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A MULTIVARIATE CENTRAL LIMIT THEOREM FOR RANDOMIZED ORTHOGONAL ARRAY SAMPLING DESIGNS IN COMPUTER EXPERIMENTS

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Let $f : [0, 1)^d \to \mathbb{R}$ be an integrable function. An objective of many computer experiments is to estimate $\int_{[0,1)^d} f(x)\,dx$ by evaluating f at a finite number of points in $[0, 1)^d$. There is a design issue in the choice of these points and a popular choice is via the use of randomized orthogonal arrays. This article proves a multivariate central limit theorem for a class of randomized orthogonal array sampling designs [Owen *Statist. Sinica* **2** (1992a) 439–452] as well as for a class of OA-based Latin hypercubes [Tang *J. Amer. Statist. Assoc.* **81** (1993) 1392–1397].

1. Introduction. Let X be a random vector uniformly distributed on the d-dimensional unit hypercube $[0, 1)^d$ and f be an integrable function from $[0, 1)^d$ to $\mathbb{R}$. An objective of many computer experiments [see, e.g., McKay, Conover and Beckman (1979), Stein (1987), Sacks, Welch, Mitchell and Wynn (1989) and Santner, Williams and Notz (2003)] is to estimate

$$\mu = Ef(X) = \int_{[0,1)^d} f(x)\,dx, \tag{1}$$

using a finite number of function evaluations. It is well known that as the dimension d increases, Monte Carlo methods and (deterministic) equidistribution methods become competitive and ultimately dominant. Indeed Davis and Rabinowitz (1984), Chapter 5.10, consider $d > 15$ to be a high enough dimensionality that sampling or equidistribution methods are indicated.

For definiteness, let n, d, q and t be positive integers such that $t \le d$. An orthogonal array of strength t is a matrix of n rows and d columns with elements taken from the set of symbols $\{0, 1, \ldots, q - 1\}$ such that in any $n \times t$ submatrix, each of the q^t possible rows occurs the same number of times. The class of all such arrays is denoted by $\mathrm{OA}(n, d, q, t)$. Comprehensive accounts of orthogonal arrays can be found in the books by Raghavarao (1971) and Hedayat, Sloane and Stufken (1999).

Owen (1992a, 1994) and Tang (1993) independently proposed the use of randomized orthogonal arrays in computer experiment sampling designs. The main

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attraction of these designs is that they, in contrast to simple random sampling, stratify on all t-variate margins simultaneously. A class of randomized orthogonal array sampling designs proposed by Owen (1992a) is as follows: Let:

(a) $A \in \mathrm{OA}(q^t, d, q, t)$ where $a_{i,j}$ denotes the (i, j)th element of A,

(b) $\pi_1, \dots, \pi_d$ be random permutations of $\{0, \dots, q-1\}$, each uniformly distributed on all the $q!$ possible permutations,

(c) $\{U_{i,j} : i = 1, \dots, q^t, j = 1, \dots, d\}$, be $[0, 1)$ uniform random variables,

(d) and all the $U_{i,j}$'s and π_k's are independent.

We randomize the symbols of A by applying the permutation π_j to the jth column of A, $j = 1, \dots, d$. This gives us another orthogonal array A^* such that its (i, j)th element satisfies $a^*_{i,j} = \pi_j(a_{i,j})$. An orthogonal array-based sample of size q^t (taken from $[0, 1)^d$) is defined to be $\{X_1, \dots, X_{q^t}\}$ where for $i = 1, \dots, q^t$, $X_i = (X_{i,1}, \dots, X_{i,d})'$,

$$X_{i,j} = \frac{a^*_{i,j} + U_{i,j}}{q} \qquad \forall j = 1, \dots, d. \tag{2}$$

For $t \geq 2$, Tang (1993) observed that the above sampling designs may not stratify well on s-variate margins if $s < t$. He suggested modified designs that stratify on t-variate margins as well as 1-variate margins simultaneously. He called these designs OA-based Latin hypercubes. Finally, Owen (1997a, 1997b), in a series of articles, proposed the use of scrambled nets. Given $t \in \mathbb{Z}^+$, the scrambled nets stratify on s-variate margins whenever t/s is a positive integer.

A class of OA-based Latin hypercubes can be constructed as follows: Let $A \in \mathrm{OA}(q^t, d, q, t)$. As before, we randomize its symbols to obtain the orthogonal array A^*. Then for each column of A^*, we replace the q^{t-1} positions with entry k by a random permutation (with each such permutation having an equal probability of being chosen) of $\{kq^{t-1}, kq^{t-1} + 1, \dots, (k+1)q^{t-1} - 1\}$, for all $k = 0, \dots, q - 1$. After the replacement is done for all d columns of A^*, the newly obtained matrix, say A^{**}, satisfies $A^{**} \in \mathrm{OA}(q^t, d, q^t, 1)$.

One version of OA-based Latin hypercubes that was considered by Owen (1997a), page 1906, is of the form $\{Y_1, \dots, Y_{q^t}\}$ where for $i = 1, \dots, q^t$, $Y_i = (Y_{i,1}, \dots, Y_{i,d})'$,

$$Y_{i,j} = \frac{a^{**}_{i,j} + U_{i,j}}{q^t} \qquad \forall j = 1, \dots, d, \tag{3}$$

$\{U_{i,j} : i = 1, \dots, q^t, j = 1, \dots, d\}$ are $U[0, 1)$ random variables independent of one another and all other permutations, and $a^{**}_{i,j}$ denotes the (i, j)th element of A^{**}. The class of OA-based Latin hypercubes proposed by Tang (1993) requires one more level of randomization where the columns of A^{**} are randomized. We denote the resulting matrix by A^{***}. Tang's OA-based Latin hypercubes can be expressed as $\{Y_1^*, \dots, Y_{q^t}^*\}$ where for $i = 1, \dots, q^t$, $Y_i^* = (Y_{i,1}^*, \dots, Y_{i,d}^*)'$,

$$Y_{i,j}^* = \frac{a_{i,j}^{***} + U_{i,j}}{q^t} \qquad \forall j = 1, \dots, d, \tag{4}$$

$\{U_{i,j} : i = 1, \dots, q^t, j = 1, \dots, d\}$ are, as before, $U[0, 1)$ random variables independent of one another and all other permutations, and $a_{i,j}^{***}$ denotes the (i, j)th element of A^{***}. We note that $\{Y_1, \dots, Y_{q^t}\}$ and $\{Y_1^*, \dots, Y_{q^t}^*\}$ are Latin hypercube samples [see, e.g., McKay, Conover and Beckman (1979) and Owen (1992b)].

The estimators for μ in (1) that we are concerned with are

$$\begin{aligned} &\hat{\mu}_{oas} = q^{-t} \sum_{i=1}^{q^t} f(X_i), \\ &\hat{\mu}_{oal} = q^{-t} \sum_{i=1}^{q^t} f(Y_i) \quad \text{and} \quad \hat{\mu}_{oal}^* = q^{-t} \sum_{i=1}^{q^t} f(Y_i^*), \end{aligned} \tag{5}$$

where the X_i's, Y_i's and Y_i^*'s are as in (2), (3) and (4) respectively. It is easily seen that $\hat{\mu}_{oas}$, $\hat{\mu}_{oal}$ and $\hat{\mu}_{oal}^*$ are all unbiased estimators for μ. For simplicity, we write $\sigma_{oas}^2 = \text{Var}(\hat{\mu}_{oas})$, $\sigma_{oal}^2 = \text{Var}(\hat{\mu}_{oal})$ and $\sigma_{oal}^{*2} = \text{Var}(\hat{\mu}_{oal}^*)$.

In this article, we shall assume that $t = 2$. This significantly simplifies the notation as well as the theoretical arguments that follow. Also as Owen (1992a) and Tang (1993) noted, orthogonal arrays of strength $t = 2$ lead to the most economical sample size q^2. This is important in practice especially when q is large. The following theorem is due to Owen (1992a) and Tang (1993):

THEOREM 1. *Let $d \geq 3$, f be a bounded continuous function on $[0, 1)^d$ and $\hat{\mu}_{oas}$, $\hat{\mu}_{oal}^*$ be as in (5) with $A \in \text{OA}(q^2, d, q, 2)$. Then as $q \to \infty$, we have*

$$q^2 \sigma_{oas}^2 = \int_{[0,1)^d} f_{rem}^2(x)\, dx + o(1) \quad \textit{and} \quad q^2 \sigma_{oal}^{*2} = \int_{[0,1)^d} f_{rem}^2(x)\, dx + o(1),$$

where for all $x = (x_1, \dots, x_d)' \in [0, 1)^d$, $1 \leq j \leq d$, $1 \leq k < l \leq d$,

$$\begin{aligned} f_j(x_j) &= \int_{[0,1)^{d-1}} [f(x) - \mu] \prod_{1 \leq k \leq d : k \neq j} dx_k, \\ f_{k,l}(x_k, x_l) &= \int_{[0,1)^{d-2}} [f(x) - \mu - f_k(x_k) - f_l(x_l)] \prod_{1 \leq j \leq d : j \neq k,l} dx_j, \\ f_{rem}(x) &= f(x) - \mu - \sum_{j=1}^{d} f_j(x_j) - \sum_{1 \leq k < l \leq d} f_{k,l}(x_k, x_l). \end{aligned} \tag{6}$$

Theorem 1 implies that: (i) the asymptotic variances of $\hat{\mu}_{oas}$ and $\hat{\mu}^*_{oal}$ are always less than or equal to the asymptotic variance of an analogous estimator based on a simple random sample of the same size, (ii) they are dramatically smaller if the integrand f can be approximated by a sum of bivariate functions, and (iii) $\sigma^2_{oas} \sim \sigma^{*2}_{oal}$ if $\int_{[0,1)^d} f^2_{rem}(x)\,dx > 0$. Tang (1993), page 1395, further showed that $\sigma^{*2}_{oal} \le \sigma^2_{oas}$ if f is additive.

The aim of this article is to study the asymptotic distributions of $\hat{\mu}_{oas}$, $\hat{\mu}_{oal}$ and $\hat{\mu}^*_{oal}$. For instance, such a result will be useful in the construction of confidence intervals for μ.

DEFINITION. A function $f:[0,1)^d \to \mathbb{R}$ is smooth with a Lipschitz continuous mixed partial of order d if there exist finite constants $B \ge 0$ and $\beta \in (0,1]$ such that

$$\sup_{j_1,\dots,j_d \in \{0,\dots,d\}:\, j_1+\cdots+j_d=d} \left| \frac{\partial^d}{\partial x_1^{j_1}\cdots\partial x_d^{j_d}} f(x) - \frac{\partial^d}{\partial x_1^{j_1}\cdots\partial x_d^{j_d}} f(y) \right| \le B\|x-y\|^\beta,$$

$\forall x, y \in [0,1)^d$ where $\|\cdot\|$ is the usual Euclidean norm. We shall now state the main result of this article, the proof of which is deferred to the Appendix.

THEOREM 2. *Suppose $d \ge 3$ and $f:[0,1)^d \to \mathbb{R}$ is smooth with a Lipschitz continuous mixed partial of order d such that $\int_{\mathbb{R}^d} f^2_{rem}(x)\,dx > 0$. Define $W_{oas} = (\hat{\mu}_{oas} - \mu)/\sigma_{oas}$, $W_{oal} = (\hat{\mu}_{oal} - \mu)/\sigma_{oal}$ and $W^*_{oal} = (\hat{\mu}^*_{oal} - \mu)/\sigma^*_{oal}$ with $A \in$* OA$(q^2, d, q, 2)$. *Then W_{oas}, W_{oal} and W^*_{oal} each converges in law to the standard normal distribution as $q \to \infty$.*

The rest of this article is organized as follows: In Section 2 we shall first establish base q expansions for $\{X_1,\dots,X_{q^2}\}$ and $\{Y_1,\dots,Y_{q^2}\}$. The main point here is that the difference between these two base q expansions is of order $O(1/q)$. Following Owen (1997a), a d-dimensional base q Haar multiresolution analysis is applied to f and an ANOVA decomposition of f is obtained. This ANOVA decomposition facilitates much of the theoretical analysis that ensues.

In Section 3, a proxy statistic W for W_{oas} and W_{oal} is introduced. Proposition 2 shows that to prove the asymptotic normality of W_{oas} and W_{oal} as $q \to \infty$, it suffices to prove that W is asymptotically normal. Stein (1972) proposed a powerful and general method for obtaining a bound for the error in the normal approximation to the distribution of a sum of dependent random variables. Since then, Stein's method has found considerable applications in combinatorics, probability and statistics [see Stein (1986)]. We shall use the multivariate normal version of Stein's method as given in Götze (1991) and Bolthausen and Götze (1993). In particular, Theorem 3 establishes a multivariate central limit theorem for the "components" of W under the conditions of Theorem 2. This result is needed in the proof of Theorem 2. Finally, the Appendix contains the proofs of all the results in this article.

We would like to add that Loh (1996) has established the asymptotic normality of $\hat{\mu}_{oas}$ when $d = 3$ and $t = 2$ under moment conditions on f. However the approach in Loh (1996), which uses directly the univariate version of Stein's method, does not seem to be extendable to $d \geq 4$. For example, the inequality (11) in Loh (1996) is valid for $d = 3$ but not for $d \geq 4$.

We conclude the Introduction with a note on notation. In this article, the indicator function is denoted by $\mathcal{I}\{\cdot\}$ and if x is a vector, then x' is its transpose. $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^p$ where p is either $d-2$ or d (depending on the context).

2. ANOVA decomposition. We shall first establish base q expansions for randomized orthogonal array samples as well as for OA-based Latin hypercubes. Let $A \in \mathrm{OA}(q^2, d, q, 2)$, $a_{i,j}$ be the (i, j)th element of A and

$$\{\pi_j, \pi_{j;b}, \pi_{i,j,k} : i = 1, \ldots, q^2, j = 1, \ldots, d, b = 0, \ldots, q-1, k = 2, 3, \ldots\} \tag{7}$$

be a set of mutually independent random permutations of $\{0, 1, \ldots, q-1\}$, where each of these permutations is uniformly distributed over its $q!$ possible values. We observe that the randomized orthogonal array sample $X_1, \ldots, X_{q^2}$ in (2) can be expressed as $X_i = (X_{i,1}, \ldots, X_{i,d})'$ where

$$X_{i,j} = \sum_{k=1}^{\infty} x_{i,j,k} q^{-k} \qquad \forall i = 1, \ldots, q^2, j = 1, \ldots, d, \tag{8}$$

$x_{i,j,1} = \pi_j(a_{i,j})$ and $x_{i,j,k} = \pi_{i,j,k}(0)$ for all $k \geq 2$. Let $A^{**} \in \mathrm{OA}(q^2, d, q^2, 1)$ be as in Section 1 with $t = 2$. Since $0 \leq a^{**}_{i,j}/q^2 < 1$, we observe that $a^{**}_{i,j}/q^2 = \sum_{k=1}^{\infty} b_{i,j,k} q^{-k}$ for suitable integers $0 \leq b_{i,j,1}, b_{i,j,2} \leq q-1$ and $b_{i,j,k} = 0$ for all $k \geq 3$. Owen (1997a), page 1907, observed that an OA-based Latin hypercube defined as in (3) has the form $\{Y_i = (Y_{i,1}, \ldots, Y_{i,d})' : i = 1, \ldots, q^2\}$, where

$$Y_{i,j} = \sum_{k=1}^{\infty} y_{i,j,k} q^{-k} \tag{9}$$

and for $1 \leq i \leq q^2$, $1 \leq j \leq d$, $y_{i,j,1} = \pi_j(a_{i,j})$, $y_{i,j,2} = \pi_{j;a_{i,j}}(b_{i,j,2})$, $y_{i,j,k} = \pi_{i,j,k}(0)$ for all $k \geq 3$. We observe from (8) and (9) that $\sup_{1 \leq i \leq q^2, 1 \leq j \leq d} |X_{i,j} - Y_{i,j}| \leq (q-1)/q^2$.

Let $f : [0, 1)^d \to \mathbb{R}$ be a square integrable function. Inspired by Owen (1997a), we apply a d-dimensional base q Haar multiresolution analysis to f. More precisely, for any integer $k \geq 0$, let $\mathcal{Y}_k$ denote the linear span of the functions

$\{\psi_{k,t,c} : t = 0, 1, \dots,$ and $c = 0, \dots, q-1\}$ where

$$\psi_{k,t,c}(x) = q^{(k+1)/2} \mathcal{I}\left\{x \in \left[\frac{qt+c}{q^{k+1}}, \frac{qt+c+1}{q^{k+1}}\right)\right\} - q^{(k-1)/2} \mathcal{I}\left\{x \in \left[\frac{t}{q^k}, \frac{t+1}{q^k}\right)\right\},$$

$\forall x \in [0, 1)$. We observe that the functions in $\mathcal{Y}_k$ are constant on $[tq^{-k-1}, (t+1)q^{-k-1})$ and integrate to zero over $[tq^{-k}, (t+1)q^{-k})$. Next let $\mathcal{U}_0$ denote the space of functions that are constant on $[0, 1)$ and

$$\mathcal{U}_k = \{g + g_0 + \cdots + g_{k-1} : g \in \mathcal{U}_0, g_j \in \mathcal{Y}_j, j = 0, \dots, k-1\} \qquad \forall k = 1, 2, \dots.$$

Then it is well known that $\bigcup_{k=0}^{\infty} \mathcal{U}_k$ is dense in $L^2([0, 1))$ and $\bigcap_{k=0}^{\infty} \mathcal{U}_k = \mathcal{U}_0$. We further observe from Owen (1997a), page 1897, that a typical basis function for $L^2([0, 1)^d)$ is of the form $\prod_{r=1}^{l} \psi_{k_{j_r}, t_{j_r}, c_{j_r}}(x_{j_r})$ for all $(x_1, \dots, x_d)' \in [0, 1)^d$, where $1 \le j_1 < \cdots < j_l \le d$, and $k_{j_r} \ge 0$, $0 \le t_{j_r} \le q^{k_{j_r}} - 1$, $0 \le c_{j_r} \le q-1$ whenever $1 \le r \le l$. Here by convention, an empty product (i.e., $l = 0$) is taken to be 1. Hence for each $f \in L^2([0, 1)^d)$, it follows from (6.6) of Owen (1997a), page 1898, that

$$f(x) = \mu + \sum_{l=1}^{d} \sum_{1 \le j_1 < \cdots < j_l \le d} \left(\sum_{k_{j_1}=0}^{\infty} \sum_{t_{j_1}=0}^{q^{k_{j_1}}-1} \sum_{c_{j_1}=0}^{q-1}\right) \cdots \left(\sum_{k_{j_l}=0}^{\infty} \sum_{t_{j_l}=0}^{q^{k_{j_l}}-1} \sum_{c_{j_l}=0}^{q-1}\right) \times \left\langle f, \prod_{r=1}^{l} \psi_{k_{j_r}, t_{j_r}, c_{j_r}} \right\rangle \prod_{r=1}^{l} \psi_{k_{j_r}, t_{j_r}, c_{j_r}}(x_{j_r}), \qquad \text{a.e. } x \in [0, 1)^d, \tag{10}$$

where μ is as in (1) and

$$\left\langle f, \prod_{r=1}^{l} \psi_{k_{j_r}, t_{j_r}, c_{j_r}} \right\rangle = \int_{[0,1)^d} f(x) \left[\prod_{r=1}^{l} \psi_{k_{j_r}, t_{j_r}, c_{j_r}}(x_{j_r})\right] dx. \tag{11}$$

Without loss of generality, we can assume that equality in (10) holds for all $x \in [0, 1)^d$ since changing the value of f on a set of Lebesgue measure zero will not alter the value of μ. For simplicity let

$$\{U[\tilde{c}_{j,1}, \dots, \tilde{c}_{j,u_j} : 1 \le j \le d] : 0 \le \tilde{c}_{j,1}, \dots, \tilde{c}_{j,u_j} \le q-1, u_j \ge 0, 1 \le j \le d\}$$

be a set of mutually independent random vectors where each $U[\tilde{c}_{j,1}, \dots, \tilde{c}_{j,u_j} : 1 \le j \le d]$ has the uniform distribution on the d-dimensional interval $\prod_{j=1}^{d} [\sum_{k=1}^{u_j} \tilde{c}_{j,k} q^{-k}, q^{-u_j} + \sum_{k=1}^{u_j} \tilde{c}_{j,k} q^{-k})$. Here $\sum_{k=1}^{u_j} \tilde{c}_{j,k} q^{-k} = 0$ if $u_j = 0$. Furthermore we assume that the above U's are independent of π's [defined as in (7)]. For nonnegative integers $u_1^*, u_1, \dots, u_d^*, u_d$, we write:

(i) $(u_1^*, \dots, u_d^*) \preceq (u_1, \dots, u_d)$ if and only if $u_j^* \le u_j$ for all $j = 1, \dots, d$,

(ii) $(u_1^*, \dots, u_d^*) \prec (u_1, \dots, u_d)$ if and only if $u_j^* \le u_j$ for all $j = 1, \dots, d$ with at least one strict inequality.

The following construction establishes an ANOVA decomposition of $Ef \circ U = Ef(U)$ where E denotes expectation. For integers $u_j \ge 0$, $0 \le \tilde{c}_{j,k} \le q-1$, $1 \le j \le d, k \ge 1$, define recursively

$$\begin{aligned}\nu_{u_1,\dots,u_d}&[\tilde{c}_{j,1}, \dots, \tilde{c}_{j,u_j} : 1 \le j \le d] \\ &= Ef \circ U[\tilde{c}_{j,1}, \dots, \tilde{c}_{j,u_j} : 1 \le j \le d] \\ &\quad - \sum_{u_1^*,\dots,u_d^* : (0,\dots,0) \preceq (u_1^*,\dots,u_d^*) \prec (u_1,\dots,u_d)} \nu_{u_1^*,\dots,u_d^*}[\tilde{c}_{j,1}, \dots, \tilde{c}_{j,u_j^*} : \\ &\qquad\qquad 1 \le j \le d],\end{aligned} \tag{12}$$

and hence

$$\begin{aligned}\lim_{u_1,\dots,u_d \to \infty} & Ef \circ U[\tilde{c}_{j,1}, \dots, \tilde{c}_{j,u_j} : 1 \le j \le d] \\ &= \sum_{u_1,\dots,u_d \ge 0} \nu_{u_1,\dots,u_d}[\tilde{c}_{j,1}, \dots, \tilde{c}_{j,u_j} : 1 \le j \le d].\end{aligned}$$

Writing $|u| = \sum_{j=1}^d \mathbb{1}\{u_j \ge 1\}$, $u = (u_1, \dots, u_d)'$, such that $1 \le j_1 < \cdots < j_{|u|} \le d$ and $u_j \ge 1$ if and only if $j \in \{j_1, \dots, j_{|u|}\}$, it follows from (10) that $\nu_{u_1,\dots,u_d}$ can be written down explicitly as $\nu_{0,\dots,0}[\cdot] = \mu$ if $|u| = 0$ and

$$\begin{aligned}\nu_{u_1,\dots,u_d}&[\tilde{c}_{j,1}, \dots, \tilde{c}_{j,u_j} : 1 \le j \le d] \\ &= \left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1} \sum_{c_{j_1}=0}^{q-1}\right) \cdots \left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1} \sum_{c_{j_{|u|}}=0}^{q-1}\right) \left\langle f, \prod_{l=1}^{|u|} \psi_{u_{j_l}-1, t_{j_l}, c_{j_l}} \right\rangle \\ &\quad \times E\left\{\prod_{l=1}^{|u|} \psi_{u_{j_l}-1, t_{j_l}, c_{j_l}} \circ U_{j_l}[\tilde{c}_{j,1}, \dots, \tilde{c}_{j,u_j} : 1 \le j \le d]\right\} \\ &= \left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1} \sum_{c_{j_1}=0}^{q-1}\right) \cdots \left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1} \sum_{c_{j_{|u|}}=0}^{q-1}\right) \left\langle f, \prod_{l=1}^{|u|} \psi_{u_{j_l}-1, t_{j_l}, c_{j_l}} \right\rangle \\ &\quad \times \prod_{l=1}^{|u|} \psi_{u_{j_l}-1, t_{j_l}, c_{j_l}} \left(\sum_{k=1}^{u_{j_l}} \tilde{c}_{j_l,k} q^{-k}\right),\end{aligned} \tag{13}$$

if $|u| \ge 1$. Here U_{j_l} denotes the j_lth coordinate of U and the last equality uses the fact that $\psi_{u_{j_l}-1, t_{j_l}, c_{j_l}}$ is constant on $[tq^{-u_{j_l}}, (t+1)q^{-u_{j_l}})$ for an arbitrary but fixed integer t. An important consequence of the ANOVA decomposition (that will be applied repeatedly in the sequel) is if $u_k \geq 1$ for some $1 \leq k \leq d$, then

$$\sum_{\tilde{c}_{k,u_k}=0}^{q-1} \nu_{u_1,\dots,u_d}[\tilde{c}_{j,1},\dots,\tilde{c}_{j,u_j} : 1 \leq j \leq d] = 0. \tag{14}$$

Writing $W_{oal} = (\hat{\mu}_{oal} - \mu)/\sigma_{oal}$, we observe from (5), (9), (12), (13) and (14) that

$$\begin{aligned} W_{oal} &= \frac{1}{q^2 \sigma_{oal}} \sum_{i=1}^{q^2} [f(Y_i) - \mu] \\ &= \frac{1}{q^2 \sigma_{oal}} \sum_{i=1}^{q^2} \sum_{u_1,\dots,u_d : (0,\dots,0) \prec (u_1,\dots,u_d)} \nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}), \pi_{j;a_{i,j}}(b_{i,j,2}), \\ &\qquad \pi_{i,j,3}(0),\dots,\pi_{i,j,u_j}(0) : 1 \leq j \leq d] \\ &= \frac{1}{q^2 \sigma_{oal}} \sum_{i=1}^{q^2} \sum_{u_1,\dots,u_d \geq 0 : u_1+\dots+u_d \geq 3} \nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}), \pi_{j;a_{i,j}}(b_{i,j,2}), \\ &\qquad \pi_{i,j,3}(0),\dots,\pi_{i,j,u_j}(0) : 1 \leq j \leq d]. \end{aligned} \tag{15}$$

Writing $W_{oas} = (\hat{\mu}_{oas} - \mu)/\sigma_{oas}$ and in a similar manner to (15), we have

$$\begin{aligned} W_{oas} &= \frac{1}{q^2 \sigma_{oas}} \sum_{i=1}^{q^2} \sum_{u_1,\dots,u_d \geq 0 : u_1+\dots+u_d \geq 3} \nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}), \\ &\qquad \pi_{i,j,2}(0),\dots,\pi_{i,j,u_j}(0) : 1 \leq j \leq d] \\ &\quad + \frac{1}{q^2 \sigma_{oas}} \sum_{i=1}^{q^2} \sum_{1 \leq k \leq d : u_k = 2, u_l = 0 \forall l \neq k} \nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}), \\ &\qquad \pi_{i,j,2}(0),\dots,\pi_{i,j,u_j}(0) : 1 \leq j \leq d]. \end{aligned} \tag{16}$$

For brevity of notation, we write in the sequel

$$\begin{aligned} &\nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}), \pi_{j;a_{i,j}}(b_{i,j,2}), \pi_{i,j,3}(0),\dots,\pi_{i,j,u_j}(0) : 1 \leq j \leq d] \\ &\quad = \nu^*_{u_{j_1},\dots,u_{j_{|u|}}}[\pi_{j_1}(a_{i,j_1}), \pi_{j_1;a_{i,j_1}}(b_{i,j_1,2}), \pi_{i,j_1,3}(0),\dots, \\ &\qquad \pi_{i,j_1,u_{j_1}}(0); \dots; \pi_{j_{|u|}}(a_{i,j_{|u|}}), \pi_{j_{|u|};a_{i,j_{|u|}}}(b_{i,j_{|u|},2}), \\ &\qquad \pi_{i,j_{|u|},3}(0),\dots,\pi_{i,j_{|u|},u_{j_{|u|}}}(0)], \end{aligned} \tag{17}$$

$\nu^*[\cdot] = \nu^*_{u_{j_1},\dots,u_{j_{|u|}}}[\cdot]$ if $u_{j_1} = \dots = u_{j_{|u|}} = 1$, and

$$\sigma^2 = E\left\{\left\{\frac{1}{q^2} \sum_{i=1}^{q^2} \sum_{0 \leq u_1,\dots,u_d \leq 1 : |u| \geq 3} \nu^*[\pi_{j_1}(a_{i,j_1}); \dots; \pi_{j_{|u|}}(a_{i,j_{|u|}})]\right\}^2\right\}, \tag{18}$$

where $1 \leq j_1 < \cdots < j_{|u|} \leq d$ are exactly those coordinates of $u = (u_1, \ldots, u_d)'$ in which $u_j \geq 1$ and $|u|$ denotes the cardinality of that set. We end this section with the following proposition whose proof can be found in the Appendix:

PROPOSITION 1. *Let $f : [0, 1)^d \to \mathbb{R}$ be smooth with a Lipschitz continuous mixed partial of order d. Then*

$$\sigma_{oal}^2 = q^{-2} \sum_{0 \leq u_1, \ldots, u_d \leq 1 \,:\, |u| \geq 3} E\{\nu^*[\pi_{j_1}(a_{1,j_1}); \ldots; \pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\} + O(q^{-3}),$$

$$\sigma_{oas}^2 = \sigma_{oal}^2 + O(q^{-3}) \quad \text{and} \quad \sigma^2 = \sigma_{oal}^2 + O(q^{-3}) \qquad \text{as } q \to \infty.$$

3. A multivariate central limit theorem. First we define the proxy statistic

$$W = \frac{1}{q^2 \sigma} \sum_{i=1}^{q^2} \sum_{0 \leq u_1, \ldots, u_d \leq 1 : |u| \geq 3} \nu^*[\pi_{j_1}(a_{i,j_1}); \ldots; \pi_{j_{|u|}}(a_{i,j_{|u|}})], \tag{19}$$

where σ^2 is as in (18). Clearly we have $E(W^2) = 1$.

PROPOSITION 2. *Let $f : [0, 1)^d \to \mathbb{R}$ be smooth with a Lipschitz continuous mixed partial of order d. Suppose W_{oal}, W_{oas} and W are as in* (15), (16) *and* (19) *respectively with $A \in \mathrm{OA}(q^2, d, q, 2)$. Then $q(\sigma_{oal} W_{oal} - \sigma W) \to 0$ and $q(\sigma_{oas} W_{oas} - \sigma W) \to 0$ in probability as $q \to \infty$.*

REMARK. As mentioned in Section 1, the above proposition implies that to prove the asymptotic normality of W_{oas} and W_{oal}, it suffices to prove that W is asymptotically normal.

Suppose $d \geq 3$. For $\ell = 1, \ldots, d - 2$, we define

$$\begin{aligned} \sigma_\ell^2 &= E\left\{\left\{\frac{1}{q^2} \sum_{i=1}^{q^2} \sum_{0 \leq u_1, \ldots, u_d \leq 1 \,:\, |u| = \ell + 2} \nu^*[\pi_{j_1}(a_{i,j_1}); \ldots; \pi_{j_{|u|}}(a_{i,j_{|u|}})]\right\}^2\right\}, \\ V_\ell &= \frac{1}{q^2 \sigma_\ell} \sum_{i=1}^{q^2} \sum_{0 \leq u_1, \ldots, u_d \leq 1 \,:\, |u| = \ell + 2} \nu^*[\pi_{j_1}(a_{i,j_1}); \ldots; \pi_{j_{|u|}}(a_{i,j_{|u|}})], \end{aligned} \tag{20}$$

and $V = (V_1, \ldots, V_{d-2})'$ where $|u| = \sum_{i=1}^d \mathbb{I}\{u_i \geq 1\}$.

We shall prove that the random vector V converges weakly to the standard $(d - 2)$-variate normal distribution Φ_{d-2} as q tends to infinity. To do so, we shall use the multivariate normal version of Stein's method [see Stein (1972, 1986)] as given in Götze (1991) and Bolthausen and Götze (1993).

Let $\mathcal{A}$ be a class of measurable functions from $\mathbb{R}^{d-2} \to \mathbb{R}$ such that $\sup_{v\in\mathbb{R}^{d-2}} |g(v)| \le 1$ for all $g \in \mathcal{A}$. For $g \in \mathcal{A}$ and $\delta > 0$, define

$$\begin{aligned} g_\delta^+(v) &= \sup\{g(v+y) : \|y\| \le \delta\} \qquad \forall v \in \mathbb{R}^{d-2}, \\ g_\delta^-(v) &= \inf\{g(v+y) : \|y\| \le \delta\} \qquad \forall v \in \mathbb{R}^{d-2}, \\ \omega(g,\delta) &= \int_{\mathbb{R}^{d-2}} [g_\delta^+(y) - g_\delta^-(y)]\Phi_{d-2}(dy). \end{aligned}$$

We further assume that $\mathcal{A}$ is closed under supremum and affine transformations, that is, $g \in \mathcal{A}$ implies that $g_\delta^+ \in \mathcal{A}$, $g_\delta^- \in \mathcal{A}$ and $g \circ T \in \mathcal{A}$ whenever $T : \mathbb{R}^{d-2} \to \mathbb{R}^{d-2}$ is affine. Finally we assume that there exists a constant $\Delta \ge 2\sqrt{d-2}$ such that

$$\sup\{\omega(g,\delta) : g \in \mathcal{A}\} \le \Delta\delta \qquad \forall \delta > 0. \tag{21}$$

We observe from Bolthausen and Götze (1993) that $\mathcal{A}$ can be taken to be the class of all indicator functions of measurable convex sets in $\mathbb{R}^{d-2}$. For $h \in \mathcal{A}$ and $0 \le t \le 1$, define

$$\begin{aligned} \chi_t(v|h) &= \int_{\mathbb{R}^{d-2}} \bigl[h(y) - h\bigl(t^{1/2}y + (1-t)^{1/2}v\bigr)\bigr]\Phi_{d-2}(dy), \\ \psi_t(v) &= \frac{1}{2}\int_t^1 \chi_s(v|h)\frac{ds}{1-s} \qquad \forall v \in \mathbb{R}^{d-2}. \end{aligned} \tag{22}$$

Then $-\chi_0(v|h) = h(v) - \Phi_{d-2}(h)$ where $\Phi_{d-2}(h) = Eh(Z)$ and Z is a random vector having distribution Φ_{d-2}. The following two lemmas are due to Götze (1991). Since the proofs are only briefly sketched in Götze (1991), we refer the reader to Loh (2007) for the detailed proofs of Lemmas 1 and 2.

LEMMA 1. *For $0 < t < 1$ and $v = (v_1, \dots, v_{d-2})' \in \mathbb{R}^{d-2}$, we have*

$$\sum_{i=1}^{d-2} \frac{\partial^2}{\partial v_i^2}\psi_t(v) - \sum_{i=1}^{d-2} v_i \frac{\partial}{\partial v_i}\psi_t(v) = -\chi_t(v|h). \tag{23}$$

There exists a constant c (depending only on d) such that

$$\sup_{1\le i,j\le d-2}\ \sup_{v\in\mathbb{R}^{d-2}} \left|\frac{\partial^2}{\partial v_i\,\partial v_j}\psi_t(v)\right| \le \|h\|_\infty \log(1/t),$$

where $\|h\|_\infty = \sup_{v\in\mathbb{R}^{d-2}} |h(v)|$ and

$$\begin{aligned} &\sup_{1\le i,j,k\le d-2} \left|\int_{\mathbb{R}^{d-2}} \left[\frac{\partial^3}{\partial v_i\,\partial v_j\,\partial v_k}\psi_t(v)\right] Q(dv)\right| \\ &\qquad \le \frac{c}{t^{1/2}} \sup\left\{\left|\int_{\mathbb{R}^{d-2}} h(sv+y)Q(dv)\right| : 0 \le s \le 1,\ y \in \mathbb{R}^{d-2}\right\} \end{aligned}$$

for all finite signed measures Q on $\mathbb{R}^{d-2}$.

LEMMA 2. *Suppose that* (21) *holds. Let* $0 < \varepsilon < 1/2$ *and* Q *be a probability distribution on* $\mathbb{R}^{d-2}$. *Then there exists a constant* c_d *depending only on* d *such that*

$$\sup_{g\in\mathcal{A}}\left|\int_{\mathbb{R}^{d-2}} g(v)[Q(dv) - \Phi_{d-2}(dv)]\right| \le c_d\left[\sup_{h\in\mathcal{A}}\left|\int_{\mathbb{R}^{d-2}} \chi_{\varepsilon^2}(v|h)Q(dv)\right| + \Delta\varepsilon\right].$$

The theorem below is the main result of this section and is needed in the proof of Theorem 2.

THEOREM 3. *Suppose* $d \ge 3$. *Let* $f:[0,1)^d \to \mathbb{R}$ *be smooth with a Lipschitz continuous mixed partial of order* d *such that* $\int_{\mathbb{R}^d} f^2_{rem}(x)\,dx > 0$ *and the* $(d-2)$*-variate random vector* V *be as in* (20). *Then* V *converges to* Φ_{d-2} *in distribution as* $q \to \infty$.

APPENDIX

PROOF OF PROPOSITION 1. Since $E(W^2_{oal}) = 1$, we observe from (15) that

$$\begin{aligned}\sigma^2_{oal} = {}&\frac{1}{q^4}\sum_{i=1}^{q^2}\sum_{u_1,\dots,u_d\ge 0:\,u_1+\cdots+u_d\ge 3} E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}), \pi_{j;a_{i,j}}(b_{i,j,2}),\\ &\pi_{i,j,3}(0),\dots,\pi_{i,j,u_j}(0): 1\le j\le d]^2\}\\ &+\frac{1}{q^4}\sum_{i_1=1}^{q^2}\sum_{i_2\ne i_1}\sum_{u_1,\dots,u_d\ge 0:\,u_1+\cdots+u_d\ge 3} E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i_1,j}),\\ &\pi_{j;a_{i_1,j}}(b_{i_1,j,2}), \pi_{i_1,j,3}(0),\dots,\pi_{i_1,j,u_j}(0): 1\le j\le d]\\ &\times \nu_{u_1,\dots,u_d}[\pi_j(a_{i_2,j}), \pi_{j;a_{i_2,j}}(b_{i_2,j,2}), \pi_{i_2,j,3}(0),\dots,\\ &\pi_{i_2,j,u_j}(0): 1\le j\le d]\}.\end{aligned}\tag{24}$$

Since $A \in \mathrm{OA}(q^2, d, q, 2)$, we have

$$\begin{aligned}&\frac{1}{q^4}\sum_{i_1=1}^{q^2}\sum_{i_2\ne i_1}\sum_{u_1,\dots,u_d\ge 0:\,u_1+\cdots+u_d\ge 3} E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i_1,j}), \pi_{j;a_{i_1,j}}(b_{i_1,j,2}),\\ &\pi_{i_1,j,3}(0),\dots,\pi_{i_1,j,u_j}(0): 1\le j\le d]\nu_{u_1,\dots,u_d}[\pi_j(a_{i_2,j}),\\ &\pi_{j;a_{i_2,j}}(b_{i_2,j,2}), \pi_{i_2,j,3}(0),\dots,\pi_{i_2,j,u_j}(0): 1\le j\le d]\}\\ &\quad= \frac{1}{q^4}\sum_{i_1=1}^{q^2}\sum_{0\le u_1,\dots,u_d\le 2:\,u_1+\cdots+u_d\ge 3}\sum_{i_2\ne i_1} E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i_1,j}),\\ &\qquad\pi_{j;a_{i_1,j}}(b_{i_1,j,2}), \pi_{i_1,j,3}(0),\dots,\pi_{i_1,j,u_j}(0): 1\le j\le d]\end{aligned}\tag{25}$$

$$\times \nu_{u_1,\dots,u_d}[\pi_j(a_{i_2,j}), \pi_{j;a_{i_2,j}}(b_{i_2,j,2}), \pi_{i_2,j,3}(0), \dots, \pi_{i_2,j,u_j}(0) : 1 \le j \le d]\}$$

$$= \frac{1}{q^4} \sum_{i_1=1}^{q^2} \sum_{0 \le u_1,\dots,u_d \le 2 : u_1+\cdots+u_d \ge 3} \sum_{i_2 \ne i_1} \sum_{l=1}^{|u|} \mathcal{I}\{a_{i_1,j_l} = a_{i_2,j_l}, u_{j_l} = 1\}$$

$$\times E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i_1,j}), \pi_{j;a_{i_1,j}}(b_{i_1,j,2}), \pi_{i_1,j,3}(0), \dots, \pi_{i_1,j,u_j}(0) : 1 \le j \le d] \nu_{u_1,\dots,u_d}[\pi_j(a_{i_2,j}), \pi_{j;a_{i_2,j}}(b_{i_2,j,2}), \pi_{i_2,j,3}(0), \dots, \pi_{i_2,j,u_j}(0) : 1 \le j \le d]\}$$

$$+ \frac{1}{q^4} \sum_{i_1=1}^{q^2} \sum_{0 \le u_1,\dots,u_d \le 2 : u_1+\cdots+u_d \ge 3} \sum_{i_2 \ne i_1} \sum_{l=1}^{|u|} \mathcal{I}\{a_{i_1,j_l} = a_{i_2,j_l}, u_{j_l} = 2\}$$

$$\times E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i_1,j}), \pi_{j;a_{i_1,j}}(b_{i_1,j,2}), \pi_{i_1,j,3}(0), \dots, \pi_{i_1,j,u_j}(0) : 1 \le j \le d] \nu_{u_1,\dots,u_d}[\pi_j(a_{i_2,j}), \pi_{j;a_{i_2,j}}(b_{i_2,j,2}), \pi_{i_2,j,3}(0), \dots, \pi_{i_2,j,u_j}(0) : 1 \le j \le d]\}$$

$$+ \frac{1}{q^4} \sum_{i_1=1}^{q^2} \sum_{0 \le u_1,\dots,u_d \le 2 : u_1+\cdots+u_d \ge 3} \sum_{i_2 \ne i_1} \mathcal{I}\{a_{i_1,j_l} \ne a_{i_2,j_l}, \forall l = 1, \dots, |u|\}$$

$$\times E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i_1,j}), \pi_{j;a_{i_1,j}}(b_{i_1,j,2}), \pi_{i_1,j,3}(0), \dots, \pi_{i_1,j,u_j}(0) : 1 \le j \le d] \nu_{u_1,\dots,u_d}[\pi_j(a_{i_2,j}), \pi_{j;a_{i_2,j}}(b_{i_2,j,2}), \pi_{i_2,j,3}(0), \dots, \pi_{i_2,j,u_j}(0) : 1 \le j \le d]\}.$$

We further note that

$$\mathcal{I}\{a_{i_1,j_l} = a_{i_2,j_l}\} = \mathcal{I}\{a_{i_1,j_l} = a_{i_2,j_l} \text{ and } a_{i_1,j_m} \ne a_{i_2,j_m}\ \forall m \ne l\}.$$

Consequently,

$$\frac{1}{q^4} \sum_{i_1=1}^{q^2} \sum_{0 \le u_1,\dots,u_d \le 2 : u_1+\cdots+u_d \ge 3} \sum_{i_2 \ne i_1} \sum_{l=1}^{|u|} \mathcal{I}\{a_{i_1,j_l} = a_{i_2,j_l}, u_{j_l} = 1\}$$

$$\times E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i_1,j}), \pi_{j;a_{i_1,j}}(b_{i_1,j,2}), \pi_{i_1,j,3}(0), \dots, \pi_{i_1,j,u_j}(0) : 1 \le j \le d] \times \nu_{u_1,\dots,u_d}[\pi_j(a_{i_2,j}), \pi_{j;a_{i_2,j}}(b_{i_2,j,2}), \pi_{i_2,j,3}(0), \dots, \pi_{i_2,j,u_j}(0) : 1 \le j \le d]\}$$

$$= \frac{1}{q^4} \sum_{i_1=1}^{q^2} \sum_{0 \le u_1,\dots,u_d \le 1 : u_1+\cdots+u_d \ge 3} \sum_{i_2 \ne i_1} \sum_{l=1}^{|u|} \mathcal{I}\{a_{i_1,j_l} = a_{i_2,j_l}\} \left(\frac{1}{q-1}\right)^{|u|-1}$$

$$\times E\Bigg\{\nu^*[\pi_{j_1}(a_{i_1,j_1});\ldots;\pi_{j_{|u|}}(a_{i_1,j_{|u|}})]$$

$$\times\Bigg[\prod_{1\le m\le|u|\,:\,m\ne l}\ \sum_{0\le\tilde c_{j_m}\le q-1:\tilde c_{j_m}\ne\pi_{j_m}(a_{i_1,j_m})}\Bigg]$$

$$\times\,\nu^*[\tilde c_{j_1};\ldots;\tilde c_{j_{l-1}};\pi_{j_l}(a_{i_1,j_l});\tilde c_{j_{l+1}};\ldots;\tilde c_{j_{|u|}}]\Bigg\}$$

$$=\sum_{0\le u_1,\ldots,u_d\le 1:|u|\ge 3}\frac{(-1)^{|u|-1}|u|}{q^2(q-1)^{|u|-2}}E\{\nu^*[\pi_{j_1}(a_{1,j_1});\ldots;\pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\},$$

$$\frac{1}{q^4}\sum_{i_1=1}^{q^2}\ \sum_{0\le u_1,\ldots,u_d\le 2\,:\,u_1+\cdots+u_d\ge 3}\ \sum_{i_2\ne i_1}\sum_{l=1}^{|u|}\mathbb{I}\{a_{i_1,j_l}=a_{i_2,j_l},u_{j_l}=2\}$$

$$\times E\{\nu_{u_1,\ldots,u_d}[\pi_j(a_{i_1,j}),\pi_{j;a_{i_1,j}}(b_{i_1,j,2}),\pi_{i_1,j,3}(0),\ldots,\pi_{i_1,j,u_j}(0):$$

$$1\le j\le d]\nu_{u_1,\ldots,u_d}[\pi_j(a_{i_2,j}),\pi_{j;a_{i_2,j}}(b_{i_2,j,2}),\pi_{i_2,j,3}(0),\ldots,$$

$$\pi_{i_2,j,u_j}(0):1\le j\le d]\}$$

$$=\frac{1}{q^4}\sum_{i_1=1}^{q^2}\ \sum_{0\le u_1,\ldots,u_d\le 2:u_1+\cdots+u_d\ge 3}\ \sum_{i_2\ne i_1}\sum_{l=1}^{|u|}\mathbb{I}\{a_{i_1,j_l}=a_{i_2,j_l},u_{j_l}=2\}$$

$$\times\,\mathbb{I}\{u_{j_k}=1,\forall k\ne l\}$$

$$\times E\Bigg\{\nu^*_{u_{j_1},\ldots,u_{j_{|u|}}}[\pi_{j_1}(a_{i_1,j_1});\ldots;\pi_{j_{l-1}}(a_{i_1,j_{l-1}});$$

$$\pi_{j_l}(a_{i_1,j_l}),\pi_{j_l;a_{i_1,j_l}}(b_{i_1,j_l,2});$$

$$\pi_{j_{l+1}}(a_{i_1,j_{l+1}});\ldots;\pi_{j_{|u|}}(a_{i_1,j_{|u|}})]$$

$$\times\left(\frac{1}{q-1}\right)^{|u|}\sum_{0\le\tilde c_{j_l}\le q-1:\tilde c_{j_l}\ne\pi_{j_l;a_{i_1,j_l}}(b_{i_1,j_l,2})}$$

$$\times\Bigg[\prod_{1\le m\le|u|\,:\,m\ne l}\ \sum_{0\le\tilde c_{j_m}\le q-1\,:\,\tilde c_{j_m}\ne\pi_{j_m}(a_{i_1,j_m})}\Bigg]$$

$$\times\,\nu^*_{u_{j_1},\ldots,u_{j_{|u|}}}[\tilde c_{j_1};\ldots;\tilde c_{j_{l-1}};\pi_{j_l}(a_{i_1,j_l}),\tilde c_{j_l};\tilde c_{j_{l+1}};\ldots;\tilde c_{j_{|u|}}]\Bigg\}$$

$$
\begin{aligned}
&= \frac{1}{q^2} \sum_{l=1}^{d} \sum_{u_1,\ldots,u_d : u_l = 2, 0 \le u_k \le 1 \forall k \ne l, |u|+1 \ge 3} \frac{(-1)^{|u|}}{(q-1)^{|u|-1}} \\
&\quad \times E\{\nu^*_{u_{j_1},\ldots,u_{j_{|u|}}} [\pi_{j_1}(a_{1,j_1}); \ldots ; \pi_{j_{l-1}}(a_{1,j_{l-1}}); \\
&\qquad \pi_{j_l}(a_{1,j_l}), \pi_{j_l; a_{1,j_l}}(b_{1,j_l,2}); \pi_{j_{l+1}}(a_{1,j_{l+1}}); \ldots ; \pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{q^4} \sum_{i_1=1}^{q^2} \sum_{0 \le u_1,\ldots,u_d \le 2 : u_1+\cdots+u_d \ge 3} \sum_{i_2 \ne i_1} \mathcal{I}\{a_{i_1,j_l} \ne a_{i_2,j_l}, \forall l = 1, \ldots, |u|\} \\
&\times E\{\nu_{u_1,\ldots,u_d}[\pi_j(a_{i_1,j}), \pi_{j;a_{i_1,j}}(b_{i_1,j,2}), \pi_{i_1,j,3}(0), \ldots, \pi_{i_1,j,u_j}(0) : \\
&\qquad 1 \le j \le d] \nu_{u_1,\ldots,u_d}[\pi_j(a_{i_2,j}), \pi_{j;a_{i_2,j}}(b_{i_2,j,2}), \pi_{i_2,j,3}(0), \ldots, \\
&\qquad\qquad \pi_{i_2,j,u_j}(0) : 1 \le j \le d]\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{q^4} \sum_{i_1=1}^{q^2} \sum_{0 \le u_1,\ldots,u_d \le 1 : |u| \ge 3} \sum_{i_2 \ne i_1} \mathcal{I}\{a_{i_1,j_l} \ne a_{i_2,j_l}, \forall l = 1, \ldots, |u|\} \\
&\quad \times E\left\{ \nu^*[\pi_{j_1}(a_{i_1,j_1}); \ldots ; \pi_{j_{|u|}}(a_{i_1,j_{|u|}})] \left(\frac{1}{q-1}\right)^{|u|} \right. \\
&\qquad \left. \times \left[\prod_{1 \le m \le |u|} \sum_{0 \le \tilde{c}_{j_m} \le q-1 : \tilde{c}_{j_m} \ne \pi_{j_m}(a_{i_1,j_m})} \right] \nu^*[\tilde{c}_{j_1}; \ldots ; \tilde{c}_{j_{|u|}}] \right\} \\
&= \sum_{0 \le u_1,\ldots,u_d \le 1 : |u| \ge 3} \frac{(-1)^{|u|}(q+1-|u|)}{q^2 (q-1)^{|u|-1}} \\
&\quad \times E\{\nu^*[\pi_{j_1}(a_{1,j_1}); \ldots ; \pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\}.
\end{aligned} \tag{26}
$$

We conclude from (24), (25), (26) and Lemma 4 that

$$
\begin{aligned}
\sigma^2_{oal} &= \frac{1}{q^2} \sum_{0 \le u_1,\ldots,u_d \le 1 : |u| \ge 3} E\{\nu^*[\pi_{j_1}(a_{1,j_1}); \ldots ; \pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\} \\
&\quad + \frac{1}{q^2} \sum_{u_1,\ldots,u_d \ge 0 : u_1+\cdots+u_d \ge 3 \vee (|u|+1)} E\{\nu_{u_1,\ldots,u_d}[\pi_j(a_{1,j}), \pi_{j;a_{1,j}}(b_{1,j,2}), \\
&\quad \pi_{1,j,3}(0), \ldots, \pi_{1,j,u_j}(0) : 1 \le j \le d]^2\} \\
&\quad + \sum_{0 \le u_1,\ldots,u_d \le 1 : |u| \ge 3} \frac{(-1)^{|u|-1}|u|}{q^2(q-1)^{|u|-2}} E\{\nu^*[\pi_{j_1}(a_{1,j_1}); \ldots , \pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=1}^{d}\sum_{u_1,\dots,u_d:u_l=2,0\le u_k\le 1\forall k\ne l,|u|\ge 2}\frac{(-1)^{|u|}}{q^2(q-1)^{|u|-1}}\\
&\quad\times E\{\nu^*_{u_{j_1},\dots,u_{j_{|u|}}}[\pi_{j_1}(a_{1,j_1});\dots;\pi_{j_{l-1}}(a_{1,j_{l-1}});\pi_{j_l}(a_{1,j_l}),\\
&\qquad\qquad \pi_{j_l;a_{1,j_l}}(b_{1,j_l,2});\pi_{j_{l+1}}(a_{1,j_{l+1}});\dots;\pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\}\\
&+\sum_{0\le u_1,\dots,u_d\le 1:|u|\ge 3}\frac{(-1)^{|u|}(q+1-|u|)}{q^2(q-1)^{|u|-1}}\\
&\quad\times E\{\nu^*[\pi_{j_1}(a_{1,j_1});\dots;\pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\}\\
=&\frac{1}{q^2}\sum_{0\le u_1,\dots,u_d\le 1:|u|\ge 3}E\{\nu^*[\pi_{j_1}(a_{1,j_1});\dots;\pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\}+O\left(\frac{1}{q^3}\right),
\end{aligned}
$$

as $q\to\infty$. The proofs of the two remaining cases, namely σ^2_{oas} and σ^2, are similar and can be found in Loh (2007). This proves Proposition 1. □

LEMMA 3. *Let $f:[0,1)^d\to\mathbb{R}$ be smooth with a Lipschitz continuous mixed partial of order d. Then for $(a_1,\dots,a_d)',(x_1,\dots,x_d)'\in[0,1)^d$, we have*

$$
\begin{aligned}
&f(x_1,\dots,x_d)\\
&\quad=(x_1-a_1)\cdots(x_d-a_d)\frac{\partial^d f(a_1,\dots,a_d)}{\partial x_1\cdots\partial x_d}\\
&\qquad+\int_{a_1}^{x_1}\cdots\int_{a_d}^{x_d}\left[\frac{\partial^d f(t_1,\dots,t_d)}{\partial x_1\cdots\partial x_d}-\frac{\partial^d f(a_1,\dots,a_d)}{\partial x_1\cdots\partial x_d}\right]dt_d\cdots dt_1\\
&\qquad+h_{1;a_1,\dots,a_d}(x_2,\dots,x_d)+h_{2;a_1,\dots,a_d}(x_1,x_3,\dots,x_d)+\cdots\\
&\qquad+h_{d;a_1,\dots,a_d}(x_1,\dots,x_{d-1}),
\end{aligned}
$$

where $h_{i,a_1,\dots,a_d}:\mathbb{R}^{d-1}\to\mathbb{R}$, $i=1,\dots,d$ are suitably chosen functions.

Lemma 3 can be proved by mathematical induction on d.

LEMMA 4. *Let $f:[0,1)^d\to\mathbb{R}$ be smooth with a Lipschitz continuous mixed partial of order d. Then $\sum_{c_j=0}^{q-1}\langle f,\psi_{u_j-1,t_j,c_j}\rangle=0$ if $u_j\ge 1$, and*

$$
\begin{aligned}
&\left\langle f,\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\right\rangle\\
&\quad=q^{|u|-\sum_{l=1}^{|u|}3u_{j_l}/2}\left[\frac{\partial^{|u|}}{\partial x_{j_1}\cdots\partial x_{j_{|u|}}}f_{j_1,\dots,j_{|u|}}\left(\frac{t_{j_1}+0.5}{q^{u_{j_1}-1}},\dots,\frac{t_{j_{|u|}}+0.5}{q^{u_{j_{|u|}}-1}}\right)\right]
\end{aligned}
$$

$$\times \prod_{l=1}^{|u|}\left(\frac{c_{j_l}}{q}+\frac{1}{2q}-\frac{1}{2}\right)+O(1)q^{|u|-\sum_{l=1}^{|u|}3u_{j_l}/2}\max_{1\le l\le |u|} q^{-\beta(u_{j_l}-1)},$$
$$\textit{as } q\to\infty$$

where $f_{j_1,\dots,j_{|u|}}(x_{j_1},\dots,x_{j_{|u|}})=\int_{[0,1)^{d-|u|}} f(x)\prod_{1\le i\le d:\, i\notin\{j_1,\dots,j_{|u|}\}} dx_i$. *Also,*

$$\begin{aligned}
&E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}),\pi_{j;a_{i,j}}(b_{i,j,2}),\pi_{i,j,3}(0),\dots,\pi_{i,j,u_j}(0):1\le j\le d]^2\}\\
&\quad= E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}),\pi_{i,j,2}(0),\dots,\pi_{i,j,u_j}(0):1\le j\le d]^2\}\\
&\quad=\left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1}\sum_{c_{j_1}=0}^{q-1}\right)\cdots\left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1}\sum_{c_{j_{|u|}}=0}^{q-1}\right)\left\langle f,\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\right\rangle^2\\
&\qquad\times\sum_{k=0}^{|u|}\binom{|u|}{k}\frac{1}{q^k}\left(1-\frac{1}{q}\right)^{|u|-k}.
\end{aligned}$$

Finally $E\{\nu^*[\pi_{j_1}(a_{i,j_1});\dots;\pi_{j_{|u|}}(a_{i,j_{|u|}})]^4\}=O(1)$ *as* $q\to\infty$.

PROOF. $\sum_{c_j=0}^{q-1}\langle f,\psi_{u_j-1,t_j,c_j}\rangle=0$ if $u_j\ge 1$ follows easily from (11). We observe that

$$\begin{aligned}
&\left\langle f,\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\right\rangle\\
&\quad=\int_{[0,1)^d} f(x)\left\{\prod_{l=1}^{|u|}q^{u_{j_l}/2}\left[\mathbb{1}\left\{x_{j_l}\in\left[\frac{qt_{j_l}+c_{j_l}}{q^{u_{j_l}}},\frac{qt_{j_l}+c_{j_l}+1}{q^{u_{j_l}}}\right)\right\}\right.\right.\\
&\qquad\qquad\left.\left.-\frac{1}{q}\mathbb{1}\left\{x_{j_l}\in\left[\frac{t_{j_l}}{q^{u_{j_l}-1}},\frac{t_{j_l}+1}{q^{u_{j_l}-1}}\right)\right\}\right]\right\}dx\\
&\quad=q^{-\sum_{l=1}^{|u|}u_{j_l}/2}\int_{[0,1)^{|u|}} f_{j_1,\dots,j_{|u|}}(x_{j_1},\dots,x_{j_{|u|}})\\
&\qquad\times\left\{\prod_{l=1}^{|u|}q^{u_{j_l}}\left[\mathbb{1}\left\{x_{j_l}\in\left[\frac{qt_{j_l}+c_{j_l}}{q^{u_{j_l}}},\frac{qt_{j_l}+c_{j_l}+1}{q^{u_{j_l}}}\right)\right\}\right.\right.\\
&\qquad\qquad\left.\left.-\frac{1}{q}\mathbb{1}\left\{x_{j_l}\in\left[\frac{t_{j_l}}{q^{u_{j_l}-1}},\frac{t_{j_l}+1}{q^{u_{j_l}-1}}\right)\right\}\right]\right\}dx_{j_1}\cdots dx_{j_{|u|}}.
\end{aligned}\tag{27}$$

We observe from Lemma 3 that the right-hand side of (27) equals

$$
\begin{aligned}
& q^{-\sum_{l=1}^{|u|} u_{j_l}/2}\left[\frac{\partial^{|u|}}{\partial x_{j_1}\cdots\partial x_{j_{|u|}}} f_{j_1,\dots,j_{|u|}}\left(\frac{t_{j_1}+0.5}{q^{u_{j_1}-1}},\dots,\frac{t_{j_{|u|}}+0.5}{q^{u_{j_{|u|}}-1}}\right)\right] \\
& \quad \times \prod_{l=1}^{|u|}\left[q^{u_{j_l}} \int_{(qt_{j_l}+c_{j_l})/q^{u_{j_l}}}^{(qt_{j_l}+c_{j_l}+1)/q^{u_{j_l}}} \left(x_{j_l}-\frac{t_{j_l}+0.5}{q^{u_{j_l}-1}}\right) dx_{j_l}\right. \\
& \qquad \left. - q^{u_{j_l}-1} \int_{t_{j_l}/q^{u_{j_l}-1}}^{(t_{j_l}+1)/q^{u_{j_l}-1}} \left(x_{j_l}-\frac{t_{j_l}+0.5}{q^{u_{j_l}-1}}\right) dx_{j_l}\right] \\
& \quad + q^{-\sum_{l=1}^{|u|} u_{j_l}/2} \int_{[0,1)^{|u|}} \int_{(t_{j_1}+0.5)/q^{u_{j_1}-1}}^{x_{j_1}} \cdots \int_{(t_{j_{|u|}}+0.5)/q^{u_{j_{|u|}}-1}}^{x_{j_{|u|}}} \\
& \quad \times \left[\frac{\partial^{|u|} f_{j_1,\dots,j_{|u|}}(s_1,\dots,s_{|u|})}{\partial x_{j_1}\cdots\partial x_{j_{|u|}}}\right. \\
& \qquad \left. - \frac{\partial^{|u|} f_{j_1,\dots,j_{|u|}}(\frac{t_{j_1}+0.5}{q^{u_{j_1}-1}},\dots,\frac{t_{j_{|u|}}+0.5}{q^{u_{j_{|u|}}-1}})}{\partial x_{j_1}\cdots\partial x_{j_{|u|}}}\right] ds_{|u|}\cdots ds_1 \\
& \quad \times \left\{\prod_{l=1}^{|u|} q^{u_{j_l}}\left[\mathcal{I}\left\{x_{j_l}\in\left[\frac{qt_{j_l}+c_{j_l}}{q^{u_{j_l}}},\frac{qt_{j_l}+c_{j_l}+1}{q^{u_{j_l}}}\right)\right\}\right.\right. \\
& \qquad \left.\left. - \frac{1}{q}\mathcal{I}\left\{x_{j_l}\in\left[\frac{t_{j_l}}{q^{u_{j_l}-1}},\frac{t_{j_l}+1}{q^{u_{j_l}-1}}\right)\right\}\right]\right\} dx_{j_1}\cdots dx_{j_{|u|}} \\
& = q^{|u|-\sum_{l=1}^{|u|} 3u_{j_l}/2}\left[\frac{\partial^{|u|}}{\partial x_{j_1}\cdots\partial x_{j_{|u|}}} f_{j_1,\dots,j_{|u|}}\left(\frac{t_{j_1}+0.5}{q^{u_{j_1}-1}},\dots,\frac{t_{j_{|u|}}+0.5}{q^{u_{j_{|u|}}-1}}\right)\right] \\
& \quad \times \prod_{l=1}^{|u|}\left[\frac{c_{j_l}}{q}-\frac{1}{2}\left(1-\frac{1}{q}\right)\right] + O(1) q^{|u|-\sum_{l=1}^{|u|} 3u_{j_l}/2} \max_{1\le l\le |u|} q^{-\beta(u_{j_l}-1)},
\end{aligned}
$$

as $q \to \infty$. Next we observe from (13) that

$$
\begin{aligned}
& \nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}), \pi_{j;a_{i,j}}(b_{i,j,2}), \pi_{i,j,3}(0),\dots,\pi_{i,j,u_j}(0) : 1 \le j \le d] \\
& = \left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1}\sum_{c_{j_1}=0}^{q-1}\right)\cdots\left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1}\sum_{c_{j_{|u|}}=0}^{q-1}\right)\left\langle f, \prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\right\rangle \\
& \quad \times \prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\left(\frac{\pi_{j_l}(a_{i,j_l})}{q}+\frac{\pi_{j_l;a_{i,j_l}}(b_{i,j_l,2})}{q^2}+\sum_{k=3}^{u_{j_l}}\frac{\pi_{i,j_l,k}(0)}{q^k}\right),
\end{aligned}
$$

and hence

$$E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}),\pi_{j;a_{i,j}}(b_{i,j,2}),\pi_{i,j,3}(0),\dots,\pi_{i,j,u_j}(0):1\le j\le d]^2\}$$

$$= \left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1}\sum_{c_{j_1}=0}^{q-1}\right)\cdots\left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1}\sum_{c_{j_{|u|}}=0}^{q-1}\right)\left(\sum_{t'_{j_1}=0}^{q^{u_{j_1}-1}-1}\sum_{c'_{j_1}=0}^{q-1}\right)\cdots$$

$$\times\left(\sum_{t'_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1}\sum_{c'_{j_{|u|}}=0}^{q-1}\right)\left\langle f,\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\right\rangle\left\langle f,\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t'_{j_l},c'_{j_l}}\right\rangle$$

$$\times\prod_{l=1}^{|u|}E\Bigg[\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\left(\frac{\pi_{j_l}(a_{i,j_l})}{q}+\frac{\pi_{j_l;a_{i,j_l}}(b_{i,j_l,2})}{q^2}+\sum_{k=3}^{u_{j_l}}\frac{\pi_{i,j_l,k}(0)}{q^k}\right)$$

$$\times\psi_{u_{j_l}-1,t'_{j_l},c'_{j_l}}\left(\frac{\pi_{j_l}(a_{i,j_l})}{q}+\frac{\pi_{j_l;a_{i,j_l}}(b_{i,j_l,2})}{q^2}+\sum_{k=3}^{u_{j_l}}\frac{\pi_{i,j_l,k}(0)}{q^k}\right)\Bigg]$$

$$= \left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1}\sum_{c_{j_1}=0}^{q-1}\sum_{c'_{j_1}=0}^{q-1}\right)\cdots\left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1}\sum_{c_{j_{|u|}}=0}^{q-1}\sum_{c'_{j_{|u|}}=0}^{q-1}\right)$$

$$\times\left\langle f,\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\right\rangle\left\langle f,\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c'_{j_l}}\right\rangle$$

$$\times\prod_{l=1}^{|u|}E\Bigg[\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\left(\frac{\pi_{j_l}(a_{i,j_l})}{q}+\frac{\pi_{j_l;a_{i,j_l}}(b_{i,j_l,2})}{q^2}+\sum_{k=3}^{u_{j_l}}\frac{\pi_{i,j_l,k}(0)}{q^k}\right)$$

$$\times\psi_{u_{j_l}-1,t_{j_l},c'_{j_l}}\left(\frac{\pi_{j_l}(a_{i,j_l})}{q}+\frac{\pi_{j_l;a_{i,j_l}}(b_{i,j_l,2})}{q^2}+\sum_{k=3}^{u_{j_l}}\frac{\pi_{i,j_l,k}(0)}{q^k}\right)\Bigg].$$

We observe that the right-hand side of the last equation can be expressed as a finite sum of terms of the following form (up to permutations): for $\alpha = 0,\dots,|u|$,

$$\left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1}\sum_{c_{j_1}=0}^{q-1}\sum_{c'_{j_1}\ne c_{j_1}}\right)\cdots\left(\sum_{t_{j_\alpha}=0}^{q^{u_{j_\alpha}-1}-1}\sum_{c_{j_\alpha}=0}^{q-1}\sum_{c'_{j_\alpha}\ne c_{j_\alpha}}\right)$$

$$\times\left(\sum_{t_{j_{\alpha+1}}=0}^{q^{u_{j_{\alpha+1}}-1}-1}\sum_{c_{j_{\alpha+1}}=0}^{q-1}\right)\cdots\left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1}\sum_{c_{j_{|u|}}=0}^{q-1}\right)\left\langle f,\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\right\rangle$$

$$\times \left\langle f, \left(\prod_{l_1=1}^{\alpha} \psi_{u_{j_{l_1}}-1, t_{j_{l_1}}, c'_{j_{l_1}}}\right)\left(\prod_{l_2=\alpha+1}^{|u|} \psi_{u_{j_{l_2}}-1, t_{j_{l_2}}, c_{j_{l_2}}}\right)\right\rangle$$

$$\times E\Bigg[\prod_{l_1=1}^{\alpha} \psi_{u_{j_{l_1}}-1, t_{j_{l_1}}, c_{j_{l_1}}}\left(\frac{\pi_{j_{l_1}}(a_{i,j_{l_1}})}{q} + \frac{\pi_{j_{l_1}; a_{i,j_{l_1}}}(b_{i,j_{l_1},2})}{q^2} + \sum_{k=3}^{u_{j_{l_1}}} \frac{\pi_{i,j_{l_1},k}(0)}{q^k}\right)$$

$$\times \psi_{u_{j_{l_1}}-1, t_{j_{l_1}}, c'_{j_{l_1}}}\left(\frac{\pi_{j_{l_1}}(a_{i,j_{l_1}})}{q} + \frac{\pi_{j_{l_1}; a_{i,j_{l_1}}}(b_{i,j_{l_1},2})}{q^2} + \sum_{k=3}^{u_{j_{l_1}}} \frac{\pi_{i,j_{l_1},k}(0)}{q^k}\right)\Bigg]$$

$$\times E\Bigg[\prod_{l_2=\alpha+1}^{|u|} \psi_{u_{j_{l_2}}-1, t_{j_{l_2}}, c_{j_{l_2}}}$$

$$\times \left(\frac{\pi_{j_{l_2}}(a_{i,j_{l_2}})}{q} + \frac{\pi_{j_{l_2}; a_{i,j_{l_2}}}(b_{i,j_{l_2},2})}{q^2} + \sum_{k=3}^{u_{j_{l_2}}} \frac{\pi_{i,j_{l_2},k}(0)}{q^k}\right)^2\Bigg]$$

$$= \left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1} \sum_{c_{j_1}=0}^{q-1} \sum_{c'_{j_1} \neq c_{j_1}}\right) \cdots \left(\sum_{t_{j_\alpha}=0}^{q^{u_{j_\alpha}-1}-1} \sum_{c_{j_\alpha}=0}^{q-1} \sum_{c'_{j_\alpha} \neq c_{j_\alpha}}\right)$$

$$\times \left(\sum_{t_{j_{\alpha+1}}=0}^{q^{u_{j_{\alpha+1}}-1}-1} \sum_{c_{j_{\alpha+1}}=0}^{q-1}\right) \cdots \left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1} \sum_{c_{j_{|u|}}=0}^{q-1}\right) \left\langle f, \prod_{l=1}^{|u|} \psi_{u_{j_l}-1, t_{j_l}, c_{j_l}}\right\rangle$$

$$\times \left\langle f, \left(\prod_{l_1=1}^{\alpha} \psi_{u_{j_{l_1}}-1, t_{j_{l_1}}, c'_{j_{l_1}}}\right)\left(\prod_{l_2=\alpha+1}^{|u|} \psi_{u_{j_{l_2}}-1, t_{j_{l_2}}, c_{j_{l_2}}}\right)\right\rangle$$

$$\times \left(-\frac{1}{q}\right)^{\alpha}\left(1 - \frac{1}{q}\right)^{|u|-\alpha}$$

$$= \left(-\frac{1}{q}\right)^{\alpha}\left(1 - \frac{1}{q}\right)^{|u|-\alpha} \left[\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1} \sum_{c_{j_1}=0}^{q-1} \left(-\sum_{c'_{j_1}=0}^{q-1} \mathbb{1}\{c'_{j_1} = c_{j_1}\}\right)\right] \cdots$$

$$\times \left[\sum_{t_{j_\alpha}=0}^{q^{u_{j_\alpha}-1}-1} \sum_{c_{j_\alpha}=0}^{q-1} \left(-\sum_{c'_{j_\alpha}=0}^{q-1} \mathbb{1}\{c'_{j_\alpha} = c_{j_\alpha}\}\right)\right]$$

$$\times \left(\sum_{t_{j_{\alpha+1}}=0}^{q^{u_{j_{\alpha+1}}-1}-1} \sum_{c_{j_{\alpha+1}}=0}^{q-1}\right) \cdots \left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1} \sum_{c_{j_{|u|}}=0}^{q-1}\right) \left\langle f, \prod_{l=1}^{|u|} \psi_{u_{j_l}-1, t_{j_l}, c_{j_l}}\right\rangle$$

$$\times \left\langle f, \left(\prod_{l_1=1}^{\alpha} \psi_{u_{j_{l_1}}-1, t_{j_{l_1}}, c'_{j_{l_1}}}\right)\left(\prod_{l_2=\alpha+1}^{|u|} \psi_{u_{j_{l_2}}-1, t_{j_{l_2}}, c_{j_{l_2}}}\right)\right\rangle$$

$$= \frac{1}{q^\alpha}\left(1-\frac{1}{q}\right)^{|u|-\alpha}\left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1}\sum_{c_{j_1}=0}^{q-1}\right)\cdots\left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1}\sum_{c_{j_{|u|}}=0}^{q-1}\right)$$

$$\times \left\langle f, \prod_{l=1}^{|u|} \psi_{u_{j_l}-1, t_{j_l}, c_{j_l}}\right\rangle^2.$$

We conclude via symmetry that

$$E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}), \pi_{j;a_{i,j}}(b_{i,j,2}), \pi_{i,j,3}(0), \dots, \pi_{i,j,u_j}(0) : 1 \le j \le d]^2\}$$

$$= \sum_{\alpha=0}^{|u|}\binom{|u|}{\alpha}\frac{1}{q^\alpha}\left(1-\frac{1}{q}\right)^{|u|-\alpha}\left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1}\sum_{c_{j_1}=0}^{q-1}\right)\cdots\left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1}\sum_{c_{j_{|u|}}=0}^{q-1}\right) \tag{28}$$

$$\times \left\langle f, \prod_{l=1}^{|u|} \psi_{u_{j_l}-1, t_{j_l}, c_{j_l}}\right\rangle^2.$$

The proof that $E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}), \pi_{i,j,2}(0), \dots, \pi_{i,j,u_j}(0) : 1 \le j \le d]^2\}$ is equal to the right-hand side of (28) is similar and is omitted. Finally since the verification that $E\{\nu^*[\pi_{j_1}(a_{i,j_1}); \dots; \pi_{j_{|u|}}(a_{i,j_{|u|}})]^4\} = O(1)$ as $q \to \infty$ is quite long and tedious, we shall refer the reader to Loh (2007) for its proof. □

PROOF OF PROPOSITION 2. We note that $q(\sigma_{oal}W_{oal} - \sigma W) = \sum_{r=1}^d \sum_{1 \le k_1 < \dots < k_r \le d} \Delta_{k_1,\dots,k_r}$, where

$$\Delta_{k_1,\dots,k_r} = \frac{1}{q}\sum_{i=1}^{q^2} \sum_{u_1,\dots,u_d \ge 0 : u_k \ge 2 \Leftrightarrow k \in \{k_1,\dots,k_r\}, u_1+\dots+u_d \ge 3} \nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}),$$
$$\pi_{j;a_{i,j}}(b_{i,j,2}), \pi_{i,j,3}(0), \dots, \pi_{i,j,u_j}(0) : 1 \le j \le d].$$

Here ⇔ denotes if and only if and that given $u_1, \dots, u_d \ge 0$, we write $u_k \ge 1 \Leftrightarrow k \in \{j_1, \dots, j_{|u|}\}$ where $|u| = \sum_{i=1}^d \mathbb{I}\{u_i \ge 1\}$. Now for $r = 1, \dots, d$, we have

$$E(\Delta^2_{1,\dots,r})$$
$$= \frac{1}{q^2}\sum_{i=1}^{q^2} \sum_{u_1,\dots,u_d \ge 0 : u_k \ge 2 \Leftrightarrow k \in \{1,\dots,r\}, u_1+\dots+u_d \ge 3} E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i,j}),$$
$$\pi_{j;a_{i,j}}(b_{i,j,2}), \pi_{i,j,3}(0), \dots, \pi_{i,j,u_j}(0) : 1 \le j \le d]^2\} \tag{29}$$

$$+\frac{1}{q^2}\sum_{i_1=1}^{q^2}\sum_{i_2\neq i_1}\sum_{u_1,\dots,u_d\geq 0:\,u_k\geq 2\Leftrightarrow k\in\{1,\dots,r\},u_1+\cdots+u_d\geq 3} E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i_1,j}),$$
$$\pi_{j;a_{i_1,j}}(b_{i_1,j,2}),\pi_{i_1,j,3}(0),\dots,\pi_{i_1,j,u_j}(0):1\leq j\leq d]$$
$$\times\,\nu_{u_1,\dots,u_d}[\pi_j(a_{i_2,j}),\pi_{j;a_{i_2,j}}(b_{i_2,j,2}),\pi_{i_2,j,3}(0),\dots,\pi_{i_2,j,u_j}(0):$$
$$1\leq j\leq d]\}.$$

Using the fact that $A\in\mathrm{OA}(q^2,d,q,2)$, we further observe that

$$\frac{1}{q^2}\sum_{i_1=1}^{q^2}\sum_{i_2\neq i_1}\sum_{u_1,\dots,u_d\geq 0:u_k\geq 2\Leftrightarrow k\in\{1,\dots,r\},u_1+\cdots+u_d\geq 3} E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i_1,j}),$$
$$\pi_{j;a_{i_1,j}}(b_{i_1,j,2}),\pi_{i_1,j,3}(0),\dots,\pi_{i_1,j,u_j}(0):1\leq j\leq d]$$
$$\times\,\nu_{u_1,\dots,u_d}[\pi_j(a_{i_2,j}),\pi_{j;a_{i_2,j}}(b_{i_2,j,2}),\pi_{i_2,j,3}(0),\dots,\pi_{i_2,j,u_j}(0):1\leq j\leq d]\}$$
$$=\frac{1}{q^2}\sum_{i_1=1}^{q^2}\sum_{0\leq u_1,\dots,u_d\leq 2:\,u_k=2\Leftrightarrow k\in\{1,\dots,r\},u_1+\cdots+u_d\geq 3}$$
$$\times\sum_{i_2\neq i_1}E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{i_1,j}),$$
$$\pi_{j;a_{i_1,j}}(b_{i_1,j,2}),\pi_{i_1,j,3}(0),\dots,\pi_{i_1,j,u_j}(0):1\leq j\leq d]$$
$$\times\,\nu_{u_1,\dots,u_d}[\pi_j(a_{i_2,j}),\pi_{j;a_{i_2,j}}(b_{i_2,j,2}),\pi_{i_2,j,3}(0),\dots,\pi_{i_2,j,u_j}(0):$$
$$1\leq j\leq d]\}$$
$$=\frac{\mathcal{I}\{r=1\}}{q^2}\sum_{i_1=1}^{q^2}\sum_{0\leq u_1,\dots,u_d\leq 2:\,u_k=2\Leftrightarrow k=1,|u|\geq 2}\sum_{i_2\neq i_1}\mathcal{I}\{a_{i_1,1}=a_{i_2,1}\}$$
$$\times\,E\{\nu^*_{u_1,u_{j_2},\dots,u_{j_{|u|}}}[\pi_1(a_{i_1,1}),\pi_{1;a_{i_1,1}}(b_{i_1,1,2});$$
$$\pi_{j_2}(a_{i_1,j_2});\dots;\pi_{j_{|u|}}(a_{i_1,j_{|u|}})]$$
$$\times\,\nu^*_{u_1,u_{j_2},\dots,u_{j_{|u|}}}[\pi_1(a_{i_2,1}),\pi_{1;a_{i_2,1}}(b_{i_2,1,2});$$
$$\pi_{j_2}(a_{i_2,j_2});\dots;\pi_{j_{|u|}}(a_{i_2,j_{|u|}})]\}$$
$$=\frac{\mathcal{I}\{r=1\}}{q^2}\sum_{i_1=1}^{q^2}\sum_{0\leq u_1,\dots,u_d\leq 2:\,u_k=2\Leftrightarrow k=1,|u|\geq 2}\sum_{i_2\neq i_1}\mathcal{I}\{a_{i_1,1}=a_{i_2,1}\}$$
$$\times\,E\Bigg\{\nu^*_{u_1,u_{j_2},\dots,u_{j_{|u|}}}[\pi_1(a_{i_1,1}),\pi_{1;a_{i_1,1}}(b_{i_1,1,2});$$
$$\pi_{j_2}(a_{i_1,j_2});\dots;\pi_{j_{|u|}}(a_{i_1,j_{|u|}})]\tag{30}$$

$$\times\left(\frac{1}{q-1}\right)^{|u|}$$

$$\times \sum_{0\le\tilde c_1\le q-1:\tilde c_1\ne\pi_{1;a_{i_1,1}}(b_{i_1,1,2})}\left[\prod_{2\le m\le|u|}\sum_{0\le\tilde c_{j_m}\le q-1:\tilde c_{j_m}\ne\pi_{j_m}(a_{i_1,j_m})}\right]$$

$$\times \nu^*_{u_1,u_{j_2},\dots,u_{j_{|u|}}}[\pi_1(a_{i_1,1}),\tilde c_1;\tilde c_{j_2};\dots;\tilde c_{j_{|u|}}]\Bigg\}$$

$$=\mathcal{I}\{r=1\}\sum_{0\le u_1,\dots,u_d\le 2:\,u_k=2\Leftrightarrow k=1,|u|\ge2}\frac{(-1)^{|u|}}{(q-1)^{|u|-1}}$$

$$\times E\{\nu^*_{u_1,u_{j_2},\dots,u_{j_{|u|}}}[\pi_1(a_{1,1}),\pi_{1;a_{1,1}}(b_{1,1,2});$$

$$\pi_{j_2}(a_{1,j_2});\dots;\pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\}.$$

It follows from (29) and (30) that

$$q^2E[(\sigma_{oal}W_{oal}-\sigma W)^2]$$

$$=\sum_{r=1}^{d}\sum_{1\le j_{k_1}<\cdots<j_{k_r}\le d}E(\Delta^2_{j_{k_1},\dots,j_{k_r}})$$

$$\begin{aligned}&=\sum_{r=1}^{d}\sum_{1\le j_{k_1}<\cdots<j_{k_r}\le d}\ \sum_{u_1,\dots,u_d\ge0:\,u_k\ge2\Leftrightarrow k\in\{j_{k_1},\dots,j_{k_r}\},u_1+\cdots+u_d\ge3}\\&\qquad\times E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{1,j}),\pi_{j;a_{1,j}}(b_{1,j,2}),\pi_{1,j,3}(0),\dots,\\&\qquad\qquad\pi_{1,j,u_j}(0):1\le j\le d]^2\}\\&\quad+\sum_{r=1}^{d}\sum_{0\le u_1,\dots,u_d\le2:\,u_k=2\Leftrightarrow k=r,|u|\ge2}\frac{(-1)^{|u|}}{(q-1)^{|u|-1}}\\&\qquad\times E\{\nu_{u_1,\dots,u_d}[\pi_j(a_{1,j}),\pi_{j;a_{1,j}}(b_{1,j,2}),\\&\qquad\qquad\pi_{1,j,3}(0),\dots,\pi_{1,j,u_j}(0):1\le j\le d]^2\}.\end{aligned}\tag{31}$$

Finally from (31) and Lemma 4, we have

$$q^2E[(\sigma_{oal}W_{oal}-\sigma W)^2]$$

$$=\sum_{r=1}^{d}\sum_{1\le j_{k_1}<\cdots<j_{k_r}\le d}\ \sum_{u_1,\dots,u_d\ge0:\,u_k\ge2\Leftrightarrow k\in\{j_{k_1},\dots,j_{k_r}\},u_1+\cdots+u_d\ge3}$$

$$\left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1}\sum_{c_{j_1}=0}^{q-1}\right)\cdots\left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1}\sum_{c_{j_{|u|}}=0}^{q-1}\right)\left\langle f,\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\right\rangle^2$$

$$\times\sum_{\alpha=0}^{|u|}\binom{|u|}{\alpha}\frac{1}{q^\alpha}\left(1-\frac{1}{q}\right)^{|u|-\alpha}$$

$$+\sum_{r=1}^{d}\sum_{0\le u_1,\dots,u_d\le 2:\,u_k=2\Leftrightarrow k=r,|u|\ge 2}\frac{(-1)^{|u|}}{(q-1)^{|u|-1}}\left(\sum_{t_{j_1}=0}^{q^{u_{j_1}-1}-1}\sum_{c_{j_1}=0}^{q-1}\right)\cdots$$

$$\left(\sum_{t_{j_{|u|}}=0}^{q^{u_{j_{|u|}}-1}-1}\sum_{c_{j_{|u|}}=0}^{q-1}\right)\left\langle f,\prod_{l=1}^{|u|}\psi_{u_{j_l}-1,t_{j_l},c_{j_l}}\right\rangle^2\sum_{\alpha=0}^{|u|}\binom{|u|}{\alpha}\frac{1}{q^\alpha}\left(1-\frac{1}{q}\right)^{|u|-\alpha}$$

$$=\sum_{r=1}^{d}\binom{d}{r}\sum_{u_1,\dots,u_d\ge 0:\,u_k\ge 2\Leftrightarrow k\in\{1,\dots,r\},u_1+\cdots+u_d\ge 3}O\big(q^{2|u|-2(u_1+\cdots+u_d)}\big)$$

$$+\sum_{0\le u_1,\dots,u_d\le 2:\,u_k=2\Leftrightarrow k=1,|u|\ge 2}\frac{(-1)^{|u|}d}{(q-1)^{|u|-1}}O\big(q^{2|u|-2(u_1+\cdots+u_d)}\big)$$

$$=O(q^{-2}),$$

as $q\to\infty$. Using Chebyshev's inequality, we conclude that $q(\sigma_{oal}W_{oal}-\sigma W)\to 0$ in probability as $q\to\infty$. The proof that $q(\sigma_{oas}W_{oas}-\sigma W)\to 0$ in probability as $q\to\infty$ is similar and can be found in Loh (2007). □

PROOF OF THEOREM 3. In this proof it suffices to take $\Delta=2\sqrt{d-2}$ in (21) and $\mathcal{A}$ to be the class of all indicator functions of measurable convex sets in $\mathbb{R}^{d-2}$. Let J be a random variable uniformly distributed over $\{1,\dots,d\}$ and (B_1,B_2) be a random vector uniformly distributed over the set $\{(b_1,b_2)\in\{0,\dots,q-1\}^2: b_1\neq b_2\}$. J and (B_1,B_2) are independent of each other and are also independent of all previously defined random quantities. Define for $j=1,\dots,d$,

$$\tilde{\pi}_j=\begin{cases}\pi_j, & \text{if } j\neq J,\\ \tau_{B_1,B_2}\circ\pi_j, & \text{if } j=J,\end{cases}$$

where τ_{B_1,B_2} denotes the permutation of $\{0,\dots,q-1\}$ that transposes B_1 and B_2 leaving all other elements fixed. We further define for $\ell=1,\dots,d-2$,

$$\begin{aligned}\tilde{V}_\ell&=\frac{1}{q^2\sigma_\ell}\sum_{i=1}^{q^2}\sum_{0\le u_1,\dots,u_d\le 1:\,|u|=\ell+2}\nu^*[\tilde{\pi}_{j_1}(a_{i,j_1});\dots;\tilde{\pi}_{j_{|u|}}(a_{i,j_{|u|}})],\\ \tilde{V}&=(\tilde{V}_1,\dots,\tilde{V}_{d-2})'.\end{aligned}\tag{32}$$

From symmetry, we observe that $(V, \tilde{V})$ is an exchangeable pair of random vectors in that $(V, \tilde{V})$ and $(\tilde{V}, V)$ possess the same $2(d-2)$-variate distribution. We now write

$$\tilde{V}_\ell - V_\ell = \tilde{S}_\ell - S_\ell, \tag{33}$$

where

$$\tilde{S}_\ell = \frac{1}{q^2\sigma_\ell} \sum_{i=1}^{q^2} \sum_{0 \le u_1,\ldots,u_d \le 1 : |u| = \ell+2} \mathcal{I}\{J \in \{j_1, \ldots, j_{|u|}\}, \pi_J(a_{i,J}) \in \{B_1, B_2\}\} \\ \times \nu^*[\tilde{\pi}_{j_1}(a_{i,j_1}); \ldots; \tilde{\pi}_{j_{|u|}}(a_{i,j_{|u|}})],$$

and

$$S_\ell = \frac{1}{q^2\sigma_\ell} \sum_{i=1}^{q^2} \sum_{0 \le u_1,\ldots,u_d \le 1 : |u| = \ell+2} \mathcal{I}\{J \in \{j_1, \ldots, j_{|u|}\}, \pi_J(a_{i,J}) \in \{B_1, B_2\}\} \\ \times \nu^*[\pi_{j_1}(a_{i,j_1}); \ldots; \pi_{j_{|u|}}(a_{i,j_{|u|}})].$$

Let $\mathcal{W}$ be the σ-field generated by the random quantities $\{\pi_j(a_{i,j}) : i = 1, \ldots, q^2, j = 1, \ldots, d\}$, $E^{\mathcal{W}}$ denote conditional expectation given $\mathcal{W}$ and $\psi_t(\cdot)$ be as in (22). From the exchangeability of $(V, \tilde{V})$, we have for $0 < \varepsilon < 1/2$,

$$\begin{aligned} 0 &= E\left\{(\tilde{V}_i - V_i)\left[\frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(V) + \frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(\tilde{V})\right]\right\} \\ &= 2E\left\{\left[\frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(V)\right]E^{\mathcal{W}}(\tilde{V}_i - V_i)\right\} \\ &\quad + E\left\{(\tilde{V}_i - V_i)\left[\frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(\tilde{V}) - \frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(V)\right]\right\}. \end{aligned}$$

We observe from Proposition 3 (see below) that

$$\begin{aligned} E\left\{(\tilde{V}_i - V_i)\left[\frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(\tilde{V}) - \frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(V)\right]\right\} &= -2E\left\{\left[\frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(V)\right]E^{\mathcal{W}}(\tilde{V}_i - V_i)\right\} \\ &= \frac{4(i+2)}{d(q-1)} E\left[V_i \frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(V)\right]. \end{aligned}$$

Now using Lemma 1, we have

$$\begin{aligned} &E[\chi_{\varepsilon^2}(V|h)] \\ &\quad = E\left[\sum_{i=1}^{d-2} V_i \frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(V) - \sum_{i=1}^{d-2} \frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\right] \\ &\quad = \sum_{i=1}^{d-2} \frac{d(q-1)}{4(i+2)} E\left\{(\tilde{V}_i - V_i)\left[\frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(\tilde{V}) - \frac{\partial}{\partial v_i}\psi_{\varepsilon^2}(V)\right]\right\} \end{aligned}$$

$$
\begin{aligned}
&- E\left[\sum_{i=1}^{d-2} \frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\right] \\
&= \sum_{i=1}^{d-2}\sum_{j=1}^{d-2} \frac{d(q-1)}{4(i+2)} E\biggl\{(\tilde{V}_i - V_i)(\tilde{V}_j - V_j) \\
&\qquad\qquad \times \int_0^1 \frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}\bigl(V + t(\tilde{V} - V)\bigr)\,dt\biggr\} \\
&\quad - E\left[\sum_{i=1}^{d-2} \frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\right] \\
&= \sum_{i=1}^{d-2}\biggl\{\biggl[\frac{d(q-1)}{4(i+2)} E(\tilde{S}_i - S_i)^2 - 1\biggr] E\biggl[\frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\biggr] \\
&\qquad - \frac{d(q-1)}{4(i+2)} E[(\tilde{S}_i - S_i)^2] E\biggl[\frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\biggr] \\
&\qquad + \frac{d(q-1)}{4(i+2)} E\biggl[(\tilde{S}_i - S_i)^2 \frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\biggr] \\
&\qquad + \frac{d(q-1)}{4(i+2)} E\biggl\{(\tilde{S}_i - S_i)^2 \int_0^1 \biggl[\frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}\bigl(V + t(\tilde{V} - V)\bigr) \\
&\qquad\qquad - \frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\biggr]\,dt\biggr\}\biggr\} \\
&\quad + \sum_{i=1}^{d-2}\sum_{j\neq i}\biggl\{\frac{d(q-1)}{4(i+2)} E\biggl\{(\tilde{S}_i - S_i)(\tilde{S}_j - S_j)\frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}(V)\biggr\} \\
&\quad + \frac{d(q-1)}{4(i+2)} E\biggl\{(\tilde{S}_i - S_i)(\tilde{S}_j - S_j)\int_0^1 \biggl[\frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}\bigl(V + t(\tilde{V} - V)\bigr) \\
&\qquad\qquad - \frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}(V)\biggr]\,dt\biggr\}\biggr\}.
\end{aligned}
$$

Hence it follows from Propositions 4 to 7 (see below) that

$$
E[\chi_{\varepsilon^2}(V|h)] = O\left(\frac{\log(1/\varepsilon)}{q^{1/2}}\right) + O\left(\frac{1}{\varepsilon q^{1/2}}\right), \tag{34}
$$

as $q \to \infty$ uniformly over $h \in \mathcal{A}$ and $\varepsilon \in (0, 1/2)$. Using (34) and Lemma 2, we have

$$
\sup_{g\in\mathcal{A}}\left|E[g(V)] - \int_{\mathbb{R}^{d-2}} g(v)\Phi_{d-2}(dv)\right| = O\left(\varepsilon + \frac{\log(1/\varepsilon)}{q^{1/2}} + \frac{1}{\varepsilon q^{1/2}}\right), \tag{35}
$$

as $q \to \infty$ uniformly over $\varepsilon \in (0, 1/2)$. By taking $\varepsilon = q^{-1/4}$, we conclude that the left-hand side of (35) tends to 0 as $q \to \infty$. This implies that V converges to Φ_{d-2} in distribution as $q \to \infty$ and Theorem 3 is proved. $\square$

PROPOSITION 3. *Let V_ℓ and $\tilde{V}_\ell, \ell = 1, \dots, d-2$, be as in (20) and (32) respectively. Then*

$$E^{\mathcal{W}}(\tilde{V}_\ell - V_\ell) = -\frac{2(\ell+2)}{d(q-1)} V_\ell.$$

PROOF. First we observe from (33) that

$$\begin{aligned}
E^{\mathcal{W}}(S_\ell) &= E^{\mathcal{W}}\Biggl\{\frac{1}{dq^2\sigma_\ell}\sum_{i=1}^{q^2}\sum_{k=1}^{d}\sum_{0\le u_1,\dots,u_d\le 1\,:\,|u|=\ell+2} \mathcal{I}\{k \in \{j_1, \dots, j_{|u|}\}\} \\
&\qquad \times \mathcal{I}\{\pi_k(a_{i,k}) \in \{B_1, B_2\}\}\nu^*[\pi_{j_1}(a_{i,j_1}); \cdots; \pi_{j_{|u|}}(a_{i,j_{|u|}})]\Biggr\} \\
&= \frac{2(\ell+2)}{dq^3\sigma_\ell}\sum_{i=1}^{q^2}\sum_{0\le u_1,\dots,u_d\le 1:|u|=\ell+2} \nu^*[\pi_{j_1}(a_{i,j_1}); \cdots; \pi_{j_{|u|}}(a_{i,j_{|u|}})]
\end{aligned}$$

and

$$\begin{aligned}
E^{\mathcal{W}}(\tilde{S}_\ell) &= E^{\mathcal{W}}\Biggl\{\frac{1}{dq^2\sigma_\ell}\sum_{i=1}^{q^2}\sum_{k=1}^{d}\sum_{0\le u_1,\dots,u_d\le 1\,:\,|u|=\ell+2} \mathcal{I}\{k \in \{j_1, \dots, j_{|u|}\}\} \\
&\qquad \times \mathcal{I}\{\pi_k(a_{i,k}) \in \{B_1, B_2\}\}\nu^*[\tilde{\pi}_{j_1}(a_{i,j_1}); \cdots; \tilde{\pi}_{j_{|u|}}(a_{i,j_{|u|}})]\Biggr\} \\
&= E^{\mathcal{W}}\Biggl\{\frac{1}{dq^2\sigma_\ell}\sum_{i=1}^{q^2}\sum_{k=1}^{d}\sum_{0\le u_1,\dots,u_d\le 1\,:\,|u|=\ell+2}\sum_{l=1}^{|u|} \mathcal{I}\{k = j_l\} \\
&\qquad \times \mathcal{I}\{\pi_{j_l}(a_{i,j_l}) = B_1\}\nu^*[\pi_{j_1}(a_{i,j_1}); \dots; \pi_{j_{l-1}}(a_{i,j_{l-1}}); \\
&\qquad\qquad B_2; \pi_{j_{l+1}}(a_{i,j_{l+1}}); \cdots; \tilde{\pi}_{j_{|u|}}(a_{i,j_{|u|}})]\Biggr\} \\
&\quad + E^{\mathcal{W}}\Biggl\{\frac{1}{dq^2\sigma_\ell}\sum_{i=1}^{q^2}\sum_{k=1}^{d}\sum_{0\le u_1,\dots,u_d\le 1\,:\,|u|=\ell+2}\sum_{l=1}^{|u|} \mathcal{I}\{k = j_l\} \\
&\qquad \times \mathcal{I}\{\pi_{j_l}(a_{i,j_l}) = B_2\}\nu^*[\pi_{j_1}(a_{i,j_1}); \dots; \pi_{j_{l-1}}(a_{i,j_{l-1}}); \\
&\qquad\qquad B_1; \pi_{j_{l+1}}(a_{i,j_{l+1}}); \cdots; \tilde{\pi}_{j_{|u|}}(a_{i,j_{|u|}})]\Biggr\}
\end{aligned}$$

$$= -\frac{2(\ell+2)}{dq^3(q-1)\sigma_\ell} \times \sum_{i=1}^{q^2} \sum_{0 \le u_1,\dots,u_d \le 1 : |u|=\ell+2} \nu^*[\pi_{j_1}(a_{i,j_1}); \dots; \pi_{j_{|u|}}(a_{i,j_{|u|}})].$$

Thus we conclude that

$$\begin{aligned} E^{\mathcal{W}}(\tilde{V}_\ell - V_\ell) &= E^{\mathcal{W}}(\tilde{S}_\ell - S_\ell) \\ &= -\frac{2(\ell+2)}{dq^2(q-1)\sigma_\ell} \\ &\quad \times \sum_{i=1}^{q^2} \sum_{0 \le u_1,\dots,u_d \le 1:|u|=\ell+2} \nu^*[\pi_{j_1}(a_{i,j_1}); \dots; \pi_{j_{|u|}}(a_{i,j_{|u|}})] \\ &= -\frac{2(\ell+2)}{d(q-1)} V_\ell. \end{aligned}$$

□

PROPOSITION 4. *Let S_i and $\tilde{S}_i$, $i = 1, \dots, d-2$, be as in (33). Then*

$$\left| \frac{d(q-1)}{4(i+2)} E[(\tilde{S}_i - S_i)^2] - 1 \right| = O(1/q) \qquad \forall i = 1, \dots, d-2,$$

and

$$\left| \sum_{i=1}^{d-2} \left\{ \left[\frac{d(q-1)}{4(i+2)} E(\tilde{S}_i - S_i)^2 - 1 \right] E\left[\frac{\partial^2}{\partial v_i^2} \psi_{\varepsilon^2}(V) \right] \right\} \right| = O\left(\frac{\|h\|_\infty}{q} \right) \log\left(\frac{1}{\varepsilon} \right), \tag{36}$$

as $q \to \infty$ uniformly over $0 < \varepsilon < 1$.

PROOF. For $\ell = 1, \dots, d-2$ and $k = 1, 2$, we write

$$\begin{aligned} \tilde{S}_{\ell,k} &= \frac{1}{q^2\sigma_\ell} \sum_{i=1}^{q^2} \sum_{0 \le u_1,\dots,u_d \le 1 : |u|=\ell+2} \mathcal{I}\{J \in \{j_1, \dots, j_{|u|}\}, \pi_J(a_{i,J}) = B_k\} \\ &\quad \times \nu^*[\tilde{\pi}_{j_1}(a_{i,j_1}); \dots; \tilde{\pi}_{j_{|u|}}(a_{i,j_{|u|}})], \\ S_{\ell,k} &= \frac{1}{q^2\sigma_\ell} \sum_{i=1}^{q^2} \sum_{0 \le u_1,\dots,u_d \le 1 : |u|=\ell+2} \mathcal{I}\{J \in \{j_1, \dots, j_{|u|}\}, \pi_J(a_{i,J}) = B_k\} \\ &\quad \times \nu^*[\pi_{j_1}(a_{i,j_1}); \dots; \pi_{j_{|u|}}(a_{i,j_{|u|}})]. \end{aligned} \tag{37}$$

Then for $i = 1, \dots, d-2$,

$$\begin{aligned}
&\frac{d(q-1)}{4(i+2)} E[(\tilde{S}_i - S_i)^2] \\
&\qquad = \frac{d(q-1)}{4(i+2)} E[(\tilde{S}_{i,1} + \tilde{S}_{i,2} - S_{i,1} - S_{i,2})^2] \\
&\qquad = \frac{d(q-1)}{4(i+2)} E(4S_{i,1}^2 + 4S_{i,1}S_{i,2} - 4\tilde{S}_{i,1}S_{i,1} - 4\tilde{S}_{i,1}S_{i,2}).
\end{aligned} \tag{38}$$

As in Proposition 1, we have for $i = 1, \dots, d-2$,

$$\sigma_i^2 = \frac{1}{q^2}\left[1 + O\left(\frac{1}{q}\right)\right] \sum_{0 \le u_1, \dots, u_d \le 1 \,:\, |u| = i+2} E\{\nu^*[\pi_{j_1}(a_{1,j_1}); \dots; \pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\},$$

as $q \to \infty$. Hence it follows from (38) and Lemma 5 that

$$\frac{d(q-1)}{4(i+2)} E[(\tilde{S}_i - S_i)^2] = 1 + O(1/q) \qquad \forall i = 1, \dots, d-2, \tag{39}$$

as $q \to \infty$. Finally, (36) is an immediate consequence of (39) and Lemma 1. □

LEMMA 5. *Let* $S_{\ell,k}$ *and* $\tilde{S}_{\ell,k}$, $\ell = 1, \dots, d-2$, $k = 1, 2$, *be as in* (37). *Then*

$$\begin{aligned}
E(\tilde{S}_{\ell,k}^2) &= E(S_{\ell,k}^2) \\
&= \frac{\ell+2}{dq^3\sigma_\ell^2} \sum_{0 \le u_1, \dots, u_d \le 1 \,:\, |u| = \ell+2} E\{\nu^*[\pi_{j_1}(a_{1,j_1}); \dots; \pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\} \\
&\quad + \frac{(-1)^{\ell-1}(\ell+2)}{dq^3(q-1)^\ell\sigma_\ell^2} \\
&\quad \times \sum_{0 \le u_1, \dots, u_d \le 1 \,:\, |u| = \ell+2} E\{\nu^*[\pi_{j_1}(a_{1,j_1}); \dots; \pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\}, \\
E(\tilde{S}_{\ell,1}\tilde{S}_{\ell,2}) &= E(S_{\ell,1}S_{\ell,2}) \\
&= \frac{(-1)^{\ell-1}\ell(\ell+2)}{dq^2(q-1)^{\ell+2}\sigma_\ell^2} \\
&\quad \times \sum_{0 \le u_1, \dots, u_d \le 1 \,:\, |u| = \ell+2} E\{\nu^*[\pi_{j_1}(a_{1,j_1}); \dots; \pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\}, \\
E(\tilde{S}_{\ell,1}S_{\ell,2}) &= E(S_{\ell,1}\tilde{S}_{\ell,2}) \\
&= \frac{(-1)^{\ell}\ell(\ell+2)}{dq^2(q-1)^{\ell+1}\sigma_\ell^2} \\
&\quad \times \sum_{0 \le u_1, \dots, u_d \le 1 \,:\, |u| = \ell+2} E\{\nu^*[\pi_{j_1}(a_{1,j_1}); \dots; \pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\},
\end{aligned}$$

$$
\begin{aligned}
E(\tilde{S}_{\ell,1}S_{\ell,1}) &= E(\tilde{S}_{\ell,2}S_{\ell,2}) \\
&= -\frac{\ell+2}{dq^3(q-1)\sigma_\ell^2} \\
&\quad \times \sum_{0\le u_1,\dots,u_d\le 1\,:\,|u|=\ell+2} E\{\nu^*[\pi_{j_1}(a_{1,j_1});\dots;\pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\} \\
&\quad + \frac{(-1)^\ell(\ell+2)}{dq^3(q-1)^{\ell+1}\sigma_\ell^2} \\
&\qquad \times \sum_{0\le u_1,\dots,u_d\le 1\,:\,|u|=\ell+2} E\{\nu^*[\pi_{j_1}(a_{1,j_1});\dots;\pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\}.
\end{aligned}
$$

PROOF. We observe that for $\ell = 1,\dots,d-2$ and $k = 1,2$,

$$
\begin{aligned}
E(S_{\ell,k}^2) &= \frac{1}{q^4\sigma_\ell^2} E\Biggl\{\sum_{i_1=1}^{q^2}\sum_{i_2=1}^{q^2}\sum_{0\le u_1,\dots,u_d\le 1\,:\,|u|=\ell+2} \mathcal{I}\{J\in\{j_1,\dots,j_{|u|}\}\} \\
&\qquad \times \mathcal{I}\{\pi_J(a_{i_1,J}) = \pi_J(a_{i_2,J}) = B_k\} \\
&\qquad \times \nu^*[\pi_{j_1}(a_{i_1,j_1});\dots;\pi_{j_{|u|}}(a_{i_1,j_{|u|}})] \\
&\qquad\qquad \times \nu^*[\pi_{j_1}(a_{i_2,j_1});\dots;\pi_{j_{|u|}}(a_{i_2,j_{|u|}})]\Biggr\} \\
&= \frac{1}{dq^5\sigma_\ell^2}\sum_{i_1=1}^{q^2}\sum_{0\le u_1,\dots,u_d\le 1\,:\,|u|=\ell+2}\sum_{k=1}^{d} \mathcal{I}\{k\in\{j_1,\dots,j_{|u|}\}\} \\
&\quad \times E\{\nu^*[\pi_{j_1}(a_{i_1,j_1});\dots;\pi_{j_{|u|}}(a_{i_1,j_{|u|}})]^2\} \\
&\quad + \frac{1}{dq^5\sigma_\ell^2}\sum_{i_1=1}^{q^2}\sum_{i_2\ne i_1}\sum_{0\le u_1,\dots,u_d\le 1\,:\,|u|=\ell+2}\sum_{k=1}^{d} \mathcal{I}\{k\in\{j_1,\dots,j_{|u|}\}\} \\
&\quad \times \mathcal{I}\{a_{i_1,k} = a_{i_2,k}\}\left(-\frac{1}{q-1}\right)^{|u|-1} \\
&\quad \times E\{\nu^*[\pi_{j_1}(a_{i_1,j_1});\dots;\pi_{j_{|u|}}(a_{i_1,j_{|u|}})]^2\} \\
&= \frac{\ell+2}{dq^3\sigma_\ell^2}\sum_{0\le u_1,\dots,u_d\le 1\,:\,|u|=\ell+2} E\{\nu^*[\pi_{j_1}(a_{1,j_1});\dots;\pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\} \\
&\quad + \frac{\ell+2}{dq^3(q-1)\sigma_\ell^2}\sum_{0\le u_1,\dots,u_d\le 1\,:\,|u|=\ell+2}\left(-\frac{1}{q-1}\right)^{|u|-3} \\
&\quad \times E\{\nu^*[\pi_{j_1}(a_{1,j_1});\dots;\pi_{j_{|u|}}(a_{1,j_{|u|}})]^2\}.
\end{aligned}
$$

The rest of the proof follows in a similar manner and we refer the reader to Loh (2007) for the details. □

PROPOSITION 5. *Let S_i and $\tilde{S}_i$, $i = 1, \ldots, d-2$, be as in* (33). *Then for* $i = 1, \ldots, d-2$,

$$\frac{d(q-1)}{4(i+2)}\left|E\left[(\tilde{S}_i - S_i)^2 \frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\right] - E[(\tilde{S}_i - S_i)^2]E\left[\frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\right]\right| = O\left(\frac{\|h\|_\infty}{q^{1/2}}\right)\log(1/\varepsilon),$$

as $q \to \infty$ *uniformly over* $0 < \varepsilon < 1$.

PROOF. Let $\tilde{S}_{i,k}$ and $S_{i,k}$, $i = 1, \ldots, d-2$, $k = 1, 2$, be as in (37). Using Lemma 5, we have

$$\begin{aligned}
&\frac{d(q-1)}{4(i+2)}\left|E\left[(\tilde{S}_i - S_i)^2 \frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\right] - E[(\tilde{S}_i - S_i)^2]E\left[\frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\right]\right| \\
&\quad = \frac{d(q-1)}{4(i+2)}\left|E\left\{\left[\frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\right]E^{\mathcal{W}}[(\tilde{S}_i - S_i)^2 - E(\tilde{S}_i - S_i)^2]\right\}\right| \\
&\quad \le \frac{d(q-1)}{4(i+2)}\left\{\sup_{v \in \mathbb{R}^{d-2}}\left|\frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(v)\right|\right\} \\
&\qquad \times E|E^{\mathcal{W}}[(\tilde{S}_{i,1} + \tilde{S}_{i,2} - S_{i,1} - S_{i,2})^2 \\
&\qquad\qquad - E(\tilde{S}_{i,1} + \tilde{S}_{i,2} - S_{i,1} - S_{i,2})^2]| \qquad (40)\\
&\quad \le \frac{d(q-1)}{2(i+2)}\left\{\sup_{v \in \mathbb{R}^{d-2}}\left|\frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(v)\right|\right\} \\
&\qquad \times \{E|E^{\mathcal{W}}[\tilde{S}_{i,1}^2 - E(\tilde{S}_{i,1}^2)]| + E|E^{\mathcal{W}}[S_{i,1}^2 - E(S_{i,1}^2)]| \\
&\qquad\quad + E|E^{\mathcal{W}}(\tilde{S}_{i,1}\tilde{S}_{i,2})| + 2E|E^{\mathcal{W}}(\tilde{S}_{i,1}S_{i,1})| \\
&\qquad\quad + 2E|E^{\mathcal{W}}(\tilde{S}_{i,1}S_{i,2})| + E|E^{\mathcal{W}}(S_{i,1}S_{i,2})| + O(q^{-2})\},
\end{aligned}$$

as $q \to \infty$ uniformly over $0 < \varepsilon < 1$. Now Proposition 5 follows from Lemma 1, Lemmas 6 and 7. □

LEMMA 6. *With the notation of* (40), *for* $\ell = 1, \ldots, d-2$,

$$\frac{d(q-1)}{2(\ell+2)}E|E^{\mathcal{W}}[S_{\ell,1}^2 - E(S_{\ell,1}^2)]| = O(q^{-1/2}),$$
$$\frac{d(q-1)}{2(\ell+2)}E|E^{\mathcal{W}}[\tilde{S}_{\ell,1}^2 - E(\tilde{S}_{\ell,1}^2)]| = O(q^{-1/2}), \qquad \text{as } q \to \infty.$$

PROOF. First we observe from (37) that for $\ell = 1, \dots, d-2$,

$$E^{\mathcal{W}}(S_{\ell,1}^2) = \frac{1}{dq^5\sigma_\ell^2} \sum_{i_1=1}^{q^2} \sum_{i_2=1}^{q^2} \sum_{0 \le u_1^{(1)}, \dots, u_d^{(1)} \le 1 : |u^{(1)}| = \ell+2} \sum_{0 \le u_1^{(2)}, \dots, u_d^{(2)} \le 1 : |u^{(2)}| = \ell+2}$$
$$\times \sum_{k=1}^{d} \mathbb{1}\{k \in \{j_{1,1}, \dots, j_{1,|u^{(1)}|}\} \cap \{j_{2,1}, \dots, j_{2,|u^{(2)}|}\}\} \mathbb{1}\{a_{i_1,k} = a_{i_2,k}\}$$
$$\times \nu^*[\pi_{j_{1,1}}(a_{i_1,j_{1,1}}); \dots; \pi_{j_{1,|u^{(1)}|}}(a_{i_1,j_{1,|u^{(1)}|}})]$$
$$\times \nu^*[\pi_{j_{2,1}}(a_{i_2,j_{2,1}}); \dots; \pi_{j_{2,|u^{(2)}|}}(a_{i_2,j_{2,|u^{(2)}|}})].$$

Here for $l = 1, \dots, 4$, given $u^{(l)} = (u_1^{(l)}, \dots, u_d^{(l)})'$, we write $k \in \{j_{l,1}, \dots, j_{l,|u^{(l)}|}\}$ if and only if $u_k^{(l)} \ge 1$. Hence

$$\begin{aligned}
&E\{[E^{\mathcal{W}}(S_{\ell,1}^2)]^2\} \\
&\quad = E\Biggl\{\frac{1}{d^2q^{10}\sigma_\ell^4} \sum_{i_1=1}^{q^2} \sum_{i_2=1}^{q^2} \sum_{i_3=1}^{q^2} \sum_{i_4=1}^{q^2} \sum_{0 \le u_1^{(1)}, \dots, u_d^{(1)} \le 1 : |u^{(1)}| = \ell+2} \\
&\qquad \times \sum_{0 \le u_1^{(2)}, \dots, u_d^{(2)} \le 1 : |u^{(2)}| = \ell+2} \sum_{0 \le u_1^{(3)}, \dots, u_d^{(3)} \le 1 : |u^{(3)}| = \ell+2} \\
&\qquad \times \sum_{0 \le u_1^{(4)}, \dots, u_d^{(4)} \le 1 : |u^{(4)}| = \ell+2} \sum_{k_1=1}^{d} \sum_{k_3=1}^{d} \mathbb{1}\{k_1 \in \{j_{1,1}, \dots, j_{1,|u^{(1)}|}\} \\
&\qquad\qquad \cap \{j_{2,1}, \dots, j_{2,|u^{(2)}|}\}\} \\
&\qquad \times \mathbb{1}\{k_3 \in \{j_{3,1}, \dots, j_{3,|u^{(3)}|}\} \cap \{j_{4,1}, \dots, j_{4,|u^{(4)}|}\}\} \\
&\qquad \times \mathbb{1}\{a_{i_1,k_1} = a_{i_2,k_1}\} \mathbb{1}\{a_{i_3,k_3} = a_{i_4,k_3}\} \\
&\qquad \times \nu^*[\pi_{j_{1,1}}(a_{i_1,j_{1,1}}); \dots; \pi_{j_{1,|u^{(1)}|}}(a_{i_1,j_{1,|u^{(1)}|}})] \\
&\qquad \times \nu^*[\pi_{j_{2,1}}(a_{i_2,j_{2,1}}); \dots; \pi_{j_{2,|u^{(2)}|}}(a_{i_2,j_{2,|u^{(2)}|}})] \\
&\qquad \times \nu^*[\pi_{j_{3,1}}(a_{i_3,j_{3,1}}); \dots; \pi_{j_{3,|u^{(3)}|}}(a_{i_3,j_{3,|u^{(3)}|}})] \\
&\qquad \times \nu^*[\pi_{j_{4,1}}(a_{i_4,j_{4,1}}); \dots; \pi_{j_{4,|u^{(4)}|}}(a_{i_4,j_{4,|u^{(4)}|}})]\Biggr\} \\
&\quad = R_{\{1,2,3,4\}} + R_{\{1,2,3\},\{4\}} + R_{\{1,2,4\},\{3\}} + R_{\{1,3,4\},\{2\}} + R_{\{2,3,4\},\{1\}} \\
&\qquad + R_{\{1,2\},\{3\},\{4\}} + R_{\{1,3\},\{2\},\{4\}} + R_{\{1,4\},\{2\},\{3\}} + R_{\{2,3\},\{1\},\{4\}} \\
&\qquad + R_{\{2,4\},\{1\},\{3\}} + R_{\{3,4\},\{1\},\{2\}} + R_{\{1,2\},\{3,4\}} + R_{\{1,3\},\{2,4\}} \\
&\qquad + R_{\{1,4\},\{2,3\}} + R_{\{1\},\{2\},\{3\},\{4\}},
\end{aligned} \tag{41}$$

where given a partition, say $A_1, \dots, A_p$, of $\{1,2,3,4\}$, (i.e., $\bigcup_{i=1}^p A_i = \{1,2,3,4\}$ and $A_i \cap A_j = \varnothing$ whenever $i \neq j$), we define

$$
\begin{aligned}
R_{A_1,\dots,A_p} = E\Biggl\{ & \frac{1}{d^2 q^{10}\sigma_\ell^4} \sum\nolimits^* \sum_{0\le u_1^{(1)},\dots,u_d^{(1)}\le 1 : |u^{(1)}|=\ell+2} \sum_{0\le u_1^{(2)},\dots,u_d^{(2)}\le 1 : |u^{(2)}|=\ell+2} \\
& \times \sum_{0\le u_1^{(3)},\dots,u_d^{(3)}\le 1 : |u^{(3)}|=\ell+2} \sum_{0\le u_1^{(4)},\dots,u_d^{(4)}\le 1 : |u^{(4)}|=\ell+2} \\
& \times \sum_{k_1=1}^d \sum_{k_3=1}^d \mathcal{I}\{k_1 \in \{j_{1,1},\dots,j_{1,|u^{(1)}|}\} \cap \{j_{2,1},\dots,j_{2,|u^{(2)}|}\}\} \\
& \times \mathcal{I}\{k_3 \in \{j_{3,1},\dots,j_{3,|u^{(3)}|}\} \cap \{j_{4,1},\dots,j_{4,|u^{(4)}|}\}\} \\
& \times \mathcal{I}\{a_{i_1,k_1} = a_{i_2,k_1}\}\mathcal{I}\{a_{i_3,k_3} = a_{i_4,k_3}\} \qquad (42) \\
& \times \nu^*[\pi_{j_{1,1}}(a_{i_1,j_{1,1}});\dots;\pi_{j_{1,|u^{(1)}|}}(a_{i_1,j_{1,|u^{(1)}|}})] \\
& \times \nu^*[\pi_{j_{2,1}}(a_{i_2,j_{2,1}});\dots;\pi_{j_{2,|u^{(2)}|}}(a_{i_2,j_{2,|u^{(2)}|}})] \\
& \times \nu^*[\pi_{j_{3,1}}(a_{i_3,j_{3,1}});\dots;\pi_{j_{3,|u^{(3)}|}}(a_{i_3,j_{3,|u^{(3)}|}})] \\
& \times \nu^*[\pi_{j_{4,1}}(a_{i_4,j_{4,1}});\dots;\pi_{j_{4,|u^{(4)}|}}(a_{i_4,j_{4,|u^{(4)}|}})]\Biggr\},
\end{aligned}
$$

and Σ^* denotes summation over $1 \le i_1, i_2, i_3, i_4 \le q^2$ such that if $k, l \in A_j$, then $i_k = i_l$, and if k, l are in different A_j's, then $i_k \neq i_l$. In order to evaluate the terms on the right-hand side of (41), it is convenient to further define the following subsets of $\{1,\dots,d\}$: for $\{l_1,l_2,l_3,l_4\} = \{1,2,3,4\}$,

$$
\begin{aligned}
\Theta_{\{1,2,3,4\}} &= \Biggl\{ l \in \bigcap_{\alpha=1}^4 \{j_{l_\alpha,1},\dots,j_{l_\alpha,|u^{(l_\alpha)}|}\}\Biggr\}, \\
\Theta_{\{l_1,l_2,l_3\}} &= \Biggl\{ l \in \bigcap_{\alpha=1}^3 \{j_{l_\alpha,1},\dots,j_{l_\alpha,|u^{(l_\alpha)}|}\}\Biggr\} \Bigg\backslash \{j_{l_4,1},\dots,j_{l_4,|u^{(l_4)}|}\}, \\
\Theta_{\{l_1,l_2\}} &= \Biggl\{ l \in \bigcap_{\alpha=1}^2 \{j_{l_\alpha,1},\dots,j_{l_\alpha,|u^{(l_\alpha)}|}\}\Biggr\} \Bigg\backslash \bigcup_{\beta=3}^4 \{j_{l_\beta,1},\dots,j_{l_\beta,|u^{(l_\beta)}|}\}, \\
\Theta_{\{l_1\}} &= \Biggl\{ l \in \{j_{l_1,1},\dots,j_{l_1,|u^{(l_1)}|}\}\Biggr\} \Bigg\backslash \bigcup_{\beta=2}^4 \{j_{l_\beta,1},\dots,j_{l_\beta,|u^{(l_\beta)}|}\}.
\end{aligned}
$$

Now we observe from Lemma 4 that as $q \to \infty$, $R_{\{1,2,3,4\}} = O(q^{-4})$, $R_{\{1,2,3\},\{4\}} = R_{\{1,2,4\},\{3\}} = R_{\{1,3,4\},\{2\}} = R_{\{2,3,4\},\{1\}} = O(q^{-3})$, $R_{\{1,3\},\{2\},\{4\}} = R_{\{1,4\},\{2\},\{3\}} =$

$R_{\{2,3\},\{1\},\{4\}} = R_{\{2,4\},\{1\},\{3\}} = O(q^{-3})$, $R_{\{1,3\},\{2,4\}} = R_{\{1,4\},\{2,3\}} = O(q^{-3})$,

$$
\begin{aligned}
&R_{\{1,2\},\{3\},\{4\}} \\
&\quad = R_{\{3,4\},\{1\},\{2\}} \\
&\quad = E\Biggl\{\frac{1}{d^2 q^{10}\sigma_\ell^4} \sum_{i_1=i_2=1}^{q^2} \sum_{1\le i_3\le q^2\,:\, i_3\ne i_1} \sum_{1\le i_4\le q^2\,:\, i_4\ne i_1,i_3} \\
&\qquad \times \sum_{0\le u_1^{(1)},\dots,u_d^{(1)}\le 1\,:\,|u^{(1)}|=\ell+2} \sum_{0\le u_1^{(2)},\dots,u_d^{(2)}\le 1\,:\,|u^{(2)}|=\ell+2} \\
&\qquad \times \sum_{0\le u_1^{(3)},\dots,u_d^{(3)}\le 1\,:\,|u^{(3)}|=\ell+2} \sum_{0\le u_1^{(4)},\dots,u_d^{(4)}\le 1\,:\,|u^{(4)}|=\ell+2} \\
&\qquad \times \sum_{k_1=1}^{d} \sum_{k_3=1}^{d} \mathbb{1}\bigl\{k_1 \in \{j_{1,1},\dots,j_{1,|u^{(1)}|}\} \cap \{j_{2,1},\dots,j_{2,|u^{(2)}|}\}\bigr\} \\
&\qquad \times \mathbb{1}\bigl\{k_3 \in \{j_{3,1},\dots,j_{3,|u^{(3)}|}\} \cap \{j_{4,1},\dots,j_{4,|u^{(4)}|}\}\bigr\} \\
&\qquad \times \mathbb{1}\{a_{i_3,k_3} = a_{i_4,k_3}\}\nu^*\bigl[\pi_{j_{1,1}}(a_{i_1,j_{1,1}});\dots;\pi_{j_{1,|u^{(1)}|}}(a_{i_1,j_{1,|u^{(1)}|}})\bigr] \\
&\qquad \times \nu^*\bigl[\pi_{j_{2,1}}(a_{i_2,j_{2,1}});\dots;\pi_{j_{2,|u^{(2)}|}}(a_{i_2,j_{2,|u^{(2)}|}})\bigr] \\
&\qquad \times \nu^*\bigl[\pi_{j_{3,1}}(a_{i_3,j_{3,1}});\dots;\pi_{j_{3,|u^{(3)}|}}(a_{i_3,j_{3,|u^{(3)}|}})\bigr] \\
&\qquad \times \nu^*\bigl[\pi_{j_{4,1}}(a_{i_4,j_{4,1}});\dots;\pi_{j_{4,|u^{(4)}|}}(a_{i_4,j_{4,|u^{(4)}|}})\bigr]\Biggr\} \\
&\quad = \frac{O(1)}{q^{10}\sigma_\ell^4} \sum_{i_1=1}^{q^2} \sum_{1\le i_3\le q^2 : i_3\ne i_1} \sum_{1\le i_4\le q^2 : i_4\ne i_1,i_3} \sum_{0\le u_1^{(1)},\dots,u_d^{(1)}\le 1\,:\,|u^{(1)}|=\ell+2} \\
&\qquad \times \sum_{0\le u_1^{(2)},\dots,u_d^{(2)}\le 1 : |u^{(2)}|=\ell+2} \sum_{0\le u_1^{(3)},\dots,u_d^{(3)}\le 1 : |u^{(3)}|=\ell+2} \sum_{0\le u_1^{(4)},\dots,u_d^{(4)}\le 1 : |u^{(4)}|=\ell+2} \\
&\qquad \times \sum_{k_3=1}^{d} \mathbb{1}\bigl\{k_3 \in \{j_{3,1},\dots,j_{3,|u^{(3)}|}\} \cap \{j_{4,1},\dots,j_{4,|u^{(4)}|}\}\bigr\}\mathbb{1}\{a_{i_3,k_3} = a_{i_4,k_3}\} \\
&\qquad \times 0^{|\Theta_{\{4\}}|}\left(\frac{1}{q}\right)^{|\Theta_{\{1,2,3,4\}}|+|\Theta_{\{1,2,4\}}|+|\Theta_{\{1,3,4\}}|+|\Theta_{\{2,3,4\}}|+|\Theta_{\{1,4\}}|+|\Theta_{\{2,4\}}|+|\Theta_{\{3,4\}}|-1} \\
&\quad = \frac{O(1)}{q^{10}\sigma_\ell^4} \sum_{i_1=1}^{q^2} \sum_{1\le i_3\le q^2\,:\, i_3\ne i_1} \sum_{1\le i_4\le q^2 : i_4\ne i_1,i_3} \sum_{0\le u_1^{(1)},\dots,u_d^{(1)}\le 1 : |u^{(1)}|=\ell+2}
\end{aligned}
$$

$$\times \sum_{0\le u_1^{(2)},\dots,u_d^{(2)}\le 1:|u^{(2)}|=\ell+2} \sum_{0\le u_1^{(3)},\dots,u_d^{(3)}\le 1:|u^{(3)}|=\ell+2} \sum_{0\le u_1^{(4)},\dots,u_d^{(4)}\le 1:|u^{(4)}|=\ell+2}$$

$$\times \sum_{k_3=1}^{d} \mathcal{I}\{k_3 \in \{j_{3,1},\dots,j_{3,|u^{(3)}|}\} \cap \{j_{4,1},\dots,j_{4,|u^{(4)}|}\}\}$$

$$\times \mathcal{I}\{a_{i_3,k_3} = a_{i_4,k_3}\}\left(\frac{1}{q}\right)^{|u^{(4)}|-1}$$

$$= O\left(\frac{1}{q^3}\right),$$

$$R_{\{1\},\{2\},\{3\},\{4\}}$$

$$= E\left\{\frac{1}{d^2 q^{10}\sigma_\ell^4} \sum_{i_1=1}^{q^2} \sum_{1\le i_2\le q^2:i_2\ne i_1} \sum_{1\le i_3\le q^2:i_3\ne i_1,i_2} \sum_{1\le i_4\le q^2:i_4\ne i_1,i_2,i_3}\right.$$

$$\times \sum_{0\le u_1^{(1)},\dots,u_d^{(1)}\le 1:|u^{(1)}|=\ell+2} \sum_{0\le u_1^{(2)},\dots,u_d^{(2)}\le 1:|u^{(2)}|=\ell+2}$$

$$\times \sum_{0\le u_1^{(3)},\dots,u_d^{(3)}\le 1:|u^{(3)}|=\ell+2} \sum_{0\le u_1^{(4)},\dots,u_d^{(4)}\le 1:|u^{(4)}|=\ell+2}$$

$$\times \sum_{k_1=1}^{d}\sum_{k_3=1}^{d} \mathcal{I}\{k_1 \in \{j_{1,1},\dots,j_{1,|u^{(1)}|}\} \cap \{j_{2,1},\dots,j_{2,|u^{(2)}|}\}\}$$

$$\times \mathcal{I}\{k_3 \in \{j_{3,1},\dots,j_{3,|u^{(3)}|}\} \cap \{j_{4,1},\dots,j_{4,|u^{(4)}|}\}\}$$

$$\times \mathcal{I}\{a_{i_1,k_1} = a_{i_2,k_1}\}\mathcal{I}\{a_{i_3,k_3} = a_{i_4,k_3}\}$$

$$\times \nu^*[\pi_{j_{1,1}}(a_{i_1,j_{1,1}});\dots;\pi_{j_{1,|u^{(1)}|}}(a_{i_1,j_{1,|u^{(1)}|}})]$$

$$\times \nu^*[\pi_{j_{2,1}}(a_{i_2,j_{2,1}});\dots;\pi_{j_{2,|u^{(2)}|}}(a_{i_2,j_{2,|u^{(2)}|}})]$$

$$\times \nu^*[\pi_{j_{3,1}}(a_{i_3,j_{3,1}});\dots;\pi_{j_{3,|u^{(3)}|}}(a_{i_3,j_{3,|u^{(3)}|}})]$$

$$\left.\times \nu^*[\pi_{j_{4,1}}(a_{i_4,j_{4,1}});\dots;\pi_{j_{4,|u^{(4)}|}}(a_{i_4,j_{4,|u^{(4)}|}})]\right\}$$

$$= \frac{O(1)}{q^{10}\sigma_\ell^4} \sum_{i_1=1}^{q^2} \sum_{1\le i_2\le q^2:\, i_2\ne i_1} \sum_{1\le i_3\le q^2:\, i_3\ne i_1,i_2} \sum_{1\le i_4\le q^2:\, i_4\ne i_1,i_2,i_3}$$

$$\times \sum_{0\le u_1^{(1)},\dots,u_d^{(1)}\le 1:|u^{(1)}|=\ell+2} \sum_{0\le u_1^{(2)},\dots,u_d^{(2)}\le 1:|u^{(2)}|=\ell+2}$$

$$\times \sum_{0\le u_1^{(3)},\dots,u_d^{(3)}\le 1:|u^{(3)}|=\ell+2} \sum_{0\le u_1^{(4)},\dots,u_d^{(4)}\le 1:|u^{(4)}|=\ell+2}$$

$$\times \sum_{k_1=1}^{d}\sum_{k_3=1}^{d} \mathscr{I}\{k_1 \in \{j_{1,1},\dots,j_{1,|u^{(1)}|}\} \cap \{j_{2,1},\dots,j_{2,|u^{(2)}|}\}\}$$

$$\times \mathscr{I}\{k_3 \in \{j_{3,1},\dots,j_{3,|u^{(3)}|}\} \cap \{j_{4,1},\dots,j_{4,|u^{(4)}|}\}\}$$

$$\times \mathscr{I}\{a_{i_1,k_1} = a_{i_2,k_1}\}\mathscr{I}\{a_{i_3,k_3} = a_{i_4,k_3}\}\mathscr{I}\{|\Theta_{\{1,2\}}| \ge |\Theta_{\{3,4\}}|\}$$

$$\times 0^{|\Theta_{\{1\}}|+|\Theta_{\{2\}}|+|\Theta_{\{3\}}|+|\Theta_{\{4\}}|}$$

$$\times \left(\frac{1}{q^2}\right)^{|\Theta_{\{1,2,3,4\}}|+|\Theta_{\{1,2,3\}}|+|\Theta_{\{1,2,4\}}|+|\Theta_{\{1,3,4\}}|+|\Theta_{\{2,3,4\}}|}$$

$$\times \left(\frac{1}{q}\right)^{|\Theta_{\{1,2\}}|+|\Theta_{\{1,3\}}|+|\Theta_{\{1,4\}}|+|\Theta_{\{2,3\}}|+|\Theta_{\{2,4\}}|+|\Theta_{\{3,4\}}|-2}$$

$$= \frac{O(1)}{q^{10}\sigma_\ell^4} \sum_{i_1=1}^{q^2} \sum_{1\le i_2\le q^2: i_2\ne i_1} \sum_{1\le i_3\le q^2: i_3\ne i_1,i_2} \sum_{1\le i_4\le q^2: i_4\ne i_1,i_2,i_3}$$

$$\times \sum_{0\le u_1^{(1)},\dots,u_d^{(1)}\le 1:|u^{(1)}|=\ell+2} \sum_{0\le u_1^{(2)},\dots,u_d^{(2)}\le 1:|u^{(2)}|=\ell+2}$$

$$\times \sum_{0\le u_1^{(3)},\dots,u_d^{(3)}\le 1:|u^{(3)}|=\ell+2} \sum_{0\le u_1^{(4)},\dots,u_d^{(4)}\le 1:|u^{(4)}|=\ell+2}$$

$$\times \sum_{k_1=1}^{d}\sum_{k_3=1}^{d} \mathscr{I}\{a_{i_1,k_1} = a_{i_2,k_1}\}\mathscr{I}\{a_{i_3,k_3} = a_{i_4,k_3}\}\left(\frac{1}{q}\right)^{|u^{(3)}|+|u^{(4)}|-2}$$

$$= O\left(\frac{1}{q^4}\right).$$

The second last equality can be obtained using the heuristic that $a_{i_1,j} \ne a_{i_2,j}$ when $i_1 \ne i_2$ etc. and $k_1 \ne k_3$. However the above bound remains valid when $k_1 = k_3$ or when $a_{i_1,j} = a_{i_2,j}$ since this additional constraint introduces a factor of q while it reduces the number of $i_1,\dots,i_4$ by a factor of $1/q$. Finally,

$$R_{\{1,2\},\{3,4\}} = \frac{1}{d^2 q^{10}\sigma_\ell^4} \sum_{i_1=1}^{q^2} \sum_{1\le i_3\le q^2: i_3\ne i_1} \sum_{0\le u_1^{(1)},\dots,u_d^{(1)}\le 1:|u^{(1)}|=\ell+2}$$

$$\times \sum_{0\le u_1^{(2)},\dots,u_d^{(2)}\le 1:|u^{(2)}|=\ell+2} \sum_{0\le u_1^{(3)},\dots,u_d^{(3)}\le 1:|u^{(3)}|=\ell+2}$$

$$
\begin{aligned}
&\times \sum_{0\le u_1^{(4)},\dots,u_d^{(4)}\le 1:|u^{(4)}|=\ell+2} \\
&\times \sum_{k_1=1}^{d}\sum_{k_3=1}^{d} \mathcal{I}\{k_1 \in \{j_{1,1},\dots,j_{1,|u^{(1)}|}\} \cap \{j_{2,1},\dots,j_{2,|u^{(2)}|}\}\} \\
&\times \mathcal{I}\{k_3 \in \{j_{3,1},\dots,j_{3,|u^{(3)}|}\} \cap \{j_{4,1},\dots,j_{4,|u^{(4)}|}\}\} \\
&\times E\{\nu^*[\pi_{j_{1,1}}(a_{i_1,j_{1,1}});\dots;\pi_{j_{1,|u^{(1)}|}}(a_{i_1,j_{1,|u^{(1)}|}})] \\
&\times \nu^*[\pi_{j_{2,1}}(a_{i_1,j_{2,1}});\dots;\pi_{j_{2,|u^{(2)}|}}(a_{i_1,j_{2,|u^{(2)}|}})] \\
&\times \nu^*[\pi_{j_{3,1}}(a_{i_3,j_{3,1}});\dots;\pi_{j_{3,|u^{(3)}|}}(a_{i_3,j_{3,|u^{(3)}|}})] \\
&\times \nu^*[\pi_{j_{4,1}}(a_{i_3,j_{4,1}});\dots;\pi_{j_{4,|u^{(4)}|}}(a_{i_3,j_{4,|u^{(4)}|}})]\} \\
={}& \frac{1}{d^2q^6\sigma_\ell^4} \sum_{0\le u_1^{(1)},\dots,u_d^{(1)}\le 1:|u^{(1)}|=\ell+2}\ \sum_{0\le u_1^{(3)},\dots,u_d^{(3)}\le 1:|u^{(3)}|=\ell+2} |u^{(1)}||u^{(3)}| \\
&\times E\{\nu^*[\pi_{j_{1,1}}(a_{1,j_{1,1}});\dots;\pi_{j_{1,|u^{(1)}|}}(a_{1,j_{1,|u^{(1)}|}})]^2\} \\
&\times E\{\nu^*[\pi_{j_{3,1}}(a_{1,j_{3,1}});\dots;\pi_{j_{3,|u^{(3)}|}}(a_{1,j_{3,|u^{(3)}|}})]^2\} + O\left(\frac{1}{q^3}\right) \\
={}& [E(S_{\ell,1}^2)]^2 + O\left(\frac{1}{q^3}\right),
\end{aligned}
$$

as $q \to \infty$. The last equality uses Lemma 5. Consequently it follows from (41) that

$$
\begin{aligned}
&\frac{d(q-1)}{2(\ell+2)} E|E^{\mathcal{W}}[S_{\ell,1}^2 - E(S_{\ell,1}^2)]| \\
&\qquad \le \frac{d(q-1)}{2(\ell+2)}\{E\{[E^{\mathcal{W}}(S_{\ell,1}^2) - E(S_{\ell,1}^2)]^2\}\}^{1/2} \\
&\qquad = \frac{d(q-1)}{2(\ell+2)}\{E\{[E^{\mathcal{W}}(S_{\ell,1}^2)]^2\} - [E(S_{\ell,1}^2)]^2\}^{1/2} \\
&\qquad = O\left(\frac{1}{q^{1/2}}\right),
\end{aligned}
$$

as $q \to \infty$. The analogous result for $\tilde{S}_{\ell,1}$ can be proved similarly and we refer the reader to Loh (2007) for the details. This proves Lemma 6. □

LEMMA 7. *Let S_ℓ, $\ell = 1,\dots,d-2$, be as in (33). Then $E(S_\ell^4) = O(1/q^2)$ as $q \to \infty$. Let $S_{\ell,1}$ and $\tilde{S}_{\ell,1}$, $\ell = 1,\dots,d-2$, be as in (37). Then for $1 \le \ell_1, \ell_2 \le$*

$d-2$,

$$\frac{d(q-1)}{2(\ell_1+2)}E|E^{\mathcal{W}}(S_{\ell_1,1}S_{\ell_2,2})| = O\left(\frac{1}{q}\right),$$
$$\frac{d(q-1)}{\ell_1+2}E|E^{\mathcal{W}}(\tilde{S}_{\ell_1,1}S_{\ell_2,2})| = O\left(\frac{1}{q}\right),$$
$$\frac{d(q-1)}{\ell_1+2}E|E^{\mathcal{W}}(\tilde{S}_{\ell_1,1}S_{\ell_2,1})| = O\left(\frac{1}{q}\right),$$
$$\frac{d(q-1)}{\ell_1+2}E|E^{\mathcal{W}}(\tilde{S}_{\ell_1,1}\tilde{S}_{\ell_2,2})| = O\left(\frac{1}{q}\right), \qquad \text{as } q \to \infty.$$

Finally for $1 \le \ell_1 \ne \ell_2 \le d-2$,

$$\frac{d(q-1)}{2(\ell_1+2)}E|E^{\mathcal{W}}(S_{\ell_1,1}S_{\ell_2,1})| = O(q^{-1/2}),$$
$$\frac{d(q-1)}{2(\ell_1+2)}E|E^{\mathcal{W}}(\tilde{S}_{\ell_1,1}\tilde{S}_{\ell_2,1})| = O(q^{-1/2}), \qquad \text{as } q \to \infty.$$

The proof of Lemma 7 is similar to Lemma 6 and can be found in Loh (2007).

PROPOSITION 6. *Let* S_i *and* $\tilde{S}_i$, $i = 1, \dots, d-2$, *be as in* (33). *Then for* $1 \le i \ne j \le d-2$,

$$\left|\frac{d(q-1)}{4(i+2)}E\left\{(\tilde{S}_i - S_i)^2 \int_0^1 \left[\frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}\big(V + t(\tilde{V} - V)\big) - \frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\right]dt\right\}\right|$$
$$= O\left(\frac{\|h\|_\infty}{\varepsilon q^{1/2}}\right),$$

and

$$\left|\frac{d(q-1)}{4(i+2)}E\left\{(\tilde{S}_i - S_i)(\tilde{S}_j - S_j)\right.\right.$$
$$\left.\left.\times \int_0^1 \left[\frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}\big(V + t(\tilde{V} - V)\big) - \frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}(V)\right]dt\right\}\right|$$
$$= O\left(\frac{\|h\|_\infty}{\varepsilon q^{1/2}}\right), \qquad \text{as } q \to \infty \text{ uniformly over } 0 < \varepsilon < 1.$$

PROOF. Using Taylor series and Lemma 1, we observe that for $0 < \varepsilon < 1$ and $i = 1, \dots, d-2$,

$$\left|\frac{d(q-1)}{4(i+2)}E\left\{(\tilde{S}_i - S_i)^2 \int_0^1 \left[\frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}\big(V + t(\tilde{V} - V)\big) - \frac{\partial^2}{\partial v_i^2}\psi_{\varepsilon^2}(V)\right]dt\right\}\right|$$

$$\leq \sum_{j=1}^{d-2} \frac{d(q-1)}{4(i+2)} E\biggl[(\tilde{S}_i - S_i)^2 |\tilde{S}_j - S_j| \sup_{v\in\mathbb{R}^{d-2}} \biggl|\frac{\partial^3}{\partial v_i^2\,\partial v_j}\psi_{\varepsilon^2}(v)\biggr|\biggr]$$

$$\leq \frac{c\|h\|_\infty}{\varepsilon} \sum_{j=1}^{d-2} \frac{d(q-1)}{4(i+2)} \{E[(\tilde{S}_i - S_i)^4]\}^{1/2} \{E[(\tilde{S}_j - S_j)^2]\}^{1/2}$$

$$\leq \frac{2^{1/2} c\|h\|_\infty}{\varepsilon} \sum_{j=1}^{d-2} \frac{d(q-1)}{4(i+2)} [E(S_i^4)]^{1/2} \{E[(\tilde{S}_j - S_j)^2]\}^{1/2},$$

where c is a constant depending only on d. In a similar fashion, we have for $1 \leq i \neq j \leq d-2$,

$$\biggl|\frac{d(q-1)}{4(i+2)} E\biggl\{(\tilde{S}_i - S_i)(\tilde{S}_j - S_j) \times \int_0^1 \biggl[\frac{\partial^2}{\partial v_i \partial v_j}\psi_{\varepsilon^2}\bigl(V + t(\tilde{V} - V)\bigr) - \frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}(V)\biggr] dt\biggr\}\biggr|$$

$$\leq \sum_{k=1}^{d-2} \frac{d(q-1)}{4(i+2)} \times E\biggl[|\tilde{S}_i - S_i||\tilde{S}_j - S_j||\tilde{S}_k - S_k| \sup_{v\in\mathbb{R}^{d-2}} \biggl|\frac{\partial^3}{\partial v_i\,\partial v_j\,\partial v_k}\psi_{\varepsilon^2}(v)\biggr|\biggr]$$

$$\leq \frac{c\|h\|_\infty}{\varepsilon} \sum_{k=1}^{d-2} \frac{d(q-1)}{4(i+2)} \{E[(\tilde{S}_i - S_i)^4]\}^{1/4} \times \{E[(\tilde{S}_j - S_j)^4]\}^{1/4} \{E[(\tilde{S}_k - S_k)^2]\}^{1/2}$$

$$\leq \frac{2^{1/2} c\|h\|_\infty}{\varepsilon} \sum_{k=1}^{d-2} \frac{d(q-1)}{4(i+2)} [E(S_i^4)]^{1/4} [E(S_j^4)]^{1/4} \{E[(\tilde{S}_k - S_k)^2]\}^{1/2}.$$

Proposition 6 now follows from Proposition 4 and Lemma 7. □

PROPOSITION 7. *Let S_i and $\tilde{S}_i$, $i = 1, \dots, d-2$, be as in* (33). *Then for $1 \leq i \neq j \leq d-2$,*

$$\biggl|\frac{d(q-1)}{4(i+2)} E\biggl[(\tilde{S}_i - S_i)(\tilde{S}_j - S_j)\frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}(V)\biggr]\biggr| = O\biggl(\frac{\|h\|_\infty}{q^{1/2}}\biggr)\log(1/\varepsilon),$$

as $q \to \infty$ uniformly over $0 < \varepsilon < 1$.

PROOF. Let $S_{i,k}$ and $\tilde{S}_{i,k}$, $i = 1, \dots, d-2$ and $k = 1, 2$, be as in (37). Then $S_i = S_{i,1} + S_{i,2}$ and $\tilde{S}_i = \tilde{S}_{i,1} + \tilde{S}_{i,2}$. For $1 \le i \ne j \le d-2$, we observe that

$$
\begin{aligned}
&\left|\frac{d(q-1)}{4(i+2)} E\left[(\tilde{S}_i - S_i)(\tilde{S}_j - S_j)\frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}(V)\right]\right| \\
&\quad = \left|\frac{d(q-1)}{4(i+2)} E\left\{\frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}(V) E^{\mathcal{W}}[(\tilde{S}_i - S_i)(\tilde{S}_j - S_j)]\right\}\right| \\
&\quad = \frac{d(q-1)}{4(i+2)}\left\{\sup_{v\in\mathbb{R}^{d-2}}\left|\frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}(v)\right|\right\} \\
&\qquad \times E|E^{\mathcal{W}}[(\tilde{S}_{i,1} + \tilde{S}_{i,2} - S_{i,1} - S_{i,2})(\tilde{S}_{j,1} + \tilde{S}_{j,2} - S_{j,1} - S_{j,2})]| \\
&\quad \le \frac{d(q-1)}{2(i+2)}\left\{\sup_{v\in\mathbb{R}^{d-2}}\left|\frac{\partial^2}{\partial v_i\,\partial v_j}\psi_{\varepsilon^2}(v)\right|\right\}\left\{E|E^{\mathcal{W}}(\tilde{S}_{i,1}\tilde{S}_{j,1})| + E|E^{\mathcal{W}}(\tilde{S}_{i,1}\tilde{S}_{j,2})|\right. \\
&\qquad + E|E^{\mathcal{W}}(\tilde{S}_{i,1}S_{j,1})| + E|E^{\mathcal{W}}(\tilde{S}_{i,1}S_{j,2})| + E|E^{\mathcal{W}}(S_{i,1}\tilde{S}_{j,1})| \\
&\qquad \left. + E|E^{\mathcal{W}}(S_{i,1}\tilde{S}_{j,2})| + E|E^{\mathcal{W}}(S_{i,1}S_{j,1})| + E|E^{\mathcal{W}}(S_{i,1}S_{j,2})|\right\}.
\end{aligned}
$$

Now Proposition 7 follows from Lemma 1 and Lemma 7. □

PROOF OF THEOREM 2. Let Z_1 be a random variable having the standard (univariate) normal distribution and $\xi = (\sigma_1/\sigma, \dots, \sigma_{d-2}/\sigma)'$. Then $\|\xi\|^2 = 1$ and $W = \xi' V$. For ease of exposition in the subsequent argument, we shall write $\xi = \xi(q)$ and $V = V(q)$.

We claim that $\xi' V \to Z_1$ in distribution as $q \to \infty$. We shall prove this claim by contraposition. Suppose the claim is false. Then there exists an interval, say $[a, b) \subset \mathbb{R}$, such that $P(\xi' V \in [a, b))$ does not converge to $P(Z_1 \in [a, b))$ as $q \to \infty$. Since $P(\xi' V \in [a, b)) \in [0, 1]$, by the compactness of $[0, 1]$, there exists a subsequence, say $\xi'(q_k)V(q_k)$, of $\xi' V$ such that $P(\xi'(q_k)V(q_k) \in [a, b))$ converges to a number, say $L \ne P(Z_1 \in [a, b))$. As $\|\xi(q_k)\| = 1$, there exists a further subsequence, say $\xi'(q_{k_l})V(q_{k_l})$, of $\xi'(q_k)V(q_k)$ such that $\xi(q_{k_l})$ converges to a point $\tilde{\xi} \in \mathbb{R}^{d-2}$ as $q_{k_l} \to \infty$. This implies that $\xi'(q_{k_l})V(q_{k_l}) - \tilde{\xi}' V(q_{k_l}) \to 0$ in probability as $q_{k_l} \to \infty$ and hence $\xi'(q_{k_l})V(q_{k_l})$ and $\tilde{\xi}' V(q_{k_l})$ have the same asymptotic distribution. Using Theorem 3 and $\|\tilde{\xi}\|^2 = 1$, we observe that $\tilde{\xi}' V(q_{k_l})$ converges in law to the standard normal distribution as $q_{k_l} \to \infty$. Hence $\xi'(q_{k_l})V(q_{k_l})$ converges in law to the same latter distribution. This is a contradiction and the claim is proved.

We observe from Theorem 1 and Proposition 1 that for $\int_{[0,1)^d} f_{rem}^2(x)\,dx > 0$, we have $\sigma_{oal}^2/\sigma^2 = 1 + O(q^{-1})$ and $\sigma_{oas}^2/\sigma^2 = 1 + O(q^{-1})$ as $q \to \infty$. Thus we conclude from Proposition 2 and Slutsky's theorem that W_{oal} and W_{oas} both tend in law to the standard (univariate) normal distribution as $q \to \infty$. Finally using Theorem 1 and Proposition 1, we have $\lim_{q\to\infty} \sigma^*_{oal}/\sigma_{oal} = 1$. Hence for $[a, b) \subset \mathbb{R}$,

$$P\big(W^*_{oal} \in [a, b)\big) = \frac{1}{d!}\sum\nolimits^* P\big(W_{oal} \in [a, b)\big) + o(1) \to P\big(Z_1 \in [a, b)\big),$$

as $q \to \infty$ where $\sum^*$ denotes summation over all the $d!$ permutations of the columns of A^{**}. This proves that W^*_{oal} converges in law to the standard normal distribution. The proof of Theorem 2 is complete. □

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