

## EFFICIENT ESTIMATION FOR A SUBCLASS OF SHAPE INVARIANT MODELS

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In this paper, we observe a fixed number of unknown  $2\pi$ -periodic functions differing from each other by both phases and amplitude. This semiparametric model appears in literature under the name “shape invariant model.” While the common shape is unknown, we introduce an asymptotically efficient estimator of the finite-dimensional parameter (phases and amplitude) using the profile likelihood and the Fourier basis. Moreover, this estimation method leads to a consistent and asymptotically linear estimator for the common shape.

**1. Introduction.** In many studies, the response of interest is not a random variable but a noisy function for each experimental unit, resulting in a sample of curves. In such studies, it is often adequate to assume that the data  $Y_{i,j}$ , the  $i$ th observation on the  $j$ th experimental unit, satisfies the regression model

$$(1.1) \quad Y_{i,j} = f_j^*(t_{i,j}) + \sigma_j^* \varepsilon_{i,j}, \quad i = 1, \dots, n_j, j = 1, \dots, J.$$

Here, the unknown regression functions  $f_j^*$  are  $2\pi$ -periodic and may depend nonlinearly on the known regressors  $t_{i,j} \in [0, 2\pi]$ . The unknown error terms  $\sigma_j^* \varepsilon_{i,j}$  are independent zero mean random variables with variance  $\sigma_j^{*2}$ .

The sample of individual regression curves will show a certain homogeneity in structure, in the sense that curves coincide if they are properly scaled and shifted. In other words, the structure would be represented by the nonlinear mathematical model

$$(1.2) \quad f_j^*(t) = a_j^* f^*(t - \theta_j^*) + v_j^* \quad \forall t \in \mathbb{R}, \forall j = 1, \dots, J,$$

where the shift  $\theta^* = (\theta_j^*)_{j=1, \dots, J}$ , the scale  $a^* = (a_j^*)_{j=1, \dots, J}$  and the level  $v^* = (v_j^*)_{j=1, \dots, J}$  are vectors of  $\mathbb{R}^J$  and the function  $f^*$  is  $2\pi$ -periodic. This semiparametric model was introduced by Lawton, Sylvestre and Maggio [7] under the name of shape invariant model. We have both a finite-dimensional parameter  $(\theta^*, a^*, v^*)$  and an infinite-dimensional nuisance parameter  $f^*$  which is a member of some given large set of functions. A general feature of semiparametric methods is to

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“eliminate” the nonparametric component  $f^*$ , thus reducing the original semiparametric problem to a suitably chosen parametric one.

Such models have been used to study child growth curves (see [6]) or to improve a forecasting methodology [8] based on speed data of vehicles on a main trunk road (see [2] for more details). Since the common shape is assumed to be periodic, the model is particularly well adapted for the study of circadian rhythms (see [15]). Our model and our estimation method are illustrated with the daily temperature of several cities.

The main goal of this paper is to present a method for the efficient estimation of the parameter  $(\theta^*, a^*, \nu^*)$  without knowing  $f^*$ . The question of estimation of parameters for the shape invariant model was studied by several authors. First, Lawton, Sylvestre and Maggio [7] proposed an empirical procedure, SEMOR, based on polynomial approximation of the common shape  $f^*$  on a compact set. The convergence and the consistency for SEMOR was proved by Kneip and Gasser [6]. Härdle and Marron [5] built a  $\sqrt{n}$ -consistent estimator and an asymptotically normal estimator using a kernel estimator for the function  $f^*$ . Similar to Guardabasso, Rodbard and Munson [4], Wang and Brown [15] and Luan and Li [9] used a smoothing spline for the estimation of  $f^*$ . The method of Gamboa, Loubes and Maza [2] provides a  $\sqrt{n}$ -consistent estimator and an asymptotically normal estimator for the shift parameter  $\theta^*$ . This procedure is based on the discrete Fourier transform of data. Our estimation method is related to the method of Gamboa, Loubes and Maza [2]: The common shape  $f^*$  is approximated by trigonometric polynomials.

The efficiency of the estimators is to be understood as asymptotic unbiasedness and minimum variance. To avoid the phenomena of super-efficiency (e.g., Hodges estimators), the efficiency is studied in a local asymptotic sense, under the local asymptotic normality (LAN) structure. The usual approach for determining the efficiency is to specify a least favorable parametric submodel of the full semiparametric model (it is a submodel for which the Fisher information is the smallest), locally in a neighborhood of  $f^*$ , and to estimate  $(\theta^*, a^*, \nu^*)$  in such a model (see [12, 13]). Here, we consider the parametric submodel where  $f^*$  is a trigonometric polynomial. The method which is used is close to the procedure of Gassiat and Lévy-Leduc [3] where the authors estimate efficiently the period of an unknown periodic function. The profile log-likelihood is used in order to “eliminate” the nuisance parameter and to build an  $M$ -estimation criterion. Moreover the efficiency of the  $M$ -estimator of  $(\theta^*, a^*, \nu^*)$  is proved by using the theory developed by McNeney and Wellner [10]: The authors develop tools for nonindependent identically distributed data that are similar in spirit to those for independent identically distributed data. Thus the notions of tangent space and of differentiability of the parameter  $(\theta^*, a^*, \nu^*)$  are used in order to specify the characteristics of an efficient estimator. Under the assumptions listed in Theorem 3.1, the estimator of  $(\theta^*, a^*, \nu^*)$  is asymptotically efficient. This follows the conclusions of Murphy and Van der Vaart [11]: Semiparametric profile likelihoods, where the nuisance

parameter has been profiled out, behave like ordinary likelihoods in that they have a quadratic expansion.

The profile log-likelihood induces the definition of an estimator for the common shape. Corollary 3.1 establishes the consistency of this estimator. The rate of the regression function estimator is the optimal rate in nonparametric estimation [12], Chapter 24. Using the theory developed by McNeney and Wellner [10], we discuss its efficiency: the estimator is asymptotically linear. But the Fourier coefficients' estimators are efficient if and only if the common shape  $f^*$  is odd or even. Even if this condition is satisfied, we can not deduce that the estimator of  $f^*$  is efficient because it is not regular.

This work is related to [14], Chapter 3, where we propose another criterion which allows us to estimate efficiently the parameter  $(\theta^*, a^*, \nu^*)$ . This criterion, which is similar by its definition to the criterion proposed by Gamboa, Loubes and Maza [2] and [14], Chapter 2, allows us to build a test procedure for the model.

The rest of the paper is organized as follows: Section 2 describes the model and the estimation method. In Section 3, we discuss the efficiency of the estimator. All technical lemmas and proofs are in Section 4.

## 2. The estimation method.

*The description of the model.* The data  $(Y_{i,j})$  are the observations of  $J$  curves at the observation times  $(t_{i,j})$ . We assume that each curve is observed at the same set of equidistant points

$$t_i = t_{i,j} = \frac{i-1}{n} 2\pi \in [0, 2\pi[, \quad i = 1, \dots, n.$$

The choice of the observation times  $t_i$  is related with the choice of quadrature formula (see Remark 2.1). The studied model is

$$(2.1) \quad Y_{i,j} = a_j^* f^*(t_i - \theta_j^*) + \nu_j^* + \sigma^* \varepsilon_{i,j}, \quad j = 1, \dots, J, i = 1, \dots, n.$$

The common shape  $f^*$  is an unknown real  $2\pi$ -periodic continuous function. We denote by  $\mathcal{F}$  the set of  $2\pi$ -periodic continuous functions. The noises  $(\varepsilon_{i,j})$  are independent standard Gaussian random variables. For the sake of simplicity, we get a common variance  $\sigma^{*2} = \sigma_j^{*2}$ ,  $j = 1, \dots, J$ . However, all our results are still valid for a general variance.

The model is semiparametric:  $\alpha^* = (\theta^*, a^*, \nu^*, \sigma^*)$  is the finite-dimensional parameter and  $f^*$  is the nuisance parameter. Our aim is to estimate efficiently the internal shift  $\theta^* = (\theta_j^*)_{j=1, \dots, J}$ , the scale parameter  $a^* = (a_j^*)_{j=1, \dots, J}$  and the external shift  $\nu^* = (\nu_j^*)_{j=1, \dots, J}$  without knowing either the shape  $f^*$  or the noise level  $\sigma^*$ . We denote  $\mathcal{A} = [0, 2\pi]^J \times \mathbb{R}^J \times [-\nu_{\max}, \nu_{\max}]^J$  as the set where the parameter  $(\theta^*, a^*, \nu^*)$  lies.

*The identifiability constraints.* Before considering the estimation of parameters, we have to study the uniqueness of their definition. Indeed, the shape invariant model has some inherent unidentifiability: for a given parameter  $(\theta_0, a_0, \nu_0) \in \mathbb{R}^3$  and a shape function  $f_0$  we can always find another parameter  $(\theta_1, a_1, \nu_1) \in \mathbb{R}^3$  and another shape function  $f_1$  such that  $a_0 f_0(t - \theta_0) + \nu_0 = a_1 f_1(t - \theta_1) + \nu_1$  holds for all  $t$ .

Then we assume that the true parameters lie in the following spaces:

$$f^* \in \mathcal{F}_0 = \left\{ f \in \mathcal{F}, c_0(f) = \int_0^{2\pi} f(t) \frac{dt}{2\pi} = 0 \right\} \quad \text{and} \quad (\theta^*, a^*, \nu^*) \in \mathcal{A}_0,$$

$$\text{where } \mathcal{A}_0 = \left\{ (\theta, a, \nu) \in \mathcal{A}, \theta_1 = 0, \sum_{j=1}^J a_j^2 = J \text{ and } a_1 > 0 \right\}.$$

The constraint on the common shape allows us to uniquely define the parameter  $\nu^* [\nu_j^* = c_0(f_j^*), j = 1, \dots, J]$  and to build asymptotically independent estimators (see Remark 3.1). The constant  $\nu_{\max}$  is a user-defined (strictly positive) parameter which reflects our prior knowledge on the level parameter. The constraints  $\theta_1 = 0$  and  $a_1 > 0$  mean that the first unit ( $j = 1$ ) is taken as “reference” to estimate the shift parameter and the scale parameter. At last, the constraint  $\sum_{j=1}^J a_j^2 = J$  means that the common shape is defined as the weighted sum of the regression functions  $f_j^*$  (1.1). This condition is well adapted to our estimation criterion (see the next paragraph on the profile likelihood).

*The profile log-likelihood.* Maximizing the likelihood function directly is not possible for higher-dimensional parameters, and fails particularly for semiparametric models. Frequently, this problem is overcome by using a profile likelihood rather than a full likelihood. If  $l_n(\alpha, f)$  is the full log-likelihood, then the profile likelihood for  $\alpha \in \mathcal{A}_0$  is defined as

$$pl_n(\alpha) = \sup_{f \in \mathcal{F}_0} l_n(\alpha, f).$$

The maximum likelihood estimator for  $\alpha$ , the first component of the pair  $(\hat{\alpha}_n, \hat{f}_n)$  that maximizes  $l_n(\alpha, f)$ , is the maximizer of the profile likelihood function  $\alpha \rightarrow pl_n(\alpha)$ . Thus we maximize the likelihood in two steps. With the assumptions on the model, we shall use the Gaussian log-likelihood,

$$(2.2) \quad l_n(\alpha, f) = \frac{-1}{2\sigma^2} \sum_{i=1}^n \sum_{j=1}^J (Y_{i,j} - a_j f(t_i - \theta_j) - \nu_j)^2 - \frac{nJ}{2} \log \sigma^2.$$

Generally, the problem of minimization on a large set is solved by the consideration of a parametric subset. Here, the semiparametric problem is reduced to a parametric one:  $f$  is approximated by its truncated Fourier series. Thus the profile

likelihood is approximated by minimizing the likelihood  $l_n$  on a subset of trigonometric polynomials. More precisely, let  $(m_n)_n$  be an increasing integer's sequence, and let  $\mathcal{F}_{0,n}$  be the subspace of  $\mathcal{F}_0$  of trigonometric polynomials whose degree is less than  $m_n$ . In order to preserve the orthogonality of the discrete Fourier basis,

$$\forall |l| < \frac{n}{2}, \forall |p| < \frac{n}{2} \quad \frac{1}{n} \sum_{r=1}^n e^{i(l-p)t_r} = \begin{cases} 1, & \text{if } l = p, \\ 0, & \text{if } l \neq p, \end{cases}$$

we choose  $m_n$  and  $n$  such that

$$(2.3) \quad 2|m_n| < n, \quad \lim_{n \rightarrow +\infty} m_n = +\infty \quad \text{and} \quad n \text{ is odd.}$$

After some computations, the likelihood maximum is reached in the space  $\mathcal{F}_{0,n}$  by the trigonometric polynomial

$$(2.4) \quad \hat{f}_\alpha(t) = \sum_{1 \leq |l| \leq m_n} \hat{c}_l(\alpha) e^{ilt} \quad \forall t \in \mathbb{R},$$

where for  $l \in \mathbb{Z}$ ,  $1 \leq |l| \leq m_n$ ,

$$(2.5) \quad \hat{c}_l(\alpha) = \left( n \sum_{j=1}^J a_j^2 \right)^{-1} \sum_{j=1}^J a_j \sum_{i=1}^n (Y_{i,j} - v_j) e^{-il(t_i - \theta_j)} \quad \forall \alpha \in \mathcal{A}_0 \times \mathbb{R}_+^*.$$

Finally, using the orthogonality of the discrete Fourier basis, the following equality holds:

$$\begin{aligned} & \sum_{j=1}^J \sum_{i=1}^n \left( Y_{i,j} - a_j \sum_{1 \leq |l| \leq m_n} \hat{c}_l(\alpha) e^{il(t_i - \theta_j)} - v_j \right)^2 \\ &= \sum_{j=1}^J \sum_{i=1}^n (Y_{i,j} - v_j)^2 - \left( n \sum_{j=1}^J a_j^2 \right) \sum_{1 \leq |l| < m_n} |\hat{c}_l(\alpha)|^2 \\ & \quad + n \sum_{1 \leq |l|, |p| < m_n, l \neq p} \hat{c}_l(\alpha) \overline{\hat{c}_p(\alpha)} \varphi_n \left( \frac{l-p}{n} \right) \sum_{j=1}^J a_j^2 e^{(p-l)\theta_j}, \end{aligned}$$

where  $\varphi_n(t) = \sum_{s=1}^n e^{2i\pi st} / n$ . Let  $M_n$  be the function of  $\alpha = (\theta, a, v)$  defined as

$$M_n(\alpha) = \frac{1}{nJ} \sum_{j=1}^J \sum_{i=1}^n (Y_{i,j} - v_j)^2 - \sum_{1 \leq |l| \leq m_n} |\hat{c}_l(\alpha)|^2.$$

With the identifiability constraints of the model, the profile log-likelihood  $pl_n$  is equal to

$$(2.6) \quad pl_n(\alpha) = -(nJ) \frac{M_n(\alpha)}{2\sigma^2} - \frac{nJ}{2} \log \sigma^2.$$

REMARK 2.1. The estimation method requires the estimation of the Fourier coefficients of the common shape. A natural approach for estimating an integral is to use a quadrature formula which is associated with the observation times  $t_j$ . In this paper, the observation times are equidistant. Therefore the quadrature formula is the well-known Newton–Cotes formula. Even if another choice of the observation times is possible (see [14], Chapter 2), this formula defines the discrete Fourier coefficients  $c_l^n(f)$  which are an accurate approximation of  $c_l(f)$ :

$$c_l^n(f^*) = \frac{1}{n} \sum_{s=1}^n f^*(t_s) e^{-ilt_s} \longrightarrow c_l(f^*) = \int_0^{2\pi} f^*(t) e^{-ilt} \frac{dt}{2\pi}.$$

Moreover, the stochastic part of the coefficients (2.5) are linear combinations of the complex variables  $w_{j,l}$ ,

$$w_{j,l} = \frac{1}{n} \sum_{r=1}^n e^{-il t_r} \varepsilon_{i,j}, \quad j = 1, \dots, J, |l| \leq m_n.$$

Due to Cochran’s theorem, these variables are independent centered complex Gaussian variables whose the variance is equal to  $1/n$ . This property is related to the convergence rate of the estimators (see [14], Chapter 2, for more details, and [3] to compare).

*The estimation procedure.* Consequently, the maximum likelihood estimator of the finite-dimensional parameter is defined as

$$\hat{\beta}_n = \arg \min_{\beta \in \mathcal{A}_0} M_n(\beta) \quad \text{or} \quad \hat{\alpha}_n = (\hat{\beta}_n, \hat{\sigma}_n) = \arg \max_{\alpha \in \mathcal{A}_0 \times \mathbb{R}_+^*} p l_n(\alpha).$$

Then, the estimators of the common shape are the trigonometric polynomials, which maximize the likelihood when  $\alpha = \hat{\alpha}_n$ :

$$\hat{f}_n(t) = \hat{f}_{\hat{\alpha}_n}(t) = \sum_{1 \leq |l| \leq m_n} \hat{c}_l(\hat{\alpha}_n) e^{ilt} \quad \forall t \in \mathbb{R}.$$

First, we study the consistency of the estimator of  $(\theta^*, a^*, \nu^*)$ . The consistency of the common shape estimator is studied in the next section.

**THEOREM 2.1 (Consistency).** *Assume that  $2\pi$  is the minimal period of  $f^*$ , and that*

$$(2.7) \quad \sum_{|l| > m} |c_l(f^*)| = o\left(\frac{1}{\sqrt{m}}\right) \quad \text{and} \quad \frac{m_n}{n} = o(1).$$

*Then  $\hat{\alpha}_n$  converges in probability to  $\alpha^*$ .*

The assumption regarding the common shape means that the function  $f^*$  is a 1/2-holder function. The assumption on the number of Fourier coefficients means

that  $m_n$  has to be small in relation to the number of observation  $n$ . Notice that Theorem 2.1 is still valid if the noises  $(\varepsilon_{i,j})$  are (centered) independent identically distributed with finite variance.

**PROOF OF THEOREM 2.1.** The proof of this theorem follows the classical guidelines of the convergence of  $M$ -estimators (see, e.g., Theorem 5.7 of Van der Vaart [12]). Indeed, to ensure consistency of  $\hat{\beta}_n$ , it suffices to show that:

(i) The uniform convergence of  $M_n$  to a contrast function  $M + \sigma^{*2}$  (Lemma 4.1):

$$\sup_{\beta \in \mathbb{A}} |M_n(\beta) - M(\beta) - \sigma^{*2}| = o_{P_\alpha^*}(1),$$

where  $M$  is defined as

$$M(\beta) = \int_0^{2\pi} \frac{1}{J} \sum_{j=1}^J (f_j^*(t) - v_j)^2 \frac{dt}{2\pi} - \int_0^{2\pi} \left( \sum_{j=1}^J a_j a_j^* f^*(t - \theta_j^* + \theta_j) \right)^2 \frac{dt}{2\pi}.$$

(ii)  $M(\cdot)$  has a unique minimum at  $\beta^*$  (Lemma 4.2). □

*The daily temperatures of cities.* The estimation method is applied to daily average temperatures (the average daily temperatures are the average of 24 hourly temperature readings). The data come from of the University of Dayton (<http://www.engr.udayton.edu/weather/>). In order to illustrate the method, we limit the study to three cities which have a temperature range of an oceanic climate: Juneau (Alaska, city  $j = 1$ ), Auckland (New Zealand, city  $j = 2$ ) and Bilbao (Spain, city  $j = 3$ ). An oceanic climate is the climate typically found along the west coasts at the middle latitudes of all the world's continents, and in southeastern Australia. Similar climates are also found on coastal tropical highlands and tropical coasts on the leeward sides of mountain ranges. Figure 1(a) plots the sample of temperature curves.

If we assume that the data fit the model (2.1), the parameters  $\theta^*$ ,  $a^*$  and  $v^*$  have the following meanings:

- $v_j^*$  is the annual temperature average of the  $i$ th city,
- $a_j^*$  indicates whether the city is in the same hemisphere as the first city ( $a_j^* > 0$ ) and measures the differences between the winter and summer temperatures,
- $\theta_j^*$  is the seasonal phase of the  $i$ th city,
- $f^*$  describes the general behavior of the temperature evolution of the oceanic climate.

The estimators of these parameters are given in Table 1.

Figure 1(b) plots the estimator of the common shape. The number of the Fourier coefficients used to estimate the common shape is  $m_n = 5$ . Further study will yield the most accurate number  $m_n$ , and leads to studying the estimation problem from the point of view of the selection model.

TABLE 1  
*Estimators of the parameters  $\theta_2^*, \theta_3^*, a_1^*, a_2^*, a_3^*, \nu_1^*, \nu_2^*$  and  $\nu_3^*$*

City $j$	$j = 1$	$j = 2$	$j = 3$
$\hat{\theta}_{j,n}$ (days)	0	12.5182	25.35381
$\hat{a}_{j,n}$	1.2421	-0.5833	1.0569
$\hat{\nu}_{j,n}$ (Fahrenheit)	43.9874	58.5312	60.1814

**3. Efficient estimation.**

3.1. *The LAN property.* Before studying the asymptotic efficiency of the estimators, we have to establish the local asymptotic normality of the model. First, let us introduce some notation. The model is semiparametric. The finite-dimensional parameter  $\alpha^*$  lies in  $\mathcal{A}_0 \times \mathbb{R}_+^*$ . The nuisance parameter  $f^*$  lies in  $\mathcal{F}_0$ . For  $(\alpha, f) \in \mathcal{A}_0 \times \mathbb{R}_+ \times \mathcal{F}_0$  and  $t \in \mathbb{R}$ , we denote by  $\mathbb{P}_{\alpha, f}(t)$  the Gaussian distribution in  $\mathbb{R}^J$  with variance  $\sigma^2 I_J$  and mean  $(a_j f(t - \theta_j) + \nu_j)_{j=1, \dots, J}$ . Then the model of the observations is

$$\mathcal{P}_n = \left\{ \mathbb{P}_{\alpha, f}^{(n)} = \bigotimes_{i=1}^n \mathbb{P}_{(\alpha, f)}(t_i), (\alpha, f) \in \mathcal{A}_0 \times \mathbb{R}_+ \times \mathcal{F}_0 \right\}.$$

To avoid the phenomenon of super efficiency, we study the model on a local neighborhood of  $(\alpha^*, f^*)$ . Let  $(\alpha_n(h), f_n(h))$  be close to  $(\alpha^*, f^*)$  in the direction  $h$ . The LAN property requires that the log-likelihood ratio for the two points  $(\alpha^*, f^*)$  and  $(\alpha_n(h), f_n(h))$  converges in distribution to a Gaussian variable which depends only on  $h$ .

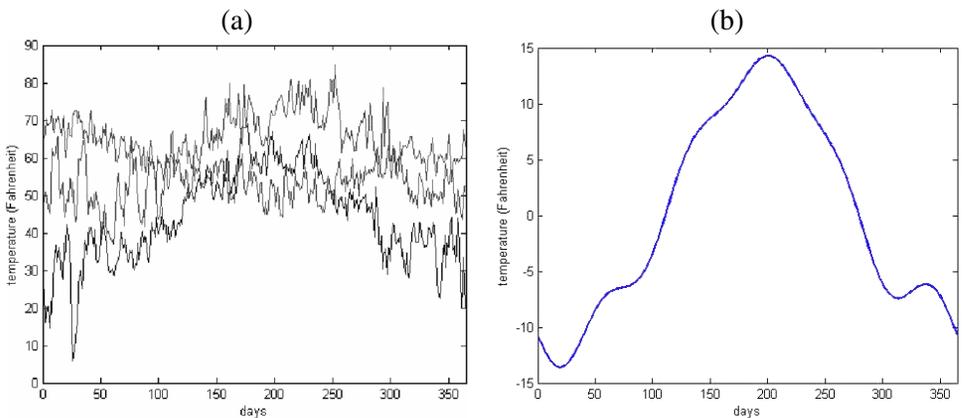


FIG. 1. (a) *Plots of the temperature curves associated with Juneau (Alaska), Auckland (New Zealand) and Bilbao (Spain) in 2004.* (b) *Plot of the estimator of the common shape  $\hat{f}_n$ .*

Since the observations of our model are not identically distributed, we shall follow the semiparametric analysis developed by McNeney and Wellner [10]. The LAN property allows identification of the least favorable direction  $h$  that approaches the model, and thus allows us to know whether the estimator is efficient. Let us denote the log-likelihood ratio for the two points  $(\alpha^*, f^*)$  and  $(\alpha, f)$

$$\Lambda_n(\alpha, f) = \log \frac{d\mathbb{P}_{\alpha, f}^{(n)}}{d\mathbb{P}_{\alpha^*, f^*}^{(n)}}.$$

PROPOSITION 3.1 (LAN property). *Assume that the function  $f^*$  is not constant and is differentiable with a continuous derivative denoted by  $\partial f^*$ . Assume that the reals  $a_j^*$ ,  $j = 1, \dots, J$ , are nonnull. Considering the vector space  $\mathcal{H} = \mathbb{R}^{J-1} \times \mathbb{R}^{J-1} \times \mathbb{R}^J \times \mathbb{R}_+ \times \mathcal{F}_0$ , the coordinates of a vector  $h \in \mathcal{H}$  are denoted as follows:*

$$h = (h_{\theta,2}, \dots, h_{\theta,J}, h_{a,2}, \dots, h_{a,J}, h_{v,2}, \dots, h_{v,J}, h_\sigma, h_f).$$

Then the space  $\mathcal{H}$  is an inner-product space endowed with the inner product  $\langle \cdot, \cdot \rangle$ ,

$$\begin{aligned} \langle h, h \rangle = & J \frac{h_\sigma h'_\sigma}{\sigma^{*2}} + \frac{1}{\sigma^{*2}} \left\langle a_1^* h_f - \sum_{j=2}^J h_{a,j} \frac{a_j^*}{a_1^*} f^* \right. \\ & \left. + v_1, a_1^* h'_f - \sum_{j=2}^J h'_{a,j} \frac{a_j^*}{a_1^*} f^* + v'_1 \right\rangle_{\mathbb{L}^2} \\ & + \frac{1}{\sigma^{*2}} \sum_{j=2}^J \langle a_1^* h_f + a_j f^* - h_{\theta,j} a_j^* \partial f^* \\ & \quad + h_{v,j}, a_1^* h'_f + h'_{a,j} f^* - h'_{\theta,j} a_j^* \partial f^* + h'_{v,j} \rangle_{\mathbb{L}^2}, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle_{\mathbb{L}^2}$  is the inner product in  $\mathbb{L}^2[0, 2\pi]$ . Moreover, the model (2.1) is LAN at  $(\alpha^*, f^*)$  indexed by the tangent space  $\mathcal{H}$ . In other words, for each  $h \in \mathcal{H}$ , there exists a sequence  $(\alpha_n(h), f_n(h))$  such that

$$\Lambda_n(\alpha_n(h), f_n(h)) = \Delta_n(h) - \frac{1}{2} \|h\|_{\mathcal{H}}^2 + o_{\mathbb{P}}(1).$$

Here, the central sequence  $\Delta_n(h)$  is linear with  $h$ ,

$$\Delta_n(h) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{j=1}^J \{ (h_\sigma / \sigma^*) (\varepsilon_{i,j}^2 - 1) + A_{i,j}^n(h) \varepsilon_{i,j} / \sigma^* \},$$

where for all  $i = 1, \dots, n$ ,

$$A_{i,j}^n(h) = \begin{cases} h_{a,1}^* f(t_i) - \sum_{k=2}^J h_{a,k} \frac{a_k^*}{a_1^*} f^*(t_i), & \text{if } j = 1, \\ a_j^* h_f(t_i - \theta_j^*) + h_{a,j} f^*(t_i - \theta_j^*) \\ \quad - h_{\theta,j} a_j^* \partial f^*(t_i - \theta_j^*) + v_j, & \text{if } j = 2, \dots, J. \end{cases}$$

Notice that for the independent identically distributed semiparametric models, the fact that the tangent space would not be complete does not imply the existence of a least favorable direction. In our model the tangent space  $\mathcal{H}$  is a subset of the Hilbert space

$$\overline{\mathcal{H}} = \mathbb{R}^{J-1} \times \mathbb{R}^{J-1} \times \mathbb{R}^J \times \mathbb{R} \times \{f \in \mathbb{L}^2[0, 2\pi], c_0(f) = 0\},$$

endowed with the inner product  $\langle \cdot, \cdot \rangle$ . Consequently, it is easier to determine the least favorable direction using the Riesz representation theorem.

3.2. *The efficiency.* The goal of this paper may be stated as the semiparametric efficient estimation of the parameter  $v_n(\mathbb{P}_{\alpha^*, f^*}^{(n)}) = (\theta_2^*, \dots, \theta_J^*, a_2^*, \dots, a_J^*, v_1^*, \dots, v_J^*)$ . This parameter is differentiable relative to the tangent space  $\mathcal{H}$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt{n}(v_n(\mathbb{P}_{\alpha_n(h), f_n(h)}^{(n)}) - v_n(\mathbb{P}_{\alpha^*, f^*}^{(n)})) \\ = (h_{\theta, 2}, \dots, h_{\theta, J}, h_{a, 2}, \dots, h_{a, J}, h_{v, 2}, \dots, h_{v, J}). \end{aligned}$$

Consequently, there exists a continuous linear map  $\dot{v}$  from  $\mathcal{H}^{3J-2}$  on to  $\mathbb{R}^{3J-2}$ . According to the Riesz representation theorem, there exist  $3J - 2$  vectors  $(\dot{v}_j^\theta)_{2 \leq j \leq J}$ ,  $(\dot{v}_j^a)_{2 \leq j \leq J}$  and  $(\dot{v}_j^v)_{1 \leq j \leq J}$  of  $\overline{\mathcal{H}}$  such that

$$\forall h \in \mathcal{H} \quad \langle \dot{v}_j^\theta, h \rangle = h_{\theta, j}, \quad \langle \dot{v}_j^a, h \rangle = h_{a, j} \quad \text{and} \quad \langle \dot{v}_j^v, h \rangle = h_{v, j}.$$

These vectors are defined in Lemma 4.3. Using the linearity with  $h$  of  $\Delta_n(h)$ , the following proposition, which is an application of Proposition 5.3 of McNeney and Wellner [10], links the notion of asymptotic linearity of an estimator and the efficiency.

PROPOSITION 3.2 (Asymptotic linearity and efficiency). *Let  $T_n$  be an asymptotically linear estimator of  $v_n(\mathbb{P}_{\alpha^*, f^*}^{(n)})$  with the central sequence*

$$(\Delta_n(\tilde{h}_2^\theta), \dots, \Delta_n(\tilde{h}_J^\theta), \Delta_n(\tilde{h}_2^a), \dots, \Delta_n(\tilde{h}_J^a), \dots, \Delta_n(\tilde{h}_j^v)).$$

*$T_n$  is regular efficient if and only if for all  $j$   $\tilde{h}_j^\theta = \dot{v}_j^\theta$ ,  $\tilde{h}_j^a = \dot{v}_j^a$  and  $\tilde{h}_j^v = \dot{v}_j^v$ .*

From Lemma 4.3, if the assumptions of Proposition 3.1 hold and if the estimator  $\hat{\beta}_n = (\hat{\theta}_n, \hat{a}_n, \hat{v}_n)$  is asymptotically linear, it is efficient if and only if

$$\begin{aligned} \sqrt{n}(\hat{\theta}_n - \theta^*) &= \frac{\sigma^*}{\|\partial f^*\|_{\mathbb{L}^2}} \sum_{i=1}^n \begin{bmatrix} \mathbb{I} \\ \vdots \\ a_1^* \\ \vdots \end{bmatrix} - D^{-1} \Big] \partial F^*(t_i) \varepsilon_{i,\cdot} + o_{\mathbb{P}}(1), \\ \sqrt{n}(\hat{a}_n - a^*) &= \frac{\sigma^*}{\|\partial f^*\|_{\mathbb{L}^2}} \sum_{i=1}^n \left\{ \begin{bmatrix} -\frac{a_1^*}{J} A \\ \vdots \\ I_{J-1} - \frac{1}{J} A {}^t A \end{bmatrix} F^*(t_i) \right\} \varepsilon_{i,\cdot} + o_{\mathbb{P}}(1), \\ \sqrt{n}(\hat{v}_n - v^*) &= \sigma^* \sum_{i=1}^n \varepsilon_{i,\cdot} + o_{\mathbb{P}}(1) \quad \text{where } {}^t \varepsilon_{i,\cdot} = {}^t(\varepsilon_{i1}, \dots, \varepsilon_{iJ}), \end{aligned}$$

where  $D$  is the diagonal matrix  $\text{diag}(a_2^*, \dots, a_j^*)$  and  $A = {}^t(a_2^*, \dots, a_j^*)$  a vector in  $\mathbb{R}^{J-1}$ .  $F^*(t)$  and  $\partial F^*(t)$  are, respectively, the diagonal matrix  $\text{diag}(f^* \times (t - \theta_1^*), \dots, f^*(t - \theta_j^*))$  and  $\text{diag}(\partial f^*(t - \theta_1^*), \dots, \partial f^*(t - \theta_j^*))$  for all  $t \in \mathbb{R}$ . We deduce the following theorem:

**THEOREM 3.1 (Efficiency).** *Assume that the assumptions of Proposition 3.1 hold and that*

$$(3.1) \quad \sum_{l \in \mathbb{Z}} |l| |c_l(f^*)| < \infty,$$

$$(3.2) \quad m_n^4/n = o(1).$$

Then  $(\hat{\theta}_n, \hat{a}_n, \hat{v}_n)$  is asymptotically efficient and  $\sqrt{n}(\hat{\theta}_n - \theta^*, \hat{a}_n - a^*, \hat{v}_n - v^*)$  converges in distribution to a Gaussian vector  $\mathcal{N}_{3J-2}(0, \sigma^{*2}H^{-1})$ , where  $H$  is the matrix defined as

$$H = \begin{pmatrix} \|\partial f^*\|_{\mathbb{L}^2}^2 \left( D^2 - \frac{1}{J} A^2 {}^tA^2 \right) & 0 & 0 \\ 0 & \|f^*\|_{\mathbb{L}^2}^2 \left( I + \frac{1}{a_1^{*2}} A {}^tA \right) & 0 \\ 0 & 0 & I_J \end{pmatrix}$$

and its inverse matrix  $H^{-1}$  is equal to

$$H^{-1} = \begin{pmatrix} \frac{1}{\|\partial f^*\|_{\mathbb{L}^2}^2} \left( D^{-2} + \frac{1}{a_1^{*2}} \mathbb{I}_{J-1} {}^t\mathbb{I}_{J-1} \right) & 0 & 0 \\ 0 & \frac{1}{\|f^*\|_{\mathbb{L}^2}^2} \left( I_{J-1} - \frac{1}{J} A {}^tA \right) & 0 \\ 0 & 0 & I_J \end{pmatrix}.$$

**PROOF.** Recall that the  $M$ -estimator is defined as the minimum of the criterion function  $M_n(\cdot)$ . Hence, we get

$$\nabla M_n(\hat{\beta}_n) = 0,$$

where  $\nabla$  is the gradient operator. Thanks to a second-order expansion, there exists  $\bar{\beta}_n$  in a neighborhood of  $\beta^*$  such that

$$\nabla^2 M_n(\bar{\beta}_n) \sqrt{n}(\hat{\beta}_n - \beta^*) = -\sqrt{n} \nabla M_n(\beta^*),$$

where  $\nabla^2$  is the Hessian operator. Now, using two asymptotic results from Proposition 4.1 and from Proposition 4.2, we obtain

$$\sqrt{n}(\hat{\theta}_n - \theta^*) = \frac{\sigma^*}{\|\partial f^*\|_{\mathbb{L}^2}^2} \left( D^{-2} + \frac{1}{a_1^{*2}} \mathbb{I}_{J-1} {}^t\mathbb{I}_{J-1} \right) G_n^\theta + o_{\mathbb{P}}(1),$$

$$\sqrt{n}(\hat{a}_n - a^*) = \frac{\sigma^*}{\|f^*\|_{\mathbb{L}^2}^2} \left( I_{J-1} - \frac{1}{J} A {}^tA \right) G_n^a + o_{\mathbb{P}}(1),$$

$$\sqrt{n}(\hat{v}_n - v^*) = \sigma^* G_n^v + o_{\mathbb{P}}(1). \quad \square$$

REMARK 3.1. The choice of the identifiability constraints is important for the relevancy of the estimation. For example, if we no longer assume that  $c_0(f)$  is null, we may consider the following parameter space:

$$\mathcal{A}_1 = \left\{ (\theta, a, \nu) \in \mathcal{A}, \text{ such that } \theta_1 = 0, \sum_{j=1}^J a_j^2 = J \text{ and } a_1 > 0 \right\} \quad \text{and} \quad f \in \mathcal{F}.$$

Consequently we have to estimate  $3J - 3$  parameters:  $\theta_2^*, \dots, \theta_J^*, a_2^*, \dots, a_J^*$ , and  $\nu_2^*, \dots, \nu_J^*$ . This choice modifies the estimation criterion and the tangent space, too. Nevertheless, if the assumptions of Theorem 3.1 hold, the estimator is asymptotically efficient. But its covariance matrix is not block diagonal any more:

$$\Gamma = \sigma^{*2} \begin{pmatrix} \frac{1}{\|\partial f^*\|_{\mathbb{L}^2}^2} \left( D^{-2} + \frac{1}{a_1^{*2}} \mathbb{I}_{J-1} {}^t \mathbb{I}_{J-1} \right) & 0 & 0 \\ 0 & \frac{1}{\|f^*\|_{\mathbb{L}^2}^2 - c_0(f^*)^2} B & \frac{-c_0(f^*)}{\|f^*\|_{\mathbb{L}^2}^2 - c_0(f^*)^2} I_{J-1} \\ 0 & \frac{-c_0(f^*)}{\|f^*\|_{\mathbb{L}^2}^2 - c_0(f^*)^2} I_{J-1} & \frac{\|f^*\|_{\mathbb{L}^2}^2}{\|f^*\|_{\mathbb{L}^2}^2 - c_0(f^*)^2} B^{-1} \end{pmatrix},$$

where  $B = I_{J-1} - \frac{1}{J} A {}^t A$  with  $B^{-1} = I_{J-1} + \frac{1}{a_1^{*2}} A {}^t A$ . In other words,  $\hat{a}_n$  and  $\hat{\nu}_n$  are not asymptotically independent: modifying the identifiability constraint  $c_0(f^*) = 0$  damages the quality of the estimation.

To illustrate this phenomenon, we present the boxplots of the estimators which are relatively associated with the parameter space  $\mathcal{A}_0$  [Figure 2(a)] and  $\mathcal{A}_1$  [Figure 2(b)]. Let  $(\alpha^*, f^*)$  be a parameter of the model. With the constraints associated with the parameter space  $\mathcal{A}_0$ , we have to estimate  $\theta_2^*, a_2^*, \nu_1^*$  and  $\nu_2^*$  for the following model ( $J = 2$ ):

$$\begin{cases} Y_{i,1} = a_1^* f^*(t_i) + \nu_1^* + \varepsilon_{i,1}, & i = 1, \dots, n, \\ Y_{i,2} = a_2^* f^*(t_i - \theta_2^*) + \nu_2^* + \varepsilon_{i,2}, & i = 1, \dots, n. \end{cases}$$

With the constraints associated with the parameter space  $\mathcal{A}_1$ , we have to estimate  $\theta_2^*, a_2^*$  and  $\nu_2$ . The data may be rewritten as

$$\begin{cases} Y_{i,1} = a_1^* g^*(t_i) + \varepsilon_{i,1}, & i = 1, \dots, n, \\ Y_{i,2} = a_2^* g^*(t_i - \theta_2^*) + \nu_2 + \varepsilon_{i,2}, & i = 1, \dots, n, \end{cases}$$

where  $g^* = f^* + \nu_1^*$  and  $\nu_2 = \nu_2^* - a_2^* \nu_1^*$ . After generating several sets of data from a parameter  $(\alpha^*, f^*)$  which we have chosen, we have computed the estimators of  $\theta_2^*, a_2^*$  and  $\nu_2^*$  for every set of data. Figure 2 presents the boxplots of the estimators of  $\theta_2^*, a_2^*$  and  $\nu_2^*$  for these two models.

As a consequence of the previous theorem, the Gaussian vector  $G_n$  converges in distribution to a centered Gaussian vector  $\mathcal{N}_{3J-2}(0, H)$ , and the equation

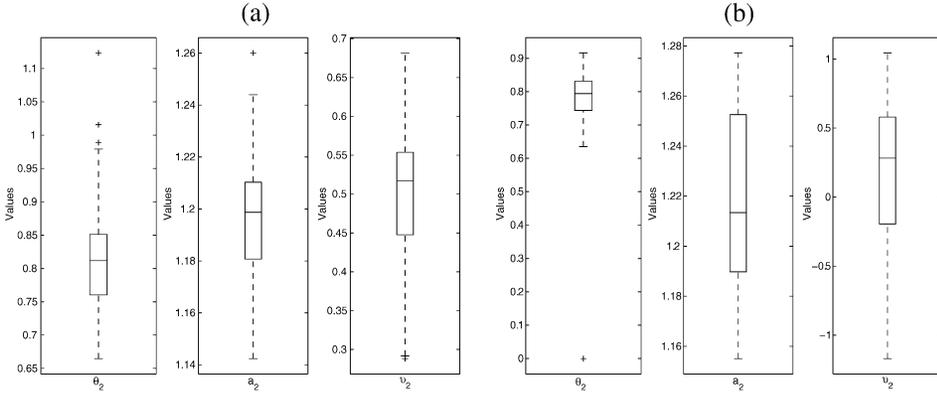


FIG. 2. Boxplots of the estimators of  $\theta_2^*$ ,  $a_2^*$  and  $v_2^*$  associated with the space parameter  $\mathcal{A}_0$  (a) and  $\mathcal{A}_1$  (b). The data are generated with  $f^*(t) = 20 * t/(2\pi)(1 - t/(2\pi))$ ,  $\theta^* = (0 \ 0.8)$ ,  $a^* = (0.75 \ 1.1990)$ ,  $v^* = (7.5/3 \ 0.5)$  and  $n = 201$ . The boxplots are computed from 100 sets of data.

holds:

$$\sqrt{n}(\hat{\beta}_n - \beta^*) = (H/\sigma^{*2})^{-1} \sigma^* G_n + o_{\mathbb{P}}(1).$$

Comparing this formula with the results of the independent identically distributed semiparametric model (see [12]), we identify the efficient information matrix as  $H/\sigma^{*2}$  and the efficient score as  $\sigma^* G_n$ .

Indeed, let  $X_1, \dots, X_n$  be a random sample from a distribution  $\mathbb{P}$  that is known to belong to a set of probabilities  $\{\mathbb{P}_{\theta, \eta}, \theta \in \Theta \subseteq \mathbb{R}^d, \eta \in \mathcal{G}\}$ . Then an estimator sequence  $T_n$  is asymptotically efficient for estimating  $\theta$  if

$$\sqrt{n}(T_n - \theta) = (\tilde{I}_{\theta, \eta})^{-1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{l}_{\theta, \eta}(X_i) \right) + o_{\mathbb{P}}(1),$$

where  $\tilde{l}_{\theta, \eta}$  is the efficient score function, and  $\tilde{I}_{\theta, \eta}$  is the efficient information matrix.

Moreover, our result follows Murphy and Van der Vaart [11]. The authors demonstrate that if the entropy of the nuisance parameters is not too large and the least favorable direction exists, the profile likelihood behaves very much like the ordinary likelihood and the profile likelihood correctly selects a least favorable direction for the independent identically distributed semiparametric model. This holds if the profile log-likelihood  $pl_n$  verifies the following equation:

$$pl_n(\hat{\theta}_n) - pl_n(\theta) = \sum_{i=1}^n \tilde{l}_{\theta, \eta}(X_i)(\hat{\theta}_n - \theta) - \frac{1}{2} n^t(\hat{\theta}_n - \theta) \tilde{I}_{\theta, \eta}(\hat{\theta}_n - \theta) + o_{\mathbb{P}}(\sqrt{n} \|\hat{\theta}_n - \theta\| + 1)^2,$$

where  $\hat{\theta}_n$  maximizes  $pl_n$ . Then, if  $\tilde{I}_{\theta,\eta}$  is invertible, and  $\hat{\theta}_n$  is consistent,  $\hat{\theta}_n$  is asymptotically efficient.

For our model, a similar asymptotic expansion holds. Indeed, by a Taylor expansion, there exists  $\tilde{\alpha}_n$  such that

$$\begin{aligned} pl_n(\hat{\alpha}_n) - pl_n(\alpha^*) &= n^{1/2} G_n(\hat{\beta}_n - \beta^*) - \frac{n}{2} {}^t(\hat{\beta}_n - \beta^*) \frac{H}{\sigma^{*2}} (\hat{\beta}_n - \beta^*) \\ &\quad + o_{\mathbb{P}}(n^{1/2} \|\hat{\beta}_n - \beta^*\| + 1)^2. \end{aligned}$$

3.3. *Asymptotic linearity of the common shape estimator.* In this subsection, we study the consistency and the characteristics of the estimator of the common shape which is defined in Section 2. We show that the convergence rate of this estimator is the optimal rate for the nonparametric estimation.

**COROLLARY 3.1.** *Assume that  $f^*$  is  $k$  times continuously differentiable with  $\int_0^{2\pi} |f^{(k)}(t)|^2 dt < \infty$  and  $k \geq 1$ . Furthermore, suppose that the assumptions of Theorem 3.1 hold; then there exists a constant  $C$  such that for a large  $m_n$*

$$\begin{aligned} \sup_{t \in \mathbb{R}} |\hat{f}_n(t) - f^*(t)| &= O_{\mathbb{P}}\left(\frac{1}{m_n^{k-1/2}} + \frac{m_n}{\sqrt{n}}\right), \\ \int_0^{2\pi} \mathbb{E}(\hat{f}_n(t) - f^*(t))^2 &\leq C\left(\frac{1}{m_n^{2k}} + \frac{m_n}{n}\right). \end{aligned}$$

Consequently, for  $m_n \sim n^{1/(2k+1)}$ , we have  $MISE_{f^*}(\hat{f}_n) = O(n^{-2k/(2k+1)})$ .

Let  $\mathbb{B}$  represent the Banach space defined as the closure of  $\mathcal{F}$  for the  $\mathbb{L}^2$ -norm

$$\mathbb{B} = \{f \in \mathbb{L}^2[0, 2\pi] \text{ such that } c_0(f) = 0\}.$$

Here, the studied sequence of parameter  $\nu_n$  is not  $(\theta^*, a^*, v^*)$  any more, but it is the truncated Fourier series of  $f^*$ :

$$\nu_n(\mathbb{P}_{\alpha^*, f^*}^{(n)}) = \sum_{|l| \leq m_n} c_l(f^*) e^{il(\cdot)}.$$

The parameter sequence  $\nu_n$  is differentiable:

$$\lim_{n \rightarrow \infty} \sqrt{n}(\nu_n(\mathbb{P}_{\alpha_n(h), f_n(h)}^{(n)}) - \nu_n(\mathbb{P}_{\alpha^*, f^*}^{(n)})) = hf.$$

Thus, there exists a continuous linear map  $\dot{\nu}$  from  $\mathcal{H}$  on to  $\mathbb{B}$ . To have a representation of the derivative  $\dot{\nu}$ , we consider the dual space  $\mathbb{B}^*$  of  $\mathbb{B}$ . In other words, for

$b^* \in \mathbb{B}^*$ ,  $b^* \dot{v}$  is represented by  $\dot{v}^{b^*} \in \overline{\mathcal{H}}$ :

$$\forall h \in \mathcal{H} \quad b^* \dot{v}(h) = \langle \dot{v}^{b^*}, h \rangle = b^* h_f.$$

Furthermore, the dual space  $\mathbb{B}^*$  is generated by the following linear real functions:

$$b_{1l}^* : f \in \mathcal{F}_0 \rightarrow \int_0^{2\pi} f(t) \cos(lt) \frac{dt}{2\pi} \quad \text{and}$$

$$b_{2l}^* : f \in \mathcal{F}_0 \rightarrow \int_0^{2\pi} f(t) \sin(lt) \frac{dt}{2\pi}, \quad l \in \mathbb{Z}^*.$$

Thus it suffices to know  $\dot{v}^{b_{1l}^*}$  and  $\dot{v}^{b_{2l}^*}$  for all  $l \in \mathbb{Z}^*$  in order to determine all  $\{\dot{v}^{b^*}, b^* \in \mathbb{B}^*\}$ . After straightforward computations, these vectors are

$$\dot{v}_{b_{1l}^*} = (0, \cos(l \cdot) / J) \quad \text{and} \quad \dot{v}_{b_{2l}^*} = (0, \sin(l \cdot) / J).$$

The estimator of the common shape is asymptotically linear. This means that for all  $b^* \in \mathbb{B}^*$  there exists  $h^{b^*} \in \overline{\mathcal{H}}$  such that

$$(3.3) \quad \sqrt{n} b^* (T_n - \nu_n(\mathbb{P}_{\alpha_n^*, f^*}^n)) = \Delta_n(h^{b^*}) + o_{\mathbb{P}}(1).$$

Since  $\{b_{1l}^*, b_{2l}^*, l \in \mathbb{Z}^*\}$  generates the dual space of  $\mathbb{B}$ , Lemma 4.4 ensures the asymptotic linearity of  $\hat{f}_n$ .

Now, we discuss the regularity and the efficiency of this estimator. We deduce from Proposition 5.4 of McNeney and Wellner [10] that:

**COROLLARY 3.2.**  *$b^* \hat{f}_n$  is a regular efficient estimator of  $b^* f^*$  for all  $b^* \in \mathbb{B}^*$  if and only if the function  $f^*$  is odd or even. In particular, in this case, the estimator of the Fourier coefficients of  $f^*$  is efficient.*

Consequently,  $\hat{f}_n$  is eventually regular and efficient if the common shape  $f^*$  is odd or even. But the fluctuations  $\sqrt{n}(T_n - \nu_n(\mathbb{P}_{\alpha_n(h), f_n(h)}^{(n)}))$  do not converge weakly under  $\mathbb{P}_{\alpha_n(h), f_n(h)}^{(n)}$  to a tight limit in  $\mathbb{B}$  for each  $\{\alpha_n(h), f_n(h)\}$  [e.g., take  $h = (0, 0)$ ]. Thus, even if  $f^*$  is odd or even,  $\hat{f}_n$  is not efficient.

**REMARK 3.2.** The model where the function  $f^*$  is assumed to be odd or even has been studied by Dalalyan, Golubev and Tsybakov [1]. In this model, the identifiability constraint “ $\theta_1 = 0$ ” is not necessary: The shift parameters are defined from the symmetric point 0. Thus the estimator of  $\theta_1^*, \dots, \theta_j^*$  would be asymptotically independent. Moreover the estimation method would be adaptive.

**4. The proofs.**

4.1. *Proof of Theorem 2.1.*

REMARK 4.1. Let us introduce some notation. First the deterministic part of  $\hat{c}_l$  (2.5) is equal to

$$\begin{aligned}
 & \frac{1}{nJ} \sum_{j=1}^J \sum_{i=1}^n a_j a_j^* f^*(t_i - \theta_j^*) e^{il(t_i - \theta_j^*)} \\
 &= \sum_{p \in \mathbb{Z}} c_p(f^*) \varphi_n\left(\frac{l-p}{n}\right) \phi(l\theta - p\theta^*, a) \\
 (4.1) \quad &= c_l(f^*) \phi(l\theta - l\theta^*, a) + g_n^l(\beta)
 \end{aligned}$$

where  $g_n^l(\beta) = \sum_{\substack{|p| \geq m \\ p-l \in n\mathbb{Z}}} c_p(f^*) \phi(l\theta - p\theta^*, a)$  and

$$\phi(\theta, a) = \sum_{j=1}^J a_j a_j^* e^{i\theta_j} / J.$$

Since assumption (2.3) holds, the term  $g_n^l$  is bounded by

$$(4.2) \quad |g_n^l(\beta)| \leq \sum_{2|p| \geq n} |c_p(f^*)|.$$

For  $j = 1, \dots, J$  and  $|l| \leq m_n$ , let us denote the variable  $\xi_{j,l}$  as  $w_{j,l} = \xi_{j,l} / \sqrt{n}$ . Then the variables  $\xi_{j,l}$  are independent standard complex Gaussian variables from Remark 2.1. Thus the stochastic part of  $\hat{c}_l$  is equal to

$$(4.3) \quad \frac{\sigma^*}{\sqrt{n}} \xi_l(\beta) = \frac{\sigma^*}{J\sqrt{n}} \sum_{j=1}^J a_j e^{il\theta_j} \xi_{j,l} \quad \text{with } |\xi_l(\beta)| \leq \frac{\sigma^*}{J\sqrt{n}} \sum_{j=1}^J |\xi_{j,l}|.$$

LEMMA 4.1 (The uniform convergence in probability). *Under the assumptions of Theorem 2.1, we have*

$$\sup_{\beta \in \mathcal{A}_0} |M_n(\beta) - M(\beta) - \sigma^{*2}| = o_{P_\beta^*}(1),$$

where  $M(\beta) = M^1(\beta) + M^2(\beta)$ ,

$$M^1(\beta) = \sum_{l \in \mathbb{Z}^*} |c_l(f)|^2 (1 - |\phi(l\theta - l\theta^*, a)|^2) \quad \text{and} \quad M^2(\beta) = \frac{1}{J} \sum_{j=1}^J (v_j^* - v_j)^2.$$

PROOF. The contrast process may be rewritten as the sum of three terms:

$$M_n(\beta) = D_n(\beta) + \sigma^* L_n(\beta) + \sigma^{*2} Q_n(\beta).$$

The term  $D_n(\beta) = D_n^1(\beta) - D_n^2(\beta)$  is the deterministic part where

$$D_n^1(\beta) = \frac{1}{Jn} \sum_{j=1}^J \left\{ \sum_{i=1}^n a_j^* f^*(t_i - \theta_j^*) + v_j^* - v_j \right\},$$

$$D_n^2(\beta) = \sum_{1 \leq |l| \leq m_n} \left| \sum_{p \in \mathbb{Z}} c_p(f^*) \varphi_n \left( \frac{l-p}{n} \right) \phi(l\theta - p\theta^*, a) \right|^2.$$

The term  $L_n(\beta) = L_n^1(\beta) - L_n^2(\beta)$  is the linear part with noise, where

$$L_n^1(\beta) = \frac{2}{nJ} \sum_{j=1}^J \sum_{i=1}^n (a_j^* f^*(t_i - \theta_j^*) + v_j^* - v_j) \sigma^* \varepsilon_{i,j},$$

$$L_n^2(\beta) = \frac{2}{\sqrt{n}} \sum_{1 \leq |l| \leq m_n} \Re \left\{ \sum_{p \in \mathbb{Z}} c_p(f^*) \varphi_n \left( \frac{l-p}{n} \right) \phi(l\theta - p\theta^*, a) \overline{\xi_l(\beta)} \right\}.$$

The term  $Q_n(\beta) = Q_n^1(\beta) - Q_n^2(\beta)$  is the quadratic part with noise:

$$Q_n^1(\beta) = \frac{1}{nJ} \sum_{j=1}^J \sum_{i=1}^n \varepsilon_{i,j}^2 \quad \text{and} \quad Q_n^2(\beta) = \frac{1}{n} \sum_{1 \leq |l| < m_n} |\xi_l(\beta)|^2.$$

From the weak law of large numbers,  $Q_n^1$  does not depend on  $\beta$  and converges in probability to 1. Furthermore,  $Q_n^2$  is bounded by

$$0 \leq Q_n^2(\beta) \leq Q_n^B \quad \text{where} \quad nJ Q_n^B = \sum_{|l| < m_n} \sum_{j=1}^J |\xi_{j,l}|^2.$$

Then assumption (2.7) induces that  $\sup_{\beta \in \mathcal{A}_0} |Q_n(\beta) - 1|$  converges to 0 in probability.

Using the fact that  $f^*$  is continuous and that  $|v_j| \leq v_{\max}$ , there exists a constant  $c > 0$  such that for all  $\beta \in \mathcal{A}_0$  we have

$$|L_n^1(\beta)| \leq c L_n^{1B} \quad \text{where} \quad L_n^{1B} = \frac{1}{nJ} \left| \sum_{j=1}^J \sum_{i=1}^n \varepsilon_{i,j} \right|.$$

Then we deduce that  $L_n^1$  converges uniformly in probability to 0. Concerning the term  $L_n^2$ , it may be written as the sum of two variables  $L_n^{21}$  and  $L_n^{22}$ :

$$\sqrt{n} L_n^{21}(\beta) = 2\Re \left\{ \sum_{1 \leq |l| \leq m_n} c_l(f^*) \phi(l\theta - l\theta^*, a) \overline{\xi_l(\beta)} \right\},$$

$$\sqrt{n} L_n^{22}(\beta) = 2\Re \left\{ \sum_{1 \leq |l| \leq m_n} g_n^l(\beta) \overline{\xi_l(\beta)} \right\}.$$

Due to assumption (2.7),  $\sqrt{n}L_n^{21}(\cdot)$  is bounded by the following variable, which is tight:

$$2\frac{\sigma^*}{J} \sum_{1 \leq |l| \leq m_n} |c_l(f^*)| \sum_{j=1}^J |\xi_{j,l}|.$$

Thus,  $L_n^{21}$  converges uniformly in probability to 0. Similarly,  $L_n^{22}$  is bounded by

$$L_n^{2B} = \frac{1}{\sqrt{n}} \left( \sum_{|2p| > n} |c_p(f^*)| \right) \sum_{|l| \leq m_n} \sum_{j=1}^J |\xi_{j,l}|.$$

Consequently, from assumption (2.7),  $L_n^{22}$  converges uniformly in probability to 0. Therefore,  $L_n$  converges uniformly in probability to 0.

It remains to prove that  $D_n$  converges uniformly to  $M$ . First it is easy to prove that  $D_n^1$  converges to  $D^1$  and  $D_n^2$  converges to  $D^2$ , where

$$D^1(\beta) = \frac{1}{J} \sum_{j=1}^J \int_0^{2\pi} (f_j^*(t) - \nu_j)^2 \frac{dt}{2\pi} \quad \text{and}$$

$$D^2(\beta) = \sum_{l \in \mathbb{Z}^*} |c_l(f^*)\phi(l\theta - l\theta^*, a)|^2.$$

Consequently,  $D_n$  pointwise converges to  $M = D^1 - D^2$ . We prove now that the convergence is uniform. For all  $\beta \in \mathcal{A}_0$ , we have

$$|D_n^1 - D^1|(\beta) \leq \frac{1}{J} \sum_{j=1}^J \left\{ \left| \int_0^{2\pi} f_j^*(t)^2 \frac{dt}{2\pi} - \frac{1}{n} \sum_{i=1}^n f^*(t_i)^2 \right| + 2\nu_{\max} \left| c_0(f_j^*) - \frac{1}{n} \sum_{i=1}^n f^*(t_i) \right| \right\},$$

$$|D_n^2 - D^2|(\beta) \leq \sum_{|l| > m_n} |c_l(f^*)|^2 + |D_n^{2B}(\beta)|$$

where  $D_n^{2B} = 2 \sum_{1 \leq |l| < m} \Re\{c_l(f^*)\phi(l\theta - l\theta^*, a)\overline{g_n^l(\beta)}\} + \sum_{1 \leq |l| < m} |g_n^l(\beta)|^2$ .

Using the Cauchy–Schwarz inequality and inequality (4.2), we have that

$$|D_n^{2B}(\beta)| \leq 2 \sum_{|l| < m} |c_l(f^*)| \sum_{|p| > m_n} |c_p(f^*)| + 2m_n \left| \sum_{|p| > m_n} |c_p(f^*)| \right|^2.$$

The assumption (2.7) ensures the uniform convergence of  $D_n^{2B}$ . Consequently, since  $f^*$  is continuous, we deduce the uniform convergence of  $D_n^1$  and  $D_n^2$ .  $\square$

LEMMA 4.2 (Uniqueness of minimum). *M has a unique minimum reached in point  $\beta = \beta^*$ .*

PROOF. First,  $M^2, M^1$  are nonnegative functions and we have that  $M(\beta^*) = 0$ . Consequently, the minimum of  $M$  is reached in  $\beta = (\theta, a, \nu) \in \mathcal{A}_0$  if and only if  $M^1(\beta) = M^2(\beta) = 0$ .

But if  $M^2$  is equal to 0, this implies that  $\nu = \nu^*$ .

Furthermore, using the Cauchy–Schwarz inequality, we have for all  $l \in \mathbb{Z}^*$  that  $|\phi(l\theta, a)| \leq 1$ . Since there exist  $l \in \mathbb{Z}^*$  such that  $c_l(f^*) \neq 0$  ( $f^*$  is not constant),  $M^1$  is equal to 0 if and only if the vectors  $(a_j^*)_{j=1, \dots, J}$  and  $(a_j e^{il(\theta_j - \theta_j^*)})_{j=1, \dots, J}$  are proportional for such  $l$ . From the identifiability constraints on the model, we deduce that

$$a = a^* \text{ and } \forall l \in \mathbb{Z} \text{ such } |c_l(f)| \neq 0 \quad l(\theta^* - \theta) \equiv 0 \pmod{2\pi}.$$

Thus it suffices that  $c_l(f) \neq 0$ , or there exist two relatively prime integers  $l, k$  such that  $c_l(f^*) \neq 0, c_k(f^*) \neq 0$  in order that  $\theta = \theta^*$ . In other words,  $2\pi$  is the minimal period of the function  $f^*$ . In conclusion,  $M^1(\beta)$  is equal to zero if and only if  $a = a^*$  and  $\theta = \theta^*$ .  $\square$

4.2. *Proof of Proposition 3.1.* The proof is divided in two parts. First, we prove that  $\langle \cdot, \cdot \rangle$  is an inner product. Next, we have to choose suitable points  $(\alpha_n(h), f_n(h))$  in order to establish the LAN property.

$\langle \cdot, \cdot \rangle$  is an inner product in  $\mathcal{H}$ . The form  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is bilinear, symmetric and positive. In order to be an inner product, the form  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  has to be definite. In other words, if  $h \in \mathcal{H}$  is such that  $\|h\|_{\mathcal{H}} = 0$ , we want to prove that  $h = 0$ . Let  $h$  be such a vector; then we have that  $h_{\sigma} = 0$  and for all  $j = 2, \dots, J$ ,

$$(4.4) \quad \|a_j^* h_f + h_{a,j} f^* - h_{\theta,j} a_j^* \partial f^* + h_{\nu,j}\|_{\mathbb{L}^2} = 0 \quad \text{and} \\ \left\| a_1^* h_f - \frac{\rho}{a_1^*} f^* + h_{\nu,1} \right\|_{\mathbb{L}^2} = 0,$$

where  $\rho = \sum_{k=2}^J h_{a,k} a_k^*$ . Since the functions  $h_f, f^*$  and  $\partial f^*$  are orthogonal to 1 in  $\mathbb{L}^2[0, 2\pi]$ , we deduce that  $h_{\nu,j} = 0$  for all  $j$ . Moreover, the functions  $h_f$  and  $f^*$  are continuous and the equation (4.4) implies that  $a_1^{*2} h_f = \rho f^*$  and that for all  $j = 2, \dots, J$  ( $f^*$  and  $\partial f^*$  are orthogonal),

$$\left\| \left( \frac{a_j^* \rho}{a_1^{*2}} + h_{a,j} \right) f^* \right\|_{\mathbb{L}^2} = 0 \quad \text{and} \quad \|h_{\theta,j} a_j^* \partial f^*\|_{\mathbb{L}^2} = 0.$$

Since  $f^*$  is not constant, we deduce that for all  $j = 2, \dots, J$  that  $h_{\theta,j} = 0$  and  $a_j^* \rho / a_1^{*2} + h_{a,j} = 0$ . Consequently,  $\rho$  verifies the equation  $\rho \frac{J - a_1^{*2}}{a_1^{*2}} + \rho = 0$ . Then  $\rho$  is equal to zero and  $h = 0$ .

*The LAN property.* Let  $h$  be in  $\mathcal{H}$ . In order to satisfy the identifiability constraints of the model, we choose the sequences  $(\alpha_n(h), f_n(h))$  [with  $\alpha_n(h) = ((\theta_n^{(j)}(h))_{1 \leq j \leq J}, (a_n^{(j)}(h))_{1 \leq j \leq J}, (v_n^{(j)}(h))_{1 \leq j \leq J}, \sigma_n(h))$ ] such that

$$\theta_n^{(j)}(h) = \theta_j^* + \frac{1}{\sqrt{n}}h_{\theta,j} \quad \text{and} \quad a_n^{(j)}(h) = a_j^* + \frac{1}{\sqrt{n}}h_{a,j} \quad \forall j = 2, \dots, J,$$

$$\theta_n^{(1)}(h) = 0, \quad a_n^{(1)}(h) = \sqrt{J - \sum_{j=2}^J a_n^{(j)}(h)^2} \quad \text{and} \quad \sigma_n(h) = \sigma^* + \frac{h_\sigma}{\sqrt{n}},$$

$$f_n(h) = f_n = f^* + \frac{1}{\sqrt{n}}h_f \quad \text{and} \quad v_n^{(j)}(h) = v_j^* + \frac{1}{\sqrt{n}}h_{v,j} \quad \forall j = 1, \dots, J.$$

Using the uniform continuity of  $\partial f^*$  and  $h_f$ , we uniformly establish for  $i = 1, \dots, n$  that

$$f_n(t_i - \theta_n^{(j)}(h)) - f_n(t_i - \theta_j^*) = \frac{h_{\theta,j}}{\sqrt{n}}\partial f^*(t_i - \theta_j^*) + o(1/\sqrt{n}) \quad \forall j = 1, \dots, J,$$

$$(a_n^{(1)}(h) - a_1^*)f_n^*(t_i) = -\frac{\sum_{j=2}^J h_{a,j}a_j^*}{a_1^*\sqrt{n}}f^*(t_i) + o(1/\sqrt{n}),$$

$$\log\left(1 + \frac{h_\sigma/\sigma^*}{\sqrt{n}}\right) = \frac{h_\sigma/\sigma^*}{\sqrt{n}} - \frac{(h_\sigma/\sigma^*)^2}{n} + o(n^{-1}).$$

Then, with the notation of the proposition, we may deduce that

$$\Delta_n(\alpha_n(h), f_n(h)) = \Delta_n(h) - \frac{1}{2n} \sum_{i=1}^n \sum_{j=1}^J A_{i,j}^n(h)^2 - \frac{J}{2} \frac{\sigma^2}{\sigma^{*2}} + o_{\mathbb{P}}(1).$$

$\sum_{i=1}^n \sum_{j=1}^J A_{i,j}^n(h)^2/n$  is a Riemann sum which converges to  $\|h\|_{\mathcal{H}}^2$ . Moreover, from the Lindeberg–Feller central limit theorem (see [12], Chapter 2)  $\Delta_n(h)$  converges in distribution to  $\mathcal{N}(0, \|h\|_{\mathcal{H}}^2)$ .

### 4.3. The efficient estimation of $\theta^*$ , $a^*$ and $v^*$ .

LEMMA 4.3 (The derivative of  $v$ ). *The representant of the  $v_n$ 's derivative is  $\dot{v} = ((\dot{v}_j^\theta)_{2 \leq j \leq J}, (\dot{v}_j^a)_{2 \leq j \leq J}, (\dot{v}_j^v)_{1 \leq j \leq J}) \in \overline{\mathcal{H}}^{3J-2}$ , where*

$$\dot{v}_j^\theta = \frac{\sigma^{*2}}{\|\partial f^*\|_{\mathbb{L}^2}} \left( \dot{\theta}^j, 0, 0, 0, \frac{1}{a_1^{*2}} \partial f^* \right) \quad \text{for } j = 2, \dots, J,$$

$$\dot{v}_j^a = \frac{\sigma^{*2}}{\|f^*\|_{\mathbb{L}^2}^2} (0, \dot{a}^j, 0, 0, 0) \quad \text{for } j = 2, \dots, J,$$

$$\dot{v}_j^v = (0, 0, e_j, 0, 0) \quad \text{for } j = 1, \dots, J,$$

where the vector  $e_j$  is the  $j$ th vector of canonical basis of  $\mathbb{R}^J$ , and the vectors  $\dot{\theta}^j = (\dot{\theta}_k^j)_{k=2, \dots, J}$  and  $\dot{a}^j = (\dot{a}_k^j)_{k=2, \dots, J}$  are defined as

$$\begin{aligned} \dot{\theta}_k^j &= \begin{cases} 1/a_1^{*2}, & \text{if } k \neq j, \\ 1/a_1^{*2} + 1/a_j^{*2}, & \text{if } k = j, \end{cases} \\ \dot{a}_k^j &= \begin{cases} -a_2^*/J, & \text{if } k \neq j, \\ 1 - a_k^{*2}/J, & \text{if } k = j. \end{cases} \end{aligned}$$

PROOF. For  $h \in \mathcal{H}$  and  $h' \in \mathcal{H}$ , we may rewrite the inner product of the tangent space under the following form:

$$\begin{aligned} \sigma^{*2} \langle h, h' \rangle &= Jh_\sigma h'_\sigma + \langle h_f, Jh'_f - \lambda \partial f^* \rangle \\ &+ \sum_{k=2}^J h_{\theta,k} \langle \partial f^*, -a_k^* h'_f + h_{\theta,k} a_k^* \partial f^* \rangle \\ &+ \sum_{k=2}^J h_{a,k} \left\langle f^*, h'_{a,k} f^* + \frac{a_k^*}{a_1^{*2}} \rho f^* \right\rangle + \sum_{k=2}^J h_{\nu,k} h'_{\nu,k}, \end{aligned}$$

where  $\lambda = \sum_{k=2}^J h'_{\theta,k} a_k^*$  and  $\rho = \sum_{k=2}^J h'_{a,k} a_k^*$ . Let  $k \in \{2, \dots, J\}$  be a fixed integer; we want to find  $h'$  such that for all  $h \in \mathcal{H}$ ,  $\langle h, h' \rangle = h_{\theta,k}$ . Consequently, such  $h'$  verifies these equations:

$$(4.5) \quad h_f = \lambda \partial f^*/J, \quad h'_\sigma = 0 \quad \text{and} \quad h'_{\nu,j} = 0, \quad \forall j = 1, \dots, J,$$

$$(4.6) \quad (h'_{a,j} + \rho a_j^*/a_1^{*2}) \|f^*\|^2 = 0, \quad \forall j = 2, \dots, J,$$

$$(4.7) \quad (-\lambda/J + h_{\theta,j}) \|\partial f^*\|^2 = \begin{cases} \sigma^{*2}, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

Combining equations (4.6) and (4.7), we have that

$$\lambda a_1^{*2} \|\partial f^*\|^2/J = \sigma^{*2} \quad \text{and} \quad \rho J \|f^*\|^2/a_1^{*2} = 0.$$

Thus we deduce that  $\rho = 0$  and  $\lambda = J\sigma^{*2}/(a_1^{*2} \|\partial f^*\|^2)$ . Consequently,  $h'$  is equal to  $\dot{v}_k^\theta$ .

We likewise solve the equation  $\langle h, h' \rangle = h_{a,k}$ . Finally, we have that  $\|f^*\|^2 \rho = \sigma^{*2} a_1^{*2} a_k^*/J$  and  $\lambda = 0$ . Hence the solution is  $h' = \dot{v}_k^a$ .  $\square$

PROPOSITION 4.1. Under the assumptions and notation of Theorem 3.1, we have that

$$\sqrt{n} \nabla M_n(\beta^*) = -\frac{2\sigma^*}{J} G_n + o_{\mathbb{P}}(1) \quad \text{where} \quad {}^t G_n = (G_n^\theta, G_n^a, G_n^\nu).$$

$G_n$  is a Gaussian vector which converges in distribution to  $\mathcal{N}_{3J-2}(0, H)$  and is defined as

$$G_n^\theta = \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{bmatrix} a_1^* A^2 \vdots \\ \vdots \\ -D + \frac{1}{J} A^2 {}^t A \end{bmatrix} \partial F^*(t_i) \varepsilon_{i,\cdot},$$

$$G_n^a = \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{bmatrix} -1 \vdots \\ \vdots \\ a_1^* A \vdots \\ \vdots \\ I_{J-1} \end{bmatrix} F^*(t_i) \varepsilon_{i,\cdot},$$

$$G_n^v = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_{i,\cdot}, \quad \text{and} \quad {}^t \varepsilon_{i,\cdot} = {}^t (\varepsilon_{i,1}, \dots, \varepsilon_{i,J}) \quad \text{for } i = 1, \dots, n.$$

PROOF. In order to prove that proposition, we proceed in two steps. First, using the notation of Proposition 4.1, we show that

$$\sqrt{n} \nabla M_n(\beta^*) = \sqrt{n} (\nabla L_n^1(\beta^*) - L_n^{21}(\beta^*)) = -\frac{2\sigma^*}{J} G_n + o_{\mathbb{P}}(1).$$

At the end, we prove that  $(G_n^\theta, G_n^a, G_n^v)$  is a Gaussian vector which converges to  $\mathcal{N}_{3J-2}(0, H)$ .

First, we study singly the gradient of  $G_n, L_n$  and  $Q_n$ . Let  $k \in \{2, \dots, J\}$  be fixed. The partial derivative with respect to the variable  $\theta_k$  is

$$\frac{\partial Q_n}{\partial \theta_k}(\beta^*) = -\frac{2}{n} \sum_{1 \leq |l| < m_n} \Re \left( \frac{i l a_k^* e^{i l \theta_k^*}}{J} \xi_{k,l} \overline{\xi_l(\beta^*)} \right).$$

It is bounded by

$$\left| \sqrt{n} \frac{\partial Q_n}{\partial \theta_k}(\beta^*) \right| \leq \frac{2}{J^2 \sqrt{n}} \sum_{1 \leq |l| < m_n} |l| |\xi_{k,l}| \sum_{j=1}^J |\xi_{j,l}|.$$

Thus  $\sqrt{n} \frac{\partial Q_n}{\partial \theta_j}(\beta^*)$  converges in probability to 0 if  $m_n^4/n = o(1)$ . Similarly, the partial derivative with respect to the variable  $a_k$  converges in probability to 0, too. Consequently,  $\sqrt{n} \nabla Q_n(\beta^*)$  converges to 0 in probability.

Concerning the deterministic part, the partial derivative with respect to  $\theta_k$  is

$$\frac{\partial D_n}{\partial \theta_k}(\beta^*) = -\frac{2}{J} \sum_{1 \leq |l| < m_n} \Re \left\{ i l \sum_{p \in \mathbb{Z}} c_p(f^*) \varphi_n \left( \frac{l-p}{n} \right) a_k^{*2} e^{i(l-p)\theta_k^*} \right. \\ \left. \times \left( \sum_{p \in \mathbb{Z}} \overline{c_p(f^*)} \varphi_n \left( \frac{p-l}{n} \right) \varphi((p-l)\theta^*, a^*) \right) \right\}.$$

Using the inequality (4.2), it is bounded by

$$\begin{aligned} \sqrt{n} \frac{\partial D_n}{\partial \theta_k}(\beta^*) &= \frac{2a_k^{*2}}{J} \sqrt{n} \left\{ 2 \sum_{|l| \leq m_n} |lc_l(f^*)| \sum_{2|p| \geq n} |c_p(f^*)| \right. \\ &\quad \left. + \sum_{|l| \leq m_n} |l| \left( \sum_{2|p| \geq n} |c_p(f^*)| \right)^2 \right\}. \end{aligned}$$

Consequently, we deduce from the assumptions of the theorem that  $\sqrt{n} \frac{\partial D_n}{\partial \theta_k}(\beta^*)$  converges in probability to 0. In like manner,  $\sqrt{n} \frac{\partial D_n}{\partial a_k}(\beta^*)$  converges in probability to 0, too. For the partial derivative with respect to  $v_k$ , we have

$$\sqrt{n} \frac{\partial D_n}{\partial v_k}(\beta^*) = -\frac{2\sqrt{n}}{Jn} \left( \sum_{i=1}^n f_k^*(t_i) - v_k^* \right) = -\frac{2a_k^*}{J} \sqrt{n} \sum_{p \in \mathbb{Z}^*} c_p(f^*) e^{-ip\theta_k^*}.$$

Thus from assumption (3.2), we deduce that  $\sqrt{n} \partial D_n / \partial v_k$  in  $\beta^*$  converges to 0. Finally,  $\sqrt{n} \nabla D_n(\beta^*)$  converges to 0 in probability.

Therefore, we have that  $\sqrt{n} \nabla M_n(\beta^*) = \sqrt{n} \nabla L_n(\beta^*) + o_{\mathbb{P}}(1)$ . With the notation of Lemma 4.1, we have

$$\sqrt{n} \frac{\partial L_n^{22}}{\partial \theta_k}(\beta^*) = \frac{2}{J} \sum_{1 \leq |l| < m_n} \Re \left\{ i l a_k^* (-e^{il\theta_k^*} \xi_{k,l} + a_k^* \bar{\xi}_l(\beta)) \sum_{\substack{2|p| \geq n \\ p-l \in n\mathbb{Z}}} c_p(f^*) \right\}.$$

The centered Gaussian variable  $\sqrt{n} \frac{L_n^{22}}{d\theta_k}(\beta^*)$  has a variance bounded by

$$\left( \sum_{2|p| > n} |c_p(f^*)| \right)^2 2m_n^3.$$

From assumption (3.2), we conclude that  $\sqrt{n} \frac{\partial L_n^{22}}{d\theta_k}(\beta^*)$  converges to 0 in probability. In like manner,  $\sqrt{n} \frac{\partial L_n^{22}}{\partial a_k}(\beta^*)$  converges in probability to 0, too. Thus we have that  $\sqrt{n} \nabla M_n(\beta^*) = \sqrt{n} \nabla L_n^1(\beta^*) - \sqrt{n} \nabla L_n^{21}(\beta^*) + o_{\mathbb{P}}(1)$ . After straightforward computations, we obtain

$$\begin{aligned} \sqrt{n} \frac{\partial M_n}{\partial \theta_k}(\beta^*) &= -\frac{2\sigma^*}{J} \sum_{1 \leq |l| \leq m_n} \Re \{ lc_l(f^*) (a_k^* \bar{\xi}_l(\beta^*) - a_k^* e^{-il\theta_k^*} \bar{\xi}_{k,l}) \}, \\ \sqrt{n} \frac{\partial M_n}{\partial a_k}(\beta^*) &= -\frac{2\sigma^*}{J} \sum_{1 \leq |l| \leq m_n} \Re \left\{ c_l(f^*) \left( e^{-il\theta_k^*} \bar{\xi}_{k,l} - \frac{a_k^*}{a_1^*} \bar{\xi}_{1,l} \right) \right\}, \\ \sqrt{n} \frac{\partial M_n}{\partial v_k}(\beta^*) &= -\frac{2\sigma^*}{J\sqrt{n}} \sum_{i=1}^n \varepsilon_{i,k} = -\frac{2\sigma^*}{J} \bar{\xi}_{k,0}. \end{aligned}$$

We can now define  $(G_n^\theta, G_n^a, G_n^v)$  as

$$G_n^\theta = \sum_{1 \leq |l| \leq m_n} \Re \left\{ ilc_l(f^*) \begin{pmatrix} a_1^* \mathbb{A}^2 \vdots \\ -D + \frac{1}{J} \mathbb{A}^2 \text{ }^t \mathbb{A} \end{pmatrix} X_l^* \right\} + o_{\mathbb{P}}(1),$$

$$G_n^a = \sum_{1 \leq |l| \leq m_n} \Re \left\{ c_l(f^*) \begin{pmatrix} -1 \mathbb{A} \vdots \\ a_1^* \vdots \\ I_{J-1} \end{pmatrix} X_l^* \right\} + o_{\mathbb{P}}(1) \quad \text{and}$$

$$G_n^v = \Re \{ X_0^* \} + o_{\mathbb{P}}(1),$$

where  $X_l^*$  denote the independent identically distributed complex Gaussian vectors defined as

$${}^t X_l^* = (e^{-il\theta_k^*} \overline{\xi_{1,l}}, \dots, e^{-il\theta_k^*} \overline{\xi_{J,l}}).$$

Since  $G_n^\theta$  and  $G_n^a$  do not depend on  $X_0^*$ ,  $G_n^v$  is independent of  $G_n^\theta$  and  $G_n^a$ . Moreover, its variance matrix is equal to the identity matrix of  $\mathbb{R}^J$ . Furthermore, the imaginary part and the real part of  $c_l(f^*)X_l^*$  are independent. Consequently,  $G_n^\theta$  and  $G_n^a$  are asymptotically independent with covariance matrix  $\|\partial f^*\|^2 \times (D^2 - A^2 \text{ }^t A^2 / J)$  and  $\|f^*\|^2 (I_{J-1} - A \text{ }^t A / a_1^2)$ , respectively.

By the definition of  $(\xi_{k,l})$  (Remark 4.1), we deduce from assumption (3.1) that for a fixed  $k = 1, \dots, J$ ,

$$\begin{aligned} \Re \left\{ \sum_{|l| \leq m_n} ilc_l(f^*) \bar{\xi}_{k,l} e^{-il\theta_k^*} \right\} &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_{i,k} \Re \left\{ \sum_{|l| \leq m_n} ilc_l(f^*) e^{il(t_i - \theta_k^*)} \right\} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_{i,k} \partial f^*(t_i - \theta_k^*) + o_{\mathbb{P}}(1). \end{aligned}$$

Thus,  $(G_n^\theta, G_n^a, G_n^v)$  are equal to the expression defined in the proposition. □

PROPOSITION 4.2. *Under the assumptions and notation of Theorem 3.1, we have*

$$\nabla^2 M_n(\bar{\beta}_n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_{\beta^*}} -\frac{2}{J^2} H.$$

PROOF. The matrix  $-2H/J^2$  is the value of the Hessian matrix of  $M$  in point  $\beta^*$ . We study locally the Hessian matrix of  $M_n$ . Consequently, we may assume that the sequences  $(\bar{\beta}_n)$  are in the following set:

$$A_n^{loc} = \{(\theta, a, v) \in \mathcal{A}_0, a_1 > r \text{ and } \|\beta - \beta^*\| \leq \|\hat{\beta}_n - \beta^*\|\},$$

where  $a_1^* > r > 0$ . Notice that for  $\varepsilon > 0$ , we have

$$\begin{aligned} & \mathbb{P}\left(\sup_{\beta \in A_n^{loc}} \|\nabla^2 M_n(\beta) - \nabla^2 M(\beta^*)\| > 2\varepsilon\right) \\ & \leq \mathbb{P}\left(\sup_{\beta \in A_n^{loc}} \|\nabla^2 M_n(\beta) - \nabla^2 M(\beta)\| > \varepsilon\right) \\ & \quad + \mathbb{P}\left(\sup_{\beta \in A_n^{loc}} \|\nabla^2 M(\beta) - \nabla^2 M(\beta^*)\| > \varepsilon\right). \end{aligned}$$

As in Lemma 4.1, assumptions (3.1) and (3.2) assure the uniform convergence in probability of  $\nabla^2 M_n$  to the Hessian matrix of  $M$  on  $A_n^{loc}$ . Thus, the first term of inequality converges to 0 with  $n$ .

Since  $\nabla^2 M$  is continuous in  $\beta^*$ , there exists  $\delta > 0$  such that

$$\nabla^2 M(B(\beta^*, \delta)) \subseteq B(\nabla^2 M(\beta^*), \varepsilon).$$

Consequently, we have the following inclusion of event:

$$\left(\sup_{\beta \in A_n^{loc}} \|\nabla^2 M(\beta) - \nabla^2 M(\beta^*)\| > \varepsilon\right) \subseteq (\|\hat{\beta}_n - \beta^*\| > \delta).$$

Thus, from Theorem 2.1, the second term of the inequality converges to 0, too.  $\square$

4.3.1. *The estimation of the common shape.*

REMARK 4.2. If the assumptions of Theorem 3.1 hold, we obtain using the Cauchy–Schwarz inequality that

$$\sum_{|l|>n} |c_l(f^*)| \leq \left\{ \sum_{|l|>n} |c_l(f^*)|^2 \right\}^{1/2} \left\{ \sum_{|l|>n} |c_l(f^*)|^2 / l^2 \right\}^{1/2} = o(1/n).$$

Similarly, if  $f^*$  is  $k$  times differentiable and  $f^{(k)}$  is squared integrable, we have

$$\sum_{|l|>n} |c_l(f^*)| = o(n^{-k+1/2}) \quad \text{and} \quad \sum_{|l|>n} |c_l(f^*)|^2 = o(n^{-2k}).$$

PROOF OF COROLLARY 3.1. Using the notation of Lemma 4.1, we have for all  $t \in \mathbb{R}$ ,

$$\begin{aligned} & f^*(t) - f_{\beta^*}(t) \\ & = \sum_{|l|>m_n} c_l(f^*)e^{ilt} + \sum_{1 \leq |l| \leq m_n} e^{ilt} \sum_{|2p|>n, p-l \in n\mathbb{Z}} c_p(f^*)\phi(l\hat{\theta} - p\theta^*, \hat{a}) \\ (4.8) \quad & + \sum_{1 \leq |l| \leq m_n} c_l(f^*)\{\phi(l(\hat{\theta} - \theta^*), \hat{a}) - 1\}e^{ilt} \end{aligned}$$

$$(4.9) \quad + \sigma^* \sum_{1 \leq |l| \leq m_n} \xi_l(\hat{\beta}) \frac{e^{ilt}}{\sqrt{n}}.$$

Since Theorem 3.1 holds and using the delta method, we have for all  $j = 1, \dots, J$ ,

$$e^{il(\hat{\theta}_j - \theta_j^*)} - 1 = il(\hat{\theta}_j - \theta_j^*) + o_{\mathbb{P}}(l/\sqrt{n}).$$

Moreover, we have

$$|\phi(l(\hat{\theta} - \theta^*), \hat{a}) - 1| \leq \frac{1}{J} \sum_{j=1}^J a_j^* |\hat{a}_j - a_j^*| + \frac{1}{J} \sum_{j=1}^J a_j^{*2} |e^{il(\hat{\theta}_j - \theta_j^*)} - 1|.$$

Then, we deduce that

$$\sup_{t \in \mathbb{R}} |(4.8)| = \mathcal{O}_{\mathbb{P}}(1/\sqrt{n}) \quad \text{and} \quad \mathbb{E} \|(4.8)\|_{\mathbb{L}^2}^2 = \mathcal{O}(1/n).$$

Using (4.3) and the Cauchy–Schwarz inequality, we have

$$W_n = \left| \sum_{1 \leq |l| \leq m_n} \xi_l(\hat{\beta}) \frac{e^{ilt}}{\sqrt{n}} \right| \leq \frac{1}{\sqrt{Jn}} \sum_{1 \leq |l| \leq m_n} \sum_{j=1}^J |\xi_{j,l}|,$$

$$\int_0^{2\pi} \mathbb{E} W_n^2 \frac{dt}{2\pi} = \frac{1}{nJ} \sum_{1 \leq |l| \leq m_n} \sum_{j=1}^J |\xi_{j,l}|^2.$$

Hence we deduce by the Markov inequality that

$$W_n = \mathcal{O}_{\mathbb{P}}(m_n/\sqrt{n}) \quad \text{and} \quad \int_0^{2\pi} \mathbb{E} W_n^2 \frac{dt}{2\pi} = \mathcal{O}(m_n/n).$$

Then, using Remark 4.1, the corollary results.  $\square$

LEMMA 4.4. *Let  $l$  be in  $\mathbb{Z}^*$ . For a large  $n$ , we have*

$$\begin{aligned} \sqrt{n} \Re(\hat{c}_l(\hat{\beta}_n) - c_l(f^*)) &= \Delta_n(-\Re(lc_l(f^*)\tilde{h}^f) + \Delta_n\left(0, 0, 0, 0, \frac{\cos(l \cdot)}{J}\right) \\ &\quad + o_{\mathbb{P}}(1), \\ \sqrt{n} \Im(\hat{c}_l(\hat{\beta}_n) - c_l(f^*)) &= \Delta_n(\Im(lc_l(f^*)\tilde{h}^f) + \Delta_n\left(0, 0, 0, 0, \frac{-\sin(l \cdot)}{J}\right) \\ &\quad + o_{\mathbb{P}}(1), \end{aligned}$$

where  $\tilde{h}^f = \frac{\sigma^{*2}}{\|\partial f^*\|_{\mathbb{L}^2}} \left( \frac{\mathbb{I}_{J-1}}{a_1^{*2}}, 0, 0, 0, \frac{J-a_1^{*2}}{a_1^{*2}} \partial f^* \right)$ .

PROOF. Let  $l$  be in  $\mathbb{Z}^*$ . For  $n$  large enough (such wise  $|l| \leq m_n$ ), from the continuous mapping theorem [12], Theorem 2.3, and from assumption (3.1) ensures

that  $c_l^n(f^*)$  converges to  $c_l(f^*)$  with a speed  $\sqrt{n}$ , we obtain

$$\begin{aligned}\sqrt{n}c_l(\hat{f}_n - f^*) &= \sqrt{n}(\hat{c}_l(\hat{\beta}_n) - c_l(f^*)), \\ &= c_l(f^*)\sqrt{n}\left(\frac{1}{J}\sum_{j=1}^J \hat{a}_{j,n}a_j^* e^{il(\hat{\theta}_{j,n} - \theta_j^*)} - 1\right) + \xi_l(\theta^*, a^*) + o_{\mathbb{P}}(1).\end{aligned}$$

Since  $\sqrt{n}(\hat{\theta}_n - \theta^*, \hat{a}_n - a^*)$  converges in distribution (Theorem 3.1), we use the delta method ([12], Chapter 3):

$$\sqrt{n}c_l(\hat{f}_n - f^*) = ilc_l(f^*)\sum_{j=2}^J \frac{a_j^{*2}}{J}\sqrt{n}(\hat{\theta}_j - \theta_j^*) + \xi_l(\theta^*, a^*) + o_{\mathbb{P}}(1).$$

Thus from Theorem 3.1 and Lemma 4.3 and due to the linearity of  $\Delta_n(\cdot)$ , we have

$$\begin{aligned}\sqrt{n}c_l(\hat{f}_n - f^*) &= ilc_l(f^*)\sum_{j=2}^J \frac{a_j^{*2}}{J}\Delta_n(\tilde{h}_j^\theta) + \xi_l(\theta^*, a^*) + o_{\mathbb{P}}(1) \\ &= \frac{ilc_l(f^*)\sigma^{*2}}{\|\partial f^*\|_{\mathbb{L}^2}}\Delta_n(\tilde{h}^f) + \xi_l(\theta^*, a^*) + o_{\mathbb{P}}(1).\end{aligned}$$

Using the definition of  $\xi_l$  (see Remark 4.1), we have

$$\Re(\xi_l(\theta^*, a^*)) = \Delta_n\left(0, \frac{\cos(l\cdot)}{J}\right) \quad \text{and} \quad \Im(\xi_l(\theta^*, a^*)) = \Delta_n\left(0, \frac{-\sin(l\cdot)}{J}\right). \quad \square$$

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