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# Pathwise differentiability for SDEs in a convex polyhedron with oblique reflection

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**Abstract.** In this paper, the object of study is a Skorohod SDE in a convex polyhedron with oblique reflection at the boundary. We prove that the solution is pathwise differentiable with respect to its deterministic starting point up to the time when two of the faces are hit simultaneously. The resulting derivatives evolve according to an ordinary differential equation, when the process is in the interior of the polyhedron, and they are projected to the tangent space, when the process hits the boundary, while they jump in the direction of the corresponding reflection vector.

**Résumé.** L'object du présent travail est l'étude d'une équation différentielle stochastique de type Skorohod dans un polyèdre convexe avec réflexions obliques au bord. Nous démontrons que pour presque toutes les trajectoires, la solution est différentiable par rapport au point de départ jusqu'au temps où deux faces sont atteintes simultanément. Les dérivées sont à l'intérieur du polyèdre solutions d'une équation différentielle ordinaire. Au bord du polyèdre elles sont projetées dans l'espace tangeant en sautant en direction du vecteur de reflection correspondant.

*MSC:* Primary 60H10; secondary 60J55; 60J50

*Keywords:* Stochastic differential equations with reflection; Oblique reflection; Polyhedral domains

## 1. Introduction

We consider a Markov process with continuous sample paths, characterized as the strong solution of a stochastic differential equation (SDE) of the Skorohod type, where the domain $G$ is a convex polyhedron in $\mathbb{R}^d$, i.e. $G$ is the intersection of a finite number of half spaces. The process is driven by a $d$-dimensional standard Brownian motion and a drift term, whose coefficient function is supposed to be continuously differentiable and Lipschitz continuous. At the boundary of the polyhedron it reflects instantaneously, the possibly oblique direction of reflection being constant along each face.

Let $G = \bigcap_{i=1}^N G_i$, where each $G_i$ is a closed half space with inward normal $n_i$. The direction of reflection on the faces $\partial G_i$ will be denoted by constant vectors $v_i$. As an example one might think of the process of a Brownian motion in an infinite two-dimensional wedge, established by Varadhan and Williams in [12] (see Fig. 1).

The study of such SDEs is motivated by several applications: For instance, these processes arise as a diffusion approximation of storage systems or of single-server queues in heavy traffic (see, e.g. Section 8.4 in [3] for details).

In [7] Lions and Sznitman established an existence and uniqueness result for solutions of SDEs with oblique reflecting boundary conditions on smooth domains, which was extended by Dupuis and Ishii in [6] for SDEs on domains that might have corners.

Burdzy and Chen proved in [2] Hölder continuity of the Neumann heat kernel in the case of normal reflection in Lipschitz domains, in order to construct a synchronous coupling of reflected Brownian motions. In [8] Mandelbaum

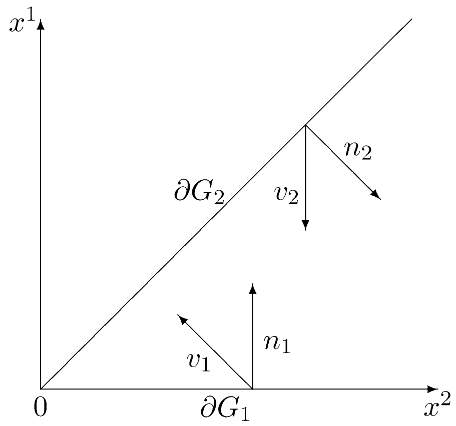


Fig. 1. Two-dimensional wegde with oblique reflection.

and Ramanan established a directional derivative of the Skorohod map along trajectories having left and right limits at every point in the case of oblique reflection on the domain $\mathbb{R}^d_+$.

The aim of the present paper is to show that the solution of the Skorohod SDE is pathwise differentiable with respect to the deterministic initial value and to characterize the pathwise derivatives up to time $\tau$, when at least two faces of $G$ are hit simultaneously for the first time. This is an addition to the results of [4], where Deuschel and Zambotti considered such a differentiability problem for SDEs on the domain $\mathbb{R}^d_+$ with normal reflection at the boundary. Our proceeding will be quite similar to that in [4], in particular we shall use the same technical lemma dealing with the minimum of a Brownian path (see Lemma 1 in [4]). The resulting derivatives are described in terms of an ODE-like equation. When the process is away from the boundary, they evolve according to a simple linear ordinary differential equation, and when it hits the boundary, they have a discontinuity; more precisely, they are projected to the tangent space and jump in direction of the corresponding reflection vector (cf. Section 3). In addition, we provide a Bismut–Elworthy formula for the gradient of the transition semigroup of the process which is stopped in $\tau$ (see Corollary 2.4).

A crucial step in the proof of the differentiability result is to show that the solution of the Skorohod SDE depends Lipschitz continuously on the initial value. To do this we shall apply a criterion given in [5]. In particular, we have to ensure that a certain static geometric property holds (cf. Assumption 2.1 in [5]), so that an additional restriction to the directions of reflection is needed.

Our result is similar to a system, which has been introduced by Airault in [1] in order to develop probabilistic representations for the solutions of linear PDE systems with mixed Dirichlet–Neumann conditions on a regular domain in $\mathbb{R}^n$. However, in contrast to [1] we study pathwise differentiability properties of a process with reflection following [4], but with possibly oblique reflection.

The paper is organized as follows: In Section 2 we state the main result and in Section 4 we prove it. In Section 3 we investigate the results in detail, while we establish a martingale problem connected with the derivatives and we check the Neumann condition.

## 2. Model and main result

Throughout the paper we denote by $\|\cdot\|$ the Euclidian norm, by $\langle\cdot,\cdot\rangle$ the canonical scalar product and by $e = (e_1, \ldots, e_d)$ the standard basis in $\mathbb{R}^d$, $d \geq 2$. We consider processes on the domain $G$, which is a convex polyhedron, i.e. $G \subseteq \mathbb{R}^d$ takes the form $G = \bigcap_{i=1}^N G_i$, where each $G_i := \{x\colon \langle x, n_i\rangle \geq c_i\}$ is a closed half space with inward normal $n_i$ and intercept $c_i$. The boundary of the polyhedron consists of the sides $\partial G_i = \{x : \langle x, n_i\rangle = c_i\}$ and with each side $\partial G_i$ we associate a constant, possibly oblique direction of reflection $v_i$, pointing into the interior of the polyhedron. We always adopt the convention that the directions $v_i$ are normalized such that $\langle v_i, n_i\rangle = 1$. For every $i \in \{1, \ldots, N\}$, let $v_i^\perp, n_i^\perp \in \operatorname{span}\{n_i, v_i\}$ be such that

$$\langle v_i, v_i^\perp\rangle = \langle n_i, n_i^\perp\rangle = 0, \qquad \langle n_i^\perp, v_i^\perp\rangle = \langle n_i, v_i\rangle = 1, \qquad \langle n_i, v_i^\perp\rangle > 0, \tag{2.1}$$

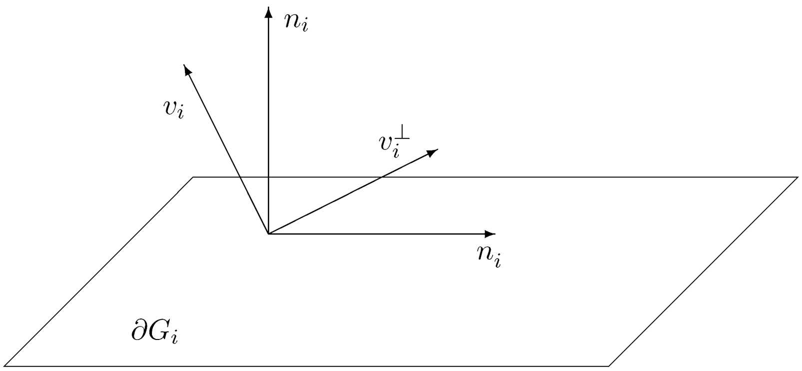


Fig. 2. Choice of $n_i^\perp$ and $v_i^\perp$.

which implies $\langle v_i, n_i^\perp\rangle = -\langle v_i^\perp, n_i\rangle$ (cf. Fig. 2). Furthermore, let $(n_i^k)_{k=3,\ldots,d}$ be a completion of $\{n_i, n_i^\perp\}$ to an orthonormal basis of $\mathbb{R}^d$.

To ensure Lipschitz continuity (see Lemma 4.1) and pathwise existence and uniqueness, a further assumption on the directions of reflection is needed, namely that either

$$n_i = v_i \quad \text{or} \quad a_i\langle n_i, v_i\rangle > \sum_{j\neq i} a_j \big|\langle n_i, v_j\rangle\big| \tag{2.2}$$

for some positive constants $a_i$ and for all $i$ (cf. Theorem 2.1 in [5]).

The set of continuous real-valued functions on $G$ is denoted by $C(G)$, and $C_b(G)$ denotes the set of those functions in $C(G)$ that are bounded on $G$. For each $k \in \mathbb{N}$, $C^k(G)$ denotes the set of real-valued functions that are $k$-times continuously differentiable in some domain containing $G$, and $C_b^k(G)$ denotes the set of those functions in $C^k(G)$ that are bounded and have bounded partial derivatives up to order $k$. Furthermore, we denote by $\Delta$ the Laplace differential operator on $C^2(G)$ and by $D_{v_i} := \langle v_i, \nabla\rangle$ the directional derivative operator associated with the direction of reflection $v_i$ on the side $\partial G_i$.

Now, for any starting point $x \in G$, we consider the following stochastic differential equation of the Skorohod type:

$$\begin{aligned} &X_t(x) = x + \int_0^t b\big(X_r(x)\big)\,\mathrm{d}r + w_t + \sum_i v_i l_t^i(x), \quad t \geq 0, \\ &X_t(x) \in G, \mathrm{d}l_t^i(x) \geq 0, \quad \int_0^\infty \mathbb{1}_{G\setminus\partial G_i}\big(X_t(x)\big)\,\mathrm{d}l_t^i(x) = 0, \quad t\geq 0, i \in \{1,\ldots,N\}, \end{aligned} \tag{2.3}$$

where $w$ is a $d$-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For every $i$, $l^i(x)$ denotes the local time of $X(x)$ in $\partial G_i$, i.e. it increases only at those times, when $X(x)$ is at the boundary $\partial G_i$. The components $b^i : G \to \mathbb{R}$ of $b$ are supposed to be in $C^1(G)$ and Lipschitz continuous. Then, existence and uniqueness of strong solutions of (2.3) are guaranteed in the case of normal reflection by the results of [11], since $G$ is convex, and in the case of oblique reflection by [6], since by condition (2.2) the assumptions of Case 2 in [6] are fulfilled (cf. Remark 3.1 in [6]).

Notice that there is one degree of freedom in defining the local times in the Skorohod SDE (2.3): Setting $\tilde{l}^i(x) = h_i l^i(x)$ for any real constants $h_i > 0$, $\tilde{l}^i(x)$ satisfies the conditions in (2.3) as well. Thus, it is possible to replace $v_i l^i(x)$ by $h_i^{-1} v_i \tilde{l}^i(x)$ in the Skorohod SDE. Consequently, the norm of the reflection vectors $v_i$ does not affect the Skorohod equation, so that the vectors can be thought to be normalized. However, we shall use the normalization $\langle v_i, n_i\rangle = 1$ chosen above, to simplify the computations in the sequel.

Furthermore, by the Girsanov theorem there exists a probability measure $\tilde{\mathbb{P}}(x)$, which is equivalent to $\mathbb{P}$, such that the process

$$W_t^i(x) := \int_0^t b^i\big(X_r(x)\big)\,\mathrm{d}r + w_t^i, \quad t \geq 0, i \in \{1,\ldots,d\}, \tag{2.4}$$

is a $d$-dimensional Brownian motion under $\tilde{\mathbb{P}}(x)$. Next we define the stopping time $\tau$ by

$$\tau := \inf\{t \geq 0\colon\ X_t(x) \in \partial G_i \cap \partial G_j, i \neq j\}, \quad x \in G, \tag{2.5}$$

to be the first time when the process hits at least two of the faces simultaneously. The following simple example shows that even under the assumption in (2.2) $\tau$ can a.s. be infinite and finite as well.

***Example 2.1.*** *Let $G = \mathbb{R}^2_+$, i.e. $G$ is a two-dimensional wedge with angle $\frac{\pi}{2}$ and inward normals $n_i = e_i$, $i \in \{1, 2\}$ (cf. Fig. 1). We choose $v_1 = n_1$ and $v_2 = (-\tan\theta, 1)$, where $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$ denotes the angle between $n_2$ and $v_2$, such that the vector $v_2$ points towards the corner if $\theta$ is positive. Then, the assumption in (2.2) holds with $a_1 = a_2 = 1$. From Theorem 2.2 in [12] we know that*

$$\mathbb{P}[\tau < \infty] = \begin{cases} 0 & \text{if } \theta \leq 0, \\ 1 & \text{if } \theta > 0, \end{cases}$$

*for any starting point $x \in G \setminus \{0\}$. Nevertheless, $\tau$ has infinite expectation for every $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (see Corollary 2.3 in [12]).*

Set

$$C^i := \{s \geq 0\colon\ X_s(x) \in \partial G_i\}, \qquad r_i(t) := \sup(C^i \cap [0, t]), \quad i \in \{1, \dots, N\},$$

with the convention $\sup\emptyset := 0$, and furthermore $C := \bigcup_{i=1}^N C^i$ and $r(t) := \max_{\{i=1,\dots,N\}} r_i(t)$. Then, for every $i$, $C^i \cap [0, \tau)$ is known to be a.s. a closed set of zero Lebesgue measure without isolated points (closed relative to $[0, \tau)$) and $t \mapsto r_i(t)$ is locally constant and right continuous. For $t \in [0, \tau)$ we define

$$s(t) := \begin{cases} 0 & \text{if } t < \inf C, \\ i & \text{if } r(t) = r_i(t), \end{cases}$$

i.e. $s(t) = i$ if the last hit of the boundary before time $t$ was in $\partial G_i$, and $s(t) = 0$ if up to time $t$ the process has not hit the boundary yet. Let $(A_n)_n$ be the family of connected components of $[0, \tau) \setminus C$. $A_n$ is open, so that there exists $q_n \in A_n \cap \mathbb{Q}$, $n \in \mathbb{N}$. We set $a_n := \inf A_n$ and $b_n := \sup A_n$ as well as $\mu(q_n) := \sup\{b_k\colon\ b_k < a_n\}$ with $\sup\emptyset := 0$, i.e. $\mu(q_n)$ denotes the last time before time $q_n$, when the process reaches the boundary of the polyhedron. Finally, we set $\mu(t) = \mu(q_n)$ for all $t \in [\mu(q_n), b_n)$. Notice that a.s. $\mu(t)$ is not the same as $r(t)$ because otherwise the set $C$ would contain isolated points.

The following theorem gives a representation of the derivatives of $X$ in terms of an ODE-like equation:

**Theorem 2.2.** *The mapping $x \mapsto X_t(x)$, $x \in G$, is differentiable a.s. for all $t \in [0, \tau) \setminus C$ and, setting $\eta_t^{ij} := \partial X_t^i(x)/\partial x^j$, $i, j \in \{1, \dots, d\}$, there exists a right continuous extension of $\eta$ on $[0, \tau)$, which has a.s. the following form:*

$$\begin{aligned} \eta_t^{\cdot j} &= \delta_{\cdot j} + \int_0^t \sum_{k=1}^d \frac{\partial b}{\partial x^k}\big(X_r(x)\big)\eta_r^{kj}\,\mathrm{d}r, \quad \text{if } s(t) = 0, \\ \eta_t^{\cdot j} &= \langle \eta_{\mu(t)-}^{\cdot j}, v_i^\perp\rangle n_i^\perp + \sum_{k=3}^d \langle \eta_{\mu(t)-}^{\cdot j}, n_i^k\rangle n_i^k + \int_{r_i(t)}^t \sum_{k=1}^d \frac{\partial b}{\partial x^k}\big(X_r(x)\big)\eta_r^{kj}\,\mathrm{d}r, \quad \text{if } s(t) = i. \end{aligned} \tag{2.6}$$

The proof of Theorem 2.2 is postponed to Section 4. If we consider the case $G = \mathbb{R}^d_+$ and normal reflection at the boundary, i.e. $v_i = n_i = e_i$, the result corresponds to that of Theorem 1 in [4].

***Remark 2.3.*** *In the special case where $N = d$ and the normals $n_i$ form an orthonormal basis of $\mathbb{R}^d$, it is also possible to provide a random walk representation for the derivatives, which is very similar to that in [4], by using essentially the same arguments as in the proof of Theorem 1 and Proposition 1 in [4].*

As soon as pathwise differentiability is established, we can immediately provide a Bismut–Elworthy formula: Define $X_t^\tau(x) := X_t(x)\mathbb{1}_{\{t<\tau\}}$ and for all $f \in C_b(G)$ the associated transition semigroup $P_t f(x) := \mathbb{E}[f(X_t^\tau(x))]$, $x \in G$, $t > 0$. Setting $\eta_t^{ij} := \partial X_t^i(x)/\partial x^j$ for $t \in [0,\tau)$ and $\eta^{ij} := 0$ on $[\tau,\infty)$, $i, j \in \{1,\dots,d\}$, we get

**Corollary 2.4.** *For all $f \in C_b(G)$, $t > 0$ and $x \in G$*:

$$\frac{\partial}{\partial x^i} P_t f(x) = \frac{1}{t}\mathbb{E}\left[f\big(X_t^\tau(x)\big)\int_0^t \sum_{k=1}^d \eta_r^{ki}\,\mathrm{d}w_r^k\right], \quad i \in \{1,\dots,d\}, \tag{2.7}$$

*and if $f \in C_b^1(G)$*:

$$\frac{\partial}{\partial x^i} P_t f(x) = \sum_{k=1}^d \mathbb{E}\left[\frac{\partial f}{\partial x^k}\big(X_t^\tau(x)\big)\eta_t^{ki}\right], \quad i \in \{1,\dots,d\}. \tag{2.8}$$

**Proof.** Formula (2.8) is straightforward from the differentiability statement in Theorem 2.2 and the chain rule. For formula (2.7) see the proof of Theorem 2 in [4]. □

## 3. Martingale problem and Neumann condition

In this section we investigate the derivatives of $X$, established in Theorem 2.2, in detail. Let $j \in \{1,\dots,d\}$ be arbitrary but fixed. From the representation of the derivatives in (2.6) it is obvious that $(\eta_t^{\cdot j})_{0\le t<\tau}$ evolves according to a linear differential equation, when the process $X$ is in the interior of the polyhedron, and that it has a discontinuity, when $X$ hits the boundary, and it jumps in the following manner: For any jump time $t_i$, when $X$ hits $\partial G_i$, $i \in \{1,\dots,N\}$, i.e. $t_i = \mu(q_n)$ for any $q_n$ satisfying $s(q_n) = i$, we have:

$$\eta_{t_i}^{\cdot j} = \langle \eta_{t_i-}^{\cdot j}, v_i^\perp\rangle n_i^\perp + \sum_{k=3}^d \langle \eta_{t_i-}^{\cdot j}, n_i^k\rangle n_i^k.$$

Recall that $\{n_i, n_i^\perp, n_i^k; k = 3,\dots,d\}$ is an orthonormal basis of $\mathbb{R}^d$, so that

$$\eta_{t_i-}^{\cdot j} = \langle \eta_{t_i-}^{\cdot j}, n_i\rangle n_i + \langle \eta_{t_i-}^{\cdot j}, n_i^\perp\rangle n_i^\perp + \sum_{k=3}^d \langle \eta_{t_i-}^{\cdot j}, n_i^k\rangle n_i^k$$

and

$$\eta_{t_i}^{\cdot j} - \eta_{t_i-}^{\cdot j} = \langle \eta_{t_i-}^{\cdot j}, v_i^\perp - n_i^\perp\rangle n_i^\perp - \langle \eta_{t_i-}^{\cdot j}, n_i\rangle n_i = -\langle \eta_{t_i-}^{\cdot j}, n_i\rangle v_i, \tag{3.1}$$

where the last equality follows from Lemma 3.1. Consequently, we observe that at each time, when $X$ reaches the boundary $\partial G_i$, $\eta^{\cdot j}$ is projected to the tangent space, since $\langle \eta_{t_i}^{\cdot j}, n_i\rangle = 0$, and jumps in the direction of $v_i$ or $-v_i$, respectively.

**Lemma 3.1.** *For all $i \in \{1,\dots,N\}$ and $\eta \in \mathbb{R}^d$*:

$$\langle \eta, v_i^\perp - n_i^\perp\rangle n_i^\perp - \langle \eta, n_i\rangle n_i = -\langle \eta, n_i\rangle v_i.$$

**Proof.** By the choice of $v_i^\perp$ and $n_i^\perp$ in (2.1) we have $v_i = n_i + \langle v_i, n_i^\perp\rangle n_i^\perp$ and $v_i^\perp = \langle v_i^\perp, n_i\rangle n_i + n_i^\perp$, which is equivalent to

$$\langle v_i, n_i^\perp\rangle n_i^\perp = v_i - n_i, \qquad v_i^\perp - n_i^\perp = -\langle v_i, n_i^\perp\rangle n_i.$$

Hence,

$$\big\langle \eta, v_i^\perp - n_i^\perp \big\rangle n_i^\perp - \langle \eta, n_i\rangle n_i = -\langle \eta, n_i\rangle \big\langle v_i, n_i^\perp \big\rangle n_i^\perp - \langle \eta, n_i\rangle n_i = -\langle \eta, n_i\rangle (v_i - n_i) - \langle \eta, n_i\rangle n_i$$
$$= -\langle \eta, n_i\rangle v_i. \qquad \square$$

From the observations above it becomes clear that the process $(X_t(x), \eta_t^{\cdot j})_{0\le t<\tau}$ is Markovian with state space $G \times \mathbb{R}^d$. Next we want to provide the infinitesimal generator for this Markov process. For that purpose we define the operator $L$ as follows: Let the domain $\mathcal{D}(L)$ be that set of continuous bounded functions $F$ on $G \times \mathbb{R}^d$ satisfying the following conditions:

(i) For every $\eta \in \mathbb{R}^d$, $F(\cdot, \eta) \in C_b^2(G)$ and the Neumann boundary condition holds:

$$D_{v_i} F(\cdot, \eta)(x) = 0 \quad \text{for } x \in \partial G_i,\ i \in \{1, \dots, N\}. \tag{3.2}$$

(ii) For every $x \in G$, we have $F(x, \cdot) \in C_b^1(\mathbb{R}^d)$, i.e. bounded and continuously differentiable with bounded partial derivatives, satisfying the following boundary conditions: If $x \in \partial G_i$ for all $\eta \in \mathbb{R}^d$:

$$F(x, \eta) = F\big(x, \eta - \langle \eta, n_i\rangle n_i\big), \qquad D_{v_i} F(x, \cdot)(\eta) = 0. \tag{3.3}$$

Note that by the jump behaviour of $\eta^{\cdot j}$, provided in (3.1), and by the boundary condition (3.3) we have for every $F \in \mathcal{D}(L)$ and $t < \tau$:

$$F\big(X_t, \eta_t^{\cdot j}\big) = F\big(X_t, \eta_{t-}^{\cdot j}\big). \tag{3.4}$$

Finally, the operator $L$ is defined by:

$$LF(x, \eta) := L^1 F(\cdot, \eta)(x) + L^2 F(x, \cdot)(\eta), \quad F \in \mathcal{D}(L),$$

where

$$L^1 F(\cdot, \eta)(x) := \frac{1}{2}\Delta F(\cdot, \eta)(x) + \sum_{i=1}^d b^i(x) \frac{\partial F}{\partial x^i}(\cdot, \eta)(x),$$
$$L^2 F(x, \cdot)(\eta) := \sum_{i=1}^d \left( \sum_{k=1}^d \frac{\partial b^i}{\partial x^k}(x) \eta^k \right) \frac{\partial F}{\partial \eta^i}(x, \cdot)(\eta).$$

**Proposition 3.2.** *For $F \in \mathcal{D}(L)$,*

$$F\big(X_t(x), \eta_t^{\cdot j}\big) - F\big(x, \eta_0^{\cdot j}\big) - \int_0^t LF\big(X_s, \eta_s^{\cdot j}\big)\,\mathrm{d}s, \quad t < \tau,$$

*is a martingale.*

**Proof.** Applying Itô's formula, in particular the version for finite variation processes (see e.g. Theorem IV.18.8 in [10]), we have

$$F\big(X_t(x), \eta_t^{\cdot j}\big) - F\big(x, \eta_0^{\cdot j}\big)$$
$$= \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x^i}\big(X_s(x), \eta_s^{\cdot j}\big)\,\mathrm{d}X_s^i(x) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial \eta^i}\big(X_s(x), \eta_s^{\cdot j}\big)\,\mathrm{d}\eta_s^{ij}(x) + \frac{1}{2}\int_0^t \Delta F\big(\cdot, \eta_s^{\cdot j}\big)\big(X_s(x)\big)\,\mathrm{d}s$$
$$+ \sum_{0<s\le t} \left\{ F\big(X_s(x), \eta_s^{\cdot j}\big) - F\big(X_s(x), \eta_{s-}^{\cdot j}\big) - \sum_{i=1}^d \frac{\partial F}{\partial \eta^i}\big(X_s(x), \eta_{s-}^{\cdot j}\big)\big(\eta_s^{ij} - \eta_{s-}^{ij}\big) \right\}$$

$$= m_t + \int_0^t LF\big(X_s(x), \eta_s^{\cdot j}\big)\,\mathrm{d}s + \sum_{i=1}^N \int_0^t D_{v_i} F\big(X_s(x), \eta_s^{\cdot j}\big)\,\mathrm{d}l_s^i(x)$$

$$+ \sum_{0<s\le t} F\big(X_s(x), \eta_s^{\cdot j}\big) - F\big(X_s(x), \eta_{s-}^{\cdot j}\big) - \sum_{0<s\le t} \sum_{i=1}^d \frac{\partial F}{\partial \eta^i}\big(X_s(x), \eta_{s-}^{\cdot j}\big)\big(\eta_s^{ij} - \eta_{s-}^{ij}\big),$$

where $(m_t)$ is a martingale. Clearly, the third and the fourth term vanish by the boundary conditions (3.2) and (3.4). Using (3.1) the last term can be rewritten as

$$- \sum_{0<s\le t} \sum_{m=1}^N \sum_{i=1}^d \frac{\partial F}{\partial \eta^i}\big(X_s(x), \eta_{s-}^{\cdot j}\big)\big(\eta_s^{ij} - \eta_{s-}^{ij}\big)\mathbb{1}_{\{s=\mu(s)\}}\mathbb{1}_{\{X_s(x)\in\partial G_m\}}$$

$$= \sum_{0<s\le t} \sum_{m=1}^N \big\langle \eta_{s-}^{\cdot j}, n_m\big\rangle \sum_{i=1}^d D_{v_m} F\big(X_s(x), \cdot\big)\big(\eta_{s-}^{\cdot j}\big)\mathbb{1}_{\{s=\mu(s)\}}\mathbb{1}_{\{X_s(x)\in\partial G_m\}},$$

which is equal to zero by (3.3). □

Since $(X_t(x), \eta_t^{\cdot j})_{t<\tau}$ is Markovian, we can conclude from Proposition 3.2 that its generator coincides with $L$ at least on the closure of $\mathcal{D}(L)$ w.r.t. the sup-norm topology.

At the end of this section we give another confirmation of the results in Theorem 2.2, namely, they imply that the Neumann condition holds for $X$:

**Corollary 3.3.** *Let $X_t^\tau(x) := X_t(x)\mathbb{1}_{\{t<\tau\}}$ again be the process stopped in $\tau$. Then, for all $f \in C_b(G)$ and $t > 0$, the transition semigroup $P_t f(x) := \mathbb{E}[f(X_t^\tau(x))]$, $x \in G$, satisfies the Neumann condition at $\partial G$:*

$$x \in \partial G_i \quad \Longrightarrow \quad D_{v_i} P_t f(x) = 0, \quad i \in \{1, \dots, N\}.$$

**Proof.** Let $x \in \partial G_i$. By a density argument it is sufficient to consider bounded functions $f$, which are continuously differentiable and have bounded derivatives. Then, for each $t > 0$ we obtain by dominated convergence and the chain rule:

$$D_{v_i} P_t f(x) = \mathbb{E}\Bigg[\sum_{k=1}^d \frac{\partial f}{\partial x^k}\big(X_t(x)\big) D_{v_i} X_t^k(x)\mathbb{1}_{\{t\in[0,\tau)\}}\Bigg].$$

Thus, it suffices to show

$$D_{v_i} X_t^k(x) = 0, \quad \forall k \in \{1, \dots, d\}, t < \tau, \tag{3.5}$$

which we shall prove now by an induction argument. First we consider $t \in [0, \min_{j\ne i} \inf C^j)$. Recall that $\inf C^i = 0$ and $\eta_0^{\cdot m} = \delta_{\cdot m}$, so we get for every $l \in \{1, \dots, d\}$ by (2.6):

$$D_{v_i} X_t^l(x) = \sum_{m=1}^d v_i^m \big\langle \eta_0^{\cdot m}, v_i^\perp\big\rangle \big(n_i^\perp\big)^l + \sum_{m=1}^d \sum_{k=3}^d v_i^m \big\langle \eta_0^{\cdot m}, n_i^k\big\rangle \big(n_i^k\big)^l$$

$$+ \int_{r_i(t)}^t \sum_{k,m=1}^d v_i^m \frac{\partial b^l}{\partial x^k}\big(X_r(x)\big)\eta_r^{km}\,\mathrm{d}r$$

$$= \int_{r_i(t)}^t \sum_{k=1}^d \frac{\partial b^l}{\partial x^k}\big(X_r(x)\big) D_{v_i} X_r^k(x)\,\mathrm{d}r,$$

where we have used that $\langle v_i, v_i^\perp\rangle = 0$ and $\langle v_i, n_i^k\rangle = 0$. By the Lipschitz continuity of $b$ we obtain

$$\sum_{k=1}^d \left|D_{v_i} X_t^k(x)\right| \le C \int_0^t \sum_{k=1}^d \left|D_{v_i} X_r^k(x)\right| \,\mathrm{d}r,$$

for some positive constant $C$ and by applying Gronwall's lemma we conclude that (3.5) holds for $t \in [0, \min_{j\neq i} \inf C^j)$.

Let now $t < \tau$ be arbitrary and assume that $D_{v_i} X^k(x) = 0$ on $[0, \mu(t))$ for all $k$. Moreover, let $j$ be such that $s(t) = j$. Then, for every $l$ we get again by (2.6):

$$\begin{aligned} D_{v_i} X_t^l(x) &= \left\langle D_{v_i} X_{\mu(t)-}, v_j^\perp\right\rangle n_j^\perp + \sum_{k=3}^d \left\langle D_{v_i} X_{\mu(t)-}, n_j^k\right\rangle n_j^k + \int_{r_j(t)}^t \sum_{k,m=1}^d v_i^m \frac{\partial b^l}{\partial x^k}\big(X_r(x)\big)\eta_r^{km} \,\mathrm{d}r \\ &= \int_{r_j(t)}^t \sum_{k=1}^d \frac{\partial b^l}{\partial x^k}\big(X_r(x)\big) D_{v_i} X_r^k(x) \,\mathrm{d}r, \end{aligned}$$

and as above we obtain

$$\sum_{k=1}^d \left|D_{v_i} X_t^k(x)\right| \le C \int_0^t \sum_{k=1}^d \left|D_{v_i} X_r^k(x)\right| \,\mathrm{d}r,$$

which completes the proof again by Gronwall's lemma. □

## 4. Proof of the main result

The first step to prove Theorem 2.2 is to show the Lipschitz continuity of $x \mapsto (X_t(x))_t$ w.r.t. the sup-norm topology on a finite time interval:

**Lemma 4.1.** *For an arbitrary but fixed $T > 0$, let $(X_t(x))$ and $(X_t(y))$, $0 \le t \le T$, be solutions of (2.3) for some $x, y \in G$. Then, there exists a positive constant $K$, only depending on $T$, such that a.s.*

(i) $\displaystyle\sup_{t\in[0,T]} \left\|X_t(x) - X_t(y)\right\| \le K\|x-y\|,$

(ii) $\displaystyle\sup_{t\in[0,T]} \left|l_t^i(x) - l_t^i(y)\right| \le K\|x-y\|$ *for all $i$.*

**Proof.** By the assumption in (2.2), Theorem 2.1 in [5] ensures that Assumption 2.1 in [5] holds. Thus, we can apply Theorem 2.2 in [5] to obtain

$$\sup_{t\in[0,T]} \left\|X_t(x) - X_t(y)\right\| \le K_1\|x-y\| + K_1 \sup_{t\in[0,T]} \left\| \int_0^t \big[b\big(X_r(x)\big) - b\big(X_r(y)\big)\big] \,\mathrm{d}r \right\|,$$

and by the Lipschitz continuity of $b$ we get

$$\sup_{t\in[0,T]} \left\|X_t(x) - X_t(y)\right\| \le K_1\|x-y\| + K_2 \int_0^T \sup_{r\le s} \left\|X_r(x) - X_r(y)\right\| \,\mathrm{d}s,$$

for some positive constants $K_1$ and $K_2$, and (i) follows by the Gronwall lemma. Using again Theorem 2.2 in [5], the Lipschitz property of $b$ and (i) we obtain (ii). □

Recall the definition of the $\tilde{\mathbb{P}}(x)$ Brownian motion $W(x)$ in (2.4); the Skorohod SDE in (2.3) can be rewritten as follows:

$$X_t(x) = x + W_t(x) + \sum_i v_i l_t^i(x), \quad t \geq 0, \tag{4.1}$$

so that

$$\langle X_t(x), n_i\rangle = \langle x, n_i\rangle + \langle W_t(x), n_i\rangle + \hat{L}_t^i(x) + l_t^i(x), \quad t \geq 0,$$

since $\langle v_i, n_i\rangle = 1$, where

$$\hat{L}_t^i(x) := \sum_{j\neq i} \langle v_j, n_i\rangle l_t^j(x), \quad t \geq 0.$$

Note that $\langle W(x), n_i\rangle$ is again a Brownian motion under $\tilde{\mathbb{P}}(x)$ by Levy's characterization theorem, since $n_i$ is a unit vector. The local time $l^i(x)$ is carried by the set of times $t$, when $\langle X_t(x), n_i\rangle - c_i = 0$, so that it can be computed directly by Skorohod's lemma (see, e.g. Lemma VI.2.1 in [9]). This yields

$$l_t^i(x) = \Bigl[-\langle x, n_i\rangle + c_i - \inf_{s\leq t}\bigl(\langle W_s(x), n_i\rangle + \hat{L}_s^i(x)\bigr)\Bigr]^+, \quad t \geq 0.$$

Fix any $q_n$. Since $\langle X_{r_i(q_n)}(x), n_i\rangle - c_i = 0$ and $t \mapsto l_t^i(x)$ is increasing, we have for all $s \leq r_i(q_n)$:

$$\begin{aligned}\langle W_{r_i(q_n)}(x), n_i\rangle + \hat{L}_{r_i(q_n)}^i(x) &= -\langle x, n_i\rangle + c_i - l_{r_i(q_n)}^i(x) \leq -\langle x, n_i\rangle + c_i - l_s^i(x)\\ &= -\langle X_s(x), n_i\rangle + c_i + \langle W_s(x), n_i\rangle + \hat{L}_s^i(x)\\ &\leq \langle W_s(x), n_i\rangle + \hat{L}_s^i(x).\end{aligned} \tag{4.2}$$

Therefore, for all $t \in A_n$:

$$l_t^i(x) = l_{r_i(q_n)}^i(x) = \bigl[-\langle x, n_i\rangle + c_i - \langle W_{r_i(t)}(x), n_i\rangle - \hat{L}_{r_i(t)}^i(x)\bigr]^+. \tag{4.3}$$

Next we compute the local times of the process with perturbed starting point. Recall that $e = (e_1, \ldots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$. Set $x_\varepsilon := x + \varepsilon e_j$, $\varepsilon \in \mathbb{R}$, $j \in \{1, \ldots, d\}$, where $|\varepsilon|$ is always supposed to be sufficiently small, such that $x_\varepsilon$ lies in $G \setminus \bigcup_{i,j:\, i\neq j}(\partial G_i \cap \partial G_j)$. We start with a preparing lemma:

**Lemma 4.2.** *Let $i \in \{1, \ldots, N\}$ and $0 \leq s < t$ be arbitrary and let $\vartheta : \Omega \to [s, t]$ be the random variable such that a.s. $\langle W_\vartheta(x), n_i\rangle < \langle W_r(x), n_i\rangle$ for all $r \in [s, t] \setminus \{\vartheta\}$. Then, there exists a random $\Delta > 0$ such that a.s. $\vartheta$ is the only time, when $\langle W(x_\varepsilon), n_i\rangle = \langle W(x), n_i\rangle + \langle W(x_\varepsilon) - W(x), n_i\rangle$ attains its minimum over $[s, t]$ for all $|\varepsilon| < \Delta$.*

**Proof.** Since $\langle W(x), n_i\rangle$ is a Brownian motion under $\tilde{\mathbb{P}}(x)$, by Lemma 1 in [4] there exists a random variable $\gamma$ such that every $\gamma$-Lipschitz perturbation of $\langle W(x), n_i\rangle$ attains its minimum only at $\vartheta$. Using Lemma 4.1 and the Lipschitz continuity of $b$ we find a $\Delta > 0$ such that $\sup_{r\in[s,t]} |\langle b(X_r(x_\varepsilon)) - b(X_r(x)), n_i\rangle| \leq \gamma$ for all $|\varepsilon| < \Delta$. This implies that $h(r) := \langle W_r(x_\varepsilon) - W_r(x), n_i\rangle = h(s) + \int_s^r \langle b(X_u(x_\varepsilon)) - b(X_u(x)), n_i\rangle \,\mathrm{d}u$ is a $\gamma$-Lipschitz perturbation for such $\varepsilon$, and the claim follows. □

**Lemma 4.3.** *For all $i$ and $q_n$, $n \in \mathbb{N}$, there exists a random $\Delta_n^i > 0$ such that for all $|\varepsilon| < \Delta_n^i$ a.s.:*

$$l_{q_n}^i(x_\varepsilon) = \bigl[-\langle x_\varepsilon, n_i\rangle + c_i - \langle W_{r_i(q_n)}(x_\varepsilon), n_i\rangle - \hat{L}_{r_i(q_n)}^i(x_\varepsilon)\bigr]^+. \tag{4.4}$$

**Proof.** We need only to consider the case $s(q_n) = i$. Indeed, if $q_n < \inf C^i$ we can use Lemma 4.1 to find a $\Delta_n^i > 0$, such that $X_t(x_\varepsilon) \notin \partial G_i$ for all $t \in [0, q_n]$ and for all $|\varepsilon| < \Delta_n^i$, which implies $l_{q_n}^i(x_\varepsilon) = l_{q_n}^i(x) = 0$. If $q_n > \inf C^i$

and $s(t) \neq i$, we set $\tilde{q}_n := \sup\{q_k\colon\, q_k < q_n, s(q_k) = i\}$ and again by Lemma 4.1 there exists a $\Delta_n^i > 0$, such that $X_t(x_\varepsilon) \notin \partial G_i$ for all $t \in [\tilde{q}_n, q_n]$ and for all $|\varepsilon| < \Delta_n^i$, which implies $l^i_{q_n}(x_\varepsilon) = l^i_{\tilde{q}_n}(x_\varepsilon)$.

Let now $q_n$ be such that $s(q_n) = i$. Using again Skorohod's lemma, we obtain for all $\varepsilon$:

$$\begin{aligned} l^i_{q_n}(x_\varepsilon) &= \Big[-\langle x_\varepsilon, n_i\rangle + c_i - \inf_{s\le q_n}\big(\big\langle W_s(x_\varepsilon), n_i\big\rangle + \hat{L}^i_s(x_\varepsilon)\big)\Big]^+ \\ &= \Big[-\langle x_\varepsilon, n_i\rangle + c_i - \inf_{s\le q_n}\big(f_\varepsilon(s) + g_\varepsilon(s)\big)\Big]^+, \end{aligned}$$

where $f_\varepsilon(s) := \langle W_s(x_\varepsilon), n_i\rangle + \hat{L}^i_s(x)$ and $g_\varepsilon(s) := \hat{L}^i_s(x_\varepsilon) - \hat{L}^i_s(x)$. From the calculation in (4.2) above we know that $\langle W(x), n_i\rangle + \hat{L}^i_s(x)$ attains its minimum over $[0, q_n]$ at $r_i(q_n)$, and we have to show that for sufficiently small $|\varepsilon|$:

$$\inf_{s\le q_n}\big(f_\varepsilon(s) + g_\varepsilon(s)\big) = f_\varepsilon\big(r_i(q_n)\big) + g_\varepsilon\big(r_i(q_n)\big). \tag{4.5}$$

Recall that $q_n < \tau$, i.e. the process $X$ hits the faces of the polyhedron $G$ only successively, and recall that $C^i$ is the support of $l^i(x)$. Thus, there exists a time $q_n^- < q_n$ such that $2d := l^i_{q_n}(x) - l^i_{q_n^-}(x) > 0$ and $X_s(x) \notin \bigcup_{j\neq i}\partial G_j$ for all $s \in [q_n^-, q_n]$ (note that we might have $q_n^- = 0$ in the case where $x \in \partial G_i$ and $q_n < \inf\bigcup_{j\neq i} C^j$). We apply Lemma 4.1 and find a $\Delta'_n > 0$ such that also $X_s(x_\varepsilon) \notin \bigcup_{j\neq i}\partial G_j$ for all $s \in [q_n^-, q_n]$ and $|\varepsilon| < \Delta'_n$. Hence, for such $\varepsilon$ it follows that $\hat{L}^i(x)$, $\hat{L}^i(x_\varepsilon)$ and $g_\varepsilon$ are constant on $[q_n^-, q_n]$, so that $f_\varepsilon + g_\varepsilon$ attains its minimum over $[q_n^-, q_n]$ at the same time as $\langle W(x_\varepsilon), n_i\rangle$. By Lemma 4.2, possibly after choosing a smaller $\Delta'_n$, we know that this time is $r_i(q_n)$, so that

$$\inf_{s\in[q_n^-, q_n]}\big(f_\varepsilon(s) + g_\varepsilon(s)\big) = f_\varepsilon\big(r_i(q_n)\big) + g_\varepsilon\big(r_i(q_n)\big), \quad \forall|\varepsilon| < \Delta'_n. \tag{4.6}$$

Proceeding as in (4.2), we get for all $s \le q_n^-$:

$$\begin{aligned} &\big\langle W_{r_i(q_n)}(x), n_i\big\rangle + \hat{L}^i_{r_i(q_n)}(x) \\ &\quad = -\langle x, n_i\rangle + c_i - l^i_{r_i(q_n)}(x) = -\langle x, n_i\rangle + c_i - l^i_{q_n}(x) \\ &\quad = -\langle x, n_i\rangle + c_i - l^i_{q_n^-}(x) - 2d \le -\langle x, n_i\rangle + c_i - l^i_s(x) - 2d \\ &\quad = -\big\langle X_s(x), n_i\big\rangle + c_i + \big\langle W_s(x), n_i\big\rangle + \hat{L}^i_s(x) - 2d \\ &\quad \le \big\langle W_s(x), n_i\big\rangle + \hat{L}^i_s(x) - 2d. \end{aligned} \tag{4.7}$$

Using the Lipschitz continuity of $b$ and Lemma 4.1, we find a $\Delta''_n > 0$ such that

$$\sup_{s\le q_n}\big|\big\langle W_s(x_\varepsilon) - W_s(x), n_i\big\rangle\big| = \sup_{s\le q_n}\left|\int_0^s \big\langle b\big(X_r(x_\varepsilon)\big) - b\big(X_r(x)\big), n_i\big\rangle\,\mathrm{d}r\right| \le \frac{d}{2}, \quad \forall|\varepsilon| < \Delta''_n,$$

i.e. for such $\varepsilon$ (4.7) implies

$$\begin{aligned} \inf_{s\le q_n^-} f_\varepsilon(s) - f_\varepsilon\big(r_1(q_n)\big) &= \inf_{s\le q_n^-}\big(\big\langle W_s(x_\varepsilon), n_i\big\rangle + \hat{L}^i_s(x)\big) - \big\langle W_{r_i(q_n)}(x_\varepsilon), n_i\big\rangle - \hat{L}^i_{r_i(q_n)}(x) \\ &\ge \inf_{s\le q_n^-}\big(\big\langle W_s(x), n_i\big\rangle + \hat{L}^i_s(x)\big) - \sup_{s\le q_n^-}\big|\big\langle W_s(x_\varepsilon) - W_s(x), n_i\big\rangle\big| \\ &\quad - \big\langle W_{r_i(q_n)}(x_\varepsilon), n_i\big\rangle - \hat{L}^i_{r_i(q_n)}(x) \\ &\ge \frac{3}{2}d - \big\langle W_{r_i(q_n)}(x_\varepsilon) - W_{r_i(q_n)}(x), n_i\big\rangle \ge d. \end{aligned} \tag{4.8}$$

By Lemma 4.1(ii) there exists a random $\Delta_n''' > 0$ such that a.s.

$$\sup_{s \le q_n} \left|g_\varepsilon(s)\right| < \frac{d}{2}, \quad \forall |\varepsilon| < \Delta_n''. \tag{4.9}$$

Now using (4.8) and (4.9) we obtain for $|\varepsilon| < \min(\Delta_n'', \Delta_n''')$:

$$\begin{aligned}
&\inf_{s \le q_n^-} \bigl(f_\varepsilon(s) + g_\varepsilon(s)\bigr) \\
&\quad \ge \inf_{s \le q_n^-} f_\varepsilon(s) - \sup_{s \le q_n^-} \left|g_\varepsilon(s)\right| > d + f_\varepsilon\bigl(r_1(q_n)\bigr) - \frac{d}{2} \\
&\quad = f_\varepsilon\bigl(r_1(q_n)\bigr) + \frac{d}{2} > f_\varepsilon\bigl(r_i(q_n)\bigr) + g_\varepsilon\bigl(r_i(q_n)\bigr),
\end{aligned} \tag{4.10}$$

so that (4.5) follows from (4.6) and (4.10) for all $|\varepsilon| < \Delta_n^i := \min(\Delta_n', \Delta_n'', \Delta_n''')$. □

Next we compute the difference quotients of $X$. For fixed $j \in \{1, \dots, d\}$ we set $x_\varepsilon = x + \varepsilon e_j$, $\varepsilon \ne 0$, and

$$\eta_t(\varepsilon) := \frac{1}{\varepsilon}\bigl(X_t(x_\varepsilon) - X_t(x)\bigr), \quad t \ge 0.$$

Let now $t \in [0, \tau) \setminus C$ be fixed and $n$ such that $t \in A_n$. We choose $\Delta_n > 0$ such that a.s. for all $|\varepsilon| < \Delta_n$ we have for all $i$ that $l^i_{q_n}(x) = l^i_{q_n}(x_\varepsilon) = 0$ if $q_n < \inf C^i$, and both of them are strictly positive if $q_n > \inf C^i$, and finally that formula (4.4) holds.

From (2.3) we deduce directly

$$X_t(x_\varepsilon) - X_t(x) = x_\varepsilon - x + W_t(x_\varepsilon) - W_t(x) + \sum_{j=1}^N v_j\bigl(l_t^j(x_\varepsilon) - l_t^j(x)\bigr). \tag{4.11}$$

If $s(t) = 0$ we get immediately

$$\eta_t(\varepsilon) = \delta_{\cdot j} + \frac{1}{\varepsilon}\int_0^t \bigl(b\bigl(X_r(x_\varepsilon)\bigr) - b\bigl(X_r(x)\bigr)\bigr)\,\mathrm{d}r. \tag{4.12}$$

Let us now consider the case $s(t) = i$, $i \in \{1, \dots, N\}$, and set $t_i := \mu(t)$. Then, possibly after choosing a smaller $\Delta_n$, we may suppose that $l^j(x_\varepsilon)$ is constant on $[t_i - \delta, q_n \vee t]$ for every $j \ne i$ and some positive $\delta$ since $t < \tau$. In (4.11) we use (4.3) and (4.4) to obtain

$$\begin{aligned}
&X_t(x_\varepsilon) - X_t(x) \\
&\quad = x_\varepsilon - x + W_t(x_\varepsilon) - W_t(x) + \sum_{j \ne i} v_j\bigl(l^j_{r_i(q_n)}(x_\varepsilon) - l^j_{r_i(q_n)}(x)\bigr) \\
&\qquad + v_i\bigl(-\langle x_\varepsilon - x, n_i\rangle - \bigl\langle W_{r_i(q_n)}(x_\varepsilon) - W_{r_i(q_n)}(x), n_i\bigr\rangle - \bigl(\hat{L}^i_{r_i(q_n)}(x_\varepsilon) - \hat{L}^i_{r_i(q_n)}(x)\bigr)\bigr).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\bigl\langle X_t(x_\varepsilon) - X_t(x), n_i\bigr\rangle \\
&\quad = \langle x_\varepsilon - x, n_i\rangle + \bigl\langle W_t(x_\varepsilon) - W_t(x), n_i\bigr\rangle + \hat{L}^i_{r_i(q_n)}(x_\varepsilon) - \hat{L}^i_{r_i(q_n)}(x) \\
&\qquad - \langle x_\varepsilon - x, n_i\rangle - \bigl\langle W_{r_i(q_n)}(x_\varepsilon) - W_{r_i(q_n)}(x), n_i\bigr\rangle - \bigl(\hat{L}^i_{r_i(q_n)}(x_\varepsilon) - \hat{L}^i_{r_i(q_n)}(x)\bigr) \\
&\quad = \int_{r_i(q_n)}^t \bigl\langle b\bigl(X_r(x_\varepsilon)\bigr) - b\bigl(X_r(x)\bigr), n_i\bigr\rangle\,\mathrm{d}r.
\end{aligned} \tag{4.13}$$

Recall the definition of $v_i^\perp$ and $n_i^\perp$ in (2.1). Since $\langle v_i, n_i^\perp\rangle = -\langle v_i^\perp, n_i\rangle$, we have

$$\begin{aligned}
&\big\langle X_t(x_\varepsilon) - X_t(x), n_i^\perp\big\rangle \\
&\quad = \big\langle x_\varepsilon - x, n_i^\perp\big\rangle + \big\langle W_t(x_\varepsilon) - W_t(x), n_i^\perp\big\rangle + \sum_{j\neq i}\big\langle v_j, n_i^\perp\big\rangle\big(l^j_{r_i(q_n)}(x_\varepsilon) - l^j_{r_i(q_n)}(x)\big) \\
&\qquad + \big\langle v_i^\perp, n_i\big\rangle\big(\langle x_\varepsilon - x, n_i\rangle + \big\langle W_{r_i(q_n)}(x_\varepsilon) - W_{r_i(q_n)}(x), n_i\big\rangle + \hat{L}^i_{r_i(q_n)}(x_\varepsilon) - \hat{L}^i_{r_i(q_n)}(x)\big) \\
&\quad = \big\langle x_\varepsilon - x, n_i^\perp\big\rangle + \big\langle W_{r_i(q_n)}(x_\varepsilon) - W_{r_i(q_n)}(x), n_i^\perp\big\rangle + \sum_{j\neq i}\big\langle v_j, n_i^\perp\big\rangle\big(l^j_{r_i(q_n)}(x_\varepsilon) - l^j_{r_i(q_n)}(x)\big) \\
&\qquad + \big\langle x_\varepsilon - x, \big\langle v_i^\perp, n_i\big\rangle n_i\big\rangle + \big\langle W_{r_i(q_n)}(x_\varepsilon) - W_{r_i(q_n)}(x), \big\langle v_i^\perp, n_i\big\rangle n_i\big\rangle \\
&\qquad + \sum_{j\neq i}\big\langle v_j, \big\langle v_i^\perp, n_i\big\rangle n_i\big\rangle\big(l^j_{r_i(q_n)}(x_\varepsilon) - l^j_{r_i(q_n)}(x)\big) + \int_{r_i(q_n)}^t \big\langle b\big(X_r(x_\varepsilon)\big) - b\big(X_r(x)\big), n_i^\perp\big\rangle\,\mathrm{d}r.
\end{aligned}$$

By the choice of $v_i^\perp$ and $n_i^\perp$, clearly $v_i^\perp = \langle v_i^\perp, n_i\rangle n_i + n_i^\perp$, so that

$$\begin{aligned}
&\big\langle X_t(x_\varepsilon) - X_t(x), n_i^\perp\big\rangle \\
&\quad = \Big\langle x_\varepsilon - x + W_{r_i(q_n)}(x_\varepsilon) - W_{r_i(q_n)}(x) + \sum_{j\neq i} v_j\big(l^j_{r_i(q_n)}(x_\varepsilon) - l^j_{r_i(q_n)}(x)\big), v_i^\perp\Big\rangle \\
&\qquad + \int_{r_i(q_n)}^t \big\langle b\big(X_r(x_\varepsilon)\big) - b\big(X_r(x)\big), n_i^\perp\big\rangle\,\mathrm{d}r.
\end{aligned}$$

Note that $W_{r_i(q_n)}(x_\varepsilon) - W_{r_i(q_n)}(x) = W_{t_i-}(x_\varepsilon) - W_{t_i-}(x)$, since $C^i$ has zero Lebesgue measure. Moreover, $l^j_{r_i(q_n)}(x_\varepsilon) - l^j_{r_i(q_n)}(x) = l^j_{t_i-}(x_\varepsilon) - l^j_{t_i-}(x)$ for all $j \neq i$ by the choice of $\Delta_n$. Using this and the fact that $\langle v_i, v_i^\perp\rangle = 0$ we obtain

$$\begin{aligned}
\big\langle X_t(x_\varepsilon) - X_t(x), n_i^\perp\big\rangle &= \Big\langle x_\varepsilon - x + W_{t_i-}(x_\varepsilon) - W_{t_i-}(x) + \sum_{j} v_j\big(l^j_{t_i-}(x_\varepsilon) - l^j_{t_i-}(x)\big), v_i^\perp\Big\rangle \\
&\qquad + \int_{r_i(q_n)}^t \big\langle b\big(X_r(x_\varepsilon)\big) - b\big(X_r(x)\big), n_i^\perp\big\rangle\,\mathrm{d}r \\
&= \big\langle X_{t_i-}(x_\varepsilon) - X_{t_i-}(x), v_i^\perp\big\rangle + \int_{r_i(q_n)}^t \big\langle b\big(X_r(x_\varepsilon)\big) - b\big(X_r(x)\big), n_i^\perp\big\rangle\,\mathrm{d}r, \qquad (4.14)
\end{aligned}$$

and for every $k \in \{3, \ldots, d\}$, we have $\langle v_i, n_i^k\rangle = 0$, so that

$$\begin{aligned}
&\big\langle X_t(x_\varepsilon) - X_t(x), n_i^k\big\rangle \\
&\quad = \Big\langle x_\varepsilon - x + W_t(x_\varepsilon) - W_t(x) + \sum_{j\neq i} v_j\big(l^j_{r_i(q_n)}(x_\varepsilon) - l^j_{r_i(q_n)}(x)\big), n_i^k\Big\rangle + \big\langle v_i, n_i^k\big\rangle\big(l^i_t(x_\varepsilon) - l^i_t(x)\big) \\
&\quad = \Big\langle x_\varepsilon - x + W_{r_i(q_n)}(x_\varepsilon) - W_{r_i(q_n)}(x) + \sum_{j} v_j\big(l^j_{r_i(q_n)}(x_\varepsilon) - l^j_{r_i(q_n)}(x)\big), n_i^k\Big\rangle \\
&\qquad + \int_{r_i(q_n)}^t \big\langle b\big(X_r(x_\varepsilon)\big) - b\big(X_r(x)\big), n_i^k\big\rangle\,\mathrm{d}r \\
&\quad = \big\langle X_{t_i-}(x_\varepsilon) - X_{t_i-}(x), n_i^k\big\rangle + \int_{r_i(q_n)}^t \big\langle b\big(X_r(x_\varepsilon)\big) - b\big(X_r(x)\big), n_i^k\big\rangle\,\mathrm{d}r. \qquad (4.15)
\end{aligned}$$

Recall that $\{n_i, n_i^{\perp}, n_i^k; k = 3, \dots, d\}$ is an orthonormal basis of $\mathbb{R}^d$. By (4.13)–(4.15) we obtain

$$\begin{aligned}\eta_t(\varepsilon) &= \langle \eta_t(\varepsilon), n_i\rangle n_i + \langle \eta_t(\varepsilon), n_i^{\perp}\rangle n_i^{\perp} + \sum_{k=3}^{d} \langle \eta_t(\varepsilon), n_i^k\rangle n_i^k \\ &= \langle \eta_{t_i-}(\varepsilon), v_i^{\perp}\rangle n_i^{\perp} + \sum_{k=3}^{d} \langle \eta_{t_i-}(\varepsilon), n_i^k\rangle n_i^k + \frac{1}{\varepsilon}\int_{r_i(t)}^{t} \bigl(b\bigl(X_r(x_\varepsilon)\bigr) - b\bigl(X_r(x)\bigr)\bigr)\,\mathrm{d}r. \end{aligned} \tag{4.16}$$

Then, from (4.12) and (4.16) we get

$$\begin{aligned}\eta_t(\varepsilon) &= \eta_{a_n}(\varepsilon) + \frac{1}{\varepsilon}\int_{a_n}^{t} \bigl[b\bigl(X_r(x_\varepsilon)\bigr) - b\bigl(X_r(x)\bigr)\bigr]\,\mathrm{d}r \\ &= \eta_{a_n}(\varepsilon) + \int_{a_n}^{t} \sum_{k=1}^{d} \biggl[\int_0^1 \frac{\partial b}{\partial x^k}\bigl(X_r^{\alpha,\varepsilon}\bigr)\,\mathrm{d}\alpha\biggr] \eta_r^{kj}(\varepsilon)\,\mathrm{d}r, \qquad t \in A_n,\end{aligned}$$

where $X_r^{\alpha,\varepsilon} := \alpha X_r(x_\varepsilon) + (1-\alpha)X_r(x)$, $\alpha \in [0,1]$. By the same arguments as in Step 5 in the proof of Theorem 1 in [4], we obtain now that (2.6) holds for $t \in [0,\tau) \setminus C$. The Lipschitz continuity of $x \mapsto (X_t(x))_t$, which is crucial for that argument, is ensured by Lemma 4.1. Since $C$ has zero Lebesgue measure, for $t \in C$ the value of $\eta_t^{\cdot j}$ can be changed arbitrarily without affecting the ODE system. Thus, by setting $\eta_t^{\cdot j} = \eta_{r(q_n)}^{\cdot j}$ for $t \in [\mu(q_n), q_n] \cap C$ we can extend $\eta^{\cdot j}$ on $[0,\tau)$, such that $\eta^{\cdot j}$ is right continuous, since $t \mapsto s(t)$ and $t \mapsto r_i(t)$ are right continuous as well, and the proof of Theorem 2.2 is complete.

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