Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2009, Vol. 45, No. 1, 201–225
DOI: 10.1214/07-AIHP162

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

On the left tail asymptotics for the limit law of supercritical Galton–Watson processes in the Böttcher case*

Klaus Fleischmann[a] and Vitali Wachtel[b]

[a]*Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, D-10117 Berlin, Germany. E-mail: fleischm@wias-berlin.de*
[b]*Technische Universität München, Zentrum Mathematik, Bereich M 5, D-85747 Garching bei München, Germany. E-mail: wachtel@ma.tum.de*

Received 19 June 2007; revised 26 November 2007; accepted 4 December 2007

Abstract. Under a well-known scaling, supercritical Galton–Watson processes Z converge to a non-degenerate non-negative random limit variable W. We are dealing with the left tail (i.e. close to the origin) asymptotics of its law. In the Böttcher case (i.e. if always at least two offspring are born), we describe the precise asymptotics exposing oscillations (Theorem 1). Under a reasonable additional assumption, the oscillations disappear (Corollary 2). Also in the Böttcher case, we improve a recent lower deviation probability result by describing the precise asymptotics under a logarithmic scaling (Theorem 7). Under additional assumptions, we even get the fine (i.e. without log-scaling) asymptotics (Theorem 8).

Résumé. Par un changement d'échelle bien connu, on obtient que les processus de Galton–Watson supercritiques sur Z convergent vers une variable aléatoire non-degénerée W. Nous considérons les estimées asymptotiques à gauche (près de l'origine) de la distribution. Dans le cas Böttcher (quand il y a au moins deux progénitures en chaque point), nous obtenons l'asymptotique exacte présentant un comportement oscillatoire (Théorème 1). Sous une autre hypothèse raisonnable, les oscillations s'annulent (Corollaire 2). Pour le cas Böttcher, nous présentons un résultat sur la probabilité des grandes déviations, amélioré en exprimant l'asymptotique exacte sous un scaling logarithmique (Théorème 7). En imposant d'autres conditions, nous obtenons des asymptotiques plus raffinées (Théorème 8), c'est-à-dire sans log-scaling.

MSC: Primary 60J80; secondary 60F10

Keywords: Lower deviation probabilities; Schröder case; Böttcher case; Logarithmic asymptotics; Fine asymptotics; Precise asymptotics; Oscillations

1. Introduction and statement of results

1.1. *Motivation and sketch of results*

Let $Z = (Z_n)_{n\geq 0}$ denote a Galton–Watson process with $Z_0 = 1$ and offspring generating function

$$f(s) = \sum_{j=0}^{\infty} p_j s^j, \quad 0 \leq s \leq 1. \tag{1}$$

We restrict our attention to the supercritical case, i.e. $\mathbf{E}Z_1 = f'(1) =: m \in (1, \infty)$. Clearly, we exclude the trivial case that Z_1 is degenerate. As is well known, one can find constants $c_n > 0$ converging to infinity such that $c_n^{-1} Z_n$

*Supported by the German Science Foundation and the German Israeli Foundation for Scientific Research and Development. This paper has essentially been written during the time the second author was a staff member of the WIAS.

converges almost surely to a non-degenerate random variable $W \geq 0$. In particular, we have the following convergence in terms of the iterated offspring generating functions f_n:

$$f_n\left(\mathrm{e}^{-u/c_n}\right) \underset{n\uparrow\infty}{\longrightarrow} \mathbf{E}\mathrm{e}^{-uW} =: \varphi(u), \quad u \geq 0. \tag{2}$$

Moreover, the variable W restricted to $(0, \infty)$ has a (strictly) positive continuous density function denoted by w, and W equals zero with (extinction) probability q, where $q \in [0, 1)$ is the smallest non-negative root of the equation $f(s) = s$. Furthermore, the Laplace transform φ of W satisfies the *Poincaré functional equation*

$$\varphi(mu) = f\left(\varphi(u)\right), \quad u \geq 0. \tag{3}$$

Up to a scaling factor, this equation has a unique (strictly) decreasing, convex solution with $\varphi(0) = 1$. In other words, (3) determines the distribution of W up to a constant factor. But only in very special cases one can solve (3) explicitly (some examples of explicit solutions can be found in Hambly [13] and Harris [14]).

However, the *left tail asymptotics* of the distribution of W, that is the asymptotics close to the origin, can be studied under quite general conditions on the offspring law. This problem was the objective of interest of many researchers. But the precise (without any log-scaling) asymptotics of $w(x)$ and $\mathbf{P}(W < x)$ as $x \downarrow 0$ remained unknown in the so-called *Böttcher case*, that is if $p_0 + p_1 = 0$. We fill this gap, see Theorem 1. This involves some multiplicatively periodic functions producing oscillations. Moreover, we give a necessary and sufficient condition implying that these multiplicatively periodic functions can be replaced by constants, consequently that the oscillations disappear, i.e. degenerate (see Corollary 2). One of the reasons we are interested in the asymptotics of the law of W near 0 in the Böttcher case is that it is closely related to the behavior of Brownian motions on fractals (see, for example, Barlow and Perkins [1] and [13]).

Besides the $x \downarrow 0$ asymptotics of the distribution of W, we investigate a more delicate problem: so-called *lower deviation probabilities* of Z, i.e. the asymptotic behavior of $\mathbf{P}(Z_n = k_n)$ when $k_n/c_n \to 0$. The main reason for studying these probabilities comes from statistical inference. Our recent paper [12] is just devoted to this lower deviation problem of supercritical Galton–Watson processes, but our result in the Böttcher case is not very satisfactory: we obtained only asymptotic bounds and this in fact only under some log-scaling. In the present note we first of all sharpen the asymptotic bounds to asymptotic *limits* (see Theorem 7). Furthermore, under two different additional assumptions on the tail of the offspring law, we find the *fine asymptotics* for lower deviation probabilities, that is without any log-scaling (see Theorem 8).

1.2. *Dichotomy for supercritical processes*

For convenience, we recall here some basic facts on supercritical Galton–Watson processes. Under our supercriticality assumption, the generating function f has two fixed points: $q \in [0, 1)$ and 1. The behavior of its iterations f_n in the vicinity of 1 is described by the convergence statement (2) and the Poincaré functional equation (3). Concerning the behavior of iterations in the vicinity of q, two cases are possible (see, e.g. [12], Section 1.3):

(a) (*Schröder case*). Here we have by definition $p_0 + p_1 > 0$, or equivalently $f'(q) =: \gamma > 0$. Then

$$\frac{f_n(s) - q}{(f'(q))^n} \underset{n\uparrow\infty}{\longrightarrow} \text{some } \mathsf{S}(s), \quad 0 \leq s \leq 1, \tag{4}$$

and S satisfies the *Schröder functional equation*

$$\mathsf{S}\left(f(s)\right) = \gamma \mathsf{S}(s), \quad 0 \leq s \leq 1. \tag{5}$$

(b) (*Böttcher case*). Here $p_0 + p_1 = 0$, that is $f'(q) = 0$. In this case, $\mu := \min\{k\colon\ p_k > 0\} \geq 2$, and one has the convergence

$$\left(f_n(s)\right)^{(\mu^{-n})} \underset{n\uparrow\infty}{\longrightarrow} \text{some } \mathsf{B}(s), \quad 0 \leq s \leq 1. \tag{6}$$

B is continuous, positive, and satisfies the *Böttcher functional equation*

$$\mathsf{B}\left(f(s)\right) = \left(\mathsf{B}(s)\right)^{\mu}, \quad 0 \leq s \leq 1. \tag{7}$$

1.3. *Left tail asymptotics for w and the law of W*

First we describe the more studied *Schröder case*. Here the *Schröder constant* $\alpha \in (0, \infty)$ is defined by the requirement $f'(q) = m^{-\alpha}$. Biggins and Bingham [4] have shown that there exists a continuous, multiplicatively periodic function $V : (0, \infty) \to (0, \infty)$ with period m (that is, $V(mx) = V(x)$ for all $x > 0$), such that

$$x^{1-\alpha} w(x) = V(x) + \mathrm{o}(1) \quad \text{as } x \downarrow 0. \tag{8}$$

Dubuc [9] has proven that the function V can be replaced by a constant $V_0 > 0$ if and only if

$$\mathsf{S}\big(\varphi(u)\big) = K_0 u^{-\alpha}, \quad u \ge 0, \tag{9}$$

for some constant $K_0 > 0$.

Now we come to the *Böttcher case.* Since here $f'(q) = 0$, we would have $\alpha = \infty$. But now one can introduce the *Böttcher constant* $\beta \in (0, 1)$ by the requirement $\mu = m^{\beta}$. It is shown in [4] that there exists an analytic, multiplicatively periodic function $M : (0, \infty) \to (0, \infty)$, with period $m^{1-\beta}$, such that

$$-\log \mathbf{P}(W < x) = x^{-\beta/(1-\beta)} M(x) + \mathrm{o}\big(x^{-\beta/(1-\beta)}\big) \quad \text{as } x \downarrow 0. \tag{10}$$

Bingham [6] observed that under the condition $-\log \varphi(u) \sim \kappa u^{\beta}$ as $u \uparrow \infty$ for some constant $\kappa > 0$, the function M can be replaced by a constant $M_0 > 0$. Since $\mathbf{P}(W < x)$ decreases exponentially as $x \downarrow 0$, one can expect that the density function w has the same rate of decrease. In fact, by Remark 7 in [12],

$$-\overline{M} < \liminf_{x\downarrow 0} x^{\beta/(1-\beta)} \log w(x) \le \limsup_{x\downarrow 0} x^{\beta/(1-\beta)} \log w(x) < -\underline{M} \tag{11}$$

for some positive constants $\overline{M}$ and $\underline{M}$.

The first theorem, our *main result*, improves the statements (10) and (11). Recall that we are in the Böttcher case.

Theorem 1 (Precise left tail asymptotics for w and the law of W). *There are positive functions M, M_1 and M_2, multiplicatively periodic with period $m^{1-\beta}$, such that as $x \downarrow 0$,*

$$w(x) = M_1(x) x^{(\beta-2)/2(1-\beta)} \exp\big\{-M(x) x^{-\beta/(1-\beta)}\big\}\big(1 + \mathrm{O}\big(x^{\beta/2(1-\beta)} \log^3 x\big)\big) \tag{12}$$

and

$$\mathbf{P}(W < x) = M_2(x) x^{\beta/2(1-\beta)} \exp\big\{-M(x) x^{-\beta/(1-\beta)}\big\}\big(1 + \mathrm{O}\big(x^{\beta/2(1-\beta)} \log^3 x\big)\big). \tag{13}$$

The multiplicatively periodic functions in (12) and (13) produce oscillations of $w(x)$ and $\mathbf{P}(W < x)$. Now the question arises of when these oscillations disappear, i.e. in which cases these functions are actually constants. Hambly [13] has given an *example* (of a class of supercritical processes in the Böttcher case), for which it is possible to calculate the density function w explicitly and for which there are indeed no oscillations. In our proof of Theorem 1 (in Section 3) we will express the functions M, M_1 and M_2 via the Legendre transform of the function

$$K(u) := -u^{-\beta} \log \mathsf{B}\big(\varphi(u)\big), \quad u > 0 \tag{14}$$

(with B from (6)). Analyzing these expressions in the case when the function K degenerates to a constant, we will see that there are actually no oscillations. Moreover, this statement can be reversed:

Corollary 2 (No oscillations). *If $K(u) \equiv \kappa > 0$, then*

$$M(x) \equiv (\kappa\beta)^{1/(1-\beta)}\big(\beta^{-1} - 1\big), \tag{15}$$

$$M_1(x) \equiv p_\mu^{-1/(\mu-1)} \left(\frac{(\kappa\beta)^{1/(1-\beta)}}{2\pi(1-\beta)}\right)^{1/2} \tag{16}$$

and

$$M_2(x) \equiv p_\mu^{-1/(\mu-1)} \left(\frac{1}{2\pi(1-\beta)(\kappa\beta)^{1/(1-\beta)}} \right)^{1/2}. \tag{17}$$

Conversely, $M(x) \equiv \text{const}$ *implies the existence of* $\lim_{u\uparrow\infty} u^{-\beta} \log \varphi(u)$, *yielding* $K(u) \equiv \text{const}$.

In the example of [13] mentioned above, $p_\mu = 2^{1-\mu}$, $\beta = 1/2$ and $K(u) \equiv \sqrt{2}$. Thus, we can apply Corollary 2 to obtain, $M(x) \equiv 1/2$ and $M_1(x) \equiv 2/\sqrt{2\pi}$. Then (12) gives

$$w(x) \sim \frac{2}{\sqrt{2\pi}} x^{-3/2} \exp\bigl\{-(2x)^{-1}\bigr\} \quad \text{as } x \downarrow 0. \tag{18}$$

This of course also follows from the exact formula for w in Hambly's example.

A classical *example* of non-trivial oscillations in the left tail of W is the process Z with offspring generating function $f(s) = s^2/(4-3s)$ considered by Barlow and Perkins [1]. They have shown that

$$\liminf_{u\uparrow\infty} u^{-\beta} \log \varphi(u) < \limsup_{u\uparrow\infty} u^{-\beta} \log \varphi(u). \tag{19}$$

Consequently, in this example the function M is not a constant by Corollary 2. But their calculations show also that here the variation of K is very small. That is, $K_* \le K(u) \le K^*$, $u > 0$, with $K^* - K_*$ small. On the other hand, Biggins and Bingham [3] have obtained some bounds for the variation of K under the restriction that the offspring law is shifted infinitely divisible, that is, $f(s) = s^r h(s)$ with h an infinitely divisible probability generating function and $r \ge 2$ a natural number. Moreover, Bingham [6] has shown that $K_* \le K(u) \le K^*$, $u > 0$, implies

$$(K_*\beta)^{1/(1-\beta)}\bigl(\beta^{-1} - 1\bigr) \le M(x) \le \bigl(K^*\beta\bigr)^{1/(1-\beta)}\bigl(\beta^{-1} - 1\bigr), \quad x > 0. \tag{20}$$

That is, a small variation of L implies a small variation of M giving *tiny* oscillations in (12) and (13).

We finish this section with some further remarks.

Remark 3 (Right tail asymptotics). *In the case when our supercritical offspring generating function f is a* polynomial *one can easily adopt our methods to find the exact asymptotics of $w(x)$ and* $\mathbf{P}(W > x)$ *as $x \uparrow \infty$. Indeed, there exist multiplicatively periodic functions N, N_1 and N_2 such that*

$$w(x) \sim N_1(x) x^{(2-\gamma)/2(\gamma-1)} \exp\bigl\{-N(x) x^{-\gamma/(\gamma-1)}\bigr\} \tag{21}$$

and

$$\mathbf{P}(W > x) \sim N_2(x) x^{-\gamma/2(\gamma-1)} \exp\bigl\{-N(x) x^{-\gamma/(\gamma-1)}\bigr\} \tag{22}$$

as $x \uparrow \infty$, where $\gamma > 1$ is defined by the relation $m^\gamma = \max\{k\colon\ p_k > 0\}$.

Remark 4 (Multi-type case). *A very interesting question is to investigate the tail behavior of the limit of* multi-type *Galton–Watson processes. Some first results in this direction can be found in Jones* [16]. *For some related limit theory for iterations of generating functions see Biggins* [2] *and Jones* [15].

Remark 5 (Continuous state case). *Bingham* [5] *has investigated the asymptotic behavior of the limit law of a supercritical* continuous-state *branching process. In the situation analogous to the Böttcher case he has obtained a version of* (10) *with a* slowly varying *function M, see Theorems* 5.4 *and* 5.6 *in* [5]. *The non-oscillating behavior of M can be understood by the smoother behavior of continuous-state branching compared to the Galton–Watson case.*

Remark 6 (Diffusions on fractals). *It would be interesting to understand whether our results allow one to obtain more precise probability bounds for diffusions on finitely ramified fractals* (*recall e.g.* [1] *and* [13]).

1.4. *Lower deviation probabilities of Z*

Here we state our results on lower deviation probabilities of Z. Recalling that $\mu = \min\{k\colon\ p_k > 0\}$ and that the offspring generating function f is said to be *of type* (d,μ), if $d \ge 1$ denotes the greatest common divisor of the set $\{j-l\colon\ j \ne l,\ p_j p_l > 0\}$, we use *from now on* the symbol d (and μ) in this sense.

For the *Schröder case*, we can simply specialize Theorem 4 of [12]. In fact, for $k_n \equiv \mu (\operatorname{mod} d)$ with $k_n \to \infty$ but $k_n = \mathrm{o}(c_n)$ we have

$$\mathbf{P}(Z_n = k_n) = \frac{d}{m^{n-a_n} c_{a_n}} w\left(\frac{k_n}{m^{n-a_n} c_{a_n}}\right)\bigl(1+\mathrm{o}(1)\bigr) \quad \text{as } n \uparrow \infty, \tag{23}$$

where $a_n := \min\{j \ge 1\colon\ c_j \ge k_n\}$. Clearly, if additionally $\mathbf{E}Z_1 \log Z_1 < \infty$ holds, then one can choose $c_n = m^n$, and (23) simplifies to

$$\mathbf{P}(Z_n = k_n) = d m^{-n} w\left(\frac{k_n}{m^n}\right)\bigl(1+\mathrm{o}(1)\bigr) \quad \text{as } n \uparrow \infty. \tag{24}$$

Now we turn to the *Böttcher case.* In [12], Theorem 6, we have found bounds for $\log[c_n \mathbf{P}(Z_n = k_n)]$, which can be rewritten, after some elementary calculations, as follows: for all large enough n,

$$C_1 \log w\left(\frac{k_n}{c_{j_n} m^{n-j_n}}\right) \le \log\bigl[c_n \mathbf{P}(Z_n = k_n)\bigr] \le C_2 \log w\left(\frac{k_n}{c_{j_n} m^{n-j_n}}\right) \tag{25}$$

for some positive constants C_1 and C_2. Now we are able to be more precise.

Theorem 7 (Precise logarithmic asymptotics of lower deviations). *Let $k_n \equiv \mu^n (\operatorname{mod} d)$ with $k_n/\mu^n \to \infty$ but $k_n = \mathrm{o}(c_n)$ as $n \uparrow \infty$. Then*

$$\log\bigl[c_n \mathbf{P}(Z_n = k_n)\bigr] \sim \log w\left(\frac{k_n}{c_{j_n} m^{n-j_n}}\right) \quad \text{as } n \uparrow \infty, \tag{26}$$

where $j_n := \max\{l \ge 1\colon\ c_l \mu^{n-l} \le k_n\}$.

Of course, under the condition $\mathbf{E}Z_1 \log Z_1 < \infty$, relation (26) simplifies to

$$\log\bigl[m^n \mathbf{P}(Z_n = k_n)\bigr] \sim \log w\left(\frac{k_n}{m^n}\right). \tag{27}$$

This reminds one of (24) except for the additional log-scaling. However, without logarithmic scaling, the behavior of lower deviation probabilities turns out to depend heavily on the tail of the offspring law:

Theorem 8 (Fine asymptotics of lower deviations). *Assume that $k_n \equiv \mu^n (\operatorname{mod} d)$ with $k_n/\mu^n \to \infty$ but $k_n = \mathrm{o}(m^n)$ as $n \uparrow \infty$. If $\mathbf{E}Z_1^2 < \infty$, then there exists a positive, multiplicatively periodic function V_2 such that*

$$\frac{m^n \mathbf{P}(Z_n = k_n)}{d\, w(k_n/m^n)} = \exp\left\{-V_2\left(\frac{k_n}{m^n}\right)\left(\frac{m^{2n\beta}}{k_n^{\beta+1}}\right)^{1/(1-\beta)}\bigl(1+\mathrm{o}(1)\bigr)\right\} \quad \text{as } n \uparrow \infty. \tag{28}$$

If instead only

$$\mathbf{P}(Z_1 \ge x) = x^{-r}\ell(x), \quad x > 0, \tag{29}$$

for some $r \in (1,2)$ and some function ℓ, slowly varying at infinity, then there exists a positive, multiplicatively periodic function V_r such that as $n \uparrow \infty$,

$$\frac{m^n \mathbf{P}(Z_n = k_n)}{d\, w(k_n/m^n)} = \exp\left\{-V_r\left(\frac{k_n}{m^n}\right)\left(\frac{m^{nr\beta}}{k_n^{r+\beta-1}}\right)^{1/(1-\beta)} \ell\left(\left(\frac{k_n}{m^{\beta n}}\right)^{1/(1-\beta)}\right)\bigl(1+\mathrm{o}(1)\bigr)\right\}. \tag{30}$$

It should be noted that from Theorem 8 we obtain fine asymptotic statements only under additional restrictions on k_n. If, for example, $\mathbf{E}Z_1^2$ is finite, then for $k_n > \varepsilon m^{2n\beta/(1+\beta)}$ with an arbitrary $\varepsilon > 0$, we get from (28) the relation

$$\mathbf{P}(Z_n = k_n) \sim \frac{dw(k_n/m^n)}{m^n} \exp\left\{-V_2\left(\frac{k_n}{m^n}\right)\left(\frac{m^{2n\beta}}{k_n^{\beta+1}}\right)^{1/(1-\beta)}\right\}. \tag{31}$$

But since the asymptotic behavior of $w(x)$ is known, this yields the fine asymptotics for $\mathbf{P}(Z_n = k_n)$. However, in the case $k_n = \mathrm{o}(m^{2n\beta/(1+\beta)})$, formula (28) says only that

$$\log\bigl[m^n\mathbf{P}(Z_n = k_n)\bigr] - \log w\left(\frac{k_n}{m^n}\right) \sim -V_2\left(\frac{k_n}{m^n}\right)\left(\frac{m^{2n\beta}}{k_n^{\beta+1}}\right)^{1/(1-\beta)} \quad \text{as } n \uparrow \infty.$$

This is more precise than the statement of Theorem 7 but not sufficient for a fine asymptotics.

However, we believe that the statements of Theorem 8 are optimal in the sense that it is impossible to obtain more information on lower deviation probabilities without an additional assumption on the offspring distribution. More precisely, we conjecture that the form of the $\mathrm{o}(1)$ in (28) depends on higher moments of Z_1.

In our Theorems 7 and 8 we assumed $k_n/\mu^n \to \infty$. Thus, it remains to consider the lower deviation problem for k_n in the case that k_n/μ^n is bounded.

Theorem 9 (Fine asymptotics for extreme lower deviations). *Assume that $k_n \equiv \mu^n \,(\mathrm{mod}\ d)$ and fix some $1 < \lambda_1 < \lambda_2 < \infty$. Then, uniformly in $k_n \in [\lambda_1\mu^n, \lambda_2\mu^n]$,*

$$\mathbf{P}(Z_n = k_n) = \frac{r p_\mu^{-1/(\mu-1)}}{\mu^{-n/2}\sqrt{2\pi(r^2 b''(r) + r b'(r))}} \exp\bigl\{\mu^n\bigl(b(r) - r b'(r)\log r\bigr)\bigr\}\bigl(1 + \mathrm{O}\bigl(\mu^{-n/2}\bigr)\bigr), \tag{32}$$

where

$$b(s) := \log s + \sum_{j=0}^{\infty} \mu^{-j-1} \log \frac{f_{j+1}(s)}{f_j^{\mu}(s)}, \quad s \in (0,1), \tag{33}$$

and r is the unique solution of

$$r b'(r) = \frac{k_n}{\mu^n}. \tag{34}$$

Let $G(s) = \sum_{j=0}^{J} g_j s^j$ with $g_j \ge 0$, $\sum_{j=0}^{J} g_j > 0$ and $J > 1$. Define the sequence of polynomials $G_n(s) = \sum_{j\ge 0} g_{n,j} s^j$ by the recurrence relation

$$G_{n+1}(s) = G\bigl(G_n(s)\bigr), \quad n \ge 0,\ G_0(s) = s,\ s \ge 0. \tag{35}$$

Flajolet and Odlyzko [11] studied the asymptotic behavior of the $g_{n,j}$ as $n \uparrow \infty$. (Actually, they studied the more general case $G_{n+1}(s) = G(s, G_n(s))$ with $G(s,y) := \sum_{j=0}^{J} g_j(s) y^j$.) Their method relies on the combination of the saddle point approximation and the following property of the sequence G_n (see Lemma 2.5 in [11]):

$$\bigl(G_n(s)\bigr)^{(J^{-n})} \underset{n\uparrow\infty}{\longrightarrow} \text{some } g(s) \tag{36}$$

for all $s > \rho := \inf\{s > 0\colon\ G_n(s) \to \infty \text{ as } n \uparrow \infty\}$. Moreover, the limit g satisfies the Böttcher equation

$$g\bigl(G(s)\bigr) = \bigl(g(s)\bigr)^J, \quad s \in (\rho, \infty). \tag{37}$$

Our problem concerning lower deviation probabilities in the Böttcher case is similar to the problem considered in [11]. Indeed, local probabilities $\mathbf{P}(Z_n = k)$ are coefficients of the iterations f_n, and, furthermore, the convergence (6) is analogous to (36). In view of this similarity we will use, following Flajolet and Odlyzko, the saddle point method in proving our Theorems 7–9. To this aim we need to adopt some technical results from [11] to our setting. This will be done in Section 2.1. After these preparations, the proof of Theorem 9 follows the pattern of the proof of Theorem 1 of [11], and we leave this to the reader.

In the case $k_n \gg \mu^n$ as in Theorems 7 and 8, the Böttcher convergence (6) turns out not to be sufficient for finding the asymptotics of $\mathbf{P}(Z_n = k_n)$. But besides (6), which describes the behavior of f_n in the vicinity of the attractive fixed point $s = 0$ (for the mapping $s \mapsto f(s)$), we have available (2) governing the behavior of f_n near the repulsive fixed point $s = 1$. The existence of the second fixed point makes our setting different from that in [11] (there the sequence G_n is assumed to have only the single fixed point $s = \infty$), and this enables us to study the behavior of $\mathbf{P}(Z_n = k_n)$ also in the case $k_n \gg \mu^n$ and to find this way the left tail asymptotics concerning W.

2. Various auxiliary results

As in our theorems, we *always assume from now on* to be in the Böttcher case.

2.1. *On a convergence of iterated offspring generating functions*

Clearly, we may extend the domain of definition of f and f_n to complex variables z with $|z| \le 1$. Set (at this stage at least formally)

$$b(z) := \log z + \sum_{j=0}^{\infty} \mu^{-j-1} \log \frac{f_{j+1}(z)}{f_j^{\mu}(z)}, \quad 0 < |z| \le 1, \tag{38}$$

and

$$\mathcal{D}(\delta, \theta) := \bigl\{z\colon\ 0 < |z| \le 1 - \delta,\ |\arg z| \le \theta\bigr\}, \quad \delta \in [0, 1), \theta \in (0, \pi). \tag{39}$$

In (38) and in what follows we take the principal value of the logarithm.

Lemma 10 (On analyticity and convergence). *For every $\delta \in (0, 1)$ there exists a constant $\theta = \theta(\delta) \in (0, \pi)$ such that b is analytic on $\mathcal{D}(\delta, \theta)$. Furthermore,*

$$f_n(z) = p_\mu^{-1/(\mu-1)} \exp\bigl\{\mu^n b(z)\bigr\}\bigl(1 + \mathrm{o}\bigl(\mathrm{e}^{-\delta\mu^n}\bigr)\bigr) \quad \textit{as } n \uparrow \infty, \tag{40}$$

uniformly in $z \in \mathcal{D}(\delta, \theta)$, for these δ and θ.

Proof. If $f_k(z) \ne 0$, then

$$\frac{f_{k+1}(z)}{p_\mu f_k^{\mu}(z)} = 1 + \sum_{j=1}^{\infty} \frac{p_{\mu+j}}{p_\mu} f_k^{j}(z). \tag{41}$$

Hence,

$$\left|\frac{f_{k+1}(z)}{p_\mu f_k^{\mu}(z)} - 1\right| \le \frac{1 - p_\mu}{p_\mu} f_k\bigl(|z|\bigr) \le C|z|^{(\mu^k)} \tag{42}$$

and

$$\bigl|f_{k+1}(z)\bigr| > p_\mu \bigl|f_k(z)\bigr|^{\mu}\bigl(1 - C|z|^{(\mu^k)}\bigr) \tag{43}$$

for some (positive) constant C, since in the Böttcher case

$$f_k(s) \le s^{(\mu^k)}, \quad k \ge 0, s \in (0, 1). \tag{44}$$

From (43) follows that there exists $k_0 = k_0(\delta)$ such that, if $f_{k_0}(z) \neq 0$ and $|z| \leq 1-\delta$, then $f_k(z) \neq 0$ for all $k > k_0$. Furthermore, since the zeros of f_k are separated points, there exists $\theta = \theta(k_0)$ such that $f_k(z) \neq 0$ for all $k \leq k_0$ and $z \in \mathcal{D}(0,\theta)$. Summarizing, for every $\delta > 0$ there exists $\theta > 0$ such that $f_k(z) \neq 0$ for all $k \geq 0$ and $z \in \mathcal{D}(\delta,\theta)$. Thus, for every $k \geq 0$ the function $z \mapsto \log(f_{k+1}(z)/p_\mu f_k^\mu(z))$ is analytic on $\mathcal{D}(\delta,\theta)$.

It is known that $\log(1+z)$ is analytic at $z=0$ and, moreover, $\log(1+z) = \sum_{j=1}^\infty (-1)^{j-1} j^{-1} z^j$ for all $|z| < 1$. Consequently,

$$\bigl|\log(1+z)\bigr| \leq \frac{|z|}{1-|z|} \leq 2|z| \quad \text{if } |z| \leq \frac{1}{2}. \tag{45}$$

Combining this inequality with (42), we conclude that for all large enough k

$$\left|\log \frac{f_{k+1}(z)}{p_\mu f_k^\mu(z)}\right| \leq C|z|^{(\mu^k)}. \tag{46}$$

Clearly, for $0 < \delta < 1$ fixed, $|z| \leq 1-\delta$ implies $|z| \leq \mathrm{e}^{-\delta}$. Hence, for $z \in \mathcal{D}(\theta,\delta)$,

$$\left|\log \frac{f_{k+1}(z)}{p_\mu f_k^\mu(z)}\right| \leq C|z|^{(\mu^k)} \leq C\mathrm{e}^{-\delta\mu^k} \leq C. \tag{47}$$

Consequently,

$$\sum_{k=0}^{n-1} \mu^{-k-1} \log \frac{f_{k+1}(z)}{p_\mu f_k^\mu(z)} \underset{n\uparrow\infty}{\longrightarrow} \sum_{k=0}^{\infty} \mu^{-k-1} \log \frac{f_{k+1}(z)}{p_\mu f_k^\mu(z)}, \tag{48}$$

uniformly in $z \in \mathcal{D}(\delta,\theta)$. Moreover, as the uniform limit of analytic functions, the right-hand side function in (48) is analytic on $\mathcal{D}(\delta,\theta)$. Noting that

$$b(z) = \log z + \frac{1}{\mu-1}\log p_\mu + \sum_{k=0}^{\infty} \mu^{-k-1} \log \frac{f_{k+1}(z)}{p_\mu f_k^\mu(z)}, \tag{49}$$

we see that b is analytic on $\mathcal{D}(\delta,\theta)$ as well.

We now turn to the proof of (40). It can easily be seen that

$$\mu^{-n}\log f_n(z) = b(z) - \sum_{k=n}^{\infty} \mu^{-k-1} \log \frac{f_{k+1}(z)}{f_k^\mu(z)}, \quad z \in \mathcal{D}(0,\theta), \tag{50}$$

for all $n \geq 0$. Note also that for $z \in \mathcal{D}(0,\theta)$,

$$\sum_{k=n}^{\infty} \mu^{-k-1} \log \frac{f_{k+1}(z)}{f_k^\mu(z)} = \frac{\mu^{-n}}{\mu-1}\log p_\mu + \sum_{k=n}^{\infty} \mu^{-k-1} \log \frac{f_{k+1}(z)}{p_\mu f_k^\mu(z)}. \tag{51}$$

From these identities and (47) we get

$$\log f_n(z) = \mu^n b(z) - \frac{1}{\mu-1}\log p_\mu + \mathrm{O}\bigl(\mathrm{e}^{-\delta\mu^n}\bigr) \quad \text{as } n\uparrow\infty, \tag{52}$$

implying (40), uniformly in $z \in \mathcal{D}(\delta,\theta)$. This completes the proof. □

Remark 11 (On the relation between** b **and** B**). *From (40) one can easily deduce that* $(f(s))^{(\mu^{-n})} \to \mathrm{e}^{b(s)}$ *as* $n\uparrow\infty$. *Thus, comparing this convergence with (6), we see that* $b(s) = \log \mathsf{B}(s)$. *Hence, using (7), we have*

$$b\bigl(f(s)\bigr) = \mu b(s), \quad 0 < s < 1. \tag{53}$$

Remark 12 (Analyticity of** b **on** $(0,1)$**). *It follows from Lemma 10 that* b *is analytic at every point* $s \in (0,1)$.

Lemma 13 (An upper bound of f_n**).** *For all* $s \in (0,1)$ *and* $n \geq 1$,

$$f_n(s) < p_\mu^{-1/(\mu-1)} \exp\{\mu^n b(s)\}. \tag{54}$$

Proof. Combining (50) and (51) gives

$$\mu^{-n} \log f_n(s) = b(s) - \frac{\mu^{-n}}{\mu-1} \log p_\mu - \sum_{k=n}^{\infty} \mu^{-k-1} \log \frac{f_{k+1}(s)}{p_\mu f_k^\mu(s)}. \tag{55}$$

Since $f_{k+1}(s) > p_\mu f_k^\mu(s)$ for all $k \geq 1$ and $s \in (0,1)$, the sum at the right-hand side of (55) is positive. This means that

$$\mu^{-n} \log f_n(s) < b(s) - \frac{\mu^{-n}}{\mu-1} \log p_\mu, \tag{56}$$

giving (54). This finishes the proof. □

Lemma 14 (Further properties of b**).** *We have*

$$sb''(s) + b'(s) = \big(sb'(s)\big)' > 0, \quad s \in (0,1). \tag{57}$$

Furthermore,

$$\lim_{s\uparrow 1} sb'(s) = \infty \quad \textit{and} \quad \lim_{s\downarrow 0} sb'(s) = 1. \tag{58}$$

Proof. We first note that in view of Lemma 10,

$$\big(sb'(s)\big)' = \lim_{n\uparrow\infty} \mu^{-n} \big(s\big(\log f_n(s)\big)'\big)' = \lim_{n\uparrow\infty} \mu^{-n} \left(\frac{sf_n'(s)}{f_n(s)}\right)'. \tag{59}$$

It was shown in [11], formula (2.37), that if $g(s) = g_1(s) + g_2(s)$, where $g_1(s)$ and $g_2(s)$ are power series with non-negative coefficients, then for all $s \in (0,1)$,

$$\left(\frac{sg'(s)}{g(s)}\right)' \geq \frac{g_1(s)}{g(s)} \left(\frac{sg_1'(s)}{g_1(s)}\right)'. \tag{60}$$

Using this inequality with $g_1(s) = p_\mu f_n^\mu(s)$ and $g_2(s) = f_{n+1}(s) - p_\mu f_n^\mu(s)$, we get for every $n \geq 0$,

$$\left(\frac{sf_{n+1}'(s)}{f_{n+1}(s)}\right)' \geq \mu \frac{p_\mu f_n^\mu(s)}{f_{n+1}(s)} \left(\frac{sf_n'(s)}{f_n(s)}\right)'. \tag{61}$$

Then after $n-1$ iterations we arrive at

$$\mu^{-n} \left(\frac{sf_n'(s)}{f_n(s)}\right)' \geq \mu^{-1} \left(\frac{sf'(s)}{f(s)}\right)' \prod_{j=1}^{n-1} \frac{p_\mu f_j^\mu(s)}{f_{j+1}(s)}. \tag{62}$$

It is easily seen that

$$\left(\frac{sf'(s)}{f(s)}\right)' = \operatorname{Var} X(s), \tag{63}$$

where the law of the random variable $X(s)$ is defined by $\mathbf{P}(X(s)=k)=p_k s^k/f(s)$. Since Z_1 is non-degenerate, $\operatorname{Var} X(s)>0$ for every $s\in(0,1)$. Consequently,

$$\left(\frac{sf'(s)}{f(s)}\right)'>0,\quad s\in(0,1). \tag{64}$$

Obviously,

$$\prod_{j=1}^{n-1}\frac{p_\mu f_j^\mu(s)}{f_{j+1}(s)}=\exp\left\{-\sum_{j=1}^{n-1}\log\left(\frac{p_\mu f_j^\mu(s)}{f_{j+1}(s)}\right)\right\}. \tag{65}$$

Then, in view of (48),

$$\lim_{n\uparrow\infty}\sum_{j=1}^{n-1}\log\left(\frac{p_\mu f_j^\mu(s)}{f_{j+1}(s)}\right)=\sum_{j=1}^{\infty}\log\left(\frac{p_\mu f_j^\mu(s)}{f_{j+1}(s)}\right)\in(0,\infty), \tag{66}$$

hence,

$$\lim_{n\uparrow\infty}\prod_{j=1}^{n-1}\frac{p_\mu f_j^\mu(s)}{f_{j+1}(s)}=\prod_{j=1}^{\infty}\frac{p_\mu f_j^\mu(s)}{f_{j+1}(s)}\in(0,1). \tag{67}$$

Combining (59), (62), (64) and (67), we obtain (57).

Next we prove the first statement in (58). Since $s\mapsto sb'(s)$ is increasing, it is enough to show that

$$s_j b'(s_j)\to\infty\quad\text{for some sequence } s_j\uparrow 1 \text{ as } j\uparrow\infty. \tag{68}$$

Fix any $s_0\in(0,1)$ and define recursively s_{j+1} by $f(s_{j+1})=s_j$, $j\geq 0$. Note that s_j increases to some s_∞ as $j\uparrow\infty$, satisfying $f(s_\infty)=s_\infty$, giving $s_\infty=1$. Then, in view of (53), $b'(s_{j+1})=b'(s_j)f'(s_{j+1})/\mu$. As $\lim_{j\uparrow\infty}f'(s_{j+1})=m>\mu$, we see that $b'(s_j)$ grows exponentially, and (68) follows.

From (41) and (49) we get

$$\begin{aligned} b'(s)&=\frac{1}{s}+\sum_{k=0}^{\infty}\mu^{-k-1}\frac{p_\mu^{-1}\sum_{j=1}^\infty jp_{\mu+j}f_k^{j-1}(s)}{1+p_\mu^{-1}\sum_{j=1}^\infty p_{\mu+j}f_k^j(s)}f_k'(s)\\ &\leq\frac{1}{s}+\frac{m}{p_\mu}\sum_{k=0}^{\infty}\mu^{-k-1}\big[f'(s)\big]^k, \end{aligned} \tag{69}$$

where in the second step we used the elementary bounds $f_k'(s)\leq[f'(s)]^k$ and $\sum_{j=1}^\infty jp_{\mu+j}f_k^{j-1}(s)<m$. Consequently, if s is so small that $f'(s)<\mu/2$, then

$$b'(s)<\frac{1}{s}+\frac{2m}{\mu p_\mu}. \tag{70}$$

This implies the second statement in (58), and the proof is finished. □

2.2. *Some statements involving the Laplace transform of W*

First we extend the definition of φ in (2) by setting $\varphi(z):=\mathbf{E}e^{-zW}$, $\Re z:=\Re(z)\geq 0$. Note that the Poincaré functional equation (3) remains valid under this extension. Recall notation $\mathcal{D}(\delta,\theta)$ from (39).

Lemma 15 (An estimate on φ). *Fix $u_0 > 0$. Then there is a constant $C = C(u_0)$ such that for all $\theta \in (0, C]$,*

$$\varphi(u - \mathrm{i}t) \in \mathcal{D}\left(\varphi(u_0), \frac{\theta}{C}\right), \quad u \geq u_0 \text{ and } |t| \leq \theta. \tag{71}$$

Proof. By the mean value theorem,

$$\varphi(u - \mathrm{i}t) - \varphi(u) = \mathrm{i}t\varphi'(u - \mathrm{i}\tau) \quad \text{for some } \tau \in (0, t). \tag{72}$$

This implies

$$\left|\Re\varphi(u - \mathrm{i}t)\right| \geq \varphi(u) - |t|\left|\varphi'(u - \mathrm{i}\tau)\right| \quad \text{and} \quad \left|\Im\varphi(u - \mathrm{i}t)\right| \leq |t|\left|\varphi'(u - \mathrm{i}\tau)\right|. \tag{73}$$

Noting that $|\varphi'(u - \mathrm{i}\tau)| \leq |\varphi'(u)|$, and using the obvious inequality $|\arg z| \leq |\Im z|/|\Re z|$, we get

$$\left|\arg\varphi(u - \mathrm{i}t)\right| \leq \frac{2|t||\varphi'(u)|}{\varphi(u)}, \quad |t| \leq \frac{\varphi(u)}{2|\varphi'(u)|}. \tag{74}$$

As φ is the Laplace transform of a non-degenerate random variable, from the Cauchy–Schwarz inequality we get

$$\left(\frac{\varphi'(u)}{\varphi(u)}\right)' = \frac{\varphi''(u)}{\varphi(u)} - \left[\frac{\varphi'(u)}{\varphi(u)}\right]^2 > 0 \quad \text{for all } u > 0. \tag{75}$$

Thus, φ'/φ is increasing, implying that

$$\frac{|\varphi'(u)|}{\varphi(u)} \leq \frac{|\varphi'(u_0)|}{\varphi(u_0)}, \quad u \geq u_0. \tag{76}$$

Combining this with (74) gives

$$\left|\arg\varphi(u - \mathrm{i}t)\right| \leq \frac{|t|}{C}, \quad u \geq u_0, |t| \leq C \tag{77}$$

with $C := \frac{\varphi(u_0)}{2|\varphi'(u_0)|}$. Finally, $|\varphi(u - \mathrm{i}t)| \leq |\varphi(u)| \leq |\varphi(u_0)|$ for $u \geq u_0$ implies the claim. □

Lemma 16 (On uniform integrability). *We have*

$$\sup_{u\geq 0} \int_{-\infty}^{\infty} \left|\varphi(u - \mathrm{i}t)\right| \mathrm{d}t < \infty. \tag{78}$$

Proof. It follows from the Poincaré functional equation that for every $j \geq 0$,

$$\begin{aligned}\int_{m^j}^{m^{j+1}} \left|\varphi(u - \mathrm{i}t)\right| \mathrm{d}t &= \int_{m^j}^{m^{j+1}} \left|f_j\left(\varphi\left(\frac{(u - \mathrm{i}t)}{m^j}\right)\right)\right| \mathrm{d}t \\ &\leq m^j \int_1^m f_j\left(\left|\varphi\left(um^{-j} - \mathrm{i}t\right)\right|\right) \mathrm{d}t.\end{aligned} \tag{79}$$

Since for $v \geq 0$ fixed, $t \mapsto \varphi(v - \mathrm{i}t)/\varphi(v)$ is the characteristic function of some absolutely continuous law (Cramér transform), we deduce that for all $v \geq 0$ and $\theta > 0$ there exists $\eta = \eta(v, \theta) \in (0, 1)$ such that

$$\left|\varphi(v - \mathrm{i}t)\right| < (1 - \eta)\varphi(v) < 1 \quad \text{for all } v \geq 0, |t| > \theta. \tag{80}$$

From this inequality and the continuity of the mapping $(v, t) \mapsto \varphi(v - \mathrm{i}t)$ we conclude that

$$\sup_{v\geq 0,\ t\in[1,m]} \left|\varphi(v - \mathrm{i}t)\right| =: s_0 < 1. \tag{81}$$

Together with inequality (79) and (44) we get

$$\sup_{u\geq 0}\int_{m^j}^{m^{j+1}} \bigl|\varphi(u-\mathrm{i}t)\bigr|\,\mathrm{d}t \leq m^{j+1}s_0^{(\mu^j)}, \quad j\geq 0. \tag{82}$$

Therefore,

$$\sup_{u\geq 0}\int_1^{\infty} \bigl|\varphi(u-\mathrm{i}t)\bigr|\,\mathrm{d}t \leq \sum_{j\geq 0} m^{j+1}s_0^{(\mu^j)} < \infty. \tag{83}$$

Analogously,

$$\sup_{u\geq 0}\int_{-\infty}^{-1} \bigl|\varphi(u-\mathrm{i}t)\bigr|\,\mathrm{d}t \leq \sum_{j\geq 0} m^{j+1}s_0^{(\mu^j)} < \infty. \tag{84}$$

Both statements imply the claim in the lemma. □

Recall notation b from (38).

Lemma 17 (Miscellaneous). *Set* $\psi(u) := b(\varphi(u))$, $u\geq 0$. *Then* ψ *is a decreasing analytic function on* $(0,\infty)$. *Moreover,*

(a) $\psi'(u)\to -\infty$ *as* $u\downarrow 0$,
(b) $\psi'(u)\to 0$ *as* $u\uparrow\infty$,
(c) $\psi''(u) > 0$ *for all* $u>0$.

Proof. As φ is analytic on $(0,\infty)$ and b (by Lemma 10) analytic on $(0,1)$, we see that ψ is analytic on $(0,\infty)$. We know that b increases and φ decreases. Then ψ decreases, i.e. $\psi'(u)<0$ for all $u\geq 0$.

(c) It follows from the definition of ψ that

$$\psi''(u) = b''\bigl(\varphi(u)\bigr)\bigl[\varphi'(u)\bigr]^2 + b'\bigl(\varphi(u)\bigr)\varphi''(u). \tag{85}$$

By Lemma 14, $\varphi(u)b'(\varphi(u))>0$. Combining this with (57), (85) and (75), we obtain (c).

(a) It was shown in [6] that

$$\psi(u) = -u^{\beta}V(u), \quad u\geq 0, \tag{86}$$

where V is a positive, multiplicatively periodic function with period m. Since $\psi(mu)=m^{\beta}\psi(u)$, differentiation gives

$$\psi'(mu) = m^{\beta-1}\psi'(u). \tag{87}$$

For $0<u<1$, we set $k_{\mathrm{a}}=k_{\mathrm{a}}(u) := \min\{j\geq 1\colon\ um^j\geq 1\}$. By (87),

$$\psi'(u) = m^{k_{\mathrm{a}}(1-\beta)}\psi'\bigl(m^{k_{\mathrm{a}}}u\bigr) \leq m^{k_{\mathrm{a}}(1-\beta)}\max_{v\in[1,m]}\psi'(v). \tag{88}$$

Recalling that $\psi'<0$ is continuous, we get (a), since $k_{\mathrm{a}}=k_{\mathrm{a}}(u)\uparrow\infty$ as $u\downarrow 0$.

(b) For $u>m$, put $k_{\mathrm{b}}=k_{\mathrm{b}}(u) := \max\{j\geq 1\colon\ u\geq m^j\}$. Using (87) once again, we have

$$\bigl|\psi'(u)\bigr| = m^{k_{\mathrm{b}}(\beta-1)}\left|\psi'\left(\frac{u}{m^{k_{\mathrm{b}}}}\right)\right| \leq m^{-k_{\mathrm{b}}(1-\beta)}\max_{v\in[1,m]}\bigl|\psi'(v)\bigr|. \tag{89}$$

From the continuity of ψ', part (b) follows, since $k_{\mathrm{b}}=k_{\mathrm{b}}(u)\uparrow\infty$ as $u\uparrow\infty$. □

2.3. *On some rates of convergence*

Put

$$\varphi_j(u) := \mathbf{E}e^{-uZ_j/m^j}, \quad j \geq 0, u \geq 0. \tag{90}$$

Note that by (2), $\varphi_j \to \varphi$ pointwise as $j \uparrow \infty$, provided that $\mathbf{E}Z_1 \log Z_1 < \infty$.

Lemma 18 (Rate of convergence of φ_j). *Assume that $\mathbf{E}Z_1^2 < \infty$. Then for each fixed $u \geq 0$,*

$$\varphi_j(u) - \varphi(u) = \frac{\varrho^2}{2} u^2 \varphi'(u) m^{-j} \big(1 + \mathrm{o}(1)\big) \quad \textit{as } j \uparrow \infty, \tag{91}$$

where we set $\varrho^2 := \mathrm{Var}\, W$. If we only assume that (29) holds, then for $u \geq 0$ fixed,

$$\varphi_j(u) - \varphi(u) = C(r,m)\, u^r \varphi'(u) m^{-j(r-1)} \ell\big(m^j\big)\big(1 + \mathrm{o}(1)\big) \quad \textit{as } j \uparrow \infty, \tag{92}$$

with constant $C(r,m) := \frac{\Gamma(2-r)}{(r-1)(m^r-m)}$ (and the slowly varying function ℓ from (29)). Moreover, both relations are uniform in u on any compact subset of $(0,\infty)$.

Proof. In view of (3) and by notation (90),

$$\varphi_j(u) - \varphi(u) = f_j\big(e^{-u/m^j}\big) - f_j\left(\varphi\left(\frac{u}{m^j}\right)\right), \quad j, u \geq 0. \tag{93}$$

Hence, by the mean value theorem,

$$\varphi_j(u) - \varphi(u) = f_j'(\theta_j)\left(e^{-u/m^j} - \varphi\left(\frac{u}{m^j}\right)\right) \tag{94}$$

for some $\theta_j \in [e^{-u/m^j}, \varphi(u/m^j)]$. Since $\mathbf{E}W = 1$ under the $Z_1 \log Z_1$-moment condition, we have

$$\varphi\left(\frac{u}{m^j}\right) = 1 - \frac{u}{m^j} + \mathrm{o}\left(\frac{1}{m^j}\right) \quad \text{as } j \uparrow \infty, \tag{95}$$

which is uniform for bounded $u \geq 0$. Thus,

$$\theta_j = \exp\left\{-\frac{u + \mathrm{o}(1)}{m^j}\right\} \quad \text{as } j \uparrow \infty, \tag{96}$$

which is uniform for bounded $u \geq 0$. Note that for $j, u \geq 0$,

$$f_j'\big(e^{-u/m^j}\big) = m^j e^{u/m^j} \mathbf{E} g_u\left(\frac{Z_j}{m^j}\right), \tag{97}$$

where we set $g_u(x) := x e^{-ux}$. It is easy to verify that for $0 < a < A < \infty$ fixed, $\mathcal{G} := \{g_u,\ u \in [a, A]\}$ is a family of uniformly bounded and equi-continuous functions. Then, by the limit theorem (2) for Z,

$$\mathbf{E} g_u\left(\frac{Z_j}{m^j}\right) \underset{j\uparrow\infty}{\longrightarrow} \mathbf{E} g_u(W) = -\varphi'(u), \quad u \geq 0, \tag{98}$$

uniformly on $\mathcal{G}$. From this and (96) we conclude that

$$f_j'(\theta_j) = -m^j \varphi'(u)\big(1 + \mathrm{o}(1)\big) \quad \text{as } j \uparrow \infty, \tag{99}$$

uniformly in u on any compact subset of $(0,\infty)$.

It is known (see, e.g., [7]) that condition $\mathbf{E}Z_1^2 < \infty$ implies $\mathbf{E}W^2 < \infty$. This then means that

$$\varphi\left(\frac{u}{m^j}\right) = 1 - \frac{u}{m^j} + \frac{\mathbf{E}W^2}{2}\frac{u^2}{m^{2j}} + \mathrm{o}\left(\frac{1}{m^{2j}}\right) \quad \text{as } j \uparrow \infty, \tag{100}$$

uniformly for bounded $u \geq 0$. Therefore,

$$\mathrm{e}^{-u/m^j} - \varphi\left(\frac{u}{m^j}\right) = -\frac{\varrho^2}{2}\frac{u^2}{m^{2j}}\big(1+\mathrm{o}(1)\big) \quad \text{as } j \uparrow \infty, \tag{101}$$

uniformly in u on any compact subset of $(0, \infty)$. Applying (99) and (101) to the right-hand side of (94), we obtain (91).

If (29) only holds, then (see [7])

$$\mathbf{P}(W \geq x) \sim \frac{x^{-r}\ell(x)}{(m^r - m)} \quad \text{as } x \uparrow \infty. \tag{102}$$

Hence, by the Abelian theorem (see, for instance, [10], Chapter XIII, Section 5), as $u \downarrow 0$,

$$\varphi(u) = 1 - u + \frac{\Gamma(2-r)}{(r-1)(m^r - m)}u^r\ell\left(\frac{1}{u}\right)\big(1+\mathrm{o}(1)\big), \tag{103}$$

and, consequently, as $j \uparrow \infty$,

$$\mathrm{e}^{-u/m^j} - \varphi\left(\frac{u}{m^j}\right) = -\frac{\Gamma(2-r)}{(r-1)(m^r - m)}u^r m^{-jr}\ell\big(m^j\big)\big(1+\mathrm{o}(1)\big), \tag{104}$$

uniformly in u on any compact subset of $(0, \infty)$. Combining now (94), (99) and (104) gives (92). Thus, the proof is complete. □

Lemma 19 (Rate of convergence of φ_j'). *Assume that $\mathbf{E}Z_1^2$ is finite. Then for each fixed $u \geq 0$,*

$$\varphi_j'(u) - \varphi'(u) = \frac{\varrho^2}{2}m^{-j}\big[2u\varphi'(u) - u^2\varphi''(u)\big]\big(1+\mathrm{o}(1)\big) \quad \textit{as } j \uparrow \infty. \tag{105}$$

If only (29) holds, then for $u \geq 0$ fixed,

$$\varphi_j'(u) - \varphi'(u) = C(r,m)\big[ru^{r-1}\varphi'(u) - u^r\varphi''(u)\big]m^{-j(r-1)}\ell\big(m^j\big)\big(1+\mathrm{o}(1)\big) \tag{106}$$

as $j \uparrow \infty$. Again, both relations are uniform in u on any compact subset of $(0, \infty)$.

Proof. Using (3) once again, we have

$$\varphi'(u) = m^{-j} f_j'\left(\varphi\left(\frac{u}{m^j}\right)\right)\varphi'\left(\frac{u}{m^j}\right). \tag{107}$$

Therefore,

$$\begin{aligned}\varphi_j'(u) - \varphi'(u) &= -m^{-j}\left(\mathrm{e}^{-u/m^j} f_j'\big(\mathrm{e}^{-u/m^j}\big) + f_j'\left(\varphi\left(\frac{u}{m^j}\right)\right)\varphi'\left(\frac{u}{m^j}\right)\right) \\ &= -\frac{f_j'(\mathrm{e}^{-u/m^j})}{m^j}\left[\mathrm{e}^{-u/m^j} + \varphi'\left(\frac{u}{m^j}\right)\right] \qquad (108)\\ &\quad + \frac{\varphi'(u/m^j)}{m^j}\left[f_j'\big(\mathrm{e}^{-u/m^j}\big) - f_j'\left(\varphi\left(\frac{u}{m^j}\right)\right)\right]. \qquad (109)\end{aligned}$$

If $\mathbf{E}Z_1^2$ is finite, then

$$\varphi'(u) = -1 + u\mathbf{E}W^2 + \mathrm{o}(u) \quad \text{as } u \downarrow 0. \tag{110}$$

Combining this with (99) gives

$$-\frac{f_j'(\mathrm{e}^{-u/m^j})}{m^j}\left[\mathrm{e}^{-u/m^j} + \varphi'\left(\frac{u}{m^j}\right)\right] = \varphi'(u)u\varrho^2 m^{-j}\big(1+\mathrm{o}(1)\big) \quad \text{as } j \uparrow \infty, \tag{111}$$

uniform in u on any compact subset of $(0, \infty)$.

Now we turn to (109). By the mean value theorem,

$$f_j'\big(\mathrm{e}^{-u/m^j}\big) - f_j'\left(\varphi\left(\frac{u}{m^j}\right)\right) = f_j''(\theta_j)\left(\mathrm{e}^{-u/m^j} - \varphi\left(\frac{u}{m^j}\right)\right). \tag{112}$$

Analogously to (99),

$$f_j''(\theta_j) = m^{2j}\varphi''(u)\big(1+\mathrm{o}(1)\big). \tag{113}$$

This together with (101) and (110) gives

$$\frac{\varphi'(u/m^j)}{m^j}\left[f_j'\big(\mathrm{e}^{-u/m^j}\big) - f_j'\left(\varphi\left(\frac{u}{m^j}\right)\right)\right] = -\frac{\varrho^2}{2}u^2\varphi''(u)m^{-j}\big(1+\mathrm{o}(1)\big), \tag{114}$$

uniform in u on any compact subset of $(0, \infty)$. Inserting now (111) into (108) and (114) into (109), we obtain (105).

In order to prove (106), only a single change is needed: Instead of (110) one has to use

$$\varphi'(u) = -1 + u^{r-1}\ell\left(\frac{1}{u}\right)\frac{r\Gamma(2-r)}{r-1}\big(1+\mathrm{o}(1)\big) \quad \text{as } u \downarrow 0, \tag{115}$$

which again follows from the Abelian theorem. This finishes the proof altogether. □

3. Precise left tail asymptotics: Proof of Theorem 1

For $0 < x \le \mu/m$, we define

$$r := r(x) := \max\left\{k \ge 1\colon \frac{\mu^k}{m^k} \ge x\right\} \quad \text{and} \quad y := y(x) := \frac{xm^{r(x)}}{\mu^{r(x)}}. \tag{116}$$

Evidently, $1 \le r(x) \uparrow \infty$ as $x \downarrow 0$. On the other hand, the function $x \mapsto y(x)$ is positive, multiplicatively periodic, with period $m/\mu = m^{1-\beta}$, since $r(xm/\mu) = r(x) - 1$. Also, $\mu^{r+1}/m^{r+1} < x \le \mu^r/m^r$ implies

$$\frac{\mu}{m} < y \le 1. \tag{117}$$

3.1. *Precise left tail asymptotics of the density function w*

By the inversion formula,

$$w(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\tau x}\varphi(-\mathrm{i}\tau)\,\mathrm{d}\tau, \quad x > 0. \tag{118}$$

Since $z \mapsto \mathrm{e}^z\varphi(z)$ is analytic on $\{z\colon \Re z > 0\}$ we can change the integration contour. In fact, for any $a > 0$ we can integrate along the line $\{z\colon \Re z = a\}$, i.e.

$$w(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{(a-\mathrm{i}\tau)x}\varphi(a-\mathrm{i}\tau)\,\mathrm{d}\tau. \tag{119}$$

Since φ satisfies the Poincaré functional equation, we have

$$\varphi(z) = f_k\bigg(\varphi\Big(\frac{z}{m^k}\Big)\bigg), \quad \Re z \geq 0, k \geq 1. \tag{120}$$

Using this with $k = r = r(x)$ from (116) gives

$$w(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{(a-\mathrm{i}\tau)x} f_r\bigg(\varphi\Big(\frac{(a-\mathrm{i}\tau)}{m^r}\Big)\bigg)\,\mathrm{d}\tau, \quad 0 < x \leq \frac{\mu}{m}. \tag{121}$$

Choose now $a = um^{r(x)}$ for any fixed $u > 0$, substitute $\tau = tm^{r(x)}$, and noting that $xm^{r(x)} = y(x)\mu^{r(x)}$, by the definition of $y = y(x)$ in (116) we get

$$w(x) = \frac{m^r}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{(u-\mathrm{i}t)y\mu^r} f_r\big(\varphi(u-\mathrm{i}t)\big)\,\mathrm{d}t, \quad 0 < x \leq \frac{\mu}{m}. \tag{122}$$

Next we want to analyze different parts of this integral.

Noting that $s \mapsto f_r(s)/s$ is increasing in the present Böttcher case, and using (80), we get for all $|t| \geq \theta$,

$$\begin{aligned} \big|f_r\big(\varphi(u-\mathrm{i}t)\big)\big| &\leq f_r\big(\big|\varphi(u-\mathrm{i}t)\big|\big) = \big|\varphi(u-\mathrm{i}t)\big| \frac{f_r(|\varphi(u-\mathrm{i}t)|)}{|\varphi(u-\mathrm{i}t)|} \\ &\leq \big|\varphi(u-\mathrm{i}t)\big| \frac{f_r(\varphi(u)(1-\eta))}{\varphi(u)(1-\eta)} \end{aligned} \tag{123}$$

with $\eta = \eta(u,\theta) \in (0,1)$. Consequently,

$$\begin{aligned} I(\theta) &:= \bigg|\int_{|t|\geq\theta} \mathrm{e}^{(u-\mathrm{i}t)y\mu^r} f_r\big(\varphi(u-\mathrm{i}t)\big)\,\mathrm{d}t\bigg| \\ &\leq \mathrm{e}^{uy\mu^r} \frac{f_r(\varphi(u)(1-\eta))}{\varphi(u)(1-\eta)} \int_{-\infty}^{\infty} \big|\varphi(u-\mathrm{i}t)\big|\,\mathrm{d}t. \end{aligned} \tag{124}$$

According to Lemma 16 the integral in (124) is finite. Further, applying Lemma 13 to $f_{r(x)}(\varphi(u)(1-\eta))$, we obtain from estimate (124),

$$I(\theta) \leq c(\theta,u) \exp\big\{\mu^r\big[uy + b\big(\varphi(u)(1-\eta)\big)\big]\big\} \tag{125}$$

for some constant $c(\theta,u)$. Since from (41) it follows that b is increasing on $(0,1]$, there exits $\varepsilon = \varepsilon(\eta,u) > 0$ such that

$$b\big(\varphi(u)(1-\eta)\big) \leq b\big(\varphi(u)\big) - \varepsilon. \tag{126}$$

Therefore, we have the *tail estimate*

$$I(\theta) \leq c(\theta,u) \exp\big\{\mu^r\big[uy + b\big(\varphi(u)\big)\big] - \varepsilon\mu^r\big\}. \tag{127}$$

Fix $u_0 > 0$. According to Lemma 15 there is a constant $C = C(u_0)$ such that for all $\theta \in (0, C]$,

$$\varphi(u-\mathrm{i}t) \in \mathcal{D}\bigg(\varphi(u_0), \frac{\theta}{C}\bigg), \quad u \geq u_0 \text{ and } |t| \leq \theta. \tag{128}$$

Furthermore, by Lemma 10, the function b is analytic on $\mathcal{D}(\varphi(u_0), \theta/C)$ for all small enough θ, say $0 < \theta \leq \theta_1$. This implies in particular, that $\frac{\partial^3}{\partial t^3} b(\varphi(u-\mathrm{i}t))$ is uniformly bounded on the set $\{u \geq u_0, |t| \leq \theta\}$. Hence, expanding in a Taylor series in the variable t,

$$b\big(\varphi(u-\mathrm{i}t)\big) = b\big(\varphi(u)\big) - b'\big(\varphi(u)\big)\varphi'(u)\mathrm{i}t - \frac{t^2}{2}\frac{\partial^2}{\partial u^2} b\big(\varphi(u)\big) + \mathrm{O}\big(\big|t^3\big|\big), \tag{129}$$

uniformly in $u \geq u_0$.

By (40), we have the following *main part* representation:

$$\int_{-\theta}^{\theta} \mathrm{e}^{(u-\mathrm{i}t)y\mu^r} f_r\bigl(\varphi(u-\mathrm{i}t)\bigr)\,\mathrm{d}t$$
$$= p_\mu^{-1/(\mu-1)}\bigl(1+\mathrm{O}\bigl(\mathrm{e}^{-\varphi(u_0)\mu^r}\bigr)\bigr)\int_{-\theta}^{\theta}\exp\bigl\{\mu^r\bigl[b\bigl(\varphi(u-\mathrm{i}t)\bigr)+(u-\mathrm{i}t)y\bigr]\bigr\}\,\mathrm{d}t \tag{130}$$

as $r=r(x)\uparrow\infty$, where the O-expression is uniform in $u\ge u_0$.

For the further analysis of the integral in (130) we want to apply now the *saddle point approximation*. For fixed $x\in(0,\mu/m]$, let $u^*:=u^*(x)>0$ denote the unique solution of the equation

$$b'\bigl(\varphi(u)\bigr)\varphi'(u)=-y(x). \tag{131}$$

The existence and the uniqueness of u^* follow from Lemma 17.

Since $u\mapsto b'(\varphi(u))\varphi'(u)$ increases (by Lemma 17(c)), if x_1,x_2 are such that $y(x_1)\le y(x_2)$, then $u^*(x_1)\ge u^*(x_2)$. But recalling (74), we have $y(x)\le 1=y(\mu/m)$. Therefore, $u^*(x)\ge u^*(\mu/m)$ for all $x\in(0,\mu/m]$. Using (129) with $u_0=u^*(\mu/m)$ and $u=u^*(x)$, we obtain for $|t|\le\theta$,

$$\begin{aligned} b\bigl(\varphi\bigl(u^*-\mathrm{i}t\bigr)\bigr)-\mathrm{i}ty &= b\bigl(\varphi\bigl(u^*\bigr)\bigr)+b\bigl(\varphi\bigl(u^*-\mathrm{i}t\bigr)\bigr)-b\bigl(\varphi\bigl(u^*\bigr)\bigr)-\mathrm{i}ty\\ &= b\bigl(\varphi\bigl(u^*\bigr)\bigr)-\bigl(b'\bigl(\varphi\bigl(u^*\bigr)\bigr)\varphi'\bigl(u^*\bigr)+y\bigr)\mathrm{i}t-\frac{\sigma^2}{2}t^2+\mathrm{O}\bigl(\bigl|t^3\bigr|\bigr)\\ &= b\bigl(\varphi\bigl(u^*\bigr)\bigr)-\frac{\sigma^2}{2}t^2+\mathrm{O}\bigl(\bigl|t^3\bigr|\bigr) \end{aligned} \tag{132}$$

as $t\to 0$, where σ is defined by

$$\sigma^2:=\sigma^2(x):=\frac{\mathrm{d}^2}{\mathrm{d}u^2}b\bigl(\varphi(u)\bigr)\Big|_{u=u^*(x)}=\psi''\bigl(u^*\bigr)>0. \tag{133}$$

The latter positivity follows from Lemma 17(c). Recall that the O is uniform in $x\in(0,\mu/m]$.

From (132) we have

$$\int_{-r\mu^{-r/2}}^{r\mu^{-r/2}}\exp\bigl\{\mu^r\bigl[b\bigl(\varphi\bigl(u^*-\mathrm{i}t\bigr)\bigr)+\bigl(u^*-\mathrm{i}t\bigr)y\bigr]\bigr\}\,\mathrm{d}t$$
$$=\exp\bigl\{\mu^r\bigl[b\bigl(\varphi\bigl(u^*\bigr)\bigr)+u^*y\bigr]\bigr\}\int_{-r\mu^{-r/2}}^{r\mu^{-r/2}}\exp\biggl\{-\mu^r\frac{\sigma^2}{2}t^2\biggr\}\,\mathrm{d}t\,\bigl(1+\mathrm{O}\bigl(r^3\mu^{-r/2}\bigr)\bigr) \tag{134}$$

as $r=r(x)\uparrow\infty$ (with O uniform in x). By the substitution $\mu^{r/2}\sigma t=:\tau$ we get

$$\begin{aligned}\int_{-r\mu^{-r/2}}^{r\mu^{-r/2}}\exp\biggl\{-\mu^r\frac{\sigma^2}{2}t^2\biggr\}\,\mathrm{d}t &= \frac{1}{\mu^{r/2}\sigma}\int_{-r\sigma}^{r\sigma}\mathrm{e}^{-\tau^2/2}\,\mathrm{d}\tau\\ &= \frac{1}{\mu^{r/2}\sigma}\biggl(\sqrt{2\pi}-2\int_{r\sigma}^{\infty}\mathrm{e}^{-\tau^2/2}\,\mathrm{d}\tau\biggr)=\frac{\sqrt{2\pi}}{\mu^{r/2}\sigma}\bigl(1+\mathrm{o}\bigl(r^3\mu^{-r/2}\bigr)\bigr)\end{aligned} \tag{135}$$

as $r=r(x)\uparrow\infty$. Inserting into (134) gives the following representation of the *central part* of the integral in (130) (with $u=u^*$):

$$\int_{-r\mu^{-r/2}}^{r\mu^{-r/2}}\exp\bigl\{\mu^r\bigl[b\bigl(\varphi\bigl(u^*-\mathrm{i}t\bigr)\bigr)+\bigl(u^*-\mathrm{i}t\bigr)y\bigr]\bigr\}\,\mathrm{d}t$$
$$=\sqrt{\frac{2\pi}{\sigma^2}}\mu^{-r/2}\exp\bigl\{\mu^r\bigl[b\bigl(\varphi\bigl(u^*\bigr)\bigr)+u^*y\bigr]\bigr\}\bigl(1+\mathrm{O}\bigl(r^3\mu^{-r/2}\bigr)\bigr) \tag{136}$$

as $r = r(x) \uparrow \infty$.

On the other hand, since $C|t^3| \le \frac{\sigma^2}{4}t^2$ for each fixed constant C and for all small enough $|t|$, relation (132) implies

$$\Re\big(b\big(\varphi\big(u^* - \mathrm{i}t\big)\big) + \big(u^* - \mathrm{i}t\big)y\big) \le b\big(\varphi\big(u^*\big)\big) + u^* y - \frac{\sigma^2}{4}t^2 \tag{137}$$

for all $|t| \le \theta$ and for small enough θ, say $\theta \le \theta_2$. Consequently, for all $\theta < \theta_2$ and all small enough x, we obtain the following estimate of an *intermediate part* of the integral in (130) (with $u = u^*$):

$$\begin{aligned} &\left| \int_{|t| \in [r\mu^{-r/2}, \theta]} \exp\big\{\mu^r \big[b\big(\varphi\big(u^* - \mathrm{i}t\big)\big) + \big(u^* - \mathrm{i}t\big)y\big]\big\}\, \mathrm{d}t \right| \\ &\quad \le 2\theta \exp\big\{\mu^r\big[b\big(\varphi\big(u^*\big)\big) + u^* y\big]\big\} \exp\left\{-\frac{\sigma^2}{4}r^2\right\} \end{aligned} \tag{138}$$

(with $r = r(x)$ and $u^* = u^*(x)$).

Putting $u = u^*(x)$ in (130) and taking into account our partial results (136) and (138), instead of (130) we get, for $\theta \le \theta_1 \wedge \theta_2$,

$$\begin{aligned} &\int_{-\theta}^{\theta} \mathrm{e}^{(u^* - \mathrm{i}t)y\mu^r} f_r\big(\varphi\big(u^* - \mathrm{i}t\big)\big)\, \mathrm{d}t \\ &\quad = p_\mu^{-1/(\mu-1)} \sqrt{\frac{2\pi}{\sigma^2}}\, \mu^{-r/2} \exp\big\{\mu^r\big[b\big(\varphi\big(u^*\big)\big) + u^* y\big]\big\}\big(1 + \mathrm{O}\big(r^3\mu^{-r/2}\big)\big) \end{aligned} \tag{139}$$

since $\mathrm{e}^{-\sigma^2 r^2/4} = \mathrm{o}(1 + \mathrm{O}(r^3\mu^{-r/2}))$ as $r \uparrow \infty$.

Applying now (127) with $u = u^*(x)$, and (139) to (122) with $u = u^*(x)$, instead of (122) we have

$$w(x) = \frac{p_\mu^{-1/(\mu-1)}}{\sqrt{2\pi\sigma^2}} m^r \mu^{-r/2} \exp\big\{\mu^r\big[b\big(\varphi\big(u^*\big)\big) + u^* y\big]\big\}\big(1 + \mathrm{O}\big(r^3\mu^{-r/2}\big)\big) \tag{140}$$

as $r = r(x) \uparrow \infty$.

It follows from the definition of $u^* = u^*(x)$ around (131), that

$$b\big(\varphi\big(u^*\big)\big) + u^* y = \min_{u \ge 0}\big\{b\big(\varphi(u)\big) + uy\big\}. \tag{141}$$

On the other hand, it is known (see Theorem 3 of [4]), that the function

$$M(v) := -v^{\beta/(1-\beta)} \min_{u \ge 0}\big\{b\big(\varphi(u)\big) + uv\big\}, \quad v > 0, \tag{142}$$

is analytic on $(0, \infty)$, positive, and multiplicatively periodic with period $m^{1-\beta}$. Therefore,

$$b\big(\varphi\big(u^*\big)\big) + u^* y = -y^{-\beta/(1-\beta)} M\big(y m^{-r(1-\beta)}\big). \tag{143}$$

Recalling that $\mu = m^\beta$, from definition (116) of $r = r(x)$ and $y = y(x)$ we have

$$y m^{-r(1-\beta)} = \frac{y m^{-r}}{\mu^{-r}} = x \quad \text{and} \quad \mu^r = \left(\frac{y}{x}\right)^{\beta/(1-\beta)}. \tag{144}$$

Applying these identities to the right-hand side of (143), we obtain

$$b\big(\varphi\big(u^*\big)\big) + u^* y = -\mu^{-r} x^{-\beta/(1-\beta)} M(x). \tag{145}$$

Moreover, the second part of (144) yields

$$\mu^{-r/2} = \left(\frac{x}{y}\right)^{\beta/2(1-\beta)} \quad \text{and} \quad m^r \mu^{-r/2} = \left(\frac{y}{x}\right)^{(2-\beta)/2(1-\beta)}. \tag{146}$$

The first of these identities gives

$$r = r(x) = \mathrm{O}(\log x), \quad \text{hence} \quad r^3 \mu^{-r/2} = \mathrm{O}\big(x^{\beta/2(1-\beta)} \log^3 x\big) \quad \text{as } x \downarrow 0. \tag{147}$$

Thus, inserting (145) and (146) into (140) we obtain (12) with

$$M_1(x) := \frac{p_\mu^{-1/(\mu-1)}}{\sqrt{2\pi\sigma^2}} y^{(2-\beta)/2(1-\beta)}. \tag{148}$$

Since $x \mapsto y(x)$ is multiplicatively periodic with period $m^{1-\beta}$, the function $x \mapsto u^*(x)$ is also multiplicatively periodic with the same period, by definition (131) of $u^*(x)$. Hence, by (133), $x \mapsto \sigma^2(x)$ is multiplicatively periodic with period $m^{1-\beta}$, too. Therefore, $x \mapsto M_1(x)$ is also multiplicatively periodic with period $m^{1-\beta}$. Thus, the proof of the first part (12) of Theorem 1 is complete.

3.2. *Precise left tail asymptotics for the law of W*

By the inversion formula for distribution functions,

$$\mathbf{P}(W < x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \mathrm{e}^{-\mathrm{i}\tau x}}{\mathrm{i}\tau} \varphi(-\mathrm{i}\tau)\, \mathrm{d}\tau. \tag{149}$$

Changing again the integration contour, we get for arbitrary $a > 0$,

$$\mathbf{P}(W < x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \mathrm{e}^{(a-\mathrm{i}\tau)x}}{a - \mathrm{i}\tau} \varphi(a - \mathrm{i}\tau)\, \mathrm{d}\tau. \tag{150}$$

After substituting $a = um^r$, $\tau = tm^r$ we have

$$\mathbf{P}(W < x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{y\mu^r(u-\mathrm{i}t)} - 1}{u - \mathrm{i}t} f_r\big(\varphi(u - \mathrm{i}t)\big)\, \mathrm{d}t \tag{151}$$

(by using (116) and (120)). Evidently,

$$\left| \int_{-\infty}^{\infty} \frac{1}{u - \mathrm{i}t} f_r\big(\varphi(u - \mathrm{i}t)\big)\, \mathrm{d}t \right| \le \frac{1}{u} \int_{-\infty}^{\infty} f_r\big(|\varphi(u - \mathrm{i}t)|\big)\, \mathrm{d}t \le \frac{f_r(\varphi(u))}{u\varphi(u)} \int_{-\infty}^{\infty} |\varphi(u - \mathrm{i}t)|\, \mathrm{d}t, \tag{152}$$

in the second step we applied the inequality $|\varphi(u - \mathrm{i}t)| \le \varphi(u)$. Using Lemmas 16 and 13 gives

$$\left| \int_{-\infty}^{\infty} \frac{1}{u - \mathrm{i}t} f_r\big(\varphi(u - \mathrm{i}t)\big)\, \mathrm{d}t \right| \le c(u) \mathrm{e}^{\mu^r b(\varphi(u))} \tag{153}$$

for some constant $c(u)$. Applying this bound to (151), we get

$$\mathbf{P}(W < x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{y\mu^r(u-\mathrm{i}t)}}{u - \mathrm{i}t} f_r\big(\varphi(u - \mathrm{i}t)\big)\, \mathrm{d}t + \mathrm{O}\big(\mathrm{e}^{\mu^r b(\varphi(u))}\big) \tag{154}$$

as $r = r(x) \uparrow \infty$. The completion of the proof of (13) follows the pattern of the proof of (12). At the end we have (13) with

$$M_2(x) := \frac{p_\mu^{-1/(\mu-1)}}{u^* \sqrt{2\pi\sigma^2}} y^{-\beta/2(1-\beta)}. \tag{155}$$

From (116), this function is multiplicatively periodic with period $m^{1-\beta}$. Thus, the proof of Theorem 1 is complete.

3.3. *No oscillations*: *Proof of Corollary* 2

Since $b(s) = \log B(s)$, condition $K(u) \equiv \kappa$ means that $b(\varphi(u)) = -\kappa u^{\beta}$. It can easily be seen that

$$u^* = \left(\frac{\kappa\beta}{y}\right)^{1/(1-\beta)}. \tag{156}$$

Thus,

$$-\kappa\left(u^*\right)^{\beta} + u^* y = y^{-\beta/(1-\beta)}(\kappa\beta)^{1/(1-\beta)}\left(\beta^{-1} - 1\right). \tag{157}$$

From this equality and definition (142) of M we conclude (15). Further, by (133) and (156),

$$\sigma^2 = \kappa\beta(1-\beta)\left(u^*\right)^{\beta-2} = (\kappa\beta)^{-1/(1-\beta)}(1-\beta)y^{(2-\beta)/(1-\beta)}. \tag{158}$$

Substituting (158) into (148) gives (16), and substituting additionally (156) into (155) yields (17).

Conversely, by a Tauberian theorem of de Bruijn (see Theorem 4.12.9 in [8]), the condition $M(x) \equiv \text{const}$ implies the existence of the limit $\lim_{u\uparrow\infty} u^{-\beta} \log\varphi(u) < 0$. Furthermore, Bingham ([6], p. 219) has shown that

$$B(s) \sim p_{\mu}^{-1/(\mu-1)} s \quad \text{as } s \downarrow 0. \tag{159}$$

From this we conclude that the limit $\lim_{u\uparrow\infty} u^{-\beta} \log B(\varphi(u))$ exists as well. But a multiplicatively periodic function has a limit if and only if this function degenerates to a constant. Summarizing, the constancy of M yields the constancy of K, finishing the proof.

4. Lower deviation probabilities: Proofs of Theorems 7 and 8

4.1. *Intermediate formula*

Fix any $y \in (0,\infty)$ and set $k = k(y, j, \ell) := y c_j \mu^{\ell}$, $j, \ell \geq 0$. By the inversion formula, for all $k \equiv \mu^{j+\ell} (\operatorname{mod} d)$ and $a > 0$,

$$\mathbf{P}(Z_{j+\ell} = k) = \frac{d}{2\pi}\int_{-\pi/d}^{\pi/d} f_{j+\ell}\left(\mathrm{e}^{-a+\mathrm{i}\tau}\right)\mathrm{e}^{(a-\mathrm{i}\tau)k}\,\mathrm{d}\tau. \tag{160}$$

Letting here $a = u/c_j$ and $\tau = t/c_j$, we get

$$\mathbf{P}(Z_{j+\ell} = k) = \frac{d}{2\pi c_j}\int_{-\pi c_j/d}^{\pi c_j/d} f_{\ell}\left(\varphi_j(u - \mathrm{i}t)\right)\mathrm{e}^{\mu^{\ell} y(u-\mathrm{i}t)}\,\mathrm{d}t. \tag{161}$$

Fix any $0 < \theta < T < \infty$. Since $\varphi_j(u - \mathrm{i}t) \to \varphi(u - \mathrm{i}t)$ uniformly in $t \in [0, T]$, from (80) we conclude that there exists $\eta = \eta(\theta) > 0$ such that

$$\left|\varphi_j(u - \mathrm{i}t)\right| \leq (1-\eta)\varphi(u) \tag{162}$$

for all $t \in [\theta, T]$ and all large enough j. On the other hand, by Lemma 9 of [12] there exists $\xi > 0$ such that for all $u \geq 0$, $j \geq 1$ and $1 \leq l \leq j$,

$$\left|\varphi_j(u - \mathrm{i}t)\right| \leq \mathrm{e}^{-\xi\mu^{j-l+1}} \quad \text{for all } \frac{\pi c_j}{d c_l} \leq |t| \leq \frac{\pi c_j}{d c_{l-1}}. \tag{163}$$

In particular, for every $l \leq j$,

$$\left|\varphi_j(u - \mathrm{i}t)\right| \leq \mathrm{e}^{-\xi\mu^{l}} \quad \text{for all } \frac{\pi c_j}{d c_{j-l}} \leq |t| \leq \frac{\pi c_j}{d}. \tag{164}$$

Choosing here l such that $\mathrm{e}^{-\xi\mu^l} \le (1-\eta)\varphi(u)$ and putting $T := \pi c_j/dc_{j-l}$, we convince ourself that the bound (162) holds for all $|t| \in [\theta, \pi c_j/d]$ and all large enough j. Therefore,

$$\begin{aligned}&\left|\int_{-\pi c_j/d}^{-\theta} \mathrm{e}^{(u-\mathrm{i}t)y\mu^\ell} f_\ell\big(\varphi_j(u-\mathrm{i}t)\big)\,\mathrm{d}t + \int_{\theta}^{\pi c_j/d} \mathrm{e}^{(u-\mathrm{i}t)y\mu^\ell} f_{n-j}\big(\varphi_j(u-\mathrm{i}t)\big)\,\mathrm{d}t\right| \\ &\quad \le \mathrm{e}^{uy\mu^\ell} \frac{f_\ell(\varphi(u)(1-\eta))}{\varphi(u)(1-\eta)} \int_{-\pi c_j/d}^{\pi c_j/d} \big|\varphi_j(u-\mathrm{i}t)\big|\,\mathrm{d}t. \end{aligned} \tag{165}$$

Using again (163), we see that

$$\int_{-\pi c_j/d}^{\pi c_j/d} \big|\varphi_j(u-\mathrm{i}t)\big|\,\mathrm{d}t \le \frac{2\pi}{d}\left(1 + \sum_{l=1}^{j} \frac{c_j}{c_{l-1}} \mathrm{e}^{-\xi\mu^{j-l+1}}\right). \tag{166}$$

From the boundedness of this integral and Lemma 13 we have

$$\begin{aligned}&\left|\int_{-\pi c_j/d}^{-\theta} \mathrm{e}^{(u-\mathrm{i}t)y\mu^\ell} f_\ell\big(\varphi_j(u-\mathrm{i}t)\big)\,\mathrm{d}t + \int_{\theta}^{\pi c_j/d} \mathrm{e}^{(u-\mathrm{i}t)y\mu^\ell} f_\ell\big(\varphi_j(u-\mathrm{i}t)\big)\,\mathrm{d}t\right| \\ &\quad \le c(\theta,u)\exp\big\{\mu^\ell\big[uy + b\big((1-\eta)\varphi(u)\big)\big]\big\} \end{aligned} \tag{167}$$

with some constant $c(\theta, u)$. In view of the monotonicity of $b(s)$,

$$\begin{aligned}&\left|\int_{-\pi c_j/d}^{-\theta} \mathrm{e}^{(u-\mathrm{i}t)y\mu^\ell} f_\ell\big(\varphi_j(u-\mathrm{i}t)\big)\,\mathrm{d}t + \int_{\theta}^{\pi c_j/d} \mathrm{e}^{(u-\mathrm{i}t)y\mu^\ell} f_\ell\big(\varphi_j(u-\mathrm{i}t)\big)\,\mathrm{d}t\right| \\ &\quad \le c(\theta,u)\exp\big\{\mu^\ell\big[uy + b\big(\varphi(u)\big)\big] - \varepsilon\mu^\ell\big\}. \end{aligned} \tag{168}$$

By Lemma 10, for all small enough θ, as $\ell \uparrow \infty$,

$$\begin{aligned}&\int_{-\theta}^{\theta} \mathrm{e}^{(u-\mathrm{i}t)y\mu^\ell} f_\ell\big(\varphi_j(u-\mathrm{i}t)\big)\,\mathrm{d}t \\ &\quad = p_\mu^{-1/(\mu-1)}\big(1 + \mathrm{O}\big(\mathrm{e}^{-\delta(u)\mu^\ell}\big)\big) \int_{-\theta}^{\theta} \exp\big\{\mu^\ell\big[b\big(\varphi_j(u-\mathrm{i}t)\big) + (u-\mathrm{i}t)y\big]\big\}\,\mathrm{d}t. \end{aligned} \tag{169}$$

Since $\varphi_j(u)$ converges to $\varphi(u)$ uniformly on the compact subsets of $(0,\infty)$, for all large enough j the equation

$$b'\big(\varphi_j(u)\big)\varphi_j'(u) = -y \tag{170}$$

has a unique solution, which will be denoted by u_j^*, and

$$\sigma_j^2 := \frac{\mathrm{d}^2}{\mathrm{d}u^2} b\big(\varphi_j(u)\big)\bigg|_{u=u_j^*} > 0. \tag{171}$$

Repeating word for word the proof of (139), we have, as $\ell \uparrow \infty$,

$$\begin{aligned}&\int_{-\theta}^{\theta} \mathrm{e}^{(u-\mathrm{i}t)y\mu^\ell} f_\ell\big(\varphi_j(u-\mathrm{i}t)\big)\,\mathrm{d}t \\ &\quad = p_\mu^{-1/(\mu-1)}\sqrt{\frac{2\pi}{\sigma_j^2}}\,\mu^{-\ell/2}\exp\big\{\mu^\ell\big[b\big(\varphi_j\big(u_j^*\big)\big) + u_j^* y\big]\big\}\big(1 + \mathrm{o}(1)\big). \end{aligned} \tag{172}$$

Applying (172) and (168) with u_j^* to (161) with u_j^*, and noting that

$$\lim_{j\uparrow\infty}\big[b\big(\varphi_j\big(u_j^*\big)\big)+u_j^* y\big]=\big[b\big(\varphi\big(u^*\big)\big)+u^* y\big] \tag{173}$$

and

$$\lim_{j\uparrow\infty}\sigma_j^2=\sigma^2 \tag{174}$$

with u^* and σ^2 defined in (131) and (133), we have, as $j,\ell\uparrow\infty$,

$$\mathbf{P}(Z_{j+\ell}=k)=\frac{dp_\mu^{-1/(\mu-1)}}{\sqrt{2\pi\sigma^2}c_j}\mu^{-\ell/2}\exp\big\{\mu^\ell\big[b\big(\varphi_j\big(u_j^*\big)\big)+u_j^* y\big]\big\}\big(1+\mathrm{o}(1)\big). \tag{175}$$

4.2. *Precise logarithmic asymptotics*: *Proof of Theorem* 7

Choosing $j=j_n$, $\ell=n-j_n$ and $y=k_n/c_{j_n}\mu^{n-j_n}$ in (175), we get, as $n\uparrow\infty$,

$$\mathbf{P}(Z_n=k_n)=\frac{dp_\mu^{-1/(\mu-1)}}{\sqrt{2\pi\sigma^2}c_{j_n}}\mu^{-(n-j_n)/2}\exp\big\{\mu^{n-j_n}\big[b\big(\varphi_{j_n}\big(u_{j_n}^*\big)\big)+u_{j_n}^* y\big]\big\}\big(1+\mathrm{o}(1)\big). \tag{176}$$

Multiplying both parts of (176) by c_n and taking logarithms, we have

$$\begin{aligned}\log\big[c_n\mathbf{P}(Z_n=k_n)\big]&=\mu^{n-j_n}\left(\big[b\big(\varphi_{j_n}\big(u_{j_n}^*\big)\big)+u_{j_n}^* y\big]+\mu^{-(n-j_n)}\log\left(\frac{c_n}{c_{j_n}}\right)+\mathrm{o}(1)\right)\\&=\mu^{n-j_n}\big(\big[b\big(\varphi\big(u^*\big)\big)+u^* y\big]+\mathrm{o}(1)\big),\end{aligned} \tag{177}$$

where in the second step we used (173) and the bound $c_n/c_j\le m^{n-j}=\mu^{(n-j)/\beta}$.

Recall definition (116) of $r=r(x)$. It is easy to see that if $x=y\mu^{n-j_n}/m^{n-j_n}$, then $r(x)=n-j_n$. Hence, in view of (140),

$$w\left(\frac{y\mu^{n-j_n}}{m^{n-j_n}}\right)=\frac{p_\mu^{-1/(\mu-1)}}{\sqrt{2\pi\sigma^2}}m^{n-j_n}\mu^{-(n-j_n)/2}\exp\big\{\mu^{n-j_n}\big[b\big(\varphi\big(u^*\big)\big)+u^* y\big]\big\}\big(1+\mathrm{o}(1)\big). \tag{178}$$

Taking logarithms, we have

$$\log w\left(\frac{y\mu^{n-j_n}}{m^{n-j_n}}\right)=\mu^{n-j_n}\big(\big[b\big(\varphi\big(u^*\big)\big)+u^* y\big]+\mathrm{o}(1)\big). \tag{179}$$

Comparing the right-hand sides of (177) and (179), we have

$$\log\big[c_n\mathbf{P}(Z_n=k_n)\big]\sim\log w\left(\frac{y\mu^{n-j_n}}{m^{n-j_n}}\right). \tag{180}$$

But by the definition of j_n,

$$\frac{y\mu^{n-j_n}}{m^{n-j_n}}=\frac{k_n}{c_{j_n}m^{n-j_n}}. \tag{181}$$

Thus, the proof of Theorem 7 is finished.

4.3. *On the asymptotic behavior of* $u_j^* - u^*$

By the definitions of u^* and u_j^*,

$$b'\big(\varphi(u^*)\big)\varphi'(u^*) = -y = b'\big(\varphi_j(u_j^*)\big)\varphi_j'(u_j^*).$$

Consequently,

$$b'\big(\varphi(u^*)\big)\varphi'(u^*) - b'\big(\varphi(u_j^*)\big)\varphi'(u_j^*) = b'\big(\varphi_j(u_j^*)\big)\varphi_j'(u_j^*) - b'\big(\varphi(u_j^*)\big)\varphi'(u_j^*). \tag{182}$$

Using the Taylor expansion, we have for the left-hand side the equality

$$b'\big(\varphi(u^*)\big)\varphi'(u^*) - b'\big(\varphi(u_j^*)\big)\varphi'(u_j^*) = -\sigma^2(u_j^* - u^*) + \mathrm{O}\big((u_j^* - u^*)^2\big) \tag{183}$$

as $j \uparrow \infty$. On the other hand,

$$b'\big(\varphi_j(u_j^*)\big) - b'\big(\varphi(u_j^*)\big) = b''\big(\varphi(u_j^*)\big)\big(\varphi_j(u_j^*) - \varphi(u_j^*)\big) + \mathrm{O}\big(\big(\varphi_j(u_j^*) - \varphi(u_j^*)\big)^2\big)$$

as $j \uparrow \infty$. Hence, applying (91) and recalling that $\varphi(u_j^*) \to \varphi(u^*)$, we get

$$b'\big(\varphi_j(u_j^*)\big) - b'\big(\varphi(u_j^*)\big) = \frac{\varrho^2}{2} b''\big(\varphi(u^*)\big)\varphi'(u^*)(u^*)^2 m^{-j}\big(1 + \mathrm{o}(1)\big) \tag{184}$$

as $j \uparrow \infty$, provided that $\mathbf{E}Z_1^2 < \infty$. From this equality and (105) we conclude that

$$b'\big(\varphi_j(u_j^*)\big)\varphi_j'(u_j^*) - b'\big(\varphi(u_j^*)\big)\varphi'(u_j^*) = \mathrm{O}\big(m^{-j}\big) \quad \text{as } j \uparrow \infty. \tag{185}$$

Combining (182), (183) and (185), we conclude that if $\mathbf{E}Z_1^2$ is finite then

$$u_j^* - u^* = \mathrm{O}\big(m^{-j}\big) \tag{186}$$

as $j \uparrow \infty$. Moreover, if (29) holds, then, proceeding analogously to the case of finite variance, we have

$$u_j^* - u^* = \mathrm{O}\big(m^{-j(r-1)}\ell(m^j)\big). \tag{187}$$

4.4. *Fine asymptotics*: *Proof of Theorem* 8

For convenience, set $Q(u) := b(\varphi(u)) + yu$ and $Q_j(u) := b(\varphi_j(u)) + yu$.

Once again, since $\mathbf{E}Z_1 \log Z_1 < \infty$ we can set $c_j = m^j$. Then from formula lines (176), (178) and (181) we get

$$\frac{m^n \mathbf{P}(Z_n = k_n)}{w(k_n/m^n)} = \exp\big\{\mu^{n-j_n}\big[Q_{j_n}(u_{j_n}^*) - Q(u^*)\big]\big\}\big(1 + \mathrm{o}(1)\big) \tag{188}$$

as $n \uparrow \infty$. Evidently,

$$Q_{j_n}(u_{j_n}^*) - Q(u^*) = \big[Q_{j_n}(u_{j_n}^*) - Q(u_{j_n}^*)\big] + \big[Q(u_{j_n}^*) - Q(u^*)\big]. \tag{189}$$

It follows from the definition of u^* that $Q'(u^*) = 0$. Thus, as $n \uparrow \infty$,

$$\big[Q(u_{j_n}^*) - Q(u^*)\big] = \frac{1}{2} Q''(u^*)(u_{j_n}^* - u^*)^2\big(1 + \mathrm{o}(1)\big). \tag{190}$$

On the other hand,

$$\big[Q_{j_n}(u_{j_n}^*) - Q(u_{j_n}^*)\big] = b\big(\varphi_{j_n}(u_{j_n}^*)\big) - b\big(\varphi(u_{j_n}^*)\big) = b'\big(\varphi(u^*)\big)\big(\varphi_{j_n}(u_{j_n}^*) - \varphi(u_{j_n}^*)\big)\big(1 + \mathrm{o}(1)\big). \tag{191}$$

If $\mathbf{E}Z_1^2 < \infty$, then applying (186) with $j = j_n$ to (190) and (91) to (191), and taking into account (189), we have

$$Q_{j_n}\big(u_{j_n}^*\big) - Q\big(u^*\big) = \frac{\varrho^2}{2}\big(u^*\big)^2 b'\big(\varphi\big(u^*\big)\big)\varphi'\big(u^*\big)m^{-j_n}\big(1 + \mathrm{o}(1)\big). \tag{192}$$

Substituting (192) into (188), and noting that

$$\mu^{n-j_n} m^{-j_n} = y^{(1+\beta)/(1-\beta)}\left(\frac{m^{2\beta n}}{k_n^{1+\beta}}\right)^{1/(1-\beta)}, \tag{193}$$

we get (28) with

$$V_2(x) := -y^{(1+\beta)/(1-\beta)}\frac{\varrho^2}{2}\big(u^*\big)^2 b'\big(\varphi\big(u^*\big)\big)\varphi'\big(u^*\big) = y^{2/(1-\beta)}\frac{\varrho^2}{2}\big(u^*\big)^2, \tag{194}$$

where $y = y(x)$ is defined as in (116).

In the case (29), using (187) and (92) instead of (186) and (91), we arrive at

$$Q_{j_n}\big(u_{j_n}^*\big) - Q\big(u^*\big) = \frac{\Gamma(2-r)}{(r-1)(m^r - m)}\big(u^*\big)^r b'\big(\varphi\big(u^*\big)\big)\varphi'\big(u^*\big)m^{-j_n(r-1)}\ell\big(m^{j_n}\big)\big(1 + \mathrm{o}(1)\big) \tag{195}$$

as $n \uparrow \infty$. Combining (195) and (188), and noting that

$$\mu^{n-j_n} m^{-j_n(r-1)} = y^{(r-1+\beta)/(1-\beta)}\left(\frac{m^{r\beta n}}{k_n^{r-1+\beta}}\right)^{1/(1-\beta)} \quad \text{and} \quad m^{j_n} = \left(\frac{k_n}{y m^{\beta n}}\right)^{1/(1-\beta)},$$

we have (30) with

$$\begin{aligned} V_r(x) &:= -y^{(r-1+\beta)/(1-\beta)}\frac{\Gamma(2-r)}{(r-1)(m^r - m)}\big(u^*\big)^r b'\big(\varphi\big(u^*\big)\big)\varphi'\big(u^*\big) \\ &= y^{r/(1-\beta)}\frac{\Gamma(2-r)}{(r-1)(m^r - m)}\big(u^*\big)^r. \end{aligned} \tag{196}$$

Note that the multiplicative periodicity of V_2 and V_r follows from the multiplicative periodicity of u^* and y. The proof of Theorem 8 is finished.

Acknowledgments

We are grateful to anonymous referees for a careful reading and their suggestions that led to an improvement of the exposition. In particular, we thank one referee for directing our attention to the paper [5].

References

[1] M. T. Barlow and E. A. Perkins. Brownian motion on the Sierpiński gasket. *Probab. Theory Related Fields* **79** (1988) 543–623. MR0966175

[2] J. D. Biggins. The growth of iterates of multivariate generating functions. *Trans. Amer. Math. Soc.* **360** (2008) 4305–4334.

[3] J. D. Biggins and N. H. Bingham. Near-constancy phenomena in branching processes. *Math. Proc. Cambridge Philos. Soc.* **110** (1991) 545–558. MR1120488

[4] J. D. Biggins and N. H. Bingham. Large deviations in the supercritical branching process. *Adv. Appl. Probab.* **25** (1993) 757–772. MR1241927

[5] N. H. Bingham. Continuous branching processes and spectral positivity. *Stochastic Process. Appl.* **4** (1976) 217–242. MR0410961

[6] N. H. Bingham. On the limit of a supercritical branching process. *J. Appl. Probab.* **25A** (1988) 215–228. MR0974583

[7] N. H. Bingham and R. A. Doney. Asymptotic properties of supercritical branching processes. I. The Galton–Watson processes. *Adv. Appl. Probab.* **6** (1974) 711–731. MR0362525

[8] N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular Variation*. Cambridge Univ. Press, 1987. MR0898871

[9] M. S. Dubuc. La densite de la loi-limite d'un processus en cascade expansif. *Z. Wahrsch. Verw. Gebiete* **19** (1971) 281–290. MR0300353

[10] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume II, 2nd edition. Wiley, New York, 1971. MR0270403

[11] P. Flajolet and A. M. Odlyzko. Limit distributions for coefficients of iterates of polynomials with applications to combinatorial enumerations. *Math. Proc. Cambridge Philos. Soc.* **96** (1984) 237–253. MR0757658

[12] K. Fleischmann and V. Wachtel. Lower deviation probabilities for supercritical Galton–Watson processes. *Ann. Inst. H. Poincaré Probab. Statist.* **43** (2007) 233–255. MR2303121

[13] B. M. Hambly. On constant tail behaviour for the limiting random variable in a supercritical branching process. *J. Appl. Probab.* **32** (1995) 267–273. MR1316808

[14] T. E. Harris. Branching processes. *Ann. Math. Statist.* **19** (1948) 474–494. MR0027465

[15] O. D. Jones. Multivariate Böttcher equation for polynomials with nonnegative coefficients. *Aequationes Math.* **63** (2002) 251–265. MR1904719

[16] O. D. Jones. Large deviations for supercritical multitype branching processes. *J. Appl. Probab.* **41** (2004) 703–720. MR2074818