s\rangle \,\mathrm{d}t\,\mathrm{d}s < \infty \quad P\text{-a.s. } \omega \in A.$$

Proof. The first part of Step 1 of the previous proof is still valid (except for the uniformity in ε of the constants). Thus in particular $E[\|\eta_\varepsilon\|^2_{L^2(\mathbb{R}^d)}] < \infty$, $I_\varepsilon(\varphi) = \langle (1-\Delta)^{\alpha/2}\varphi, \eta_\varepsilon\rangle$, $|I_\varepsilon(\varphi)| \le C_{\alpha,\varepsilon}\|\varphi\|^2_{H_2^\alpha(\mathbb{R}^d)}$, $I_\varepsilon(\varphi) \in L^2(\Omega)$ for every $\varphi \in S(\mathbb{R}^d)$ and the mapping $\varphi \mapsto I_\varepsilon(\varphi)$ extends to a linear continuous mapping from $H_2^\alpha(\mathbb{R}^d)$ to $L^2(\Omega)$.

We know, by Condition 10 and the existence of ξ, that for every $\varphi \in S(\mathbb{R}^d)$

$$\big\langle (1-\Delta)^{\alpha/2}\varphi, \eta_\varepsilon(\omega)\big\rangle_{L^2(\Omega)} \underset{\varepsilon\to 0}{\to} \big(\xi(\omega)\big)(\varphi) \quad \text{for } P\text{-a.e. } \omega \in A.$$

One can find a countable set $\mathcal{D} \subset S(\mathbb{R}^d)$ with the following two properties: (i) $(1-\Delta)^{\alpha/2}\mathcal{D}$ is dense in $L^2(\mathbb{R}^d)$ and (ii) for P-a.e. $\omega \in A$

$$\big\langle (1-\Delta)^{\alpha/2}\varphi, \eta_\varepsilon(\omega)\big\rangle_{L^2(\Omega)} \underset{\varepsilon\to 0}{\to} \big(\xi(\omega)\big)(\varphi) \quad \text{for every } \varphi \in \mathcal{D}. \tag{17}$$

Let us prove the claim by contradiction. Assume there is $A' \in \mathcal{F}$, $A' \subset A$, $P(A') > 0$, such that

$$\limsup_{\varepsilon\to 0}\int_0^T\int_0^T K_\alpha(X_t - X_s)\langle D_\varepsilon X_t, D_\varepsilon X_s\rangle \,\mathrm{d}t\,\mathrm{d}s = \infty \quad \text{for every } \omega \in A'.$$

By Lemma 8, $\limsup_{\varepsilon\to 0}\|\eta_\varepsilon(\omega)\|^2_{L^2(\mathbb{R}^d)} = \infty$ for every $\omega \in A'$.

Consequently, there is a subset A'' of A' with $P(A'') > 0$ such that (17) holds true for every $\omega \in A''$. Thus, given $\omega \in A''$, there is an infinitesimal sequence $\{\varepsilon_n(\omega)\}$ such that $\lim_{n\to\infty}\|\eta_{\varepsilon_n(\omega)}(\omega)\|^2_{L^2(\mathbb{R}^d)} = \infty$ but at the same time

$$\lim_{n\to\infty}\big\langle (1-\Delta)^{\alpha/2}\varphi, \eta_{\varepsilon_n(\omega)}(\omega)\big\rangle_{L^2(\Omega)} = \big(\xi(\omega)\big)(\varphi)$$

for every $\varphi \in \mathcal{D}$. This is impossible because of the density of $(1-\Delta)^{\alpha/2}\mathcal{D}$ in $L^2(\mathbb{R}^d)$. The proof is complete. □

Definition 15. *We say that a.s. a stochastic current I does not belong to V' if there is no $A \in \mathcal{F}$, $P(A) > 0$ such that $\{I(\varphi)\}_{\varphi\in V}$ has a pathwise redefinition on V restricted to A.*

Corollary 16. *Under the conditions of Theorem 14, if*

$$\limsup_{\varepsilon\to 0}\int_0^T\int_0^T K_\alpha(X_t-X_s)\langle D_\varepsilon X_t, D_\varepsilon X_s\rangle\,\mathrm{d}t\,\mathrm{d}s=+\infty \quad P\text{-}a.s.$$

the current I does not belong to $H_2^\alpha(\mathbb{R}^d)$.

3. Application to the fractional Brownian motion

We recall here that a fractional Brownian motion (fBm) $B=(B_t)$ with Hurst index $H\in(0,1)$, is a Gaussian mean-zero real process whose covariance function is given by

$$\operatorname{Cov}(B_s,B_t)=\frac{1}{2}\big(|s|^{2H}+|t|^{2H}-|s-t|^{2H}\big),\quad (s,t)\in\mathbb{R}_+^2.$$

This process has been widely studied: for some recent developments, we point to [3] as a relevant monograph. For instance we recall that when $H=\frac{1}{2}$, B is a classical Wiener process. Its trajectories are Hölder continuous with respect to any parameter $\gamma<H$. Recall that

$$E\big(|B_t-B_s|^2\big)=|t-s|^{2H}.$$

For our stochastic integral redefinition, we have chosen the framework of stochastic calculus via regularization; for a survey about the topic and recent developments, see [21].

Let $X=(X^1,\dots,X^d)$ be a d-dimensional fractional Brownian motion with Hurst index $H\in(0,1)$, i.e., an $\mathbb{R}^d$-valued process whose components $X^1,\dots,X^d$ are real independent fractional Brownian motions. In this section we study the regularity of the current generated by X using the results of the previous section and in particular Theorem 11 which gives sufficient conditions for regularity in the Hilbert spaces $H_2^\alpha(\mathbb{R}^d)$.

We will consider the *symmetric* and *forward* integrals, respectively defined as the limit in probability, as $\varepsilon\to 0$, of

$$I_\varepsilon^\circ(\varphi):=\int_0^T\left\langle\varphi(X_t),\frac{X_{t+\varepsilon}-X_{t-\varepsilon}}{2\varepsilon}\right\rangle\mathrm{d}t$$

and

$$I_\varepsilon^-(\varphi):=\int_0^T\left\langle\varphi(X_t),\frac{X_{t+\varepsilon}-X_t}{\varepsilon}\right\rangle\mathrm{d}t.$$

Whenever they exist we will write

$$I^\circ(\varphi)=\int_0^T\big\langle\varphi(X_t),d^0X_t\big\rangle\quad\text{and}\quad I^-(\varphi)=\int_0^T\big\langle\varphi(X_t),\mathrm{d}^-X_t\big\rangle.$$

In [1] the authors show that the symmetric integral exists for fBm with any $H>1/4$ (the proof is about the $d=1$ case but it extends without problem to higher dimensions). For $H>1/6$ necessary and sufficient conditions for the existence of the symmetric integral are given in [12]; see also [11] for a complete characterization when $H>\frac{1}{4}$.

As far as the forward integral is concerned, it is known that, in dimension 1, it does not exist at least for some (very smooth) functions φ. Existence of the forward integral in dimension one is guaranteed if and only if $H\geq 1/2$ [20,21]. Observe that, when $H<\frac{1}{2}$ the forward integral $\int_0^T B\,\mathrm{d}^-B$ does not exist since B is not of finite quadratic variation process.

When $H\geq 1/2$ and for $d>1$ the forward integral (equal to the Young integral in the case $H>1/2$), is equal to the symmetric integral minus the covariation $[\varphi(X),X]/2$. This exists if X has all its *mutual covariations* and the ith component of that bracket gives

$$\big[f(X),X^i\big]_t=\int_0^t\sum_{j=1}^d\partial_j f(X_s)\,\mathrm{d}\big[X^i,X^j\big]_s,$$

see again [20].

Remark 17. *Let X be a fBm with Hurst index H and denote again by $I^0(\varphi)$ (resp. $I^-(\varphi)$) the corresponding symmetric (resp. forward) integral. In fact,* [1] *allows us to replace the convergence in probability with a convergence in $L^2(\Omega)$ of $I^\circ_\varepsilon(\varphi)$ to $I^\circ(\varphi)$. Using again* [20], *if $H \geq \frac{1}{2}$, $I^-_\varepsilon(\varphi) \to I^-(\varphi)$ in $L^2(\Omega)$.*

Before entering into details concerning stochastic integration, we state an important preliminary result.

Proposition 18. *If $\alpha > \max(0, d/2 - 1/(2H))$, then $E(\int_0^T\int_0^T K_\alpha(X_t - X_s)\,\mathrm{d}s\,\mathrm{d}t) < +\infty$. Therefore Condition 6 holds.*

Proof. Since the corresponding random variable is non-negative, previous expectation equals

$$\int_0^T\int_0^T E\big(K_\alpha(X_t - X_s)\big)\,\mathrm{d}s\,\mathrm{d}t = 2\int_0^T\int_0^t E\big(K_\alpha(X_t - X_s)\big)\,\mathrm{d}s\,\mathrm{d}t. \tag{18}$$

Next we use the bound on K_α given in Lemma 2:

- Suppose first that $d/2 - 1/(2H) < \alpha < \frac{d}{2}$. Then the previous expression is bounded by $\mathrm{const}\int_0^T\int_0^t E(|X_t - X_s|^{2\alpha-d})\,\mathrm{d}s\,\mathrm{d}t$. For any $\gamma > -d$, the scaling property of fractional Brownian motion gives $E[|X_t - X_s|^\gamma] = E[|N|^\gamma](t-s)^{\gamma H}$, where N is a standard d-dimensional Gaussian random variable. Then the right member of (18) equals
$$\mathrm{const}\, E\big[|N|^{2\alpha-d}\big]\int_0^T\int_0^t |t-s|^{2\alpha H - dH}\,\mathrm{d}s\,\mathrm{d}t, \tag{19}$$
which is finite when $\alpha > \max(d/2 - 1/(2H), 0)$.
- Suppose now that $\frac{d}{2} < \alpha$. In this case $K_\alpha \leq \mathrm{const}$ and (18) is trivially bounded.
- Finally, in the case $\alpha = d/2$ we have $K_\alpha(x) \leq c\log|x|$ and using again the scaling, it is easy to prove the boundedness of (18). □

The proposition above will allow us to verify the condition required by Lemma 8 of the previous section.

3.1. *Symmetric integral*

Let $D^0_\varepsilon X_t = (X_{t+\varepsilon} - X_{t-\varepsilon})/2\varepsilon$ and let $f:\mathbb{R}^d - \{0\} \to \mathbb{R}_+$ such that

$$\int_0^T\int_0^T f(X_t - X_s)\,\mathrm{d}t\,\mathrm{d}s < \infty \quad P\text{-a.s.} \tag{20}$$

and denote

$$Z_\varepsilon(f) := \int_0^T\int_0^T f(X_t - X_s)\big\langle D^0_\varepsilon X_t, D^0_\varepsilon X_s\big\rangle\,\mathrm{d}s\,\mathrm{d}t.$$

In this section we will study the r.v. $Z_{\alpha,\varepsilon} = Z_\varepsilon(K_\alpha)$ in order to obtain necessary and sufficient conditions for the regularity of the symmetric current $I(\varphi)$ based on fBm.

The following lemma is proved in the Appendix.

Lemma 19. (1) *We have the following estimates*:

$$\big|\mathrm{Cov}\big(D^0_\varepsilon X^i_t, D^0_\varepsilon X^i_s\big)\big| \leq \mathrm{const}\,|t-s|^{2H-2}; \tag{21}$$

and

$$\left|\mathrm{Cov}\left(D_\varepsilon^0 X_t^i, X_t^i - X_s^i\right)\right| = \left|\mathrm{Cov}\left(D_\varepsilon^0 X_s^i, X_t^i - X_s^i\right)\right| \le \mathrm{const}\,|t-s|^{2H-1} \tag{22}$$

uniformly in $\varepsilon > 0$.

(2) *If* $s \neq t$, *one has*

$$\lim_{\varepsilon\to 0}\left|\mathrm{Cov}\left(D_\varepsilon^0 X_t^1, X_t^1 - X_s^1\right)\right| = 2H|t-s|^{2H-1}.$$

The main results of this section are contained in the next two theorems. Let $\alpha_H = d/2 - 1 + 1/(2H)$.

Theorem 20. *For any* $H > 1/4$ *and any* $\alpha > \alpha_H$ *we have* $\sup_\varepsilon EZ_{\alpha,\varepsilon} < \infty$.

Remark 21. *We observe that* $d/2 - 1/2H < \alpha_H$ *so that for the range value for* α *in Theorem* 20, *Condition* 6 *is verified.*

Theorem 22. *For any* $H > 1/4$ *and any* $\alpha < \alpha_H$ *we have* $\liminf_\varepsilon E(Z_{\alpha,\varepsilon}) = +\infty$.

Remark 23. *The statement of Theorem* 22 *and Lemma* 26 *allow us to formulate the following conjecture that we have not been able to prove*: *for* $H > 1/4$ *and* $\alpha < \alpha_H$ *we should have*

$$\liminf_{\varepsilon\to 0} Z_{\alpha,\varepsilon} = +\infty \quad a.s.$$

For $\{I(\varphi)\}_{\varphi\in H_2^\alpha(\mathbb{R}^d)}$ *to have a pathwise redefinition on* $H_2^\alpha(\mathbb{R}^d)$ *restricted to some* $A \in \mathcal{F}$, $P(A) > 0$ *a necessary condition is that* $\limsup_{\varepsilon\to 0} Z_{\alpha,\varepsilon} < \infty$ *on* A (*Theorem* 14).

If this conjecture were true we could establish that a.s. $I(\varphi)$ *does not have a pathwise redefinition on* $H_2^\alpha(\mathbb{R}^d)$ *when* $\alpha < \alpha_H$.

Before proving the theorems we deduce Sobolev regularity of fBm with any Hurst parameter between $(1/4, 1)$.

Corollary 24 (Regularity of symmetric currents). *The symmetric integral of a fractional Brownian motion with Hurst parameter* $H > 1/4$ *admits a pathwise redefinition on the space* $H_2^\alpha(\mathbb{R}^d)$ *for any* $\alpha > \alpha_H$.

Proof. By Theorem 20 we know that $E[Z_{\alpha,\varepsilon}]$ is uniformly bounded in ε when $\alpha > \alpha_H$. Since the regularized integrals $I_\varepsilon(\varphi)$ converge in $L^2(\Omega)$, as $\varepsilon \to 0$ for any $H > 1/4$, see Remark 12. So Condition 10 holds and we can apply Theorem 11 to obtain a pathwise current with values in $H_2^{-\alpha}(\mathbb{R}^d)$ for any $\alpha > \alpha_H$. □

In the following proof we will use a basic result about Gaussian random variables recalled here:

Lemma 25 (Wick's theorem). *Let* $Z = (Z_\ell)_{1\le\ell\le N}$ *be a mean-zero Gaussian random vector and* $f \in C^1(\mathbb{R}^N;\mathbb{R})$, *then we have*

$$E\left[Z_\ell f(Z)\right] = \sum_{j=1}^N \mathrm{Cov}(Z_\ell, Z_j) E\left[\nabla_j f(Z)\right]. \tag{23}$$

Proof. The conclusion follows easily taking $f(z_1,\dots,z_N) = \exp(\mathrm{i}\sum_{j=1}^N t_j z_j)$ for any $t = (t_1,\dots,t_N) \in \mathbb{R}^N$. In fact, in that case $E(f(Z))$ is provided by the characteristic function. Therefore one has

$$E\left(\exp(\mathrm{i}t\cdot Z)\right) = \exp\left(-\frac{t\Gamma t'}{2}\right),$$

where t' stands for the transposition and Γ is the covariance matrix of Z. Differentiating the previous expression with respect to t_ℓ provides the result (23) for the particular case of f. The general result follows by usual density arguments. □

Proof of Theorem 20. Using Lemma 25, independence and equal distribution of different coordinates we have

$$
\begin{aligned}
EZ_{\alpha,\varepsilon} &= E\int_0^T\int_0^T K_\alpha(X_t-X_s)\sum_i \operatorname{Cov}\bigl(D_\varepsilon^0 X_t^i, D_\varepsilon^0 X_s^i\bigr)\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + E\int_0^T\int_0^T \Delta K_\alpha(X_t-X_s)\operatorname{Cov}\bigl(D_\varepsilon^0 X_t^1, X_t^1-X_s^1\bigr)\operatorname{Cov}\bigl(D_\varepsilon^0 X_s^1, X_t^1-X_s^1\bigr)\,\mathrm{d}t\,\mathrm{d}s \\
&= E\int_0^T\int_0^T K_\alpha(X_t-X_s)\sum_i \operatorname{Cov}\bigl(D_\varepsilon^0 X_t^i, D_\varepsilon^0 X_s^i\bigr)\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + E\int_0^T\int_0^T (-\Delta)K_\alpha(X_t-X_s)\bigl|\operatorname{Cov}\bigl(D_\varepsilon^0 X_t^1, X_t^1-X_s^1\bigr)\bigr|^2\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}
\tag{24}
$$

where we used the fact that

$$
\operatorname{Cov}\bigl(D_\varepsilon^0 X_t^1, X_t^1-X_s^1\bigr) = -\operatorname{Cov}\bigl(D_\varepsilon^0 X_s^1, X_t^1-X_s^1\bigr) = \frac{1}{2\varepsilon}\bigl(|t-s+\varepsilon|^{2H}-|t-s-\varepsilon|^{2H}\bigr)
$$

which can be verified by a straightforward computation.

Consider first the case $H>1/2$. Assume $\alpha<d/2$ and note that when $H>1/2$ we have $\alpha_H<d/2$. By Lemma 19 we have

$$
\begin{aligned}
EZ_{\alpha,\varepsilon} &\le \mathrm{const}\,E\int_0^T\int_0^T K_\alpha(X_t-X_s)|t-s|^{2H-2}\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + \mathrm{const}\,E\int_0^T\int_0^T \bigl|(-\Delta)K_\alpha(X_t-X_s)\bigr||t-s|^{4H-2}\,\mathrm{d}t\,\mathrm{d}s
\end{aligned}
$$

and then, using Lemma 2, (1) and (4), we get

$$
\begin{aligned}
EZ_{\alpha,\varepsilon} &\le \mathrm{const}\,E\int_0^T\int_0^T |X_t-X_s|^{2\alpha-d}|t-s|^{2H-2}\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + \mathrm{const}\,E\int_0^T\int_0^T |X_t-X_s|^{2\alpha-d-2}|t-s|^{4H-2}\,\mathrm{d}t\,\mathrm{d}s \\
&= \mathrm{const}\int_0^T\int_0^T |t-s|^{(2\alpha+2-d)H-2}\,\mathrm{d}t\,\mathrm{d}s + \mathrm{const}\int_0^T\int_0^T |t-s|^{(2\alpha-d+2)H-2}\,\mathrm{d}t\,\mathrm{d}s
\end{aligned}
\tag{25}
$$

which are uniformly bounded in ε if $(2\alpha+2-d)H>1$, i.e., when $\alpha>\alpha_H$ as required. We have established the uniform bound when $\alpha\in(\alpha_H,d/2)$ but now recall that $EZ_{\alpha,\varepsilon}$ is a decreasing function of α so that this bound extends to all $\alpha>\alpha_H$.

Let us now consider the case $H\le 1/2$ (so that now $\alpha_H>d/2$) and assume $\alpha>d/2$. Rewrite $Z_{\alpha,\varepsilon}$ as

$$
Z_{\alpha,\varepsilon} = Z_\varepsilon(h_\alpha) + Z_\varepsilon\bigl(K_\alpha(0)\bigr), \tag{26}
$$

where $h_\alpha(x)=K_\alpha(x)-K_\alpha(0)$. Note that $0<K_\alpha(0)<\infty$ when $\alpha>d/2$. Moreover

$$
Z_\varepsilon\bigl(K_\alpha(0)\bigr) = \int_0^T\int_0^T K_\alpha(0)\langle D_\varepsilon^0 X_s, D_\varepsilon^0 X_t\rangle = K_\alpha(0)\left|\int_0^T D_\varepsilon^0 X_s\,\mathrm{d}s\right|^2
$$

and

$$\int_0^T D_\varepsilon^0 X_t \, \mathrm{d}t = \int_0^T \frac{X_{t+\varepsilon} - X_{t-\varepsilon}}{\varepsilon} \, \mathrm{d}t = \varepsilon^{-1} \int_T^{T+\varepsilon} X_t \, \mathrm{d}t$$

so that by the continuity of the process X we have the limit

$$\lim_{\varepsilon \to 0} Z_\varepsilon \big(K_\alpha(0) \big) = K_\alpha(0) |X_T|^2$$

exists almost surely and $Z_\varepsilon(K_\alpha(0))$ is uniformly in L^1 (actually in all L^p).

So it remains to consider $Z_\varepsilon(h_\alpha)$. For h_α we have the estimate (9), provided $0 < 2\alpha - d \le 2$, so that we obtain an upper bound similar to Eq. (25) and the same condition on α follows. Note that α must satisfy

$$\frac{d}{2} < \alpha_H < \alpha < \frac{d}{2} + 1$$

so that we must require $H > 1/4$. Again by monotonicity of $\alpha \mapsto Z_{\alpha,\varepsilon}$ we have uniform boundedness for any $\alpha > \alpha_H$. □

Proof of Theorem 22. We will perform a decomposition of $Z_{\alpha,\varepsilon}$ as follows: Write

$$Z_{\alpha,\varepsilon} = A_{\alpha,\varepsilon} + B_{\alpha,\varepsilon} + Q_{\alpha,\varepsilon},$$

where

$$A_{\alpha,\varepsilon} = \int_0^T \int_0^T K_\alpha(X_t - X_s) \sum_i \mathrm{Cov}\big(D_\varepsilon^0 X_t^i, D_\varepsilon^0 X_s^i \big) \, \mathrm{d}t \, \mathrm{d}s,$$

$$B_{\alpha,\varepsilon} = \int_0^T \int_0^T (-\Delta) K_\alpha(X_t - X_s) \big| \mathrm{Cov}\big(D_\varepsilon^0 X_t^1, X_t^1 - X_s^1 \big) \big|^2 \, \mathrm{d}t \, \mathrm{d}s$$

and $Q_{\alpha,\varepsilon}$ is the remainder. Note that, by comparing this decomposition with Eq. (24), we have

$$E Z_{\alpha,\varepsilon} = E A_{\alpha,\varepsilon} + E B_{\alpha,\varepsilon}$$

so that $E Q_{\alpha,\beta} = 0$. This is a kind of Wick product decomposition, but not quite, since the terms A, B are not constants, but still random variables.

A useful remark is that $A_{\alpha,\varepsilon} \ge 0$ since we can write

$$A_{\alpha,\varepsilon} = \hat{E} \int_0^T \int_0^T K_\alpha(X_t - X_s) \big\langle D_\varepsilon^0 \hat{X}_t, D_\varepsilon^0 \hat{X}_s \big\rangle \mathrm{d}t \, \mathrm{d}s,$$

where we introduced an auxiliary independent d-dimensional fBm $\hat{X}$ with the same distribution of X and where $\hat{E}$ denotes expectation with respect to this auxiliary fBm. So we have the formula $A_{\alpha,\varepsilon} = \hat{E} \|\hat{\eta}_\varepsilon\|^2$, where $\hat{\eta}_\varepsilon(x) = \int_0^T K_{\alpha/2}(x - X_t) D_\varepsilon^0 \hat{X}_t \, \mathrm{d}t$ which shows that $A_{\alpha,\varepsilon} > 0$.

Next, using the equality $-\Delta K_\alpha(x) = K_{\alpha-1}(x) - K_\alpha(x)$ we rewrite $B_{\alpha,\varepsilon}$ as $B_{\alpha,\varepsilon}^{(1)} - B_{\alpha,\varepsilon}^{(2)}$, where

$$B_{\alpha,\varepsilon}^{(1)} = \int_0^T \int_0^T K_{\alpha-1}(X_t - X_s) \big| \mathrm{Cov}\big(D_\varepsilon^0 X_t^1, X_t^1 - X_s^1 \big) \big|^2 \, \mathrm{d}t \, \mathrm{d}s$$

and

$$B_{\alpha,\varepsilon}^{(2)} = \int_0^T \int_0^T K_\alpha(X_t - X_s) \big| \mathrm{Cov}\big(D_\varepsilon^0 X_t^1, X_t^1 - X_s^1 \big) \big|^2 \, \mathrm{d}t \, \mathrm{d}s.$$

Let us show first that $E|B_\varepsilon^{(2)}|$ is uniformly bounded in ε when $\alpha > \alpha_H - 1$ and $H > 1/4$. Indeed when $\alpha \le d/2$ by computations similar to those of Theorem 20 we have that $E|B_\varepsilon^{(2)}|$ is uniformly bounded if $\alpha > \alpha_H - 1$ and $\alpha > 0$. On the other hand, when $\alpha > d/2$, the kernel K_α is bounded, so Lemma 19, (1) allows to write

$$E\big|B_{\alpha,\varepsilon}^{(2)}\big| \le \text{const} \int_0^T \int_0^T \big|\mathrm{Cov}\big(D_\varepsilon^0 X_t^1, X_t^1 - X_s^1\big)\big|^2 \,\mathrm{d}t\,\mathrm{d}s \le \text{const} \int_0^T \int_0^T |t-s|^{4H-2}\,\mathrm{d}t\,\mathrm{d}s \le C$$

uniformly in ε provided $H > 1/4$.

Moreover below we will show the following lemma.

Lemma 26. *If $H > 1/4$ and $\alpha < \alpha_H$ then $\liminf_{\varepsilon\to 0} B_{\alpha,\varepsilon}^{(1)} = +\infty$ a.s.*

Then if we admit the result of previous lemma we can conclude with the use of the Fatou lemma. In fact, for any $\alpha \in (\alpha_H - 1, \alpha_H)$ and for some positive constant c we have

$$\liminf_{\varepsilon\to 0} E(Z_{\alpha,\varepsilon}) \ge \liminf_{\varepsilon\to 0} E(B_{\alpha,\varepsilon}) \ge E\Big(\liminf_{\varepsilon\to 0} B_{\alpha,\varepsilon}^{(1)}\Big) - \sup_\varepsilon E\big(B_{\alpha,\varepsilon}^{(2)}\big) \ge E\Big(\liminf_{\varepsilon\to 0} B_{\alpha,\varepsilon}^{(1)}\Big) - c = +\infty.$$

Moreover observe that this is enough since, from Lemma 9, we have that $Z_{\alpha,\varepsilon}$ is a decreasing function of α so the result will hold for any $\alpha < \alpha_H$. □

Proof of Lemma 26. By the Fatou lemma we have

$$\liminf_{\varepsilon\to 0} B_{\alpha,\varepsilon}^{(1)} \ge 2\int_0^T \int_0^t K_{\alpha-1}(X_t - X_s) \liminf_{\varepsilon\to 0} \big|\mathrm{Cov}\big(D_\varepsilon^0 X_t^1, X_t^1 - X_s^1\big)\big|^2 \,\mathrm{d}t\,\mathrm{d}s.$$

But, when $t \ne s$,

$$\liminf_{\varepsilon\to 0} \big|\mathrm{Cov}\big(D_\varepsilon^0 X_t^1, X_t^1 - X_s^1\big)\big|^2 = 4H^2|t-s|^{4H-2}.$$

Now assume

$$\alpha < \frac{d}{2} + 1, \tag{27}$$

then by Lemma 2 there exists a small constant $r > 0$ such that $K_{\alpha-1}(x) \ge C|x|^{2\alpha-d-2} 1_{B(0,r)}(x)$. This allows us to bound from below as follows:

$$\begin{aligned}\liminf_{\varepsilon\to 0} B_{\alpha,\varepsilon}^{(1)} &\ge \text{const} \int_0^T \int_0^t K_{\alpha-1}(X_t - X_s)|t-s|^{4H-2}\,\mathrm{d}t\,\mathrm{d}s \\ &\ge \text{const} \int_0^T \int_0^t |X_t - X_s|^{2\alpha-d-2} 1_{B(0,r)}(X_t - X_s)|t-s|^{4H-2}\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}$$

Since the paths of fBm are Hölder continuous with parameters strictly smaller than H, for any $\gamma < H$ there exists a random constant $C_{X,\gamma}$ such that

$$|X_t - X_s| \le C_{X,\gamma}|t-s|^\gamma, \quad t, s \in [0, T].$$

By choosing a random time $S > 0$ small enough such that $\sup_{t,s\in[0,S]} |X_t - X_s| < r$ we have

$$\begin{aligned}\liminf_{\varepsilon\to 0} B_{\alpha,\varepsilon}^{(1)} &\ge \text{const} \int_0^S \int_0^t |X_t - X_s|^{2\alpha-d-2}|t-s|^{4H-2}\,\mathrm{d}t\,\mathrm{d}s \\ &\ge \text{const}\, C_{X,\gamma}^{d-2\alpha+2} \int_0^S \int_0^t |t-s|^{2H(\alpha-\alpha_H)-1+\delta}\,\mathrm{d}t\,\mathrm{d}s,\end{aligned} \tag{28}$$

where $\delta = (d + 2 - 2\alpha)(H - \gamma)$ is an arbitrarily small positive constant since $\gamma < H$ can be chosen arbitrarily near to H and

$$d + 2 - 2\alpha > 4 - \frac{1}{H} > 0$$

when $H > 1/4$. Then when $\alpha < \alpha_H$ we can choose δ small enough to make the double integral in Eq. (28) diverge. Summing up, we must have $\alpha < \min(\alpha_H, d/2 + 1)$ and $H > 1/4$. But when $H > 1/4$ we have $\alpha_H < d/2 - 1$ so that sufficient conditions are $\alpha < \alpha_H$ and $H > 1/4$. This observation concludes the proof. □

3.2. *The forward integral*

Let $D_\varepsilon^- X_t = (X_{t+\varepsilon} - X_t)/\varepsilon$, take $f : \mathbb{R}^d - \{0\} \to \mathbb{R}_+$ satisfying (20) and denote

$$Z_\varepsilon^-(f) = \int_0^T \int_0^T f(X_t - X_s)\langle D_\varepsilon^- X_t, D_\varepsilon^- X_s\rangle \mathrm{d}s\, \mathrm{d}t.$$

Lemma 27. 1. *We have the following estimates*:

$$\left|\mathrm{Cov}\left(D_\varepsilon^- X_t^i, D_\varepsilon^- X_s^i\right)\right| \le \mathrm{const}\, |t - s|^{2H-2} \tag{29}$$

and

$$\left|\mathrm{Cov}\left(D_\varepsilon^- X_t^i, X_t^i - X_s^i\right)\right| = \left|\mathrm{Cov}\left(D_\varepsilon^- X_s^i, X_t^i - X_s^i\right)\right| \le \mathrm{const}\, |t - s|^{2H-1}. \tag{30}$$

2. *If $s \ne t$, one has*

$$\lim_{\varepsilon \to 0}\left|\mathrm{Cov}\left(D_\varepsilon^- X_t^1, X_t^1 - X_s^1\right)\right| = 2H|t - s|^{2H-1}.$$

This lemma is proved in the Appendix. Then we can state similar theorems as in the previous section.

Theorem 28. *For any $H \ge 1/2$ and any $\alpha > \alpha_H$ we have* $\sup_\varepsilon E Z_{\alpha,\varepsilon} < \infty$.

Theorem 29. *For $H \ge 1/2$ and $\alpha < \alpha_H$ we have* $\liminf_{\varepsilon\to 0} E(Z_{\alpha,\varepsilon}) = +\infty$.

The proofs follow the same line as the corresponding theorems about symmetric integrals and will be omitted. We just note that Lemma 27 is used instead of Lemma 19 and according to Lemma 27 if $s \ne t$ one has

$$\liminf_{\varepsilon\to 0}\left|\mathrm{Cov}\left(D_\varepsilon^- X_t^1, X_t^1 - X_s^1\right)\right|^2 = 4H^2|t - s|^{4H-2}$$

when $H \ge \frac{1}{2}$. This can be used to prove Theorem 29.

In particular we have the following corollary:

Corollary 30 (Regularity of forward currents). *The forward integral of a fractional Brownian motion with Hurst parameter $H \ge 1/2$ admits a pathwise redefinition on the space $H_2^\alpha(\mathbb{R}^d)$ for any $\alpha > \alpha_H$.*

Proof. By Theorem 28 we know that $E Z_{\alpha,\varepsilon}$ is uniformly bounded in ε when $\alpha > \alpha_H$. Since the regularized integrals $I_\varepsilon^-(\varphi)$ converges in $L^2(\Omega)$, as $\varepsilon \to 0$ for any $H \ge 1/2$, Condition 10 holds and we can apply Theorem 11 to obtain a pathwise current with values in $H_2^{-\alpha}(\mathbb{R}^d)$ for any $\alpha > \alpha_H$. □

4. Brownian regularity in H_p^α, $p \neq 2$

In this section we restrict ourselves to the case when X is a d-dimensional classical Brownian motion, that we denote by W. The key ingredient is the following lemma:

Lemma 31. *If the dimension $d \geq 2$ and the real numbers $\alpha > 1$ and $p' > 1$ satisfy*

$$(d - \alpha + 1)p' < d$$

then

$$\int_{\mathbb{R}^d} E\left[\left(\int_0^T \frac{\exp(-\varepsilon|x - W_t|)}{|x - W_t|^{2d-2\alpha}}\,\mathrm{d}t\right)^{p'/2}\right]\mathrm{d}x < \infty$$

for every $\varepsilon > 0$.

We shall prove below this lemma. Let us first describe its consequences.
From the bounds on $K_{\alpha/2}$, see Lemma 2, we have

$$\int_{\mathbb{R}^d} E\left[\left(\int_0^T K_{\alpha/2}^2(x - W_t)\,\mathrm{d}t\right)^{p'/2}\right]\mathrm{d}x < \infty$$

and thus for a.e. $x \in \mathbb{R}^d$ we have $P(\int_0^T K_{\alpha/2}^2(x - W_t)\,\mathrm{d}t < \infty) = 1$ which implies that the Itô integral

$$\eta(x) := \int_0^T K_{\alpha/2}(x - W_t)\,\mathrm{d}W_t$$

is well defined, for a.e. $x \in \mathbb{R}^d$, as a limit in probability of

$$\int_0^T K_{\alpha/2}(x - W_t)\frac{W_{t+\varepsilon} - W_t}{\varepsilon}\,\mathrm{d}t;$$

see for instance [21]. These approximation integrals are measurable in the pair (x, ω), hence they are measurable in x as a mapping with values in the space of random variables with the metric of convergence in probability, and this way one can see that the limit object $\eta(x)$ is measurable in the pair (x, ω). From the Burkhoder–Davies–Gundy (BDG) inequality we have $\int_{\mathbb{R}^d} E[|\eta(x)|^{p'}]\,\mathrm{d}x < \infty$ and thus $\eta \in L^{p'}(\Omega \times \mathbb{R}^d)$ and

$$P\big(\omega \in \Omega : x \mapsto \eta(x, \omega) \in L^{p'}\big(\mathbb{R}^d\big)\big) = 1.$$

To minimize the subtleties related to a direct use of $\eta(x)$, we introduce a regularization. Let $T(t)$ be the semigroup on $L^2(\mathbb{R}^d)$ generated by $(\Delta - 1)$ and we set

$$K_{\alpha/2}^{(\delta)}(x) := \big(T(\delta)K_{\alpha/2}\big)(x) = (4\pi\delta)^{-d/2}\int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4\delta)-\delta}K_{\alpha/2}(y)\,\mathrm{d}y.$$

We have $K_{\alpha/2}^{(\delta)} \in S(\mathbb{R}^d)$. Set

$$\eta^{(\delta)}(x) := \int_0^T K_{\alpha/2}^{(\delta)}(x - W_t)\,\mathrm{d}W_t$$

which is obviously well defined for every x and has a measurable version in the pair (x, ω). It is not difficult to justify that $\eta^{(\delta)}$ is square integrable in (x, ω) and that $\eta^{(\delta)} = T(\delta/2)\eta^{(\delta/2)}$ hence $\eta^{(\delta)} \in S(\mathbb{R}^d)$ with probability one.

We have the following regularized version of (4).

Lemma 32. *For* $\delta > 0$,

$$\int_0^T \langle (T(\delta)\varphi)(W_t), \mathrm{d}W_t \rangle = \int_{\mathbb{R}^d} \langle (1-\Delta)^{\alpha/2}\varphi(x), \eta^{(\delta)}(x) \rangle \,\mathrm{d}x. \tag{31}$$

Proof. Given a vector field $\phi \in S(\mathbb{R}^d)$ and a continuous exponentially decreasing function ψ on $\mathbb{R}^d$, we have the Fubini-type identity

$$\int_0^T \left\langle \int_{\mathbb{R}^d} \psi(W_t - x)\phi(x)\,\mathrm{d}x, \mathrm{d}W_t \right\rangle = \int_{\mathbb{R}^d} \left\langle \phi(x)\,\mathrm{d}x, \int_0^T \psi(W_t - x)\,\mathrm{d}W_t \right\rangle$$

with probability one. We omit the details of the proof. We have

$$T(\delta)\varphi = T(\delta)(1-\Delta)^{-\alpha/2}(1-\Delta)^{\alpha/2}\varphi = \int_{\mathbb{R}^d} K^{(\delta)}_{\alpha/2}(\cdot - x)\big[(1-\Delta)^{\alpha/2}\varphi\big](x)\,\mathrm{d}x$$

and thus

$$\int_0^T \langle (T(\delta)\varphi)(W_t), \mathrm{d}W_t \rangle = \int_0^T \left\langle \int_{\mathbb{R}^d} K^{(\delta)}_{\alpha/2}(W_t - x)\big[(1-\Delta)^{\alpha/2}\varphi\big](x)\,\mathrm{d}x, \mathrm{d}W_t \right\rangle.$$

Here we can apply the Fubini rule because $T(\delta)\varphi \in S(\mathbb{R}^d)$ for $\delta > 0$ and $(1-\Delta)^{\alpha/2}\varphi \in S(\mathbb{R}^d)$. This implies (31) and completes the proof. □

Lemma 33. *For* $d \geq 2$, $\alpha > 1$, $p' > 1$, *such that* $(d - \alpha + 1)p' < d$ *we have*

$$\sup_{\delta > 0} \int_{\mathbb{R}^d} E\big[|\eta^{(\delta)}(x)|^{p'}\big]\,\mathrm{d}x < \infty. \tag{32}$$

Proof. Let us restrict the argument to the most difficult case $0 < \alpha < d$, where $K_{\alpha/2}$ has a singularity at zero. From the BDG inequality we have

$$\int_{\mathbb{R}^d} E\big[|\eta^{(\delta)}(x)|^{p'}\big]\,\mathrm{d}x \leq C_{p'} \int_{\mathbb{R}^d} E\left[\left(\int_0^T |K^{(\delta)}_{\alpha/2}(x - W_t)|^2\,\mathrm{d}t\right)^{p'/2}\right]\mathrm{d}x.$$

From the definition of $K^{(\delta)}_{\alpha/2}$ in terms of $K_{\alpha/2}$ and estimate (7) we get the inequality

$$K^{(\delta)}_{\alpha/2}(x) \leq C_{\alpha,d}(4\pi\delta)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4\delta)}|y|^{\alpha-d}\mathrm{e}^{-|y|/8}\,\mathrm{d}y.$$

Let us show that this implies

$$K^{(\delta)}_{\alpha/2}(x) \leq C_{\alpha,d}|x|^{\alpha-d}\mathrm{e}^{-|x|/8} \tag{33}$$

for a new constant $C_{\alpha,d}$, uniformly in $\delta \in (0,1)$. The proof of this result for $|x| > 1$ is rather easy, so let us only deal with $|x| \leq 1$. Write $x = re$ with $|e| = 1$ and change variable $y = rz$ in the integral, to get

$$K^{(\delta)}_{\alpha/2}(x) \leq r^{\alpha-d}C_{\alpha,d}(4\pi\delta)^{-d/2}r^d \int_{\mathbb{R}^d} \mathrm{e}^{-r^2|e-z|^2/(4\delta)}|z|^{\alpha-d}\,\mathrm{d}z = r^{\alpha-d}C_{\alpha,d}\left[T\left(\frac{\delta}{r^2}\right)|\cdot|^{\alpha-d}\right](e)$$

[see the definition of the semigroup $T(t)$]. It is now easy to see that $[T(t)|\cdot|^{\alpha-d}](e)$ is bounded above by a constant, uniformly in $t \geq 0$. This proves (33).

Having this estimate, it is sufficient to apply Lemma 31. The proof is complete. □

We can now prove the main result of this section.

Theorem 34. *The Itô integral $\int_0^T \langle \varphi(W_t), \mathrm{d}W_t\rangle$ has a pathwise redefinition on the space $V = H_p^\alpha(\mathbb{R}^d)$ for every dimension d and real numbers $\alpha > 1$ and $p > 1$ satisfying*

$$p > \frac{d}{\alpha - 1}.$$

In particular, in any dimension d, given $\varepsilon > 0$, for every $p > \frac{d}{\varepsilon}$ the integral $\int_0^T \langle \varphi(W_t), \mathrm{d}W_t\rangle$ has a pathwise redefinition on the space $H_p^{1+\varepsilon}(\mathbb{R}^d)$.

Proof. *Step 1*. In the case $d = 1$ we have $H_p^\alpha(\mathbb{R}) \subset C^1(\mathbb{R})$ by the Sobolev embedding theorem (see [22], Section 2.8.1, Remark 2). Thus

$$\int_0^T \varphi(W_t)\,\mathrm{d}W_t = \Phi(W_T) - \Phi(0) - \frac{1}{2}\int_0^T \varphi'(W_t)\,\mathrm{d}t,$$

where $\Phi' = \varphi$. This implies the result. We restrict now to the case $d \geq 2$.

Step 2. We pass to the limit in (31). Let us treat the left-hand side. With easy manipulations we see that

$$\bigl(T(\delta)\varphi\bigr)(x) = (2\pi)^{-d/2}\mathrm{e}^{-\delta}\int_{\mathbb{R}^d} \mathrm{e}^{-|z|^2/2}\varphi\bigl(x - z\sqrt{2\delta}\bigr)\,\mathrm{d}z.$$

Hence, splitting the integral in a sufficiently large ball and the complementary, since $\varphi \in S(\mathbb{R}^d)$, we see that $T(\delta)\varphi \to \varphi$ uniformly over all $\mathbb{R}^d$ as $\delta \to 0$. Thus $\int_0^T \langle (T(\delta)\varphi)(W_t), \mathrm{d}W_t\rangle$ easily converges to $\int_0^T \langle \varphi(W_t), \mathrm{d}W_t\rangle$, in mean square.

Given the value of p in the statement of the theorem, under the assumption $\alpha > 1$ the inequality $p > \frac{d}{\alpha-1}$ is equivalent to $(d - \alpha + 1)p' < d$, where $1/p + 1/p' = 1$, so the previous lemma applies. From (32) there is a sequence $\delta_n \to 0$ and an element $\eta^{(0)} \in L^{p'}(\Omega \times \mathbb{R}^d)$ such that $\eta^{(\delta_n)} \rightharpoonup \eta^{(0)}$ weakly in $L^{p'}(\Omega \times \mathbb{R}^d)$, when $n \to +\infty$. From (31), for a given $\varphi \in S(\mathbb{R}^d)$, we thus have, in the limit as $n \to \infty$,

$$E\left[X\int_0^T \bigl\langle \varphi(W_t), \mathrm{d}W_t\bigr\rangle\right] = E\left[X\int_{\mathbb{R}^d} \bigl\langle (1 - \Delta)^{\alpha/2}\varphi(x), \eta^{(0)}(x)\bigr\rangle\,\mathrm{d}x\right]$$

for every bounded r.v. X and thus

$$\int_0^T \bigl\langle \varphi(W_t), \mathrm{d}W_t\bigr\rangle = \int_{\mathbb{R}^d} \bigl\langle (1 - \Delta)^{\alpha/2}\varphi(x), \eta^{(0)}(x)\bigr\rangle\,\mathrm{d}x$$

with probability one.

Step 3. Therefore, given $\varphi \in S(\mathbb{R}^d)$, with probability one we have

$$\left|\int_0^T \bigl\langle \varphi(W_t), \mathrm{d}W_t\bigr\rangle\right| \leq \bigl\|(1 - \Delta)^{\alpha/2}\varphi\bigr\|_{L^p(\mathbb{R}^d)}\bigl\|\eta^{(0)}\bigr\|_{L^{p'}(\mathbb{R}^d)} \leq C\|\varphi\|_{H_p^\alpha(\mathbb{R}^d)}\bigl\|\eta^{(0)}\bigr\|_{L^{p'}(\mathbb{R}^d)}.$$

The proof is complete. □

Remark 35. *The same result is true for the stopped Brownian motion*

$$W_t^R = W_{t\wedge\tau_R}, \quad \tau_R = \inf\bigl\{t > 0\colon |W_t| \geq R\bigr\},$$

with given $R > 0$. The statement is that the Itô integral $\int_0^T \langle \varphi(W_t^R), \mathrm{d}W_t^R\rangle$ has a pathwise redefinition on the space $H_p^\alpha(\mathbb{R}^d)$ under the same conditions on d, α, p as in the theorem. The proof is the same (*even easier, since in the proof of Lemma 31 we do not have to care about the exponential term*).

Remark 36. *The same result is true in the Sobolev–Slobodeckij spaces $W_p^\alpha(\mathbb{R}^d)$ defined in* [22], *Section* 2.3. *The statement is that the Itô integral $\int_0^T \langle\varphi(W_t), \mathrm{d}W_t\rangle$ has a pathwise redefinition on the space $W_p^\alpha(\mathbb{R}^d)$ under the same conditions on d, α, p as in the theorem. Indeed, given a triple d, α, p as in the theorem, let $\alpha' < \alpha$ be such that also the triple d, α', p satisfies the assumption of the theorem. Then $\int_0^T \langle\varphi(W_t), \mathrm{d}W_t\rangle$ has a pathwise redefinition on $H_p^{\alpha'}(\mathbb{R}^d)$; by definition of pathwise redefinition, we see that this implies that $\int_0^T \langle\varphi(W_t), \mathrm{d}W_t\rangle$ has a pathwise redefinition on the space $W_p^\alpha(\mathbb{R}^d)$, because we have the continuous embedding*

$$W_p^\alpha\big(\mathbb{R}^d\big) \subset H_p^{\alpha'}\big(\mathbb{R}^d\big);$$

see [22], *Remark* 4 *of Section* 2.3.3. *The same result is of course true for the Itô integral $\int_0^T \langle\varphi(W_t^R), \mathrm{d}W_t^R\rangle$.*

We can now elaborate the previous results in the direction of the Hölder topology. Given $\varepsilon \in (0,1)$, denote by $C^{1+\varepsilon}(\mathbb{R}^d)$ the space of all continuously differentiable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{C^{1+\varepsilon}} = \sup_{x\in\mathbb{R}^d}\big(\big|f(x)\big| + \big|Df(x)\big|\big) + \sup_{x\neq y}\frac{|Df(x) - Df(y)|}{|x-y|^\varepsilon} < \infty;$$

see [22], Section 2.7. Endowed with the norm $\|\cdot\|_{C^{1+\varepsilon}}$, the space $C^{1+\varepsilon}(\mathbb{R}^d)$ is a Banach space.

Theorem 37. *In any dimension d, for every $\varepsilon \in (0,1)$ the Itô integral $\int_0^T \langle\varphi(W_t), \mathrm{d}W_t\rangle$ has a pathwise redefinition on the space $C^{1+\varepsilon}(\mathbb{R}^d)$.*

Proof. *Step 1.* This preliminary step is devoted to a few details used below. Recall that the classical Sobolev space $W_p^1(\mathbb{R}^d)$ is defined as the space of all $f \in L^p(\mathbb{R}^d)$ having distributional derivative $Df \in L^p(\mathbb{R}^{d\times d})$. Recall also (see Remark 4 of Section 2.5.1 of [22]) that, for every $\varepsilon \in (0,1)$, the space $W_p^{1+\varepsilon}(\mathbb{R}^d)$ of Remark 36 is characterized as the space of all $f \in W_p^1(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|Df(x) - Df(y)|^p}{|x-y|^{d+\varepsilon p}}\,\mathrm{d}x\,\mathrm{d}y < \infty$$

and as a norm on $W_p^{1+\varepsilon}(\mathbb{R}^d)$ one can take the following one:

$$\|f\|^p_{W_p^{1+\varepsilon}} = \|f\|^p_{L_p} + \|Df\|^p_{L_p} + \int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|Df(x) - Df(y)|^p}{|x-y|^{d+\varepsilon p}}\,\mathrm{d}x\,\mathrm{d}y.$$

Then it is easy to verify that for every $\varepsilon, \varepsilon' \in (0,1)$ with $\varepsilon > \varepsilon'$ the following assertion is true, where $B(0,R)$ denotes the ball of center 0 and radius $R > 0$:

$$f \in C^{1+\varepsilon}\big(\mathbb{R}^d\big), \quad f \text{ with support in } B(0,R) \quad\Longrightarrow\quad f \in W_p^{1+\varepsilon'}\big(\mathbb{R}^d\big)$$

and

$$\|f\|^p_{W_p^{1+\varepsilon'}} \le C\big(R, \varepsilon, \varepsilon', p, d\big)\|f\|^p_{C^{1+\varepsilon}}, \tag{34}$$

where $C(R, \varepsilon, \varepsilon', p, d)$ is a constant depending only on $R, \varepsilon, \varepsilon', p, d$.

Indeed, we have

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|Df(x) - Df(y)|^p}{|x-y|^{d+\varepsilon' p}}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad= \int_{B(0,R)\times B(0,R)}\frac{|Df(x) - Df(y)|^p}{|x-y|^{d+\varepsilon' p}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

$$\leq \int_{|x-y|\leq 1, |x|\leq R, |y|\leq R} \frac{|Df(x)-Df(y)|^p}{|x-y|^{d+\varepsilon' p}} \,\mathrm{d}x\,\mathrm{d}y + \int_{|x-y|>1, |x|\leq R, |y|\leq R} \frac{|Df(x)-Df(y)|^p}{|x-y|^{d+\varepsilon' p}} \,\mathrm{d}x\,\mathrm{d}y$$

$$\leq \int_{|x-y|\leq 1, |x|\leq R, |y|\leq R} \frac{\|f\|^p_{C^{1+\varepsilon}}}{|x-y|^{d+(\varepsilon'-\varepsilon)p}} \,\mathrm{d}x\,\mathrm{d}y + C(p,d)\|f\|^p_{C^{1+\varepsilon}} R^d,$$

where $C(p,d)$ is a constant depending only on p, d. The claim (34) easily follows from this inequality.

Step 2. Let d and ε be given, as in the claim of the theorem. Choose $\varepsilon' \in (0,\varepsilon)$ and $p > \frac{d}{\varepsilon'}$. Let W be defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Remark 36 states that there exists a random variable $C>0$ such that, for every $\varphi \in W_p^{1+\varepsilon'}(\mathbb{R}^d)$,

$$\left|\int_0^T \langle \varphi(W_t), \mathrm{d}W_t\rangle\right| \leq C\|\varphi\|_{W_p^{1+\varepsilon'}}$$

on a full probability set Ω_φ.

For every $R>0$, let $\theta_R : \mathbb{R}^d \to [0,\infty)$ be a C^∞ function such that $\theta_R(x) = 1$ for $|x| \leq R+1$, $\theta_R(x) = 0$ for $|x| \geq R+2$. Given $\varphi \in C^{1+\varepsilon}(\mathbb{R}^d)$, we have $\varphi\cdot\theta_R \in C^{1+\varepsilon}(\mathbb{R}^d)$ and thus $\varphi\cdot\theta_R \in W_p^{1+\varepsilon'}(\mathbb{R}^d)$. Therefore

$$\left|\int_0^T \langle (\varphi\cdot\theta_R)(W_t), \mathrm{d}W_t\rangle\right| \leq C\|\varphi\cdot\theta_R\|_{W_p^{1+\varepsilon'}}$$

on a full probability set $\Omega_{\varphi\cdot\theta_R}$.

From Step 1, there exists a random variable $C_R > 0$, independent of φ, such that

$$\left|\int_0^T \langle (\varphi\cdot\theta_R)(W_t), \mathrm{d}W_t\rangle\right| \leq C_R\|\varphi\|_{C^{1+\varepsilon}} \quad \text{on } \Omega_{\varphi\cdot\theta_R},$$

where we have also used the fact that $\|\varphi\cdot\theta_R\|_{C^{1+\varepsilon}} \leq C_\theta\|\varphi\|_{C^{1+\varepsilon}}$ for some constant $C_\theta > 0$ depending on the function θ (and thus on R again). Redefine, if necessary, C_R in such a way that $R \mapsto C_R$ is non-decreasing, with probability one.

Let A_R be the set

$$A_R = \{\tau_R > T\},$$

where τ_R is defined in Remark 35. The sets A_R increase with R. Given the family of events A_R and random variables C_R, we can define a new random variable $C' > 0$ such that $C_R \leq C'$ on A_R (it is sufficient to put $C' = C_{N+1}$ on $A_{N+1} \setminus A_N$). Thus, given $\varphi \in C^{1+\varepsilon}(\mathbb{R}^d)$, we have

$$\left|\int_0^T \langle (\varphi\cdot\theta_R)(W_t), \mathrm{d}W_t\rangle\right| \leq C'\|\varphi\|_{C^{1+\varepsilon}} \quad \text{on } \Omega_{\varphi\cdot\theta_R} \cap A_R.$$

For every $R>0$ and $\varphi \in C^{1+\varepsilon}(\mathbb{R}^d)$, there is a P-null set $N_{R,\varphi}$ such that

$$\int_0^T \langle (\varphi\cdot\theta_R)(W_t), \mathrm{d}W_t\rangle = \int_0^T \langle \varphi(W_t), \mathrm{d}W_t\rangle \quad \text{on } A_R \setminus N_{R,\varphi}.$$

Therefore, given $R>0$ and $\varphi \in C^{1+\varepsilon}(\mathbb{R}^d)$, we have

$$\left|\int_0^T \langle \varphi(W_t), \mathrm{d}W_t\rangle\right| \leq C'\|\varphi\|_{C^{1+\varepsilon}} \quad \text{on } \Omega_{\varphi\cdot\theta_R} \cap A_R \setminus N_{R,\varphi}.$$

It follows that

$$\left|\int_0^T \langle \varphi(W_t), \mathrm{d}W_t\rangle\right| \leq C'\|\varphi\|_{C^{1+\varepsilon}} \quad \text{on } \bigcup_{R>0} (\Omega_{\varphi\cdot\theta_R} \cap A_R \setminus N_{R,\varphi}).$$

Since $P(\bigcup_{R>0} A_R) = 1$ we have $P(\bigcup_{R>0}(\Omega_{\varphi\cdot\theta_R} \cap A_R \setminus N_{R,\varphi})) = 1$. This means $\int_0^T \langle\varphi(W_t), \mathrm{d}W_t\rangle$ has a pathwise redefinition on the space $C^{1+\varepsilon}(\mathbb{R}^d)$. The proof is complete. □

Remark 38. *The strategy of Step* 2 *in the previous proof can be used to deal with function spaces of Fréchet type that are not Banach spaces*: *By localization of the stochastic process*, *one can restrict the attention to compact support test functions and then prove the existence of a pathwise redefinition in topologies without decay at infinity. For this reason*, *even the uniformity in $x \in \mathbb{R}^d$ in the definition of $C^{1+\varepsilon}(\mathbb{R}^d)$ is not necessary.*

Remark 39. *In rough path theory* (*see* [16]), *for every rough path γ of a certain class which includes a.e. the path of Brownian motion, a notion of integral $\int_0^T \langle\varphi(\gamma_t), \mathrm{d}\gamma_t\rangle$ is defined for every function φ with ε-Hölder first derivative* (*for arbitrary $\varepsilon > 0$*). *The previous theorem is conceptually similar*; *a closer comparison*, *however*, *requires further investigation*.

Finally, we have to prove Lemma 31.

Proof of Lemma 31. If $\max_{[0,T]}|W_t| \le |x|/2$ then, for every $t \in [0, T]$,

$$\frac{1}{2}|x| \le |x - W_t| \le \frac{3}{2}|x|,$$

$$\exp\bigl(-\varepsilon|x - W_t|\bigr) \le \exp\left(-\frac{\varepsilon|x|}{2}\right),$$

$$\frac{1}{|x - W_t|^{2d-2\alpha}} \le \frac{(2/3)^{2d-2\alpha}}{|x|^{2d-2\alpha}} \quad \text{if } 2d - 2\alpha \le 0,$$

$$\frac{1}{|x - W_t|^{2d-2\alpha}} \le \frac{2^{2d-2\alpha}}{|x|^{2d-2\alpha}} \quad \text{if } 2d - 2\alpha > 0$$

and thus

$$\frac{\exp(-\varepsilon|x - W_t|)}{|x - W_t|^{2d-2\alpha}} \le \exp\left(-\frac{\varepsilon|x|}{2}\right)\frac{C_{\alpha,d}}{|x|^{2d-2\alpha}}$$

for a suitable constant $C_{\alpha,d} > 0$. Therefore

$$\int_{\mathbb{R}^d} E\left[\left(\int_0^T \frac{\exp(-\varepsilon|x - W_t|)}{|x - W_t|^{2d-2\alpha}}\,\mathrm{d}t\right)^{p'/2}\right]\mathrm{d}x \le I_1 + I_2,$$

$$I_1 := \int_{\mathbb{R}^d} \exp\left(-\frac{\varepsilon p'|x|}{4}\right)\left(\int_0^T \frac{C_{\alpha,d}}{|x|^{2d-2\alpha}}\,\mathrm{d}t\right)^{p'/2}\mathrm{d}x,$$

$$I_2 := \int_{\mathbb{R}^d} E\left[1_{\max_{[0,T]}|W_t| > |x|/2}\left(\int_0^T \frac{1}{|x - W_t|^{2d-2\alpha}}\,\mathrm{d}t\right)^{p'/2}\right]\mathrm{d}x.$$

Obviously $I_1 < \infty$, being $(d - \alpha + 1)p' < d$. Moreover

$$I_2 \le \int_{\mathbb{R}^d} P\left(\max_{[0,T]}|W_t| > \frac{|x|}{2}\right)^{\delta/(1+\delta)} E\left[\left(\int_0^T \frac{1}{|x - W_t|^{2d-2\alpha}}\,\mathrm{d}t\right)^{(1+\delta)p'/2}\right]^{1/(1+\delta)}\mathrm{d}x$$

for every $\delta > 0$. Recall the exponential inequality (see [19], Proposition 1.8)

$$P\left(\max_{t\in[0,T]} W_t \ge \beta\right) \le \mathrm{e}^{-\beta/(2T)}.$$

It easily implies, by symmetry, that

$$P\left(\max_{t\in[0,T]}|W_t|\geq\frac{|x|}{2}\right)\leq 2\mathrm{e}^{-|x|/(4T)}$$

and thus there exist $C_\delta,\lambda_\delta>0$ (depending also on T) such that

$$P\left(\max_{t\in[0,T]}|W_t|\geq\frac{|x|}{2}\right)^{\delta/(1+\delta)}\leq C_\delta\mathrm{e}^{-\lambda_\delta|x|}.$$

Moreover since $1/(1+\delta)<1$; for any $a\in[0,1]$ we have $a^{1/(1+\delta)}\leq a+1$. Thus, for every $\delta>0$,

$$I_2\leq C_\delta+C_\delta\int_{\mathbb{R}^d}\mathrm{e}^{-\lambda_\delta|x|}E\left[\left(\int_0^T\frac{1}{|x-W_t|^{2d-2\alpha}}\,\mathrm{d}t\right)^{(1+\delta)p'/2}\right]\mathrm{d}x$$

for some constant $C_\delta>0$.

The following lemma is inspired by the proof of Corollary 2.4 of [2] and in fact it was suggested to us by K. D. Elworthy.

Lemma 40. *For every $d\geq 2$, $q>1$, $\theta\in\mathbb{R}$, $x\in\mathbb{R}^d$, we have*

$$E\left[\left(\int_0^T\frac{\mathrm{d}t}{|x+W_t|^{2(1-\theta)}}\right)^{q/2}\right]\leq c_{q,\theta,T}\left\{E\left[|x+W_T|^{\theta q}\right]+|x|^{\theta q}+\int_0^T E\left[\frac{\mathrm{d}t}{|x+W_t|^{(2-\theta)q}}\right]\right\}.$$

Proof. Consider the process $Z_t=|x+W_t|^2$ (squared Bessel process of dimension d). From the Itô formula we have

$$\mathrm{d}Z_t=2\langle x+W_t,\mathrm{d}W_t\rangle+\mathrm{d}t,\quad Z_0=|x|^2.$$

Introducing an auxiliary one-dimensional Brownian motion (β_t) we may also write

$$\mathrm{d}Z_t=2\sqrt{Z_t}\,\mathrm{d}\beta_t+\mathrm{d}t.$$

Since $d\geq 2$, the one-point sets are polar sets for a d-dimensional Brownian motion; see Proposition 2.7, p. 191 of [19]. Therefore $P\{Z_t>0,t\in[0,T]\}=1$ and we can develop $Z_t^{\theta/2}$ using the Itô formula for any $\theta\in\mathbb{R}$. We obtain

$$\mathrm{d}\big(Z_t^{\theta/2}\big)=\frac{\theta}{2}Z_t^{(\theta-2)/2}\big(2\sqrt{Z_t}\,\mathrm{d}\beta_t+\mathrm{d}t\big)+\frac{1}{2}\frac{\theta}{2}\frac{\theta-2}{2}Z_t^{(\theta-4)/2}4Z_t\,\mathrm{d}t=\theta Z_t^{(\theta-1)/2}\,\mathrm{d}\beta_t+c_\theta Z_t^{(\theta-2)/2}\,\mathrm{d}t,$$

where $c_\theta=\frac{\theta(\theta-1)}{2}$. Therefore

$$\int_0^T\theta Z_t^{(\theta-1)/2}\,\mathrm{d}\beta_t=Z_T^{\theta/2}-Z_0^{\theta/2}-\int_0^T c_\theta Z_t^{(\theta-2)/2}\,\mathrm{d}t$$

and thus, from the BDG inequality, for every $q>1$

$$\begin{aligned}E\left[\left(\int_0^T\theta^2Z_t^{\theta-1}\,\mathrm{d}t\right)^{q/2}\right]&\leq c_q E\left[\left(\int_0^T\theta Z_t^{(\theta-1)/2}\,\mathrm{d}\beta_t\right)^q\right]\\&\leq c_{q,\theta}\left\{E\big[Z_T^{\theta q/2}\big]+E\big[Z_0^{\theta q/2}\big]+E\left[\left(\int_0^T Z_t^{(\theta-2)/2}\,\mathrm{d}t\right)^q\right]\right\}.\end{aligned}$$

This implies, by Hölder inequality,

$$E\left[\left(\int_0^T\frac{1}{Z_t^{1-\theta}}\,\mathrm{d}t\right)^{q/2}\right]\leq c_{q,\theta,T}\left\{E\big[Z_T^{\theta q/2}\big]+E\big[Z_0^{\theta q/2}\big]+E\left[\int_0^T\frac{1}{Z_t^{(2-\theta)q/2}}\,\mathrm{d}t\right]\right\}$$

and the proof is complete. □

We go on with the proof of Lemma 31. Simply by taking $1-\theta = d-\alpha$ and $q=(1+\delta)p'$ we have:

$$E\left[\left(\int_0^T \frac{1}{|x-W_t|^{2d-2\alpha}}\,\mathrm{d}t\right)^{(1+\delta)p'/2}\right] \le C\left\{E\left[\frac{1}{|x+W_T|^{(d-\alpha-1)(1+\delta)p'}}\right]+\frac{1}{|x|^{(d-\alpha-1)(1+\delta)p'}}\right\}+C\int_0^T E\left[\frac{1}{|x+W_t|^{(d-\alpha+1)(1+\delta)p'}}\right]\mathrm{d}t.$$

With the notation $p_t(y)=\frac{1}{\sqrt{(2\pi)^d t^d}}\exp(-\frac{|y|^2}{2t})$, and the bound $\int_0^T p_t(y)\,\mathrm{d}t \le \frac{C_d\exp(-|y|)}{|y|^{d-2}}$ we have

$$E\left[\frac{1}{|x+W_T|^{(d-\alpha-1)(1+\delta)p'}}\right]=\int_{\mathbb{R}^d}\frac{1}{|x+y|^{(d-\alpha-1)(1+\delta)p'}}p_T(y)\,\mathrm{d}y$$

and

$$\int_0^T E\left[\frac{1}{|x+W_t|^{(d-\alpha+1)(1+\delta)p'}}\right]\mathrm{d}t = \int_{\mathbb{R}^d}\frac{1}{|x+y|^{(d-\alpha+1)(1+\delta)p'}}\left(\int_0^T p_t(y)\,\mathrm{d}t\right)\mathrm{d}y \le \int_{\mathbb{R}^d}\frac{1}{|x+y|^{(d-\alpha+1)(1+\delta)p'}}\frac{C_d\exp(-|y|)}{|y|^{d-2}}\,\mathrm{d}y.$$

Thus, with a new constant $C>0$ depending on δ and the other parameters,

$$I_2 \le C\big(1+I_2^{(1)}+I_2^{(2)}+I_2^{(3)}\big),$$

where

$$I_2^{(1)} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{-\lambda_\delta|x|}\frac{1}{|x+y|^{(d-\alpha-1)(1+\delta)p'}}p_T(y)\,\mathrm{d}x\,\mathrm{d}y,$$

$$I_2^{(2)} := \int_{\mathbb{R}^d} \mathrm{e}^{-\lambda_\delta|x|}\frac{1}{|x|^{(d-\alpha-1)(1+\delta)p'}}\,\mathrm{d}x,$$

$$I_2^{(3)} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{-\lambda_\delta|x|}\frac{1}{|x+y|^{(d-\alpha+1)(1+\delta)p'}}\frac{C_d\exp(-|y|)}{|y|^{d-2}}\,\mathrm{d}x\,\mathrm{d}y.$$

Choose $\delta>0$ such that $(d-\alpha+1)(1+\delta)p'<d$. Since

$$(d-\alpha-1)(1+\delta)p' < (d-\alpha+1)(1+\delta)p' < d$$

the term $I_2^{(2)}$ is finite. For $I_2^{(1)}$ and $I_2^{(3)}$ it is sufficient to integrate first in x, bound the result uniformly in y, then integrate in y; one proves that $I_2^{(1)}$ and $I_2^{(3)}$ are finite. The proof is complete. □

5. The energy of a random vortex filament

In [4,5,7] with the purpose of modeling turbulence in 3D fluids, the authors introduce and study a model of random vortex filaments based on Brownian motion. This model has been extended to the fBm with $H>1/2$ by [8,17]. Here we recall briefly the model, emphasize the relationship of the vortex energy with the pathwise regularity of the current associated with the vortex *core* and obtain new conditions for the integrability of the vortex energy for the case $H\in(1/4,1/2)$.

For simplicity we consider only a single vortex since extension to a linear superposition of different vortexes is straightforward (and even to a random field of Poissonian vortexes; see, for example, [6]). Let $(X_t)_{t\in[0,T]}$, $T>0$ be a 3d fBm with Hurst parameter $H\in(1/4,1)$ and consider the associated vector current, formally written as

$$\xi_0(x)=\int_0^T \delta(x-X_t)\,\mathrm{d}X_t,$$

where the integral is a symmetric (Stratonovich) integral. This object should be understood according to Theorem 11; that is, as a random distribution in the Sobolev space $H_p^{-\alpha}(\mathbb{R}^d)$ of sufficiently large negative order. The vorticity field is then built by superposing translates of this core weighted according to a compactly supported signed measure ρ with finite mass which determines the intensity of vorticity. For more details about those considerations, the reader can consult [4]. Then we end up with

$$\xi(x)=\int_{\mathbb{R}^3}\xi_0(x-y)\rho(\mathrm{d}y)$$

which is again a random distribution.

The velocity field u is generated from ξ according to the Biot–Savart relation

$$u(x)=\int_{\mathbb{R}^3}\mathcal{K}(x-y)\wedge\xi(y)\,\mathrm{d}y=\int_{\mathbb{R}^3}\mathcal{K}*\rho(x-y)\wedge\xi_0(y)\,\mathrm{d}y, \tag{35}$$

where $\wedge$ is the vector product in $\mathbb{R}^3$ and the vector kernel $\mathcal{K}(x)$ is defined as $\mathcal{K}(x):=(4\pi)^{-1}x/|x|^3$ and $\mathcal{K}*\rho$ denote the convolution $(\mathcal{K}*\rho)(x)=\int_{\mathbb{R}^3}\mathcal{K}(x-z)\rho(\mathrm{d}z)$. The kinetic energy of the fluid is then defined as the $L^2(\mathbb{R}^3)$ norm of u:

$$\mathcal{E}=\int_{\mathbb{R}^3}\bigl|u(x)\bigr|^2\,\mathrm{d}x=\|u\|^2. \tag{36}$$

It is then interesting to find conditions on ρ such that the kinetic energy of the fluid is finite. Abstractly we have $u=\Phi\xi_0$ where we introduced an operator Φ whose kernel is $\mathcal{K}*\rho$ having Fourier transform

$$\mathcal{F}(\mathcal{K}*\rho)(q)=\frac{\mathrm{i}q}{|q|^2}\widehat{\rho}(q),$$

where we denoted $\widehat{\rho}$ the Fourier transform of the measure ρ.

From now on L^2 will stand for $L^2(\mathbb{R}^3)$. Since, by Corollary 24, ξ_0 belongs a.s. to the space $H_2^{-\alpha}(\mathbb{R}^d)$ for any $\alpha>\alpha_H=1/(2H)+1/2$ (since $d=3$), the condition $u\in L^2$ a.s. can be satisfied if $\Phi: H_2^{-\alpha}(\mathbb{R}^3)\to L^2$. Then going in Fourier variables is sufficient to require that

$$\|\Phi\|_{H_2^{-\alpha}\to L^2}=\bigl\|\Phi(1-\Delta)^{\alpha/2}\bigr\|_{L^2\to L^2}=\operatorname*{ess\,sup}_{q\in\mathbb{R}^3}\frac{|\widehat{\rho}(q)|}{|q|}\bigl(1+|q|^2\bigr)^{\alpha/2}<\infty$$

for some $\alpha>\alpha_H$.

We can now formulate the following result:

Corollary 41. *The kinetic energy of the vortex filament ξ built upon a 3d fractional Brownian motion of Hurst index $H>1/4$ is a.s. finite and in L^1 if the measure ρ satisfies*

$$\operatorname*{ess\,sup}_{q}\bigl|\widehat{\rho}(q)\bigr||q|^{-1}\bigl(1+|q|^2\bigr)^{\alpha/2}<\infty \tag{37}$$

for some $\alpha>\alpha_H$.

Remark 42. 1. *Known conditions on ρ which guarantee the integrability of the energy are given in* [4,5] *for the case of Brownian motion and Itô Brownian processes, and in* [17] *for the case of fractional Brownian motions with Hurst parameter $H > 1/2$. From* [17] *it can be deduced that a sufficient condition for the integrability of the energy is*

$$\int_{\mathbb{R}^3} \mathrm{d}q \, \frac{|\widehat{\rho}(q)|^2}{|q|^{4-1/H}} < \infty \tag{38}$$

or, written in a different but equivalent form,

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(\mathrm{d}x)\rho(\mathrm{d}y)}{|x-y|^{1/H-1}} < \infty.$$

2. *Condition* (37) *implies condition* (38) *when $H > 1/2$.*
In fact the left-hand side of condition (37) *gives*

$$\begin{aligned}\int_{\mathbb{R}^3} \mathrm{d}q \, \big|\widehat{\rho}(q)\big|^2 |q|^{1/H-2} \big(1+|q|^2\big)^{-\alpha} &\le \int_{\mathbb{R}^3} \mathrm{d}q \, |q|^2 \big(1+|q|^2\big)^{-\alpha} |q|^{1/H-4} \operatorname*{ess\,sup}_q \big|\widehat{\rho}(q)\big|^2 |q|^{-2} \big(1+|q|^2\big)^{\alpha} \\ &\le A^2 \int_0^{+\infty} r^{1/H} \big(1+r^2\big)^{-\alpha} \, \mathrm{d}r,\end{aligned}$$

where A is the finite quantity of (37). *Clearly, the previous expression is bounded for $\alpha > \alpha_H$. Of course the converse is not true.*

3. *A way of finding similar conditions to* (38) *is to follow the steps of Section* 2 *for the Hilbert space regularity of the stochastic currents and rewrite* (*formally*) *the kinetic energy as*

$$\mathcal{E} = \int_0^T \int_0^T \big\langle \mathrm{d}X_t, g(X_t - X_s)\,\mathrm{d}X_s \big\rangle, \tag{39}$$

where g is a vector kernel with the following Fourier transform:

$$\widehat{g}(q) = \frac{|\widehat{\rho}(q)|^2}{|q|^2} \Pi_q$$

and Π_q is the following matrix:

$$(\Pi_q)_{\alpha\beta} = \delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{|q|^2}, \qquad \alpha, \beta = 1, \ldots, 3,$$

which projects in directions orthogonal to q. Formula (39) *can be understood formally, according to Theorem* 11, *as being the limit of the expectations of ε-approximations*

$$\mathcal{E}_\varepsilon = \int_0^T \int_0^T g(X_t - X_s) \langle D_\varepsilon X_t, D_\varepsilon X_s \rangle \, \mathrm{d}t \, \mathrm{d}s.$$

To obtain conditions for its finiteness in the spirit of Eq. (38), *we need to follow again the computations involved in the proof of Theorem* 20 *and use a different strategy in bounding some terms. Then we can prove the following*:

Theorem 43. *Let $H > 1/4$ and let $\rho : \mathbb{R}^3 \to \mathbb{R}$ be a function with Fourier transform $\widehat{\rho}$ satisfying*

$$\int_{\mathbb{R}^3} \mathrm{d}q \, \frac{|\widehat{\rho}(q)|^2}{|q|^{4-1/H}} < \infty; \tag{40}$$

then the family of random fields $\{u_\varepsilon\}_{\varepsilon \in (0,1)}$ defined as

$$u_\varepsilon(x) = \int_0^1 (\mathcal{K} * \rho)(x - X_t) \wedge D_\varepsilon^0 X_t \, \mathrm{d}t, \quad x \in \mathbb{R}^3,$$

weakly converges in $L^2(\Omega; L^2(\mathbb{R}^3;\mathbb{R}^3))$ to a random field u with paths in $L^2(\mathbb{R}^3;\mathbb{R}^3)$.

Proof. We will prove that $\sup_{\varepsilon\in(0,1)} E\|u_\varepsilon\|^2<\infty$ following the lines of the proof of Theorem 20, then the conclusion follows by adapting the proof of Theorem 11. Indeed for any smooth φ we can define

$$J_\varepsilon(\varphi)=\langle\varphi,u_\varepsilon\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)}=\int_0^T(\Phi*\rho*\varphi)(X_t)\wedge D_\varepsilon^0 X_t\,\mathrm{d}t$$

which converges in $L^2(\Omega)$ to $J(\varphi)$. Since $|J_\varepsilon(\varphi)|^2\le\|\varphi\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\|u_\varepsilon\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}$ we can deduce that $\varphi\mapsto J(\varphi)$ extends by continuity to a mapping $L^2(\mathbb{R}^3;\mathbb{R}^3)\to L^2(\Omega)$ and that moreover $J_\varepsilon(\varphi)\to J(\varphi)$ in $L^2(\Omega)$ for any $\varphi\in L^2(\mathbb{R}^3)$. Then following Step 2 in the proof of Theorem 11 we can prove that $u_\varepsilon\in L^2(\Omega,L^2(\mathbb{R}^3;\mathbb{R}^3))$ is a pathwise redefinition of J_ε and it converges weakly in $u\in L^2(\Omega,L^2(\mathbb{R}^3;\mathbb{R}^3))$.

It remains to prove that $\sup_{\varepsilon\in(0,1)} E\|u_\varepsilon\|^2<\infty$. Let $\mathcal{E}_\varepsilon=\|u_\varepsilon\|^2$. We start by treating the case $H>1/2$.

Using Theorem 25 (Wick's theorem) and independence of different coordinates we have

$$\begin{aligned}E\mathcal{E}_\varepsilon&=E\int_0^T\int_0^T\sum_i g_{ii}(X_t-X_s)\,\mathrm{Cov}\big(D_\varepsilon^0X_t^1,D_\varepsilon^0X_s^1\big)\,\mathrm{d}t\,\mathrm{d}s\\&\quad-E\int_0^T\int_0^T\sum_{ij}\nabla_i\nabla_j g_{ij}(X_t-X_s)\big|\mathrm{Cov}\big(D_\varepsilon^0X_t^1,X_t^1-X_s^1\big)\big|^2\,\mathrm{d}t\,\mathrm{d}s\\&=E\int_0^T\int_0^T\mathrm{Tr}\,g(X_t-X_s)\,\mathrm{Cov}\big(D_\varepsilon^0X_t^1,D_\varepsilon^0X_s^1\big)\,\mathrm{d}t\,\mathrm{d}s\end{aligned}\tag{41}$$

since a direct computation shows that $\sum_i\nabla_i g_{ik}(x)=0$. Using the first bound in Lemma 19 we get

$$\begin{aligned}E\mathcal{E}_\varepsilon&\le\mathrm{const}\,E\int_0^T\int_0^t\mathrm{Tr}\,g(X_t-X_s)|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t\\&=\mathrm{const}\int_{\mathbb{R}^3}\mathrm{d}q\;\mathrm{Tr}\,\widehat{g}(q)\int_0^T\int_0^t|t-s|^{2H-2}E\mathrm{e}^{-\mathrm{i}\langle q,X_t-X_s\rangle}\,\mathrm{d}s\,\mathrm{d}t\\&=\mathrm{const}\int_{\mathbb{R}^3}\mathrm{d}q\;\mathrm{Tr}\,\widehat{g}(q)\int_0^T\int_0^t|t-s|^{2H-2}\mathrm{e}^{-|q|^2(t-s)^{2H}/2}\,\mathrm{d}s\,\mathrm{d}t\\&\le\mathrm{const}\int_{\mathbb{R}^3}\mathrm{d}q\,\big|\widehat{g}(q)\big|\int_0^T\mathrm{d}t\int_0^\infty\mathrm{d}\tau\,\tau^{2H-2}\mathrm{e}^{-|q|^2\tau^{2H}/2}\\&=\mathrm{const}\int_{\mathbb{R}^3}\mathrm{d}q\,\big|\widehat{g}(q)\big||q|^{1/H-2}\int_0^T\mathrm{d}t\int_0^\infty\mathrm{d}y\,y^{1-1/H}\mathrm{e}^{-y^2/2}\\&\le\mathrm{const}\,T\int_{\mathbb{R}^3}\mathrm{d}q\,\big|\widehat{g}(q)\big||q|^{1/H-2}\end{aligned}\tag{42}$$

since

$$\int_0^\infty\mathrm{d}y\,y^{1-1/H}\mathrm{e}^{-y^2/2}<\infty$$

for $1-1/H>-1$, that is $H>1/2$. Sufficient condition for uniform boundedness of $E\mathcal{E}_\varepsilon$ is that

$$\int_{\mathbb{R}^3}\mathrm{d}q\,\big|\widehat{g}(q)\big||q|^{1/H-2}<\infty.$$

Let us now consider the case $H \le 1/2$ and rewrite the approximated energy as

$$\mathcal{E}_\varepsilon = \|u_\varepsilon\|^2 = -\int_0^1\int_0^1 \big\langle h(X_t - X_s)D_\varepsilon X_s, D_\varepsilon X_t\big\rangle + \int_0^1\int_0^1 \big\langle g(0)D_\varepsilon X_s, D_\varepsilon X_t\big\rangle,$$

where $h(x) = g(0) - g(x) \ge 0$. Note that $g(0)$ is well defined using the hypothesis of the theorem about the integrability of its Fourier transform, moreover, as in Theorem 20 (in the $H < 1/2$ part) we have the limit

$$\int_0^1\int_0^1 \big\langle g(0)D_\varepsilon X_s, D_\varepsilon X_t\big\rangle \to \big\langle (X_1 - X_0)g(0), (X_1 - X_0)\big\rangle.$$

So let us focus on the double integral with the kernel h. Proceeding as in the $H > 1/2$ case we have

$$\begin{aligned} J &= -\int_0^T\int_0^T \big\langle h(X_t - X_s)D_\varepsilon X_s, D_\varepsilon X_t\big\rangle \\ &\le -\text{const}\int_0^T\int_0^T\int_{\mathbb{R}^3} \mathrm{d}q\ \operatorname{Tr}\widehat{h}(q)E\big[\mathrm{e}^{\mathrm{i}\langle q, X_t - X_s\rangle}\big]|t-s|^{2H-2}\,\mathrm{d}t\,\mathrm{d}s \\ &= -\text{const}\int_0^T\int_0^T\int_{\mathbb{R}^3} \mathrm{d}q\ \operatorname{Tr}\widehat{h}(q)\mathrm{e}^{-|q|^2/2(t-s)^{2H}}|t-s|^{2H-2}\,\mathrm{d}t\,\mathrm{d}s \end{aligned}$$

but since $\widehat{h}(q) = g(0)\delta(q) - \widehat{g}(q)$ we have

$$\begin{aligned} \int_{\mathbb{R}^3} \mathrm{d}q\ \operatorname{Tr} h(q)\mathrm{e}^{-|q|^2/2(t-s)^{2H}} &= \int_{\mathbb{R}^3} \mathrm{d}q\ \operatorname{Tr}\big[g(0)\delta(q) - \widehat{g}(q)\big]\mathrm{e}^{-|q|^2/2(t-s)^{2H}} \\ &= \int_{\mathbb{R}^3} \mathrm{d}q\ \operatorname{Tr}\big[g(0)\delta(q) - \widehat{g}(q)\big]\big[\mathrm{e}^{-|q|^2/2(t-s)^{2H}} - 1\big] \\ &= -\int_{\mathbb{R}^3} \mathrm{d}q\ \operatorname{Tr}\widehat{g}(q)\big[\mathrm{e}^{-|q|^2/2(t-s)^{2H}} - 1\big]. \end{aligned}$$

Then

$$\begin{aligned} |J| &\le \text{const}\int_0^T\int_0^T\int_{\mathbb{R}^3} \mathrm{d}q\ \big|\widehat{g}(q)\big|\big(1 - \mathrm{e}^{-|q|^2/2(t-s)^{2H}}\big)|t-s|^{2H-2}\,\mathrm{d}t\,\mathrm{d}s \\ &\le \text{const}\int_{\mathbb{R}^3} \mathrm{d}q\ \big|\widehat{g}(q)\big|\int_0^T \mathrm{d}t\int_0^t \mathrm{d}s\,(t-s)^{2H-2}\big(1 - \mathrm{e}^{-2|q|^2(t-s)^{2H}}\big) \\ &\le \text{const}\int_{\mathbb{R}^3} \mathrm{d}q\ \big|\widehat{g}(q)\big|\int_0^T \mathrm{d}t\int_0^\infty \mathrm{d}\tau\,\tau^{2H-2}\big(1 - \mathrm{e}^{-|q|^2\tau^{2H}/4}\big) \\ &= \text{const}\int_{\mathbb{R}^3} \mathrm{d}q\big|\widehat{g}(q)\big||q|^{1/H-2}\int_0^1 \mathrm{d}t\int_0^\infty \mathrm{d}y\,y^{1-1/H}\big(1 - \mathrm{e}^{-y^2/4} - 1\big) \\ &\le \text{const}\int_{\mathbb{R}^3} \mathrm{d}q\ \big|\widehat{g}(q)\big||q|^{1/H-2}, \end{aligned}$$

where we made a change of variables $y = |q|\tau^H$ and we used the fact that

$$\int_0^\infty \mathrm{d}y\,y^{1-1/H}\big(1 - \mathrm{e}^{-y^2/4}\big) \le \int_0^\infty \mathrm{d}y\,y^{1-1/H}\min\big(y^2, 1\big) < \infty$$

since $-3 < 1 - 1/H < -1$ when $1/4 < H < 1/2$. So we obtain the uniform boundedness of $E\mathcal{E}_\varepsilon$ when Eq. (40) is satisfied.

Analogously, the case $H = 1/2$ does not pose any additional problem. □

Remark 44. *Note that for $H \geq 1/2$ we recover condition* (38). *However while in* [4,17] *only the existence and the integrability properties of the energy are studied, here we have also information about convergence of ε-approximations of the velocity field generated by the random vortexes.*

Appendix: Some proofs and auxiliary results

Proof of Lemma 2. Denote $\frac{1}{\Gamma(\alpha)(4\pi)^{d/2}}$ by γ, for shortness. Notice that for $x = 0$ we have

$$K_\alpha(0) = \gamma \int_0^\infty t^{\alpha-d/2} e^{-t} \frac{dt}{t} < \infty \quad \text{if and only if } \alpha > \frac{d}{2}.$$

For $x \neq 0$ we may use the change of variables $t = |x|^2 s$ and get

$$K_\alpha(x) = |x|^{2\alpha-d} \rho(x),$$
$$\rho(x) := \gamma \int_0^\infty s^{\alpha-d/2} e^{-1/(4s)-|x|^2 s} \frac{ds}{s},$$

where the integral converges for every value of the parameters, thanks to the exponentials. For $0 < \alpha < \frac{d}{2}$, we have

$$c_{\alpha,d} e^{-2|x|^2} \leq \gamma \int_1^2 s^{\alpha-d/2} e^{-1/(4s)-|x|^2 s} \frac{ds}{s} \leq \rho(x)$$

for a positive constant $c_{\alpha,d}$. Moreover,

$$\begin{aligned}\rho(x) &\leq \gamma \int_0^{1/|x|} s^{\alpha-d/2} e^{-1/(4s)} \frac{ds}{s} + \gamma \int_{1/|x|}^\infty s^{\alpha-d/2} e^{-1/(4s)-|x|} \frac{ds}{s} \\ &\leq C_{\alpha,d} \int_0^{1/|x|} e^{-1/(8s)} \frac{ds}{s^2} + \gamma e^{-|x|} \int_{1/|x|}^\infty s^{\alpha-d/2} \frac{ds}{s} \leq C_{\alpha,d} e^{-|x|/8} + C_{\alpha,d} e^{-|x|} |x|^{d/2-\alpha} \leq C_{\alpha,d} e^{-|x|/8}\end{aligned}$$

for a positive constant $C_{\alpha,d}$ that we do not rename at every step.

If $\alpha > \frac{d}{2}$, we directly have from the original formula

$$\begin{aligned}c'_{\alpha,d} e^{-|x|^2/4} &\leq \gamma \int_1^2 t^{\alpha-d/2} e^{-|x|^2/(4t)-t} \frac{dt}{t} \leq K_\alpha(x) \\ &\leq \gamma \int_0^{|x|} t^{\alpha-d/2} e^{-|x|^2/(4t)} \frac{dt}{t} + \gamma e^{-|x|/2} \int_{|x|}^\infty t^{\alpha-d/2} e^{-t/2} \frac{dt}{t} \leq C'_{\alpha,d} e^{-|x|/8}\end{aligned}$$

for positive constants $c'_{\alpha,d}$, $C'_{\alpha,d}$. To estimate $K_\alpha(0) - K_\alpha(x)$ we write

$$K_\alpha(0) - K_\alpha(x) = \gamma \int_0^\infty t^{\alpha-d/2} e^{-t} \left(1 - e^{-|x|^2/(4t)}\right) \frac{dt}{t} = \gamma |x|^{2\alpha-d} \int_0^\infty s^{\alpha-d/2} e^{-s|x|^2} \left(1 - e^{-1/(4s)}\right) \frac{ds}{s}$$

and use the same arguments as above. In the case $\alpha = d/2$ we simply split the integral as above and by straightforward estimation we can prove that $K_\alpha(x) \leq \text{const} \log|x|$ for small $|x|$ and that $K_\alpha(x)$ decay exponentially for large $|x|$. The proof is complete. □

Proof of Lemma 4. We observe that, since K_α is the kernel of the operator $(1-\Delta)^{-\alpha}$ we have the identity $K_{\alpha-1}(x) = (1-\Delta) K_\alpha(x)$ so that

$$-\Delta K_\alpha(x) = K_{\alpha-1}(x) - K_\alpha(x), \quad x \neq 0.$$

Then $|-\Delta K_\alpha(x)| \le |K_{\alpha-1}(x)| + |K_\alpha(x)|$ which gives the required bound using Lemma 2. □

Proof of Lemma 19. We start with the first estimate in (1). A direct computation shows

$$\mathrm{Cov}\big(D_\varepsilon^0 X_t^i, D_\varepsilon^0 X_s^i\big) = \frac{|t-s|^{2H-2}}{2}\Phi\left(\frac{2\varepsilon}{t-s}\right),$$

where

$$\Phi(x) = \frac{|1+x|^{2H} + |1-x|^{2H} - 2}{x^2}.$$

The function Φ is continuous in $(0,\infty)$, $\lim_{x\to 0}\Phi(x) = 2H-1$ so, when $|t-s| > 2\varepsilon$ we have

$$\frac{|t-s|^{2H-2}}{2}\Phi\left(\frac{2\varepsilon}{t-s}\right) \le \mathrm{const}\,|t-s|^{2H-2}.$$

Moreover $\lim_{x\to\pm\infty}|x|^{2-2H}\Phi(x) = 2$, so when $|t-s| \le 2\varepsilon$ there exists a constant not depending on ε such that

$$\frac{|t-s|^{2H-2}}{2}\Phi\left(\frac{2\varepsilon}{t-s}\right) \le \mathrm{const}\,\varepsilon^{2H-2} \le \mathrm{const}\,|t-s|^{2H-2}$$

which proves the first claim.

We discuss now the second estimate in (1) and point (2). A direct computation gives

$$\begin{aligned}\mathrm{Cov}\big(D_\varepsilon^0 X_t^i, X_t^i - X_s^i\big) &= -\mathrm{Cov}\big(D_\varepsilon^0 X_s^i, X_t^i - X_s^i\big) = \frac{1}{2\varepsilon}\big(|t-s+\varepsilon|^{2H} - |t-s-\varepsilon|^{2H}\big)\\ &= |t-s|^{2H-1}\psi\left(\frac{\varepsilon}{t-s}\right),\end{aligned}$$

where $\psi(x) = \frac{|1+x|^{2H} - |1-x|^{2H}}{2x}$. It is easy to show that ψ is continuous and $\psi(0+) = 2H$, moreover $|x|^{2-2H}\psi(x) \to 2H$ when $x \to \pm\infty$. This allows us to conclude the proof. □

Proof of Lemma 27. The first estimate in (1) is very similar to the previous lemma. A direct computation shows

$$\mathrm{Cov}\big(D_\varepsilon^- X_t^i, D_\varepsilon^- X_s^i\big) = \frac{|t-s|^{2H-2}}{2}\Phi\left(\frac{\varepsilon}{t-s}\right), \tag{43}$$

where Φ is the same as previously. As for the other points a direct computation gives

$$\begin{aligned}\mathrm{Cov}\big(D_\varepsilon^- X_t^i, X_t^i - X_s^i\big) &= -\mathrm{Cov}\big(D_\varepsilon^- X_s^i, X_t^i - X_s^i\big) = \frac{1}{2\varepsilon}\big(|t-s+\varepsilon|^{2H} - |t-s-\varepsilon|^{2H} - \varepsilon^{2H}\big)\\ &= (t-s)^{2H-1}\tilde\psi\left(\frac{\varepsilon}{t-s}\right),\end{aligned}$$

where $\tilde\psi(x) = \frac{x^{2H} + 1 - (1-x)^{2H}}{2x}$. Then, it is not difficult, arguing as in Lemma 19 to conclude. In particular one can evaluate the limit in (43). □

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