

Dynamical sensitivity of the infinite cluster in critical percolation

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Abstract. In dynamical percolation, the status of every bond is refreshed according to an independent Poisson clock. For graphs which do not percolate at criticality, the dynamical sensitivity of this property was analyzed extensively in the last decade. Here we focus on graphs which percolate at criticality, and investigate the dynamical sensitivity of the infinite cluster. We first give two examples of bounded degree graphs, one which percolates for all times at criticality and one which has exceptional times of nonpercolation. We then make a nearly complete analysis of this question for spherically symmetric trees with spherically symmetric edge probabilities bounded away from 0 and 1. One interesting regime occurs when the expected number of vertices at the n th level that connect to the root at a fixed time is of order $n(\log n)^\alpha$. R. Lyons (1990) showed that at a fixed time, there is an infinite cluster a.s. if and only if $\alpha > 1$. We prove that the probability that there is an infinite cluster at all times is 1 if $\alpha > 2$, while this probability is 0 if $1 < \alpha \leq 2$. Within the regime where a.s. there is an infinite cluster at all times, there is yet another type of “phase transition” in the behavior of the process: if the expected number of vertices at the n th level connecting to the root at a fixed time is of order n^θ with $\theta > 2$, then the number of connected components of the set of times in $[0, 1]$ at which the root does not percolate is finite a.s., while if $1 < \theta < 2$, then the number of such components is infinite with positive probability.

Résumé. La percolation dynamique est un modèle dans lequel le statut de chaque arête est renouvelé aux temps de saut d'un processus de Poisson indépendant. Lorsque le graphe ne possède pas de composante infinie pour le paramètre critique de la percolation, la sensibilité dynamique de cette propriété a été étudiée en détail au cours des dix dernières années. Nous nous intéressons ici au cas des graphes pour lesquels il existe une composante infinie à la valeur critique. Tout d'abord, nous donnons deux exemples de graphes dont le degré est borné, l'un pour lequel il y a percolation à tout instant à la valeur critique, et l'autre pour lequel il existe des instants exceptionnels de non-percolation. Nous faisons ensuite une analyse quasiment complète de la question pour des arbres à symétrie sphérique, dans le cas où les probabilités d'arêtes sont également à symétrie sphérique et restent uniformément bornées loin de 0 et 1. Lorsque le nombre de sommets à distance n de la racine est de l'ordre de $n(\log n)^\alpha$, un résultat de R. Lyons affirme que, pour un instant fixé, il y a percolation si et seulement si $\alpha > 1$. Nous montrons qu'il y a une composante infinie à tout instant avec probabilité 1 lorsque $\alpha > 2$, tandis que cette probabilité vaut 0 lorsque $1 < \alpha \leq 2$. Dans le cas où il y a percolation à tout instant, nous mettons en lumière l'existence d'une autre forme de transition de phase. Si le nombre moyen de sommets qui sont connectés à la racine à un instant fixé est de l'ordre de n^θ avec $\theta > 2$, le nombre de composantes connexes de l'ensemble des instants auquel la racine ne percole pas est fini presque sûrement, mais il est infini avec probabilité strictement positive quand $1 < \theta < 2$.

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1. Introduction

Consider bond percolation on an infinite connected locally finite graph G , where for some $p \in [0, 1]$, each edge (bond) of G is, independently of all others, open with probability p and closed with probability $1 - p$. Write π_p

for this product measure. Some of the main questions in percolation theory (see [5]) deal with the possible existence of infinite connected components (clusters) in the random subgraph of G consisting of all sites and all open edges. Write \mathcal{C} for the event that there exists such an infinite cluster. By Kolmogorov’s 0–1 law, the probability of \mathcal{C} is, for fixed G and p , either 0 or 1. Since $\pi_p(\mathcal{C})$ is nondecreasing in p , there exists a critical probability $p_c = p_c(G) \in [0, 1]$ such that

$$\pi_p(\mathcal{C}) = \begin{cases} 0 & \text{for } p < p_c, \\ 1 & \text{for } p > p_c. \end{cases}$$

At $p = p_c$, we can have either $\pi_p(\mathcal{C}) = 0$ or $\pi_p(\mathcal{C}) = 1$, depending on G .

Häggström, Peres and Steif [6] initiated the study of dynamical percolation. In this model, with p fixed, the edges of G switch back and forth according to independent 2-state continuous time Markov chains where closed switches to open at rate p and open switches to closed at rate $1 - p$. Clearly, π_p is a stationary distribution for this Markov process. The general question studied in [6] was whether, when we start with distribution π_p , there could exist atypical times at which the percolation structure looks markedly different than that at a fixed time. As the results in [6] suggest, it is most interesting to consider things at criticality; that is, when $p = p_c$.

Write Ψ_p for the underlying probability measure of this Markov process, and write \mathcal{C}_t for the event that there is an infinite cluster of open edges (somewhere in the graph) at time t .

There have been a number of papers on dynamical percolation after [6], namely [8,12] and [13], but all of the results (except one, see the comment after Theorem 1.1) in these papers have been concerned with the case where the graph does not percolate at criticality (and for which there may or may not exist exceptional times). The present paper deals with the case where the graph percolates at criticality at a fixed time.

Our first theorem gives examples where exceptional times exist, and other examples where they do not exist.

Theorem 1.1. (i) *There is a bounded degree graph which, at criticality, percolates at all times; i.e.,*

$$\Psi_{p_c}(\mathcal{C}_t \text{ occurs for all } t) = 1. \tag{1.1}$$

(ii) *There is a bounded degree graph which percolates at criticality but has exceptional times, i.e.,*

$$\Psi_{p_c}(\neg\mathcal{C}_t \text{ occurs for some } t) = 1. \tag{1.2}$$

Remarks. *An example of an unbounded degree graph which percolates at criticality but for which there are exceptional times of nonpercolation can be found in [6].*

Although Theorem 1.1 follows from our Theorem 1.2, we find it instructive to treat it separately, since the proof is easier and self-contained.

We now discuss spherically symmetric trees with spherically symmetric edge probabilities. These are trees in which every vertex on a given level has the same number of offspring and the edge probabilities may vary but are constant on a given level.

Denote the root of the tree by ρ , the edge probability for edges going from level $n - 1$ to level n by p_n , the set of vertices at level n by T_n and the subtree of T rooted at some vertex x by T^x .

Standing assumption. *We assume throughout the paper that $0 < \inf_n p_n \leq \sup_n p_n < 1$.*

By a result of Lyons [9], percolation occurs (at a fixed time) if and only if

$$\sum_n \frac{(\prod_{i=1}^n p_i)^{-1}}{|T_n|} < \infty.$$

If we let $W_n := |\{x \in T_n: \rho \leftrightarrow x\}|$ and $w_n := \mathbf{E}[W_n]$, this is equivalent to

$$\sum_n \frac{1}{w_n} < \infty. \tag{1.3}$$

In fact, it follows from [9] that

$$P(\rho \leftrightarrow T_n) \asymp \left(\sum_{k=1}^n \frac{1}{w_k} \right)^{-1}. \tag{1.4}$$

(The relation \asymp means that the ratio between the two sides is bounded between two positive constants which may depend on $\inf_n p_n$ and $\sup_n p_n$.)

Dynamical percolation for a graph with edge-dependent probabilities is defined in the obvious way. To be able to see the crossover between having exceptional times of nonpercolation and not having such times, we need to look at things at the right scale. It turns out that the proper parameterization is to assume that $w_n \asymp n(\log n)^\alpha$ for some $\alpha > 0$. Lyons’ criterion (1.3) easily yields that percolation occurs (at a fixed time) if and only if $\alpha > 1$.

Theorem 1.2. *Consider a spherically symmetric tree with spherically symmetric edge probabilities.*

(i) *If*

$$\lim_n \frac{w_n}{n(\log n)^\alpha} = \infty$$

for some $\alpha > 2$, then there are no exceptional times of nonpercolation.

(ii) *If*

$$w_n \asymp n(\log n)^\alpha$$

for some $1 < \alpha \leq 2$, then there are exceptional times of nonpercolation.

Remarks. (1) *To see a concrete example, if we have a tree with $|T_n| \asymp 2^n n(\log n)^\alpha$ and $p = 1/2$ for all edges, and if $\alpha > 2$, we are in case (i) while if $\alpha \leq 2$, we are in case (ii). (Note that Lyons’ theorem tells us that $p_c = 1/2$ in these cases.)*

(2) *The theorem implies that if $w_n \asymp n^\alpha$ with $\alpha > 1$, there are no exceptional times of nonpercolation, while if $w_n \asymp n$, then (1.3) implies that there is no percolation at a fixed time. Hence, if we only look at the case where $w_n \asymp n^\alpha$ for some $\alpha \geq 1$, we do not see the dichotomy we are after. Rather, Theorem 1.2 tells us that one needs to look at a “finer logarithmic scale” to see this “phase transition.”*

Interestingly, it turns out that even within the regime where there are no exceptional times of nonpercolation, there are still two very distinct dynamical behaviors of the process.

Theorem 1.3. *Consider a spherically symmetric tree T , with spherically symmetric edge probabilities. Let d_j denote the number of children that a vertex in T_j has.*

(i) *When $\sum_{k=1}^\infty k w_k^{-1} < \infty$, a.s. the set of times $t \in [0, 1]$ at which the root percolates has finitely many connected components. (This holds, for example, if $w_k \asymp k^\theta$ with $\theta > 2$ as well as for supercritical percolation on a homogeneous tree.)*

(ii) *If $\sup_j d_j < \infty$ and $w_k \asymp k^\theta$, where $1 < \theta < 2$, then with positive probability the set of times $t \in [0, 1]$ at which the root percolates has infinitely many connected components. The same occurs if $w_k \asymp k(\log k)^\theta$ for any $\theta > 1$.*

Remarks. (1) *There is some gap between cases (i) and (ii), in particular, the case $w_k \asymp k^2$. In Theorem 5.2 we give more general conditions under which (ii) holds, but we do not close this gap.*

(2) *It is easy to show (see, for example, Lemma 3.2) that for any graph, if there are exceptional times of nonpercolation, then the set of times $t \in [0, 1]$ at which a fixed vertex percolates is totally disconnected and hence has infinitely many connected components with positive probability.*

From the proof of Theorem 1.3(i), it is easy to see that for any graph, any edge-dependent probabilities and any fixed vertex x , if I_n is the sum of the influences (see Section 5 for the definition of influence) for the event

$\{x$ percolates to distance n away $\}$, then $\liminf_n I_n < \infty$ implies that the set of times $t \in [0, 1]$ at which x percolates has finitely many connected components a.s. Next, if $I_x(e)$ is the influence of the edge e for the event $\{x \leftrightarrow \infty\}$, it is easy to see from Fatou's lemma that

$$\sum_e I_x(e) \leq \liminf_n I_n. \quad (1.5)$$

The next result tells us what we can conclude under the assumption that $\sum_e I_x(e) < \infty$.

Theorem 1.4. *Consider dynamical percolation on any connected graph with possibly edge-dependent probabilities which percolates at a fixed time and let $x \in V$. Assume that*

$$\sum_e I_x(e) < \infty. \quad (1.6)$$

Then a.s. $f(t) := \mathbf{1}_{\{x \leftrightarrow \infty\}}^t$ is equal a.e. to a function of bounded variation on $[0, 1]$. Moreover, this implies that there are no exceptional times of nonpercolation.

Remarks. (1) *Note that this result is applicable even in the supercritical case.*

(2) *While it is easy to check that when the graph is a tree the summability above does not depend on x , interestingly, this is false in the general context of connected graphs, even in the case of bounded degree.*

In [6], it was argued that the events discussed in the above theorems are measurable; a similar comment applies to all of our results. Thus, measurability issues will not concern us here.

As far as motivation, the questions that we look at give us a better understanding of the stability properties of a critical infinite cluster while at the same time they fall into the general framework of studying polar sets for stationary reversible Markov processes.

The dynamical percolation results in [6] were extended in [12] and then further refined in [8]. In [13], it was shown that there are exceptional times at criticality on the triangular lattice, yielding the first example of a transitive graph with this property. We mention a few other papers where analogous dynamical sensitivity questions have been studied for other models. Analogous questions for the Boolean model, where the points undergo independent Brownian motions, were studied in [3] and for certain interacting particle lattice systems (where updates are therefore not done in an independent fashion) are studied in [4]. In [2], it is shown that there are exceptional two-dimensional slices for the Boolean model in four dimensions and finally, in [7], dynamical versions of Dvoretzky's circle covering problem are studied.

Notation. (1) *For subsets A and B of the vertices and t , we let $\{A \overset{t}{\leftrightarrow} B\}$ be the event that at time t there is an open path from A to B and $\{A \leftrightarrow B\}$ be the analogous event for ordinary percolation. (If $B = \infty$, this has the obvious meaning.) In the context of trees with a distinguished root, $A \mapsto B$ will mean that there is a path of open edges connecting A to B along which the distance to the root is monotone increasing. The notation $A \overset{t}{\mapsto} B$ is similarly defined.*

(2) *We use \asymp to denote the relationship between two quantities whose ratio is bounded away from both 0 and ∞ .*

(3) *$O(1)$ will denote a function bounded away from ∞ , $o(1)$ will denote a function approaching 0, and $\Omega(1)$ will denote a function bounded away from 0.*

Convention. *The edges are defined to be on at the times at which they change state; in this way, the set of times an edge is on is a closed set. As explained in [6], this modification is of no significance, but allows some notational simplification in some topological arguments.*

The rest of the paper is organized as follows: In Section 2, we prove Theorem 1.1. In Section 3, we prove two lemmas which will be needed for the proof of Theorem 1.2. We prove Theorem 1.2 in Section 4, Theorem 1.3 in Section 5 and Theorem 1.4 in Section 6. In Section 7, we prove a certain 0–1 law for the evolution of the process and finally we list some open questions in Section 8.

2. Two examples

The idea in the construction of the examples is rather simple; we take the planar square lattice \mathbb{Z}^2 and replace each edge by an appropriate graph, with different graphs for different edges. For the example without exceptional times, we will want the connection along the corresponding graphs to be rather stable, while for the example with exceptional times, we will want the connections to switch quickly. The following lemma gives the existence of the necessary building blocks for both examples. It contains a variant of Lemma 2.3 in [6] with the crucial difference being that the degrees are now bounded.

Lemma 2.1. *There is a sequence of finite graphs G_j and pairs of vertices x_j and y_j in G_j , such that the following properties hold:*

1. $\mathbb{P}_{1/2}^{G_j}(x_j \leftrightarrow y_j) > \frac{2}{3}$ for all j ,
2. $\lim_{j \rightarrow \infty} \mathbb{P}_p^{G_j}(x_j \leftrightarrow y_j) = 0$ for all $p < \frac{1}{2}$,
3. for every $\varepsilon > 0$ we have

$$\lim_{j \rightarrow \infty} \Psi_{1/2}^{G_j} \left(\bigcap_{t \in [0, \varepsilon]} \{x_j \overset{t}{\leftrightarrow} y_j\} \right) = 0,$$

4. and there is some finite upper bound for the degrees of the vertices in G_j (the bound does not depend on j).

Proof. Let H be obtained from the square grid in the plane by replacing each edge by m parallel edges, where m is chosen so that the probability that the origin percolates in H at $p = 1/2$ is at least 0.99. Let v_i denote the vertex $(i, 0)$ of H . Then for every i we have $\mathbb{P}_{1/2}^H(v_0 \leftrightarrow v_i) \geq (0.99)^2 > 0.98$. Hence, there is a finite subgraph H_j of H such that $\mathbb{P}_{1/2}^{H_j}(v_0 \leftrightarrow v) \geq 0.98$ holds for every $v \in A_j$, where $A_j := \{v_i : 1 \leq i \leq 9 \cdot 2^j\}$. The graph G_j is obtained by taking two disjoint copies of H_j and connecting each of the vertices corresponding to $v_i \in A_j$ in one copy to the vertex corresponding to v_i in the other copy by a path of length j , where the paths are of course disjoint. The vertex x_j is chosen as v_0 in one copy of H_j , while y_j is v_0 in the other copy. The paths of length j in G_j connecting one copy of H_j to the other will be called *bridges*.

We now verify that G_j satisfies the required properties. Let B_j denote the set of vertices in A_j connected to v_0 by an open path in H_j . Since $\mathbb{P}_{1/2}^{H_j}(v_0 \leftrightarrow v) \geq 0.98$ for all $v \in A_j$, we have $\mathbb{P}_{1/2}^{H_j}(|B_j| < (2/3)|A_j|) < 0.9$. This implies that in G_j at $p = 1/2$ with probability at least $(0.9)^2$ we have that the endpoints of at least $1/3$ of the bridges are connected to x_j within x_j 's copy of H_j and to y_j within y_j 's copy of H_j . On this event, the conditional probability that x_j and y_j are not connected is at most

$$(1 - 2^{-j})^{|A_j|/3} \leq \exp(-2^{-j})^{|A_j|/3} = e^{-3}.$$

Thus, we get $\mathbb{P}_{1/2}^{G_j}(x_j \leftrightarrow y_j) \geq (0.9)^2(1 - e^{-3}) > 2/3$, proving 1.

If $p < 1/2$, then the expected number of bridges that are open in G_j is $|A_j|p^j = 9 \cdot 2^j \cdot p^j \rightarrow 0$ as $j \rightarrow \infty$, which proves 2.

In order to prove 3, fix some $\varepsilon > 0$, and consider dynamical percolation at $p = 1/2$ on G_j . Let $t, s \in [0, \varepsilon]$ satisfy $s \neq t$, and let X_t^j denote the event that at time t there is some bridge in G_j that is open. Fix some ordering of the bridges in G_j , and let $X_t^j(i)$ denote the event that the i th bridge is open at time t . Also let $\hat{X}_t^j(i)$ be the event that the i th bridge is open at time t and this does not hold for any smaller i . Note that for every fixed i ,

$$\Psi_{1/2}^{G_j}(X_t^j \setminus X_t^j(i) | \hat{X}_s^j(i)) \leq \Psi_{1/2}^{G_j}(X_t^j).$$

Therefore,

$$\begin{aligned} \Psi_{1/2}^{G_j}(X_t^j, \hat{X}_s^j(i)) &= \Psi_{1/2}^{G_j}(X_t^j \setminus X_t^j(i), \hat{X}_s^j(i)) + \Psi_{1/2}^{G_j}(X_t^j(i), \hat{X}_s^j(i)) \\ &\leq \Psi_{1/2}^{G_j}(X_t^j) \Psi_{1/2}^{G_j}(\hat{X}_s^j(i)) + \Psi_{1/2}^{G_j}(X_t^j(i), \hat{X}_s^j(i)). \end{aligned}$$

On the other hand, the conditional probability of $X_t^j(i)$ given $\hat{X}_s^j(i)$ does not depend on i and goes to zero as $j \rightarrow \infty$ while $s \neq t$ are held fixed. Thus,

$$\Psi_{1/2}^{G_j}(X_t^j, \hat{X}_s^j(i)) \leq \Psi_{1/2}^{G_j}(X_t^j) \Psi_{1/2}^{G_j}(\hat{X}_s^j(i)) + o(1) \Psi_{1/2}^{G_j}(\hat{X}_s^j(i)).$$

As X_s^j is the disjoint union of the events $\hat{X}_s^j(i)$, by summing the above over i , we obtain

$$\Psi_{1/2}^{G_j}(X_t^j, X_s^j) \leq \Psi_{1/2}^{G_j}(X_t^j) \Psi_{1/2}^{G_j}(X_s^j) + o(1),$$

as $j \rightarrow \infty$.

Set $X^j := \int_0^\varepsilon \mathbf{1}_{X_t^j} dt$. Fubini and the dominated convergence theorem now imply that $\limsup_{j \rightarrow \infty} \mathbf{E}[(X^j)^2] - \mathbf{E}[X^j]^2 \leq 0$; that is, the variance of X^j tends to 0. Since

$$\mathbf{E}[X^j] = \varepsilon \Psi_{1/2}^{G_j}(X_0^j) = \varepsilon(1 - (1 - 2^{-j})^{|A_j|}) \xrightarrow{j \rightarrow \infty} \varepsilon(1 - e^{-9}),$$

and the right-hand side is smaller than ε , it follows that $\Psi_{1/2}^{G_j}(X^j = \varepsilon)$ tends to 0 as $j \rightarrow \infty$. This proves 3.

Claim 4 is obvious from the construction. □

Proof of Theorem 1.1. Both examples are obtained by replacing each edge $[x, y]$ in the square lattice \mathbb{Z}^2 by a copy of some G_j , with x_j identified with x and y_j identified with y . The difference between the two examples has to do with the choice of j for the different edges.

We start by proving (i). By property 1 of Lemma 2.1, it follows that for each j there is some positive integer $n_j > 0$ such that

$$\Psi_{1/2}^{G_j} \left(\bigcap_{t \in [0, 1/n_j]} \{x_j \overset{t}{\leftrightarrow} y_j\} \right) > \frac{3}{5}.$$

We may assume without loss of generality that the sequence $\{n_j\}$ is increasing in j . We now define inductively an increasing sequence $\{R_j\}$. Set $n_j^* := n_{j+2}$. For any two radii $0 < r < r'$, let $\mathcal{A}(r, r')$ denote the event that there is an open cycle in \mathbb{Z}^2 separating $\partial B(0, r)$ from $\partial B(0, r')$, where $\partial B(0, r) := \{x: |x|_\infty = r\}$ and $|x|_\infty$ denotes the L_∞ norm of x . Let R_0 be so large that

$$\mathbb{P}_{3/5}(B(0, R_0) \leftrightarrow \infty) \geq \frac{1}{2}.$$

For all $j > 0$, given R_{j-1} , we choose $R_j > R_{j-1}$ sufficiently large so that

$$\mathbb{P}_{3/5}(B(0, R_j) \leftrightarrow \infty, \mathcal{A}(R_{j-1}, R_j)) \geq 1 - 2^{-j} (n_j^*)^{-1}.$$

Let G be obtained from \mathbb{Z}^2 by replacing, for each $j > 0$, each edge e in the annulus $B(0, R_j) \setminus B(0, R_{j-1})$ by a new copy of G_j , where x_j and y_j are identified with the endpoints of e . By property 2 of the lemma, it follows that at every $p < 1/2$, Bernoulli percolation on G a.s. has no infinite cluster. Hence $p_c(G) \geq 1/2$.

We now consider dynamical percolation on G with parameter $p = \frac{1}{2}$, and show that $\Psi_{1/2}^G$ -a.s. there is an infinite percolation cluster at all times. This, in particular, implies that $p_c(G) \leq 1/2$; and hence $p_c(G) = 1/2$.

For $I \subseteq [0, \infty)$, let $\mathcal{A}_j(I)$ denote the event that at all times $t \in I$ there is an open cycle in G separating $\partial B(0, R_j)$ from $\partial B(0, R_{j-1})$ and an open path in G connecting $\partial B(0, R_{j-1})$ with $\partial B(0, R_{j+1})$. Then $\Psi_{1/2}^G\{\mathcal{A}_j([0, 1/n_{j+1}])\} \geq 1 - 2^{-j+2}/n_{j-1}^*$, whence $\Psi_{1/2}^G(\mathcal{A}_j([0, 1])) \geq 1 - 2^{-j+2}$. Now note that if $\bigcap_{j>k} \mathcal{A}_j([0, 1])$ holds for some k , then there is percolation in G for every $t \in [0, 1]$. Since $\Psi_{1/2}^G(\bigcap_{j>k} \mathcal{A}_j([0, 1])) \geq 1 - 2^{-k+2}$, this gives $\Psi_{1/2}^G(\bigcap_{t \in [0, 1]} \mathcal{C}_t) = 1$, which implies (i).

We now turn to the proof of (ii). Using Lemma 2.1 together with the proof of the second part of Theorem 1.2 in [6], it is easily seen that if we replace the i th edge by G_{j_i} with the sequence $\{j_i\}$ growing to infinity sufficiently fast, we obtain an example of the desired form. □

3. Some lemmas

We now consider a spherically symmetric tree with spherically symmetric edge probabilities. As in the introduction, W_n will denote the number of vertices in T_n that are connected to the root, and w_n denotes the expectation of W_n .

By Theorem 2.3 of [9] (together with the proof of Theorem 2.4 in that paper and the fact that for a spherically symmetric kernel, the measure that minimizes energy is the uniform measure, a fact which in turn is obtained using convexity of energy together with symmetry), it follows that

$$\frac{w_n^2}{E[W_n^2]} \leq P(W_n > 0) \leq \frac{2w_n^2}{E[W_n^2]}. \tag{3.1}$$

The second inequality yields

$$E[W_n^2 | W_n > 0] \leq 2E[W_n | W_n > 0]^2, \tag{3.2}$$

which will be useful below.

Lemma 3.1. *Consider an indexed collection $\{X_{i,j}\}_{i \geq 1, 1 \leq j \leq N_i}$ of nonnegative mean 1 random variables such that (1) for each i , $\{X_{i,j}\}_{1 \leq j \leq N_i}$ are i.i.d. and (2) the entire family of random variables is uniformly integrable. Then for each $\varepsilon > 0$, there is $c > 0$ such that for each i ,*

$$P\left(\sum_{j=1}^{N_i} X_{i,j} \leq N_i(1 - \varepsilon)\right) \leq e^{-cN_i}.$$

Proof. Let $\varepsilon > 0$. By uniform integrability, there exists $h = h(\varepsilon)$ such that for all i and j ,

$$E(X_{i,j} \wedge h) \geq 1 - \frac{\varepsilon}{2}.$$

We then have

$$P\left(\sum_{j=1}^{N_i} X_{i,j} \leq N_i(1 - \varepsilon)\right) \leq P\left(\sum_{j=1}^{N_i} X_{i,j} \wedge h \leq N_i(1 - \varepsilon)\right) \leq P\left(\sum_{j=1}^{N_i} X_{i,j} \wedge h \leq N_i\left(E(X_{i,j} \wedge h) - \frac{\varepsilon}{2}\right)\right).$$

As we now have bounded random variables, the standard Chernoff bound arguments allow us to bound the latter by e^{-cN_i} for some fixed $c = c(\varepsilon, h) > 0$. □

Lemma 3.2. *Fix a connected graph G and $x \in V(G)$. Let $B_M := \{y: d_G(x, y) \leq M\}$ where d_G is the graph distance. Then the following are equivalent.*

- (i) $\Psi_p(\mathcal{C}_t \text{ occurs for every } t) = 1$.
- (ii) $P(\exists M: B_M \overset{t}{\leftrightarrow} \infty \forall t \in [0, 1]) = 1$.
- (iii) $P(x \overset{t}{\leftrightarrow} \infty \forall t \in [0, 1]) > 0$.

Proof. The implication (iii) \Rightarrow (i) is immediate from Kolmogorov’s 0–1 law. The implication (ii) \Rightarrow (iii) is easy and left to the reader. We now show that (i) implies (ii). If (ii) is false, Kolmogorov’s 0–1 law implies that the event in (ii) has probability 0. Positive association of the process and the above 0–1 law then would yield that for all $\delta > 0$,

$$P(\exists M: B_M \overset{t}{\leftrightarrow} \infty \forall t \in [0, \delta]) = 0. \tag{3.3}$$

Now, for each vertex v , let U_v be the open set of times in $[0, 1]$ at which v is not percolating. Countable additivity and (3.3) easily imply that a.s. each U_v is dense. The Baire category theorem implies that a.s.

$$\bigcap_v U_v \neq \emptyset.$$

However, this intersection is exactly the set of nonpercolating times in $[0, 1]$ and hence (i) is false. □

Remarks. Observe that given any graph which percolates at criticality and for which there are exceptional nonpercolating times, using the U_v 's as above, the Baire Category Theorem gives that the set of nonpercolating times in $[0, 1]$ is a dense G_δ set of zero measure. An additional use of the Baire Category Theorem tells us that if we hook up a finite number of such graphs at a common vertex, there will still be nonpercolating times and they will also form a dense G_δ of zero measure. This situation is very different from the case where one looks at time sets corresponding to the times at which a tree, which does not percolate at criticality (in static percolation), percolates; such time sets do not necessarily intersect each other.

4. Proof of Theorem 1.2

We now begin with the proof.

Proof of Theorem 1.2(i). Recall that ρ denotes the root of the tree. Fix an $\alpha > 2$, and assume that $\lim_n \frac{w_n}{n(\log n)^\alpha} = \infty$. Choose $\varepsilon > 0$ such that $2 + 2\varepsilon < \alpha$. Let $n_k := 2^{2^k}$. (So $n_0 = 2$ and $n_{k+1} = n_k^2$.) For each k and each $i \in \{1, \dots, n_k^2\}$, let $I_i^k = [(i - 1)/n_k^2, i/n_k^2]$. Let $A_i^k := \{x \in T_{n_k} : \rho \overset{t}{\leftrightarrow} x \ \forall t \in I_i^k\}$, and let G_k denote the event that $|A_i^k| \geq w_{n_k}/(\log n_k)^\varepsilon$ holds for every $i \in \{1, 2, \dots, n_k^2\}$. We need to obtain a good bound on $P(G_{k+1}^c | \mathcal{F}_{n_k})$ on the event G_k , where \mathcal{F}_n is the σ -algebra generated by the evolution of the first n levels of the tree. The key proposition, whose proof we give afterwards, is the following proposition.

Proposition 4.1. *There exists $\gamma > 1$ so that for all large k , if $A \subseteq T_{n_k}$ is fixed with $|A| \geq w_{n_k}/(\log n_k)^\varepsilon$, then*

$$P\left(\left|\{x \in T_{n_{k+1}} : A \overset{t}{\mapsto} x \ \forall t \in I_1^{k+1}\}\right| \leq \frac{w_{n_{k+1}}}{(\log n_{k+1})^\varepsilon}\right) \leq e^{-(\log n_k)^\gamma}.$$

We now first complete the proof of Theorem 1.2(i) by noting that it is easy to see that Proposition 4.1 implies that for large k , we have that on G_k

$$P(G_{k+1}^c | \mathcal{F}_{n_k}) \leq n_{k+1}^2 e^{-(\log n_k)^\gamma}.$$

Since $\gamma > 1$, we have

$$\sum_k n_{k+1}^2 e^{-(\log n_k)^\gamma} < \infty.$$

For any finite k' , we have $P(\bigcap_{k \leq k'} G_k) > 0$. Hence, the above implies that $P(G_k \ \forall k) > 0$, and since $\bigcap_k G_k \subseteq \{\rho \overset{t}{\leftrightarrow} \infty \ \forall t \in [0, 1]\}$, this implies

$$P(\rho \overset{t}{\leftrightarrow} \infty \ \forall t \in [0, 1]) > 0.$$

This yields the required result by Lemma 3.2. □

Before starting the proof of Proposition 4.1, we first need the following lemma.

Lemma 4.2. *Consider a spherically symmetric tree with spherically symmetric edge probabilities, and assume that for some $\beta > 1$, $w_n \geq \Omega(1)n(\log n)^\beta$ holds for every n . If $x \in T_{n_k}$, then*

$$P(x \mapsto T_{n_{k+1}})w_{n_k} \geq \Omega(1)(\log n_k)^{\beta-1}.$$

Proof. It is easy to see that for $x \in T_{n_k}$, the expected number of vertices in T_ℓ connected to x within T^x is w_ℓ/w_{n_k} for $\ell \geq n_k$. Hence by (1.4), if $x \in T_{n_k}$, we have that

$$P(x \mapsto T_{n_{k+1}}) \asymp \left(\sum_{\ell=n_{k+1}}^{n_{k+1}} \frac{w_{n_k}}{w_\ell} \right)^{-1} \geq \Omega(1) \frac{1}{w_{n_k}} \left(\sum_{\ell=n_{k+1}}^{n_{k+1}} \frac{1}{\ell(\log \ell)^\beta} \right)^{-1}.$$

Next

$$\sum_{\ell=n_{k+1}}^{n_{k+1}} \frac{1}{\ell(\log \ell)^\beta} \asymp \int_{n_k}^{n_{k+1}} \frac{1}{x(\log x)^\beta} dx = \int_{\log n_k}^{\log(n_{k+1})} \frac{1}{u^\beta} du \asymp (\log n_k)^{1-\beta},$$

since $n_k = 2^{2^k}$, completing the proof. □

Proof of Proposition 4.1. For $x \in T_{n_k}$, let R_x be the number of vertices at level n_{k+1} which are connected to x within T^x throughout $[0, 1/n_{k+1}^2]$ and let R_k denote a random variable which has distribution R_x . The expected number of vertices at level n_{k+1} which are connected to x within T^x at time 0 is $w_{n_{k+1}}/w_{n_k}$. Since a given path of length $n_{k+1} - n_k$ is updated during $[0, 1/n_{k+1}^2]$ with probability $o(1)$, we have

$$E[R_k] = \frac{w_{n_{k+1}}}{w_{n_k}} (1 - o(1)), \tag{4.1}$$

as $k \rightarrow \infty$.

Lemma 4.3. *Let \tilde{R}_k have distribution R_k conditioned on $\{R_k > 0\}$. Then*

$$E[(\tilde{R}_k)^2] \leq O(1)E[\tilde{R}_k]^2.$$

Proof. Fix some $x \in T_{n_k}$, and let $R'_x := |\{y \in T_{n_{k+1}} : x \stackrel{0}{\mapsto} y\}|$. We have argued above that $E[R_k] \geq (1 - o(1))E[R'_x]$. This implies $E[R_k] \asymp E[R'_x]$. A similar argument gives $P(R'_x > 0) \asymp P(R_k > 0)$. Since $R'_x \geq R_x$, this together with (3.2) easily leads to the statement; the details are left to the reader. □

Lemma 4.4. *There exists $\gamma > 1$ so that for all $\delta > 0$, we have that for large k , if $A \subseteq T_{n_k}$ with $|A| \geq w_{n_k}/(\log n_k)^\varepsilon$, then*

$$P\left(\left|\{x \in A : R_x > 0\}\right| \leq (1 - \delta) \frac{P(R_k > 0)w_{n_k}}{(\log n_k)^\varepsilon}\right) \leq e^{-(\log n_k)^\gamma}.$$

Proof. The random variable $X := |\{x \in A : R_x > 0\}|$ has a binomial distribution with parameters $|A|$ and $P(R_k > 0)$. The probability in the statement of the lemma is at most

$$P(X \leq E[X](1 - \delta)).$$

By standard large deviations (see for example Corollary A.1.14 in [1]), the latter is at most $2e^{-c_\delta E(X)}$. Lemma 4.2 and our choice of ε imply that $E[X] \geq \Omega(1)(\log n_k)^{1+\varepsilon}$, proving the claim. □

Lemma 4.5. *There exists $\delta > 0$ and $\gamma > 1$ such that for all large k , if*

$$M \geq (1 - \delta) \frac{P(R_k > 0)w_{n_k}}{(\log n_k)^\varepsilon}$$

and Y_1, \dots, Y_M are i.i.d. with the distribution of \tilde{R}_k (defined in Lemma 4.3), then

$$P\left(\sum_{i=1}^M Y_i \leq \frac{w_{n_{k+1}}}{(\log n_{k+1})^\varepsilon}\right) \leq e^{-(\log n_k)^\gamma}. \tag{4.2}$$

Proof. Choose δ so that

$$\frac{1}{2^\varepsilon(1-\delta)} < 1. \tag{4.3}$$

Our lower bound on M and an easy calculation shows that the left-hand side of (4.2) is bounded by

$$P\left(\frac{1}{M} \sum_{i=1}^M \frac{Y_i}{E[Y_i]} \leq S_k\right), \quad \text{where } S_k := \frac{w_{n_{k+1}}(\log n_k)^\varepsilon}{w_{n_k}(\log n_{k+1})^\varepsilon(1-\delta)E[R_k]}.$$

The expression (4.1) for $E[R_k]$ implies that $\lim_{k \rightarrow \infty} S_k = 1/(2^\varepsilon(1-\delta))$. Since a family of random variables which have a uniform bound on their second moments is uniformly integrable, Lemmas 3.1 and 4.3 and (4.3) imply that

$$P\left(\frac{1}{M} \sum_{i=1}^M \frac{Y_i}{E[Y_i]} \leq S_k\right) \leq e^{-cM},$$

for some $c > 0$ and all large k . Lemma 4.2 insures that $M \geq \Omega(1)(\log n_k)^{1+\varepsilon}$, completing the proof. □

One finally notes that Proposition 4.1 is a consequence of Lemmas 4.4 and 4.5. □

Remark. In the proof of Theorem 1.2(ii), we separate things into the two cases $\alpha < 2$ and $\alpha = 2$ but we emphasize that this is done for presentational purposes only.

We now move to the following proof.

Proof of Theorem 1.2(ii); case $\alpha < 2$. Let $A := \{\rho \stackrel{t}{\leftrightarrow} \infty \forall t \in [0, 1]\}$. By Lemma 3.2, it suffices to show that $P(A) = 0$ and for this it suffices to show that for every $M > 0$, there is an event $G = G(M)$ so that $P(G) \geq 1 - 2/M$ and $P(A|G) = 0$. We now fix such an M . The $O(1)$ terms appearing below may (and will) depend on M (but they will of course be independent of the level of the tree under discussion).

For the moment, we consider our percolation at a fixed time. It is well known that $\{W_n/w_n\}$ (recall W_n is the number of vertices on the n th level connected to the root) is a nonnegative martingale and hence converges a.s. to a random variable denoted W_∞ with $E[W_\infty] \leq 1$. Doob’s inequality tells us that

$$P\left(\frac{W_n}{w_n} \geq M \text{ for some } n \geq 0\right) \leq \frac{1}{M}. \tag{4.4}$$

Returning to our dynamical model, we let $W_{n,t}$ be the analogue of W_n above but at time t . We now define

$$G := \left\{ \mu \left\{ t \in [0, 1]: \frac{W_{n,t}}{w_n} \geq M \text{ for some } n \geq 0 \right\} < \frac{1}{2} \right\},$$

where μ denotes Lebesgue measure. Fubini’s theorem, Markov’s inequality and (4.4) easily yield that $P(G) \geq 1 - 2/M$. We will show that $P(A|G) = 0$, completing the proof.

Set $m_n := \lfloor M w_n \rfloor$. For all $B \subseteq T_n$ with $|B| \leq m_n$, let \tilde{B} be a subset of T_n containing B such that $|\tilde{B}| = m_n$, and such that \tilde{B} is a deterministic function of B . Of course, this can only be done for $n \geq N = N(M) := \min\{k: |T_k| \geq m_k\}$. If $|B| > m_n$, we take \tilde{B} to be the leftmost m_n elements of B .

Let $S_{n,t}$ be the set of vertices in T_n that are connected to ρ by open paths at time t . Then $W_{n,t} = |S_{n,t}|$. For each $n \geq N = N(M)$, define the random variable

$$X_n := \mu \left\{ t \in [0, 1]: W_{n,t} \leq m_n, \tilde{S}_{n,t} \not\stackrel{t}{\leftrightarrow} \infty \right\}.$$

The key step is to carry out a conditional second moment argument on X_n conditioned on the evolution of the first n levels on that part of the probability space where something “good” happens. The following proposition will be the consequence of this conditional second moment argument.

Proposition 4.6. *There exists $c = c(M) > 0$ such that for all n sufficiently large*

$$P(X_n > 0 | \mathcal{F}_n) \geq c \quad \text{on } G,$$

where \mathcal{F}_n is the σ -algebra generated by the evolution of the first n levels of the tree.

We postpone the proof of the proposition, and continue with the proof of the theorem. It is clear that $\{X_n > 0\} \subseteq A^c$ and hence

$$P(A^c | \mathcal{F}_n) \geq c \quad \text{on } G.$$

Letting $n \rightarrow \infty$, Levy's 0-1 law implies that the left-hand side approaches $\mathbf{1}_{A^c}$ a.s. As $c > 0$, we conclude that $P(A|G) = 0$, as desired. \square

Before starting the proof of Proposition 4.6, we need a lemma. Let

$$q_n := P(x \overset{0}{\mapsto} \infty) \quad \text{and} \quad q_n(t) := P(\{x \overset{t}{\mapsto} \infty\} \cap \{x \overset{0}{\mapsto} \infty\}),$$

where $x \in T_n$.

It is easy to check that the proof of Lemma 4.2 shows that

$$q_n \asymp \frac{1}{n \log n}. \tag{4.5}$$

Lemma 4.7.

$$q_n(t) \leq \frac{O(1)q_n^2}{t}.$$

Proof. Fix $x \in T_n$ and $t \in (0, 1]$. Suppose that $x \overset{0}{\mapsto} \infty$, and condition on the leftmost open path $\pi = (\pi_0, \pi_1, \dots)$ from x to ∞ inside T_x at time 0. Let K_j be the event that at time t there is an open path from x to ∞ that shares exactly j edges with π . Because in the complement of π the conditional law of the dynamical percolation is dominated by the unconditional law, we clearly have

$$P(K_j | x \overset{0}{\mapsto} \infty) \leq P(\pi_j \overset{t}{\mapsto} \infty) P(x \overset{t}{\mapsto} \pi_j | x \overset{0}{\mapsto} \infty) = q_{n+j} \prod_{i=1}^j (p_{n+i}(1 - e^{-t}) + e^{-t}).$$

Since $P(K_\infty) = 0$, we get

$$q_n(t) = q_n P(x \overset{t}{\mapsto} \infty | x \overset{0}{\mapsto} \infty) \leq q_n \sum_{j=0}^{\infty} P(K_j | x \overset{0}{\mapsto} \infty) \leq q_n \sum_{j=0}^{\infty} q_{n+j} \prod_{i=1}^j (p_{n+i}(1 - e^{-t}) + e^{-t}).$$

As the p_i 's are bounded away from 1, there exists a constant $\varepsilon_0 \in (0, 1)$ such that each factor in the product on the right is at most $1 - \varepsilon_0 t$ (regardless of the choice of t in $(0, 1]$). Hence, the above gives

$$\begin{aligned} q_n(t) &\leq q_n \sum_{j=0}^{\infty} q_{n+j} (1 - \varepsilon_0 t)^j \leq q_n \sup\{q_{n+j} : j = 0, 1, \dots\} \sum_{j=0}^{\infty} (1 - \varepsilon_0 t)^j \\ &= q_n \sup\{q_{n+j} : j = 0, 1, \dots\} (\varepsilon_0 t)^{-1}. \end{aligned}$$

Now an appeal to (4.5) completes the proof. \square

Let

$$\tilde{q}_n := 1 - q_n. \tag{4.6}$$

Next, letting $\tilde{q}_n(t)$ be the probability that a given vertex at level n does not percolate to ∞ both at time 0 and at time t , we easily have that

$$\tilde{q}_n(t) = 1 - 2q_n + q_n(t). \tag{4.7}$$

We use (4.6) and (4.7), to obtain

$$\frac{\tilde{q}_n(t)}{\tilde{q}_n^2} = \frac{1 - 2q_n + q_n(t)}{(1 - q_n)^2} = 1 + \frac{q_n(t) - q_n^2}{(1 - q_n)^2} \leq 1 + \frac{q_n(t)}{(1 - q_n)^2}.$$

By Lemma 4.7 and (4.5) we therefore get

$$\frac{\tilde{q}_n(t)}{\tilde{q}_n^2} \leq 1 + O\left(\frac{q_n^2}{t}\right). \tag{4.8}$$

We can now carry out the following proof.

Proof of Proposition 4.6. We apply a conditional second moment argument. First, it is immediate that for any $n \geq N$

$$E[X_n | \mathcal{F}_n] \geq \frac{1}{2}(\tilde{q}_n)^{m_n} \mathbf{1}_G.$$

In order to estimate $E[X_n^2 | \mathcal{F}_n]$, we note that

$$\mathbf{P}[\tilde{S}_{n,s} \not\rightarrow \infty, \tilde{S}_{n,t} \not\rightarrow \infty | \mathcal{F}_n] = \tilde{q}_n (|t - s|)^{|\tilde{S}_{n,s} \cap \tilde{S}_{n,t}|} \tilde{q}_n^{|\tilde{S}_{n,s} \setminus \tilde{S}_{n,t}| + |\tilde{S}_{n,t} \setminus \tilde{S}_{n,s}|}.$$

Since $\tilde{q}_n(t) \geq \tilde{q}_n^2$, this gives for every $n \geq N$ a.s.

$$E[X_n^2 | \mathcal{F}_n] \leq \int_0^1 \int_0^1 \tilde{q}_n (|t - s|)^{m_n} dt ds \leq 2 \int_0^1 \tilde{q}_n(t)^{m_n} dt. \tag{4.9}$$

Using the trivial bound $\tilde{q}_n(t) \leq \tilde{q}_n$ for $t \leq 1/n$ and the bound (4.8) for larger values of t , we get that on G

$$\frac{E[X_n^2 | \mathcal{F}_n]}{E[X_n | \mathcal{F}_n]^2} \leq 8 \int_0^{1/n} \left(\frac{1}{\tilde{q}_n}\right)^{m_n} dt + 8 \int_{1/n}^1 \left(1 + O\left(\frac{q_n^2}{t}\right)\right)^{m_n} dt. \tag{4.10}$$

Using (4.5) and (4.6), if $\alpha < 2$, then the first integrand is easily checked to be at most $O(1)n^\sigma$ for some $\sigma < 1$ (and in fact for any $\sigma < 1$ with the $O(1)$ term then of course depending on σ) and hence the first integral goes to 0. If $\alpha \leq 2$, then, using (4.5), it is easy to check that the second integrand, when $t \geq \frac{1}{n}$, is at most $O(1)$. So the ratio of the conditional second moment and the conditional first moment squared on G is bounded above and so the (conditional) Cauchy–Schwarz inequality yields the claim of the proposition. \square

Proof of Theorem 1.2(ii); case $\alpha = 2$. For any integers $n \geq L \geq 1$, and any $v \in T_L$, let W_n^v be the number of vertices at level n connected to ρ which are in T^v .

Lemma 4.8. *Letting $E_{L,\varepsilon} := \{W_n^v \leq \varepsilon w_n \ \forall n \geq L, \forall v \in T_L\}$, we have that for all $\varepsilon > 0$,*

$$\lim_{L \rightarrow \infty} P(E_{L,\varepsilon}) = 1.$$

Proof. Fix $\varepsilon > 0$ and $v \in T_L$. Since $W_n^v/\mathbf{E}[W_n^v]$ is a martingale with respect to n (for $n \geq L$), we have

$$P(W_n^v \geq \varepsilon w_n \text{ for some } n \geq L) = P(W_n^v \geq \varepsilon E[W_n^v] | T_L \text{ for some } n \geq L) \leq \frac{1}{\varepsilon^2 |T_L|^2} \sup_{n \geq L} \frac{E[(W_n^v)^2]}{E[W_n^v]^2}, \tag{4.11}$$

by Doob's L_2 martingale inequality. The estimate (3.1) gives for $n \geq L$

$$\frac{E[(W_n^v)^2]}{E[W_n^v]^2} \leq \frac{O(1)}{P(W_n^v > 0)} \leq \frac{O(1)}{P(\rho \leftrightarrow v)q_L} = \frac{O(|T_L|)}{w_L q_L}. \tag{4.12}$$

We sum (4.11) over $v \in T_L$ and use (4.12) as well as (4.5), to obtain

$$P(E_{L,\varepsilon}^c) \leq \frac{O(1)L \log L}{w_L \varepsilon^2}$$

which approaches 0 as $L \rightarrow \infty$, since $\alpha > 1$. □

Next, using $w_n \asymp n(\log n)^2$ and (4.5), choose an $\varepsilon > 0$ sufficiently small so that $(1/\tilde{q}_n)^{\varepsilon w_n - 1} \leq n$ for all n sufficiently large, and set $m_n := \lfloor \varepsilon w_n \rfloor$. Let $E_{L,\varepsilon,t}$ denote the event that $E_{L,\varepsilon}$ occurs at time t , let $\mathcal{G}_{L,\varepsilon} := \{t \in [0, 1] : E_{L,\varepsilon,t}\}$ and let $\tilde{\mathcal{G}}_{L,\varepsilon}$ be the (closed) support of the restriction of the Lebesgue measure μ to $\mathcal{G}_{L,\varepsilon}$. Finally, let $G_{L,\varepsilon} := \{\tilde{\mathcal{G}}_{L,\varepsilon} \neq \emptyset\} = \{\mu(\mathcal{G}_{L,\varepsilon}) \neq 0\}$. Lemma 4.8 easily implies that $\lim_{L \rightarrow \infty} P(G_{L,\varepsilon}) = 1$.

For any vertex v , let

$$T^v := \{t \in [0, 1] : \rho \not\stackrel{t}{\leftrightarrow} v\} \cup \{t \in [0, 1] : v \not\stackrel{t}{\leftrightarrow} \infty\},$$

which is the set of times in $[0, 1]$ in which ρ does not connect to ∞ through v . Note that T^v is open.

Proposition 4.9. *With the above choice of $\varepsilon > 0$, for all L and $v \in T_L$,*

$$P(T^v \cap \tilde{\mathcal{G}}_{L,\varepsilon} \text{ is dense in } \tilde{\mathcal{G}}_{L,\varepsilon}) = 1.$$

Given this proposition, the Baire category theorem (or an easy induction) yields that

$$P\left(\tilde{\mathcal{G}}_{L,\varepsilon} \cap \bigcap_{v \in T_L} T^v \text{ is dense in } \tilde{\mathcal{G}}_{L,\varepsilon}\right) = 1$$

and hence

$$P(A^c | G_{L,\varepsilon}) = 1.$$

Since $\lim_{L \rightarrow \infty} P(G_{L,\varepsilon}) = 1$, we are done. □

Proof of Proposition 4.9. Fix L and $v \in T_L$. By countable additivity, it suffices to show that for all open intervals I with rational endpoints,

$$P(\mu(I \cap \mathcal{G}_{L,\varepsilon}) = 0 \text{ or } \mu(T^v \cap I \cap \mathcal{G}_{L,\varepsilon}) > 0) = 1. \tag{4.13}$$

Set $Y := \mu(I \cap \mathcal{G}_{L,\varepsilon})$ and $Y_n := \mathbf{E}[Y | \mathcal{F}_n]$. We claim that for some constant $c > 0$, depending only on I and L , and for all sufficiently large n , we have

$$P(\mu(T^v \cap I \cap \mathcal{G}_{L,\varepsilon}) > 0 | \mathcal{F}_n) \geq c Y_n^2. \tag{4.14}$$

Clearly, $Y_n \rightarrow Y$ a.s., while Levy's 0–1 law implies that the left-hand side converges a.s. to $\mathbf{1}_{\{\mu(T^v \cap I \cap \mathcal{G}_{L,\varepsilon}) > 0\}}$. Therefore, (4.14) implies (4.13) and the proposition.

For all $B \subseteq T_n \cap T^v$ with $|B| \leq m_n$, let \tilde{B} be a subset of $T_n \cap T^v$ containing B such that $|\tilde{B}| = m_n$ and \tilde{B} is a deterministic function of B . (This only works for large enough n so that $|T^v \cap T_n| \geq m_n$.) If $|B| > m_n$, let \tilde{B} be the subset of B consisting of the leftmost m_n elements of B . Let $S_{n,t}^v$ denote the set of vertices in $T^v \cap T_n$ that are connected to ρ at time t , and define

$$X_n := \mu(\{t \in I \cap \mathcal{G}_{L,\varepsilon} : \tilde{S}_{n,t}^v \not\stackrel{t}{\nearrow} \infty\}).$$

Then

$$E[X_n | \mathcal{F}_n] = \int_I P(t \in \mathcal{G}_{L,\varepsilon} | \mathcal{F}_n) P(\tilde{S}_{n,t}^v \not\stackrel{t}{\nearrow} \infty | t \in \mathcal{G}_{L,\varepsilon}, \mathcal{F}_n) dt.$$

Since our process is positively associated even when conditioned on \mathcal{F}_n , the second factor in the integrand is at least as large as $P(\tilde{S}_{n,t}^v \not\stackrel{t}{\nearrow} \infty | \mathcal{F}_n) = (\tilde{q}_n)^{m_n}$, and hence the above gives

$$E[X_n | \mathcal{F}_n] \geq Y_n (\tilde{q}_n)^{m_n}.$$

For the conditional second moment, let

$$X_n^* := \mu(\{t \in I : \tilde{S}_{n,t}^v \stackrel{t}{\nearrow} \infty\}).$$

Then $X_n^* \geq X_n$. Arguing as in the case $\alpha < 2$, we get

$$\mathbf{E}[X_n^2 | \mathcal{F}_n] \leq \mathbf{E}[(X_n^*)^2 | \mathcal{F}_n] \leq 2\mu(I) \int_0^{\mu(I)} \tilde{q}_n(t)^{m_n} dt.$$

We take n larger than $1/\mu(I)$, and use the bounds $\tilde{q}_n(t) \leq \tilde{q}_n$ and (4.8), to get

$$\frac{E[X_n^2 | \mathcal{F}_n]}{E[X_n | \mathcal{F}_n]^2} \leq \frac{2\mu(I)}{Y_n^2} \int_0^{1/n} (\tilde{q}_n)^{-m_n} dt + \frac{2\mu(I)}{Y_n^2} \int_{1/n}^{\mu(I)} \left(1 + O\left(\frac{q_n^2}{t}\right)\right)^{m_n} dt.$$

By our choice of ε and m_n , the left integral is bounded. As we have seen in the previous case, the integrand of the right integral is also bounded. The (conditional) Cauchy–Schwarz inequality therefore gives (4.14). \square

5. Proof of Theorem 1.3

We first recall the definitions of pivotality and influence.

Definition. An edge e is pivotal for an event A if changing the status of e changes whether or not A occurs. The influence of e on the event A , $I_A(e)$, is the probability that e is pivotal for A .

Next we need the definition of a “flip time.”

Definition. Given a graph and a vertex x , a time t is called a flip time for x if x percolates at time t but there is an edge e which is pivotal for the event $\{x \leftrightarrow \infty\}$ at time t and which changes its status at time t . (Note in this case, there is a $\delta > 0$ such that either (1) x does not percolate during $(t - \delta, t)$ or (2) x does not percolate during $(t, t + \delta)$.)

Lemma 5.1. In a spherically symmetric tree with spherically symmetric edge probabilities

$$\mathbf{E}[W_n] \mathbf{P}[W_n = 1] \leq \mathbf{P}[W_n > 0]^2.$$

As we will later see in Lemma 5.4, the reverse inequality holds up to a multiplicative constant under some reasonable assumptions.

Proof. Let Q be the set of vertices in T_n that are connected to ρ . For $v \in T_n$, let \mathcal{L}_v denote the event that $v \in Q$ and v is the leftmost vertex in Q . Likewise, let \mathcal{R}_v denote the event that $v \in Q$ and v is the rightmost vertex in Q . Then

$$\mathbf{P}[Q = \{v\}] = \mathbf{P}[\mathcal{L}_v, \mathcal{R}_v] = \frac{\mathbf{P}[\mathcal{L}_v]\mathbf{P}[\mathcal{R}_v]}{\mathbf{P}[v \in Q]},$$

by the independence of what happens to the right of the path from ρ to v and what happens to the left of this path. Applying the arithmetic-geometric means inequality, we find

$$\mathbf{P}[Q = \{v\}]^{1/2}\mathbf{P}[v \in Q]^{1/2} \leq \frac{1}{2}\mathbf{P}[\mathcal{L}_v] + \frac{1}{2}\mathbf{P}[\mathcal{R}_v].$$

When $Q \neq \emptyset$, there is precisely one vertex v satisfying \mathcal{L}_v and precisely one vertex satisfying \mathcal{R}_v . Hence, by summing the above over all $v \in T_n$, we get

$$\sum_{v \in T_n} \mathbf{P}[Q = \{v\}]^{1/2}\mathbf{P}[v \in Q]^{1/2} \leq \mathbf{P}[W_n > 0].$$

Now note that for every $v \in T_n$ we have $\mathbf{P}[Q = \{v\}] = \mathbf{P}[W_n = 1]/|T_n|$ and $\mathbf{P}[v \in Q] = \mathbf{E}[W_n]/|T_n|$. The lemma follows. \square

Proof of Theorem 1.3(i). We will estimate from above the expected number of pivotal edges for the event $\{\rho \leftrightarrow T_n\}$ in a static configuration. For each $m \in \{1, \dots, n\}$, let v_m be the leftmost vertex in T_m , and let $u(m, n)$ be the expected number of edges between T_{m-1} and T_m that are pivotal for $\{\rho \leftrightarrow T_n\}$. Also let $a(m, n)$ be the probability that v_m is connected to T_n within its subtree; that is, $a(m, n) = \mathbf{P}[v_m \mapsto T_n]$. To estimate $u(m, n)$, we consider a different tree T' which is identical to T until level m , but each vertex at level m in T' has only one child at level $m + 1$, and the edge probability for the edges between levels m and $m + 1$ in T' is $a(m, n) = a(m, n; T)$ (and the $m + 1$ level is the last level of T'). The probability that the edge $[v_{m-1}, v_m]$ is pivotal for $\{\rho \leftrightarrow T_n\}$ and $\rho \leftrightarrow T_n$ holds is the probability that in T' the child of v_m is the only vertex at level $m + 1$ connected to ρ . By Lemma 5.1, the latter is bounded by

$$\mathbf{P}[\rho \leftrightarrow T_n]^2 (|T_m| w_m a(m, n))^{-1}$$

(where the notations all relate to the tree T). Therefore,

$$p_m u(m, n) \leq (w_m a(m, n))^{-1}.$$

Observe that the expected number of vertices $v \in T_k$ satisfying $v_m \mapsto v$ is w_k/w_m . Therefore (1.4) applied to the tree T^{v_m} gives

$$a(m, n)^{-1} \asymp w_m \sum_{k=m+1}^n w_k^{-1}.$$

Plugging this into the above, we get

$$p_m u(m, n) \leq O(1) \sum_{k=m+1}^n w_k^{-1}. \tag{5.1}$$

We now move to the dynamical setting. Let Z_n be the set of times in $[0, 1]$ at which $\rho \leftrightarrow T_n$, and let $Z = \bigcap_{n>0} Z_n$ be the percolation times of the root in $[0, 1]$. It is clear that $\partial Z = \limsup_n \partial Z_n$. (By definition, $\limsup_n A_n :=$

$\bigcap_{n>0} \overline{\bigcup_{j>n} A_j}$.) Note that the set ∂Z_n is the set of times at which a pivotal edge for $\{\rho \leftrightarrow T_n\}$ switches its value. Hence,

$$\mathbf{E}[|\partial Z_n|] = \sum_{m=1}^n 2p_m(1-p_m)u(m,n) \stackrel{(5.1)}{\leq} O(1) \sum_{k=1}^n kw_k^{-1}.$$

Our assumptions therefore imply that $\sup_n \mathbf{E}[|\partial Z_n|] < \infty$. Consequently, $\liminf_{n \rightarrow \infty} |\partial Z_n| < \infty$ a.s. Since $|\partial Z| \leq \liminf_{n \rightarrow \infty} |\partial Z_n|$, this proves (i) of Theorem 1.3. \square

Part (ii) of Theorem 1.3 is an easy consequence of the following theorem.

Theorem 5.2. *Suppose that $\sup_j d_j < \infty$, (1.3) and the following assumptions hold:*

$$\sum_{m=1}^n \frac{1}{m} \leq O(1) \sum_{m=1}^n \sum_{k=m}^{\infty} \frac{1}{w_k}, \tag{5.2}$$

$$\sum_{n=0}^{\infty} \left(\sum_{m=n+1}^{\infty} \frac{w_n}{w_m} \right)^{-2} < \infty, \tag{5.3}$$

$$\sum_{k=0}^{\infty} \left((k+1)w_k \left(\sum_{j=k}^{\infty} w_j^{-1} \right)^2 \right)^{-1} < \infty. \tag{5.4}$$

Then with positive probability there are infinitely many flip times for the event $\{\rho \leftrightarrow \infty\}$ in the time interval $[0, 1]$.

Let b_j denote the probability that a vertex at level j percolates to ∞ (at time 0) through its leftmost child.

Lemma 5.3. *Assume $\sup_j d_j < \infty$, (1.3) and (5.3). Then*

$$\prod_{j=0}^{n-1} (1-b_j)^{d_j-1} \asymp \left(\sum_{m=n}^{\infty} \frac{1}{w_m} \right)^2, \tag{5.5}$$

where the implied constants may depend on the tree and on the sequence $\{p_j\}$.

Proof. We start by deriving a rough estimate for b_n . If $v \in T_n$ and $m > n$, then the expected number of vertices $u \in T_m$ such that $v \mapsto u$ is w_m/w_n . Therefore, (1.4) gives

$$b_n \asymp \frac{1}{d_n w_n} \left(\sum_{m=n+1}^{\infty} \frac{1}{w_m} \right)^{-1}. \tag{5.6}$$

This estimate in itself will not be fine enough to yield (5.5), but will be a useful first step.

For each node at level j in the tree, we order its children according to some fixed linear order (e.g., left to right, if we think of the tree as embedded in the plane). If v is a vertex at level n and $j \in \{1, \dots, n\}$, let $u_j(v)$ denote the vertex at level j that has v in its subtree, and let $i_j(v)$ be the position of $u_j(v)$ among its siblings in the above order. This induces an ordering on the vertices at level n : We say that $v' < v$ if at the minimal j such that $i_j(v') \neq i_j(v)$ we have $i_j(v') < i_j(v)$. Fix some $v \in T_n$. Let \mathcal{L}_v denote the event that v is the minimal vertex at level n such that ρ percolates to ∞ through v . Note that the probability that v percolates to ∞ within its subtree is b_{n-1}/p_n and that $\mathbf{P}[\rho \leftrightarrow v] = w_n/|T_n|$. Hence

$$\mathbf{P}[\mathcal{L}_v] = \frac{w_n}{|T_n|} \frac{b_{n-1}}{p_n} \prod_{j=0}^{n-1} (1-b_j)^{i_{j+1}(v)-1}.$$

Since $\mathbf{P}[\rho \leftrightarrow \infty] = \sum_{v \in T_n} \mathbf{P}[\mathcal{L}_v]$, this gives

$$\frac{p_n \mathbf{P}[\rho \leftrightarrow \infty]}{b_{n-1} w_n} = \frac{1}{|T_n|} \sum_{v \in T_n} \prod_{j=0}^{n-1} (1 - b_j)^{i_{j+1}(v)-1}.$$

We now use $|T_n| = \prod_{j=0}^{n-1} d_j$, and get

$$\frac{p_n \mathbf{P}[\rho \leftrightarrow \infty]}{b_{n-1} w_n} = \sum_{v \in T_n} \prod_{j=0}^{n-1} \frac{(1 - b_j)^{i_{j+1}(v)-1}}{d_j} = \prod_{j=0}^{n-1} \sum_{i=1}^{d_j} \frac{(1 - b_j)^{i-1}}{d_j} = \prod_{j=0}^{n-1} \frac{1 - (1 - b_j)^{d_j}}{b_j d_j}.$$

If we compare the factor corresponding to j on the right with $(1 - b_j)^{(d_j-1)/2}$, we find that they agree up to a factor of $\exp(O(b_j^2))$, where the implied constant may depend on $\sup_j d_j$ and on $\sup_j b_j \leq \sup_j p_j < 1$. Hence,

$$\frac{p_n \mathbf{P}[\rho \leftrightarrow \infty]}{b_{n-1} w_n} = \left(\prod_{j=0}^{n-1} (1 - b_j)^{(d_j-1)/2} \right) \exp\left(O(1) \sum_{j=0}^{n-1} b_j^2 \right).$$

Now (5.5) follows by squaring both sides, using the estimate (5.6) for b_{n-1} , using $p_n d_{n-1} w_{n-1} = w_n$ and noting that $\sum_j b_j^2 < \infty$ by (5.6) and (5.3). \square

The following lemma can be seen as a partial converse to Lemma 5.1, but for convenience it is stated in a slightly different setting.

Lemma 5.4. *Let U_n denote the number of edges joining T_{n-1} to T_n through which ρ percolates to ∞ . Then under the assumptions of Lemma 5.3, we have*

$$\mathbf{P}[U_n = 1] \mathbf{E}[U_n] \asymp 1.$$

Proof. By (5.6) and (5.5), we have

$$\prod_{j=0}^{n-1} (1 - b_j)^{d_j-1} \asymp (b_{n-1} w_{n-1} d_{n-1})^{-2} = \mathbf{E}[U_n]^{-2}.$$

Now multiply the left-hand side by $|T_n| p_1 p_2 \cdots p_{n-1} b_{n-1}$ and the right-hand side by its equal, $\mathbf{E}[U_n]$. On the left-hand side we then get $\mathbf{P}[U_n = 1]$, as required. \square

Proof of Theorem 5.2. The proof is based on a second moment argument. For an edge e let $X(e)$ denote the number of flips (for $\rho \leftrightarrow \infty$) occurring at times in $[0, 1]$ when e switches. Let $m = m(e) := |e|$ denote the level of e ; that is e connects T_m and T_{m-1} . Set $\mathfrak{X}(e) := \mathbf{1}_{\{X(e) > 0\}}$, $X_n := \sum_{|e| \leq n} X(e)$ and $\mathfrak{X}_n := \sum_{|e| \leq n} \mathfrak{X}(e)$. The second moment argument will be applied to \mathfrak{X}_n : We will show that $\lim_{n \rightarrow \infty} \mathbf{E}[\mathfrak{X}_n] = \infty$, and that $\sup_n \mathbf{E}[\mathfrak{X}_n^2] / \mathbf{E}[\mathfrak{X}_n]^2 < \infty$.

At this point, we use an equivalent version of the dynamics in which at rate 1, an edge is *refreshed* and when refreshed, it chooses to be in state 1 with probability p_e . Let now Y_e be the set of times in which e refreshed, and let A_e be the set of times $t \in [0, 1]$ at which e is pivotal for $\{\rho \leftrightarrow \infty\}$. Since $2p_m(1 - p_m)$ is the probability a refresh time is a switch time, and Y_e is a Poisson point process with rate 1 independent from A_e , we have

$$1 - \exp(-\mu(A_e)) \geq \mathbf{E}[\mathfrak{X}(e) | A_e] \geq 2p_m(1 - p_m)(1 - \exp(-\mu(A_e))),$$

where μ denotes Lebesgue measure. It follows that

$$\mathbf{E}[\mathfrak{X}(e) | A_e] \asymp \mu(A_e).$$

Moreover, Fubini gives

$$\mathbf{E}[\mu(A_e)] = \mathbf{P}[e \text{ pivotal for } \{\rho \leftrightarrow \infty\} \text{ at time } 0].$$

Hence,

$$\mathbf{E}[\mathfrak{X}_n] \asymp \sum_{m=1}^n \sum_{|e|=m} \mathbf{P}[e \text{ pivotal for } \{\rho \leftrightarrow \infty\} \text{ at time } 0].$$

Lemma 5.4 easily implies that if $|e| = m$, then

$$\mathbf{P}[e \text{ pivotal for } \{(\rho \leftrightarrow \infty)\} \text{ at time } 0] \asymp \frac{1}{|T_m| w_{m-1} d_{m-1} b_{m-1}}.$$

The above together with (5.6) gives

$$\mathbf{E}[\mathfrak{X}_n] \asymp \sum_{m=1}^n \sum_{k=m}^{\infty} \frac{1}{w_k}. \quad (5.7)$$

We now turn to estimating $\mathbf{E}[\mathfrak{X}_n^2]$. Let e, e' be two different edges at levels m and m' , respectively, where $m, m' \leq n$. Then $X(e)X(e') \leq |Y_e \cap A_e| \cdot |Y_{e'} \cap A_{e'}|$. Let $\nu_{e,e'}$ denote the counting measure on the set $(Y_e \cap A_e) \times (Y_{e'} \cap A_{e'}) \subseteq [0, 1]^2$, and let $I, I' \subseteq [0, 1]$ be disjoint time intervals. Note that $Y_e \cap I, Y_{e'} \cap I'$ and $(A_e \cap I, A_{e'} \cap I')$ are independent. (Note however that $A_e \cap I$ is usually not independent from $A_{e'} \cap I'$.) Therefore

$$\mathbf{E}[\nu_{e,e'}(I \times I') | A_e \cap I, A_{e'} \cap I'] = \mu(A_e \cap I) \mu(A_{e'} \cap I').$$

Hence

$$\mathbf{E}[\nu_{e,e'}(I \times I')] = \int_{I \times I'} \mathbf{P}[t \in A_e, s \in A_{e'}] dt ds.$$

For $e \neq e'$, $\nu_{e,e'}$ gives no mass to the diagonal, and hence we can conclude that

$$\mathbf{E}[X(e)X(e')] \leq \mathbf{E}[\nu_{e,e'}([0, 1] \times [0, 1])] = \int_0^1 \int_0^1 \mathbf{P}[t \in A_e, s \in A_{e'}] dt ds.$$

Since $\sum_{|e| \leq n} \mathfrak{X}(e)\mathfrak{X}(e) = \mathfrak{X}_n$, we have

$$\mathfrak{X}_n^2 = \mathfrak{X}_n + \sum_{|e|, |e'| \leq n} \mathbf{1}_{\{e \neq e'\}} \mathfrak{X}(e)\mathfrak{X}(e') \leq \mathfrak{X}_n + \sum_{|e|, |e'| \leq n} \mathbf{1}_{\{e \neq e'\}} X(e)X(e').$$

Consequently,

$$\mathbf{E}[\mathfrak{X}_n^2] \leq \mathbf{E}[\mathfrak{X}_n] + \sum_{|e|, |e'| \leq n} \mathbf{1}_{\{e \neq e'\}} \int_0^1 \int_0^1 \mathbf{P}[t \in A_e, s \in A_{e'}] dt ds.$$

At this point, we break up the pairs (e, e') for which $e \neq e'$ into two sets, those where e and e' do not lie on the same path from the root to ∞ (which is the generic case) and those where they do lie on the same path. Call the first class \mathcal{E}_1 and the second class \mathcal{E}_2 . We consider now pairs (e, e') in \mathcal{E}_1 .

Let $v_0 = \rho, v_1, \dots, v_m$ denote the path from the root ρ to the endpoint of e at level $m = |e|$, and let $v'_0, v'_1, \dots, v'_{m'}$ denote the path from the root to the endpoint of e' at level $m' = |e'|$. Let $k \leq (m-1) \wedge (m'-1)$ be maximal such that $v_k = v'_k$. Also, fix $s, t \in [0, 1]$ and set $r := |s - t|$. Note that for every $j \in \mathbb{N}_+$ and any edge at level j , the probability that the edge is open at time s and at time t is $p_j^2 + (1 - p_j)p_j \exp(-r)$. For $j = 0, \dots, m-1$, let \mathcal{U}_j denote the event

that at time t we have $v_j \leftrightarrow \infty$ inside $T^{v_j} \setminus v_{j+1}$, and let \mathcal{U}'_j denote the corresponding event with each v_i replaced by v'_i , with t replaced by s and with m replaced by m' . Note that the event $\{t \in A_e, s \in A_{e'}\}$ is contained in the intersection of the following events: $\mathcal{L} := \{\rho \overset{t}{\leftarrow} v_k, \rho \overset{s}{\leftarrow} v_k\}$, $\mathcal{Q}_1 := \{v_k \overset{t}{\mapsto} v_{m-1}\}$, $\mathcal{Q}'_1 := \{v_k \overset{s}{\mapsto} v'_{m'-1}\}$, $\mathcal{Q}_2 := \{v_m \overset{t}{\mapsto} \infty\}$, $\mathcal{Q}'_2 := \{v'_{m'} \overset{s}{\mapsto} \infty\}$, $\mathcal{Z}_1 := \bigcap_{j=0}^{k-1} \neg \mathcal{U}_j$, $\mathcal{Z}_2 := \bigcap_{j=k+1}^{m-1} \neg \mathcal{U}_j$, $\mathcal{Z}'_2 := \bigcap_{j=k+1}^{m'-1} \neg \mathcal{U}'_j$, and that these events are all independent. Consequently,

$$\begin{aligned} \mathbf{P}[t \in A_e, s \in A_{e'}] &\leq \prod_{j=1}^k (p_j^2 + (1-p_j)p_j \exp(-r)) \times \prod_{j=k+1}^{m-1} p_j \times \prod_{j=k+1}^{m'-1} p_j \times \frac{b_{m-1}}{p_m} \times \frac{b_{m'-1}}{p_{m'}} \\ &\quad \times \prod_{j=0}^{k-1} (1-b_j)^{d_j-1} \times \prod_{j=k+1}^{m-1} (1-b_j)^{d_j-1} \times \prod_{j=k+1}^{m'-1} (1-b_j)^{d_j-1}. \end{aligned}$$

Setting $\delta := 1 - \sup_j p_j$ and noting that $r \leq 1$, we may estimate the first product as

$$\leq \left(1 - \frac{\delta r}{3}\right)^k \prod_{j=1}^k p_j \leq \exp\left(-\frac{\delta k r}{3}\right) \prod_{j=1}^k p_j.$$

Using the above and Lemma 5.3, we arrive at the estimate

$$\mathbf{P}[t \in A_e, s \in A_{e'}] \leq \mathbf{O}(1) \exp\left(-\frac{\delta k r}{3}\right) \frac{(\prod_{j=1}^{m-1} p_j)(\prod_{j=1}^{m'-1} p_j) b_{m-1} b_{m'-1} (\sum_{j=m}^{\infty} w_j^{-1})^2 (\sum_{j=m'}^{\infty} w_j^{-1})^2}{(\prod_{j=1}^k p_j) (1-b_k)^{2d_k-2} (\sum_{j=k}^{\infty} w_j^{-1})^2}.$$

Since we are assuming $\sup_j d_j < \infty$ and since $b_j \leq p_{j+1} \leq 1 - \delta$, we have $(1-b_k)^{2-2d_k} = \mathbf{O}(1)$, and that factor may be dropped. Now note that when (t, s) is uniform in $[0, 1]^2$, the probability that r is in any interval $I \subseteq [0, 1]$ is at most twice the length of I . Since $\int_0^1 \exp(-\delta k r/3) dr \leq \mathbf{O}(1/(\delta(k+1))) = \mathbf{O}(1/(k+1))$, we get

$$\int_0^1 \int_0^1 \mathbf{P}[t \in A_e, s \in A_{e'}] dt ds \leq \mathbf{O}(1) \frac{(\prod_{j=1}^{m-1} p_j)(\prod_{j=1}^{m'-1} p_j) b_{m-1} b_{m'-1} (\sum_{j=m}^{\infty} w_j^{-1})^2 (\sum_{j=m'}^{\infty} w_j^{-1})^2}{(k+1) (\prod_{j=1}^k p_j) (\sum_{j=k}^{\infty} w_j^{-1})^2}.$$

If we fix m, m' and v_k , there are at most $|T_m|/|T_k|$ possible choices for e and $|T_{m'}|/|T_k|$ possible choices for e' . Thus, there are at most $|T_m||T_{m'}|/|T_k|^2$ possible choices for pairs (e, e') . Since $|T_j| = d_{j-1}|T_{j-1}|$ and $T_j \prod_{i=1}^j p_i = w_j$, the sum of the above over all such pairs (e, e') is

$$\begin{aligned} &\leq \mathbf{O}(1) \frac{d_{m-1} d_{m'-1} w_{m-1} w_{m'-1} b_{m-1} b_{m'-1} (\sum_{j=m}^{\infty} w_j^{-1})^2 (\sum_{j=m'}^{\infty} w_j^{-1})^2}{(k+1) |T_k| w_k (\sum_{j=k}^{\infty} w_j^{-1})^2} \\ &\stackrel{(5.6)}{\asymp} \frac{(\sum_{j=m}^{\infty} w_j^{-1}) (\sum_{j=m'}^{\infty} w_j^{-1})}{(k+1) |T_k| w_k (\sum_{j=k}^{\infty} w_j^{-1})^2}. \end{aligned}$$

We now sum over all possible choices for v_k , which eliminates the $|T_k|^{-1}$ factor. Next, we bound the sum of the resulting expression for $m \in \{k+1, k+2, \dots, n\}$ and $m' \in \{k+1, k+2, \dots, n\}$ by summing over all $m, m' = 1, 2, \dots, n$. Finally, we sum over $k = 0, 1, \dots, n-1$, to obtain

$$\sum_{|e|, |e'| \leq n} \mathbf{1}_{\{(e, e') \in \mathcal{E}_1\}} \int_0^1 \int_0^1 \mathbf{P}[t \in A_e, s \in A_{e'}] dt ds \leq \mathbf{O}(1) \left(\sum_{m=1}^n \sum_{j=m}^{\infty} w_j^{-1} \right)^2 \sum_{k=0}^{\infty} \left((k+1) w_k \left(\sum_{j=k}^{\infty} w_j^{-1} \right) \right)^{-1}.$$

By (5.4) and (5.7), this is at most $\mathbf{O}(1) \mathbf{E}[\mathfrak{X}_n]^2$.

We now explain the necessary modifications for the case $(e, e') \in \mathcal{E}_2$. Let $m = |e| < |e'| = m'$. Using the same notations as above, it is easy to see that the event $\{t \in A_e, s \in A_{e'}\}$ is contained in the intersection of the following independent events: $\{\rho \xrightarrow{t} v_{m-1}, \rho \xrightarrow{s} v_{m-1}\}$, $\{v_m \xrightarrow{s} v_{m'-1}\}$, $\{v_{m'} \xrightarrow{s} \infty\}$ and $\bigcap_{j=0}^{m'-1} \neg \mathcal{U}'_j$. This leads, after a computation exactly as before, to

$$\int_0^1 \int_0^1 \mathbf{P}[t \in A_e, s \in A_{e'}] dt ds \leq O(1) \frac{(\prod_{j=1}^{m'-1} p_j) b_{m'-1} \prod_{j=0}^{m'-1} (1 - b_j)^{d_j-1}}{m+1}.$$

With e and m' fixed, there are at most $|T_{m'}|/|T_m|$ possible choices for e' and so the sum of the above over such e' is at most

$$O(1) \frac{w_{m'-1} b_{m'-1} \prod_{j=0}^{m'-1} (1 - b_j)^{d_j-1}}{m|T_m|} \leq O(1) \frac{\sum_{k=m'}^{\infty} 1/w_k}{m|T_m|},$$

by (5.5) and (5.6). At level m , there are $|T_m|$ choices for e . As $m' \geq m+1$, we can sum over m' from 1 to n and then sum over m from 1 to n to yield

$$\begin{aligned} & \sum_{|e|, |e'| \leq n} \mathbf{1}_{\{(e, e') \in \mathcal{E}_2\}} \int_0^1 \int_0^1 \mathbf{P}[t \in A_e, s \in A_{e'}] dt ds \\ & \leq O(1) \left(\sum_{m=1}^n \sum_{j=m}^{\infty} w_j^{-1} \right) \sum_{m=1}^n \frac{1}{m} = O(1) \left(\sum_{m=1}^n \sum_{j=m}^{\infty} w_j^{-1} \right)^2 \frac{\sum_{m=1}^n 1/m}{\left(\sum_{m=1}^n \sum_{j=m}^{\infty} w_j^{-1} \right)}. \end{aligned}$$

By (5.2) and (5.7), this is also at most $O(1) \mathbf{E}[\mathfrak{X}_n]^2$.

All of the above therefore yields $\mathbf{E}[\mathfrak{X}_n^2] \leq \mathbf{E}[\mathfrak{X}_n] + O(1) \mathbf{E}[\mathfrak{X}_n]^2$. Since $\lim_{n \rightarrow \infty} \mathbf{E}[\mathfrak{X}_n] = \infty$ by (5.2) and (5.7), this gives $\mathbf{E}[\mathfrak{X}_n^2] \leq O(1) \mathbf{E}[\mathfrak{X}_n]^2$. A one-sided Chebyshev inequality (see, e.g., Lemma 5.4 in [6]) or alternatively the Paley–Zygmund inequality yields that there is some $c > 0$, which does not depend on n , such that $\mathbf{P}[\mathfrak{X}_n \geq c \mathbf{E}[\mathfrak{X}_n]] \geq c$. Hence $\mathbf{P}[\lim_{n \rightarrow \infty} \mathfrak{X}_n = \infty] \geq c$, which completes the proof. \square

Proof of Theorem 1.3(ii). This easily follows from Theorem 5.2. \square

6. Proof of Theorem 1.4

We start with a lemma connecting the concepts of flip time and influence.

Lemma 6.1. *Fix a vertex x . Then*

$$2 \sum_e I_x(e) p_e (1 - p_e) = E[|\mathcal{S}|],$$

where \mathcal{S} is the set of flip times for x during $[0, 1]$.

Proof. Fix e . The probability that during $[t, t + dt]$ the edge e switches its state precisely once is easily seen to be $2p_e(1 - p_e) dt + O(dt^2)$. Conditioning on that time, the probability that e is pivotal for $\{x \leftrightarrow \infty\}$ at that time is $I_x(e)$. Hence, the probability that there is a flip associated to e during $[t, t + dt]$ is $2I_x(e)p_e(1 - p_e) dt + O(dt^2)$. It follows that $E[\mathcal{S}_e] = 2I_x(e)p_e(1 - p_e)$, where \mathcal{S}_e is the set of flip times associated to e during $[0, 1]$. Summing over e yields the result. \square

Proof of Theorem 1.4. Fix x . Let \mathcal{E}_n be the set of edges which are within graph distance n of x and let \mathcal{F}_n be the σ -algebra generated by the evolution of the edges in \mathcal{E}_n during the time interval $[0, 1]$. Let

$$X_n(t) = X_n(\omega, t) := P(x \xrightarrow{t} \infty | \mathcal{F}_n).$$

While conditional probabilities are usually only defined a.s., it is clear that there is a canonical version of these conditional probabilities and these will always be used. Let V_n denote the total variation of $X_n(t)$ on $[0, 1]$.

The following two lemmas are left to the reader.

Lemma 6.2.

$$E[V_n] = 2 \sum_{e \in \mathcal{E}_n} I(e) p_e (1 - p_e).$$

Lemma 6.3. $\{V_n\}_{n \geq 1}$ is a submartingale.

By our assumption (1.6) and by Lemma 6.2, we have $\sup_n E(V_n) < \infty$. Since $\{V_n\}_{n \geq 1}$ is a nonnegative submartingale, this implies that there is an a.s. limit $V := \lim_{n \rightarrow \infty} V_n$ satisfying $E(V) < \infty$. Now, for all t , the Martingale convergence theorem tells us that $X_n(t)$ converges a.s. to $\mathbf{1}_{\{x \overset{t}{\leftrightarrow} \infty\}}$. By Fubini's theorem, for a.e. ω , there exists $A_\omega \subseteq [0, 1]$ such that $\mu(A_\omega) = 1$ (μ is Lebesgue measure here) and

$$\lim_{n \rightarrow \infty} X_n(\omega, t) = \mathbf{1}_{\{x \overset{t}{\leftrightarrow} \infty\}} \quad \text{for all } t \in A_\omega. \quad (6.1)$$

Now define

$$\tilde{X}(\omega, t) := \begin{cases} \mathbf{1}_{\{x \overset{t}{\leftrightarrow} \infty\}} & \text{if } t \in A_\omega, \\ \limsup_{s \uparrow t, s \in A_\omega} \mathbf{1}_{\{x \overset{s}{\leftrightarrow} \infty\}} & \text{if } t \notin A_\omega. \end{cases}$$

Statement (6.1) implies that the total variation of \tilde{X} restricted to time points in A_ω is at most V for a.e. ω . It is then easy to check that the total variation of \tilde{X} over $[0, 1]$ is then at most V for a.e. ω as well. We conclude that a.s. $\mathbf{1}_{\{x \overset{t}{\leftrightarrow} \infty\}}$ is equal a.s. to a function of bounded variation.

We now show that the fact that a.s. $\mathbf{1}_{\{x \overset{t}{\leftrightarrow} \infty\}}$ is equal a.e. to a function of bounded variation implies that there are no exceptional times. Let X be the Lebesgue measure of the amount of time that x percolates during $[0, 1]$. By Fubini's theorem, $E(X)$ is the probability that x percolates. It follows that with positive probability, $X > 0$. If there were exceptional times of nonpercolation, an easy application of Kolmogorov's 0–1 law tells us that a.s. there would be such times in every nonempty interval. However, the latter together with the fact that the set of times at which x does not percolate is open and that $X > 0$ contradicts the fact that $\mathbf{1}_{\{x \overset{t}{\leftrightarrow} \infty\}}$ is equal a.s. to a function of bounded variation. \square

7. A 0–1 law

In this section, we present a 0–1 law concerning the process. In addition to being of interest in itself, we believe it might be useful for obtaining a better understanding of the path behavior of our process and might be relevant to some of the problems at the end of the paper.

Theorem 7.1. *Consider dynamical percolation $(\omega_t: t \in \mathbb{R})$ on a spherically symmetric tree T with spherically symmetric edge probabilities, and let \mathcal{Q} be the set of times $t \in \mathbb{R}$ such that the cluster of the root is infinite in ω_t . If $\mathbf{P}[0 \in \partial \mathcal{Q}] > 0$, then a.s. $\mathcal{Q} = \partial \mathcal{Q}$ (and hence by Lemma 3.2 there is a.s. a dense set of times $t \in \mathbb{R}$ in which there is no infinite cluster in ω_t).*

Now consider an arbitrary locally finite tree T with root ρ and a vertex v of T . For any $\omega \subseteq 2^{E(T)}$, we may start dynamical percolation ω_t with $\omega_0 = \omega$. It is easy to see that for this Markov process, the probability that there is a positive ε such that $v \overset{t}{\rightarrow} \infty$ for all times $t \in [0, \varepsilon]$ is 0 or 1. Let $h_v(\omega) \in \{0, 1\}$ denote this probability.

Lemma 7.2. *With the above notation, let v_1, \dots, v_m denote the children of v ; that is, the neighbors of v within T_v . Then*

$$h_v(\omega) = \max\{1_{[v, v_j] \in \omega} h_{v_j}(\omega) : j = 1, 2, \dots, m\}$$

holds for a.e. ω with respect to the invariant measure of the Markov process ω_t .

We point out that the lemma does not need to assume that T is spherically symmetric.

Proof. It is certainly clear that h_v is at least as large as the max on the right-hand side. We therefore only need to prove the reverse inequality. Let U_j be the set of times $t \in [0, \infty)$ such that v does not percolate to ∞ in $[v, v_j] \cup T^{v_j}$ at time t . Then U_j is a relatively open set.

Set $Q_k := \bigcap_{j=1}^k U_j$, and $Q'_k := \bigcup_{j=1}^k ([0, \infty) \setminus \overline{U_j})$. Note that the max on the right-hand side in the statement of the lemma is equal to $1_{0 \in Q'_m}$. We prove by induction on k that $0 \in \overline{Q_k} \cup Q'_k$ a.s. holds for $k = 0, 1, \dots, m$. The case $k = m$ then implies the statement of the lemma. The base of the induction, $k = 0$, is clear, because $Q_0 = [0, \infty)$, by convention. Now suppose that $0 < k < m$ and $0 \in \overline{Q_k} \cup Q'_k$. If $0 \in Q'_k$, then $0 \in Q'_{k+1}$. Therefore, suppose that $0 \in \overline{Q_k}$. Hence, there is a sequence $(t_n : n \in \mathbb{N})$ in Q_k such that $t_n \rightarrow 0$. Moreover, it is easy to see that we may choose the sequence to depend only on Q_k and in such a way that each t_n is measurable. In particular, the sequence $\{t_n\}$ is independent from the restriction of $(\omega_t : t \geq 0)$ to $[v, v_{k+1}] \cup T^{v_{k+1}}$. Fix some $n \in \mathbb{N}$, and suppose for the moment that t_n is in the closure of U_{k+1} . Then we can find a point t' in U_{k+1} arbitrarily close to t_n . Since $t_n \in Q_k$, and Q_k is relatively open, there is a point t' arbitrarily close to t_n that is in $Q_{k+1} = Q_k \cap U_{k+1}$. Therefore, in the case that $\{n : t_n \in \overline{U_{k+1}}\}$ is infinite a.s., we have $0 \in \overline{Q_{k+1}}$ a.s. and the inductive claim follows.

For every measurable $S \subseteq [0, 1]$ we have by elementary Fourier analysis that $1_S(t) - 1_S(t + t_n)$ tends to zero in L^2 as $n \rightarrow \infty$. Therefore, there is some infinite $Y \subseteq \mathbb{N}$ such that $1_S(t) - 1_S(t + t_n)$ tends to zero a.e. as $n \rightarrow \infty$ within Y . Consequently, a.e. $t \in S$ satisfies $|\{n : t + t_n \in S\}| = \infty$. We may apply this to the set $S := \overline{U_{k+1}} \cap [0, 1]$. However, given the sequence $\{t_n\}$, the distribution of $\overline{U_{k+1}}$ is invariant under translations. Consequently, a.s. either $0 \in Q'_{k+1}$ or $|\{n : t_n \in \overline{U_{k+1}}\}| = \infty$. This proves $0 \in \overline{Q_{k+1}} \cup Q'_{k+1}$ a.s., and completes the induction. The statement of the lemma follows immediately. \square

Lemma 7.3. *Consider stationary percolation on a spherically symmetric tree with spherically symmetric edge probabilities (and, as usual, assume that the edge probabilities are bounded away from 0 and 1). Then a.s. $W_\infty = \lim_{n \rightarrow \infty} W_n/w_n$ exists and $W_\infty < \infty$. Moreover, a.s. $W_\infty > 0$ if and only if $\rho \leftrightarrow \infty$.*

Proof. As we have noted before, W_n/w_n is a non-negative martingale, which implies the a.s. existence and finiteness of W_∞ . Let X_n be the set of vertices v at level n satisfying $\rho \leftrightarrow v$, and let $U_n := \{v \in X_n : v \mapsto \infty\}$. Fix some $v \in T_n$. With no loss of generality, assume that $\mathbf{P}[\rho \leftrightarrow \infty] > 0$, and hence $\mathbf{P}[v \in U_n] > 0$. For $m \geq n$, let $X_m^v := \{u \in T_m : v \mapsto u\}$ and $W_m^v := |X_m^v|$. The inequality (3.2) applied to T^v implies that there is a universal constant $\delta > 0$ such that

$$\mathbf{P}[W_m^v \geq \delta \mathbf{E}[W_m^v | W_m^v > 0] | W_m^v > 0] \geq \delta.$$

Since $\mathbf{1}_{\{W_m^v > 0\}} \rightarrow \mathbf{1}_{\{v \mapsto \infty\}}$ a.s. as $m \rightarrow \infty$, and $\mathbf{E}[W_m^v | W_m^v > 0] \geq w_m/|T_n|$, this implies

$$\liminf_{m \rightarrow \infty} \mathbf{P}[W_m^v \geq \delta w_m | T_n|^{-1} | v \mapsto \infty] \geq \delta.$$

Hence,

$$\mathbf{P}\left[\lim_{m \rightarrow \infty} \frac{W_m^v}{w_m} > 0 \mid v \mapsto \infty\right] \geq \delta.$$

By conditioning on the set U_n and using conditional independence on the various trees T^v , $v \in U_n$, we therefore get

$$\mathbf{P}[W_\infty > 0 | U_n] \geq 1 - (1 - \delta)^{|U_n|}.$$

By Lemma 4.2 in [11], a.s. on the event $\rho \leftrightarrow \infty$ we have $\lim_{n \rightarrow \infty} |U_n| = \infty$. Hence, for every finite N we have $\mathbf{P}[|U_n| > N | \rho \leftrightarrow \infty] \rightarrow 1$ as $n \rightarrow \infty$. The lemma follows. \square

Proof of Theorem 7.1. Let ω be a sample from the stationary measure of the Markov process ω_t . Let $q_n := \mathbf{E}[h_{u_n}(\omega)]$, where u_n is a vertex at level n (since the tree is spherically symmetric, the choice of u_n does not affect q_n). Let \mathcal{F}_n denote the σ -field generated by the restriction of ω to the ball of radius n about the root u_0 . Lemma 7.2 easily implies by induction that $h_{u_0}(\omega) = 1$ if and only if there is a vertex v at level n that is connected in ω to u_0 and satisfies $h_v(\omega) = 1$. Therefore,

$$\mathbf{E}[h_{u_0}(\omega) | \mathcal{F}_n] = 1 - (1 - q_n)^{W_n} = 1 - \exp(\log(1 - q_n)W_n).$$

Since $\mathbf{E}[h_{u_0}(\omega) | \mathcal{F}_n]$ tends to $h_{u_0}(\omega)$ as $n \rightarrow \infty$, we conclude that a.s. $\log(1 - q_n)W_n$ tends to 0 or $-\infty$. If

$$\mathbf{P}\left[\lim_{n \rightarrow \infty} \log(1 - q_n)W_n = -\infty\right] > 0,$$

then Lemma 7.3 implies

$$\mathbf{P}\left[\lim_{n \rightarrow \infty} \log(1 - q_n)W_n = -\infty \mid \rho \leftrightarrow \infty\right] = 1.$$

Therefore, we get either $h_{u_0}(\omega) = 0$ a.s., or else $h_{u_0}(\omega) = \mathbf{1}_{\{\rho \leftrightarrow \infty\}}$ a.s. The theorem follows. \square

8. Some open questions

Following are a few questions and open problems suggested by the present paper.

1. In the spherically symmetric tree case, if $w_k \asymp k^2$, is it the case that with positive probability the set of times $t \in [0, 1]$ at which the root percolates has infinitely many connected components? In this case $\mathbf{E}[\mathfrak{X}_n] \asymp \log n$ grows to ∞ but the second moment method fails.
2. Under the assumption of Theorem 1.4, is it the case that $\{t \in [0, 1]: \rho \overset{t}{\leftrightarrow} \infty\}$ has finitely many connected components a.s.? (From an earlier remark, this would be true if in this setting finiteness of the left-hand term in (1.5) implies finiteness of the right-hand term.)
3. Does the conclusion of Theorem 1.2(ii) hold under the weaker assumptions that

$$\limsup_n \frac{w_n}{n(\log n)^\alpha} < \infty$$

for some $\alpha \leq 2$ and the tree percolates with positive probability at a fixed time? We describe a natural approach which does *not* work. Note that under the above assumption, one can find a new tree which dominates the original tree (in the sense that the number of vertices at the n th level level is larger for any n) for which the new w_n 's satisfy the assumption of Theorem 1.2(ii) and hence would have exceptional times. In [10], it is shown that this domination has a number of implications. However, one cannot conclude that the set of times at which the original tree percolates is dominated by the set of times at which the new tree percolates. An example is T_1 being a tree with degrees $d_1 = 1$ and $d_2 = 2$, T_2 being a tree with degrees $d_1 = 2$ and $d_2 = 1$ and the edge probabilities are very small. Then T_2 dominates T_1 , but the probability that the root is connected to level 2 throughout the time interval $[0, 10]$ is larger for T_1 .

There are various questions concerning the path behavior of the process which might be interesting to pursue. In the following questions, we consider a spherically symmetric tree in which the root percolates with positive probability at a fixed time. Let $Z := \{t \in \mathbb{R}: \rho \overset{t}{\leftrightarrow} \infty\}$.

4. Are the boundary points of the connected components of $\mathbb{R} \setminus Z$ always flip times?
5. If Z has connected components of positive length, do the boundary points of these intervals have to also be boundary points of intervals in $\mathbb{R} \setminus Z$?
6. If there are exceptional times of nonpercolation, is Z the closure of the flip times?

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References

- [1] N. Alon and J. Spencer. *The Probabilistic Method*, 2nd edition. Wiley, New York, 2000. [MR1885388](#)
- [2] I. Benjamini and O. Schramm. Exceptional planes of percolation. *Probab. Theory Related Fields* **111** (1998) 551–564. [MR1641830](#)
- [3] J. van den Berg, R. Meester and D. G. White. Dynamic Boolean models. *Stochastic Process. Appl.* **69** (1997) 247–257. [MR1472953](#)
- [4] E. I. Broman and J. E. Steif. Dynamical stability of percolation for some interacting particle systems and ε -movability. *Ann. Probab.* **34** (2006) 539–576. [MR2223951](#)
- [5] G. Grimmett. *Percolation*, 2nd edition. Springer-Verlag, New York, 1999. [MR1707339](#)
- [6] O. Häggström, Y. Peres and J. E. Steif. Dynamical percolation. *Ann. Inst. H. Poincaré Probab. Statist.* **33** (1997) 497–528. [MR1465800](#)
- [7] J. Jonasson and J. E. Steif. Dynamical models for circle covering: Brownian motion and Poisson updating. *Ann. Probab.* **36** (2008) 739–764. [MR2393996](#)
- [8] D. Khoshnevisan. Dynamical percolation on general trees. *Probab. Theory Related Fields.* **140** (2008) 169–193. [MR2357674](#)
- [9] R. Lyons. Random walks, capacity, and percolation on trees. *Ann. Probab.* **20** (1992) 2043–2088. [MR1188053](#)
- [10] R. Pemantle and Y. Peres. Domination between trees and application to an explosion problem. *Ann. Probab.* **22** (1994) 180–194. [MR1258873](#)
- [11] R. Pemantle and Y. Peres. Critical random walk in random environment on trees. *Ann. Probab.* **23** (1995) 105–140. [MR1330763](#)
- [12] Y. Peres and J. E. Steif. The number of infinite clusters in dynamical percolation. *Probab. Theory Related Fields* **111** (1998) 141–165. [MR1626782](#)
- [13] O. Schramm and J. E. Steif. Quantitative noise sensitivity and exceptional times for percolation. *Ann. Math.* To appear.