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One-dimensional finite range random walk in random medium and invariant measure equation

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Abstract. We consider a model of random walks on $\mathbb{Z}$ with finite range in a stationary and ergodic random environment. We first provide a fine analysis of the geometrical properties of the central left and right Lyapunov eigenvectors of the random matrix naturally associated with the random walk, highlighting the mechanism of the model. This allows us to formulate a criterion for the existence of the absolutely continuous invariant measure for the environments seen from the particle. We then deduce a characterization of the non-zero-speed regime of the model.

Résumé. Nous considérons un modèle de marche aléatoire sur $\mathbb{Z}$ à pas bornés en environnement aléatoire stationnaire ergodique. Dans une première partie, nous détaillons les propriétés géométriques des vecteurs propres de Lyapunov centraux pour la matrice aléatoire naturellement associée à la marche, mettant en lumière le mécanisme du modèle. Nous formulons alors un critère, vectoriel dans les situations transientes, pour l'existence de la mesure invariante absolument continue pour les environnements vus depuis la particule. En corollaire, nous obtenons une caractérisation du régime avec vitesse non nulle.

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1. Introduction

1.1. *Model*

We describe a one-dimensional model of random walk in random environment, called the (L, R)-model in the sequel. Let $(\Omega, \mathcal{F}, \mu, T)$ be an invertible dynamical system, where $(\Omega, \mathcal{F}, \mu)$ is a probability space and T is an invertible and bi-measurable transformation preserving μ. We assume that $(\Omega, \mathcal{F}, \mu, T)$ is *ergodic*.

We fix integers $L \geq 1$, $R \geq 1$ and define an interval $\Lambda = [-L, +R]$ in $\mathbb{Z}$, as space of jumps. We next assume to be given positive random variables $(p_z)_{z\in\Lambda}$ on $(\Omega, \mathcal{F})$, such that for some $\varepsilon > 0$:

$$\forall z \in \Lambda \setminus \{0\}, \quad p_z \geq \varepsilon \quad \text{and} \quad \sum_{z\in\Lambda} p_z = 1, \quad \mu\text{-a.s.} \tag{1}$$

The iid case corresponds to $(\Omega, \mathcal{F}, \mu) = (X^{\mathbb{Z}}, \mathcal{A}^{\otimes\mathbb{Z}}, \nu^{\otimes\mathbb{Z}})$ for some probability space $(X, \mathcal{A}, \nu)$ with the left shift T and a vector $(p_z)_{z\in\Lambda}$ depending on a single coordinate in Ω.

Random walk $(\xi_n(\omega))_{n\geq 0}$
Fixing $\omega \in \Omega$, for each $k \in \mathbb{Z}$ the collection $(p_z(T^k\omega))_{z\in\Lambda}$ defines a transition law from k to $k + z$, $z \in \Lambda$. To the *environment* $[(p_z(T^k\omega))_{z\in\Lambda}]_{k\in\mathbb{Z}}$ on $\mathbb{Z}$ we associate the canonical trajectorial Markovian measures $(\mathcal{P}_k^{\omega})_{k\in\mathbb{Z}}$, where k

stands for the departure point. Let $(\xi_n(\omega))_{n\geq 0}$ be the Markov chain with law $\mathcal{P}_0^\omega$. In other words, $\xi_0(\omega)=0$ and for $n\geq 0$:

$$\mathcal{P}_0^\omega\big(\xi_{n+1}(\omega)=k+z|\xi_n(\omega)=k\big)=p_z\big(T^k\omega\big),\quad k\in\mathbb{Z}, z\in\Lambda.$$

The point of view adopted in this text is *quenched*. More precisely we are interested in the description of the properties of $(\xi_n(\omega))_{n\geq 0}$ for μ-typical $\omega\in\Omega$.

Conventions
In the whole article, the probability measures $\mathcal{P}_k^\omega$ are simplified into P_k (omitting the dependence in ω) with corresponding expectations E_k, except when stating results. Also, if f is a scalar or vectorial random variable on $(\Omega,\mathcal{F})$, we write $T^k f$ for $f\circ T^k$, $k\in\mathbb{Z}$.

1.2. *Presentation*

An essential feature of the $(1,1)$-model is the possibility of explicit computations. This contrasts with the multi-dimensional model and we refer to [25] and [26] for detailed surveys of the general model in any dimension. The (L,R)-model with $\max\{L,R\}\geq 2$ is one step higher in terms of complexity than the $(1,1)$-model. Its analysis involves random matrix products and Lyapunov exponents.

A criterion for recurrence/transience was first given by Key [14] via a random square $(L+R)$-matrix. Reformulated by Letchikov [17], it necessitates a matrix M of size $d:=L+R-1$.

Definition 1.1. *Let $M\in GL_d(\mathbb{R})$ be the random matrix (the first line is $(b_L\cdots b_1)$ if $R=1$):*

$$M=\begin{pmatrix} -a_1 & \cdots & -a_{R-1} & b_L & \cdots & b_1\\ 1 & 0 & \cdots & \cdots & \cdots & 0\\ 0 & 1 & 0 & \cdots & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & \cdots & \cdots & 1 & 0\end{pmatrix},\tag{2}$$

where $M_{i,j}=1_{i=j+1}$ for $2\leq i\leq d$ and:

$$M_{1,j}=\begin{cases} -a_j=-\big(\frac{p_{R-j}+\cdots+p_R}{p_R}\big), & 1\leq j\leq R-1,\\ b_{L+R-j}=\big(\frac{p_{R-1-j}+\cdots+p_{-L}}{p_R}\big), & R\leq j\leq d.\end{cases}$$

When $L=R=1$, then M reduces to the well-known quantity p_{-1}/p_1. The matrix M is extracted from the analysis of the Dirichlet problem in any finite interval in $\mathbb{Z}$. We make it more precise now.

For integers $a<b$, let $[a,b]$ be the corresponding interval in $\mathbb{Z}$. As the model is not nearest-neighbour, when starting a random walk in $[a,b]$ we need to specify exit points.

Definition 1.2. *Let integers $a<b$ and $k\in[a-L+1,b+R-1]$. We define boundaries $\partial_-[a,b]=\{a-l\}_{0\leq l\leq L-1}$ and $\partial_+[a,b]=\{b+r\}_{0\leq r\leq R-1}$ and introduce:*

$$\begin{cases} P_k(a,b,+)=P_k\big\{\textit{leave }]a+1,b-1[\textit{ in }[b,+\infty[\big\},\\ P_k(a,b,-)=P_k\big\{\textit{leave }]a+1,b-1[\textit{ in }]-\infty,a]\big\},\\ P_k(a,b,\zeta)=P_k\big\{\textit{leave }]a+1,b-1[\textit{ at }\zeta\big\},\quad \textit{for }\zeta\in\partial_-[a,b]\cup\partial_+[a,b].\end{cases}$$

The definitions are naturally extended to half-infinite intervals, when it has sense. Set next, for $\zeta\in\partial_-[a,b]\cup\partial_+[a,b]\cup\{\pm\}$:

$$V_k(a,b,\zeta)=\big(P_{k+R-i}(a,b,\zeta)-P_{k+R+1-i}(a,b,\zeta)\big)_{1\leq i\leq d}\in\mathbb{R}^d.$$

Fixing $a < b$ and ζ as above, the Markov property is equivalent to the harmonicity of the map $k \longmapsto P_k(a, b, \zeta)$ (with respect to the transition weights at each site) in $[a, b]$. The $(k \longmapsto P_k(a, b, \zeta))_{\zeta \in \partial_-[a,b] \cup \partial_+[a,b]}$ form the canonical basis of the space of harmonic functions on $[a, b]$.

The harmonic character of $k \longmapsto P_k(a, b, \zeta)$ can be reformulated via gradients. The role of gradients is to keep only the essential information, by eliminating the trivial harmonic function equal to 1.

Lemma 1.3 (See [7,17]). *For any integers $a < k < b$ and $\zeta \in \partial_-[a, b] \cup \partial_+[a, b] \cup \{\pm\}$, we have*:

$$V_k(a, b, \zeta) = T^k M V_{k-1}(a, b, \zeta). \tag{3}$$

Recall that M is defined independently of any interval $[a, b]$ and exit condition ζ. Iterating (3), $V_k(a, b, \zeta)$ can be expressed in terms of the gradients at the boundary of $[a, b]$, via random products of M. The matrix M can thus be seen as a transmitting matrix. The properties of the random walk are then naturally determined by that of M with respect to the dynamical system $(\Omega, \mathcal{F}, \mu, T)$.

Introduce the Lyapunov exponents $\gamma_1(M, T) \geq \cdots \geq \gamma_d(M, T)$ of the couple (M, T). Precise definitions are given in Proposition 2.1. Due to (3), the structure of the Lyapunov spectrum of (M, T) is rather special. Some known facts are collected in the next theorem.

Theorem 1.4 (See [7,17], for (i) and [7] for (ii)). (i) *We always have $\gamma_1(M, T) \geq \cdots \geq \gamma_{R-1}(M, T) > 0$ and $0 > \gamma_{R+1}(M, T) \geq \cdots \geq \gamma_d(M, T)$.*

(ii) *The Lyapunov exponent $\gamma_R(M, T)$ is simple, namely $\gamma_{R-1}(M, T) > \gamma_R(M, T) > \gamma_{R+1}(M, T)$.*

In the sequel $d = (R - 1) + 1 + (L - 1)$ is symmetrically understood with respect to L and R and $\gamma_R(M, T)$ is seen as the *central exponent* of (M, T). We now explain why this exponent is particular. For example, the nature of the dynamical system plays no role in the proof of (ii) and $\gamma_R(M, T)$ is simple for geometrical reasons.

This fact was clarified in [7] as follows. For simplicity, if $x \neq 0$ belongs to some space $\mathbb{R}^m$, denote by $Dir(x)$ its direction in the projective space of $\mathbb{R}^m$. When considering recurrence criteria, one focuses on the exit probabilities of an interval $[a, b]$ and this naturally leads to considering the family $(V_k(a, b, \zeta))$. Fixing k, these vectors are well understood when grouped in left and right packets, more precisely when considering the two subspaces $L_k(a, b)$ and $R_k(a, b)$ of $\mathbb{R}^d$ respectively spanned by $(V_k(a, b, \zeta))_{\zeta \in \partial_-[a,b]}$ and $(V_k(a, b, \zeta))_{\zeta \in \partial_+[a,b]}$. Computations involve exterior products.

Definition 1.5. *Let integers $a < b$ and $k \in [a - L + 1, b + R - 1]$. Define a global right-gradient and a global left-gradient respectively by*:

$$\begin{cases} \mathcal{R}_k(a, b) = V_k(a, b, b + R - 1) \wedge \cdots \wedge V_k(a, b, b) \in \bigwedge^R \mathbb{R}^d, \\ \mathcal{L}_k(a, b) = V_k(a, b, a) \wedge \cdots \wedge V_k(a, b, a - L + 1) \in \bigwedge^L \mathbb{R}^d. \end{cases}$$

The matrices $(-1)^{R-1} \bigwedge^R M$ and $(-1)^{L-1} \bigwedge^L M^{-1}$, acting respectively on the R-vector $\mathcal{R}_{-1}(a, b)$ and the L-vector $\mathcal{L}_{-1}(a, b)$, appear to be the natural objects for the study of the model. Focusing on $(-1)^{R-1} \bigwedge^R M$, this matrix has *directional contraction properties* in a non-trivial deterministic and explicit polyhedral convex cone of $\bigwedge^R \mathbb{R}^d$, exactly in the same way as a $m \times m$-matrix with positive entries in the positive cone of $\mathbb{R}^m$.

This key property comes from the remarks that $Dir(\mathcal{R}_{-1}(a, b))$ is independent on b, for $b \geq 0$, and that, due to its shape, the R-vector $\mathcal{R}_{-1}(a, 0)$ has a very rigid geometry. The edges of a cone stable by $(-1)^{R-1} \bigwedge^R M$ can be described by R-vectors $\mathcal{R}_{-1}(a, 0)$ corresponding to "extremal" environments in a left-neighbourhood of 0, in the sense that the transition at each site of this neighbourhood is deterministic. The cone stability property of $(-1)^{R-1} \bigwedge^R M$ naturally implies the simplicity of the top exponent of this matrix. As the same is true for $(-1)^{L-1} \bigwedge^L M^{-1}$, the simplicity of $\gamma_R(M, T)$ is then a relatively easy consequence. Numerical experiments show that the other exterior powers of M do not have this cone stability property. Nothing is known on the simplicity of the other exponents of (M, T) at such a level of generality.

Roughly speaking, the R and L-dimensional random subspaces $L_{-1}(a, b)$ and $R_{-1}(a, b)$ of $\mathbb{R}^d$ "reflect the influences" of both sides of the environment at 0. The behavior of the random walk is then related to the properties of the intersection of the previous two subspaces. The latter is one-dimensional and spanned by $V_{-1}(a,b,+)$. When a and b become infinite, $V_{-1}(a,b,+)$ has a limit direction, that of a vector with exponent $\pm\gamma_R(M,T)$ when iterating the cocycle of M in the future or in the past. This explains the role of $\gamma_R(M,T)$.

As a corollary, the previous analysis gave in [7] another proof, more algebraic, of Key's theorem. The following formulation first appeared in [17].

Theorem 1.6 (Key).

- *If* $\gamma_R(M,T)<0$, *then* $\xi_n(\omega)\to+\infty$, $\mathcal{P}_0^\omega$*-a.s*, μ*-a.s.*
- *If* $\gamma_R(M,T)>0$, *then* $\xi_n(\omega)\to-\infty$, $\mathcal{P}_0^\omega$*-a.s*, μ*-a.s.*
- *If* $\gamma_R(M,T)=0$, *then* $\liminf\xi_n(\omega)=-\infty<+\infty=\limsup\xi_n(\omega)$, $\mathcal{P}_0^\omega$*-a.s*, μ*-a.s.*

To emphasize the interest in this approach, we next discuss the efficiency of the criterion. Recall first that matrices with positive entries, as contracting the positive cone, are praised in the problem of evaluating a top Lyapunov exponent. See the discussion at the end of [21]. Cone contraction, measured for instance via Hilbert's distance, simplifies the computation and provides error estimates.

We detail a way of proceeding. Under broad hypotheses a central tool for a random matrix H with positive entries is the existence of a main Lyapunov eigenvector, or generalized eigenvector in the sense of [9]. Similar to the classical Perron eigenvector, this is a positive random vector U with $\|U\|=1$ (we fix the Euclidean norm) and such that there exists some positive random λ verifying $HU=\lambda TU$. In this case, necessarily $\int\log\lambda\,\mathrm{d}\mu=\gamma_1(H,T)$. The direction of U is uniquely determined and can be simply defined as the decreasing limit of compact sets $Dir(U)=\lim Dir(T^{-1}H\cdots T^{-n}H(\mathcal{C}))$, where $\mathcal{C}$ is the positive cone. The last convergence is exponential, with rate given by that of the cone contraction (see for instance Hennion [12], Lemma 3.3). A natural way of computing $\gamma_1(H,T)$ is then to evaluate V, giving λ. Remark that if the $(T^nH)_{n\in\mathbb{Z}}$ are iid, then λ and V only depend on one-half of the sequence, with exponential decay of the correlations.

Back to our problem, $(-1)^{R-1}\bigwedge^R M$ and $(-1)^{L-1}\bigwedge^L M^{-1}$ contract explicit cones, also with explicit contraction rates (see [7]), and the above remarks all apply. These matrices thus behave like matrices with positive entries and their top exponent is as easily evaluable. As a result (see Section 7.2 in [7]), the accessibility of $\gamma_R(M,T)$ is *exactly* that of the top Lyapunov exponent of a positive random matrix depending on a single site. The cost of dimension due to the consideration of exterior powers is very low, since in practice exclusively limited to the use of Gauss pivot.

Another approach to recurrence criteria is presented by Bolthausen and Goldsheid in [3]. As the (L,R)-model can be seen as a model of random walk on a strip $\mathbb{Z}\times\{1,\dots,m\}$ in a random environment, a recurrence criterion is available in [3], via the sign of the top Lyapunov exponent of a non-negative random matrix A. A difficulty is that the entries of A are abstract quantities. For example in an iid setup, A involves a matrix ζ whose law is the invariant measure of a rather non-trivial Markov chain in the space of stochastic matrices and the computation of this law is at least as complex as evaluating the top Lyapunov exponent of an iid product of random matrices. One may observe that ζ is an analogue of the auxiliary non-negative square matrices G and D of respective sizes L and R, presented in [7] and used for analyzing the two subsequences of best records to the left and best records to the right of the random walk. It would be interesting to provide a direct link between Key's theorem and the recurrence criterion of [3]. Theorem (6.3) in [7] connecting M to G and D via their Lyapunov spectrum goes in this direction.

We now discuss the validity of the Law of Large Numbers. The LLN was shown to hold for the (L,R)-model under a rather restrictive hypothesis (as discussed in Section 3) by Letchikov [19], next under Kalikow's condition by Rassoul-Agha [22] (in a study centered on the model on $\mathbb{Z}^d$) and then in full generality in [7]. This last result is in fact a corollary of the analysis developed in [6], via a classical hitting times approach. The LLN for the strip model in the transient case was recently proved by Roitershtein [24] using hitting times, as well as a criterion for positive speed. Other results were independently obtained by Goldsheid [11] via developing the methods from [3].

1.3. *Content of the article*

The main purpose of this text is to study in complete generality the existence of the absolutely continuous invariant measure for "the environments seen from the particle" for the (L,R)-model and then to characterize the situations when the average speed in the LLN is not 0. Our main tools are relevant from exterior algebra, combined with classical arguments from Ergodic Theory.

As detailed in the next section, we use a corollary of Oseledec's theorem giving the existence of a measurable basis $(V_i)_{1\le i\le d}$ of $\mathbb{R}^d$ such that $\|V_i\|=1$ for all i with:

$$\lim_{n\to\pm\infty}\frac{1}{|n|}\log\|M_nV_i\|=\pm\gamma_i(M,T),\quad 1\le i\le d,$$

where we introduce cocycle notations for a random invertible matrix H:

$$H_n=\begin{cases}T^{n-1}H\cdots H, & n\ge 1,\\ I, & n=0,\\ T^nH^{-1}\cdots T^{-1}H^{-1}, & n\le -1.\end{cases}\tag{4}$$

A basis $(V_i)_{1\le i\le d}$ as above is not unique. However we recall in proposition (2.6) that the simplicity of $\gamma_R(M,T)$ implies that $Dir(V_R)$ is uniquely determined. In fact V_R is naturally defined as a vector spanning the intersection of two subspaces and, concretely, is directly obtained via the canonical main Lyapunov eigenvectors of $(-1)^{R-1}\bigwedge^R M$ and $(-1)^{L-1}\bigwedge^L M^{-1}$. As a result, the cost of this definition is not more than that of the main Lyapunov eigenvector of a positive random matrix depending on a single site.

An important non-trivial point detailed in proposition (2.6) is the existence of some random $\lambda_R>0$, with $\log\lambda_R$ bounded, verifying:

$$MV_R=\lambda_R TV_R\quad\text{and}\quad\int\log\lambda_R\,\mathrm{d}\mu=\gamma_R(M,T).$$

These properties induce that V_R is uniquely defined up to multiplication by the constant -1. Indeed, if $\delta\in\{\pm1\}$ is any random change of sign, when replacing V_R by δV_R the positivity condition implies $\delta=T\delta$. Thus δ is constant, as $(\Omega,\mathcal{F},\mu,T)$ is ergodic.

Remark that the recurrence criterion, Theorem 1.6, can be reformulated in terms of λ_R. Finer properties of the random walk will involve the couple (λ_R,V_R).

We consider the invariant measure equation. Fixing $\omega\in\Omega$, define as in [15] the Markov chain "environments seen from the particle" as the sequence $(\omega_n)_{n\ge0}$, where $\omega_n=T^{\xi_n(\omega)}\omega$, $n\ge0$. Its transition operator on Ω is:

$$Pf(\omega)=\sum_{z\in\Lambda}p_z(\omega)f\big(T^z\omega\big).$$

A tool for proving *quenched* limit theorems for $(\xi_n(\omega))_{n\ge0}$ is the existence of a P-invariant probability measure ν on $(\Omega,\mathcal{F})$ equivalent to μ. Writing $\mathrm{d}\nu=\pi\,\mathrm{d}\mu$, the condition $\nu=P\nu$ is equivalent to the equality $P^*\pi=\pi$, where the adjoint operator P^* can be written in the form $P^*f(\omega)=\sum_{z\in\Lambda}p_z(T^{-z}\omega)f(T^{-z}\omega)$. This leads to the following definition:

Definition 1.7. *We call (IM) the existence of a measurable π with $\pi\ge0$, $\int\pi\,\mathrm{d}\mu=1$, $\pi=P^*\pi$, μ-a.s.*

We now mention known results. Kozlov [15] proved that if π realizes (IM), then $\pi>0$, μ-a.s, and is unique in $L^1(\mu)$. Then, under (IM) and using Birkhoff's Ergodic Theorem, the (quenched) LLN was shown to hold. A complete analysis of the equation $\nu=P\nu$, including (IM), was given by Conze–Guivarc'h [8] in the case $L=R=1$. The study of condition (IM) when $\min\{L,R\}=1$ was treated in [6]. We extend this last result as follows.

Theorem 1.8. (i) *If $\gamma_R(M,T)=0$, then: $(IM)\Leftrightarrow\exists\varphi\in L^1(\mu)$, $\varphi>0$, μ-a.s, with $\lambda_R=\varphi/T\varphi$.*

(ii) *If $\gamma_R(M,T)<0$, then: $(IM)\Leftrightarrow\|\sum_{n\ge0}(\lambda_R\cdots T^{n-1}\lambda_R)T^nV_R\|\in L^1(\mu)$. If (IM) is not satisfied, then there exists a unique non-integrable σ-finite density $\pi>0$ verifying $\pi=P^*\pi$.*

(iii) *If $\gamma_R(M,T)>0$, then: $(IM)\Leftrightarrow\|\sum_{n\ge1}(T^{-1}\lambda_R\cdots T^{-n}\lambda_R)^{-1}T^{-n}V_R\|\in L^1(\mu)$. If (IM) is not satisfied, then there exists a unique non-integrable σ-finite density $\pi>0$ verifying $\pi=P^*\pi$.*

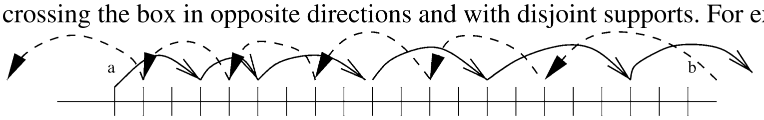
Mention that the behaviour of a random walk on a strip in a recurrent iid medium was recently clarified by Bolthausen and Goldsheid [4] and the previous result in the recurrent case is thus mainly interesting for non-iid environments. In this situation, the characterization of (IM) was a preliminary step in [6] for the analysis of the CLT when $L \geq 1$ and $R = 1$. Extending a work by Letchikov [18], it was shown in [6] (Theorem 4.5) that there is a non-degenerate invariance principle if and only if $\lambda_R = \varphi/T\varphi$ for some $\varphi > 0$ with φ and $1/\varphi$ in $L^1(\mu)$. The result was the central tool in a delicate study for proving a CLT under sharp conditions in a recurrent environment given by an irrational rotation on the Circle with regular data (Theorem (5.7) of [6]). Providing similar results for the general model is delicate and can be considered as a separate problem.

We focus next on the transient cases. In view of Theorem 1.8, it is important to understand the geometrical properties of V_R. Suppose for instance that $\gamma_R(M, T) < 0$. If $\min\{L, R\} = 1$, then it is a simple remark that V_R lies in the positive cone of $\mathbb{R}^d$, since M (resp. M^{-1}) is non-negative for $R = 1$ (resp. $L = 1$). The characterization of (IM) then reduces to $\sum_{n\geq 0}(\lambda_R \cdots T^{n-1}\lambda_R) \in L^1(\mu)$, which is the condition obtained in [6]. Indeed, reminding that $\|V_R\| = 1$, it is enough to take the scalar product of $\sum_{n\geq 0}(\lambda_R \cdots T^{n-1}\lambda_R)T^n V_R$ with the vector ${}^t(1 \cdots 1)$.

We have simply used that the dual cone of $(\mathbb{R}_+)^d$ is not reduced to $\{0\}$ and a similar property is valid if $L = R = 2$, also leading to the simplified criterion $\sum_{n\geq 0}(\lambda_2 \cdots T^{n-1}\lambda_2) \in L^1(\mu)$. In the general case however, such a reasoning cannot occur. We shall show that if $\min\{L, R\} \geq 2$ and $\max\{L, R\} \geq 3$, then there exists an example of iid environment, where the convex cone generated by the support of the law of V_R is $\mathbb{R}^d$. This gives a negative answer to a conjecture by Letchikov [17]. In such an example, the dual cone of the cone where V_R naturally lies is reduced to $\{0\}$. As a result, the characterization of (IM) in the general transient case seems not any more to be of scalar type and to involve some cumulative vector. It would be interesting to exhibit when $\min\{L, R\} \geq 2$ and $\max\{L, R\} \geq 3$ an iid environment with $\gamma_R(M, T) < 0$ such that:

$$\left\| \sum_{n\geq 0} \big(\lambda_R \cdots T^{n-1}\lambda_R\big) T^n V_R \right\| \in L^1(\mu), \quad \text{but } \sum_{n\geq 0} \big(\lambda_R \cdots T^{n-1}\lambda_R\big) \notin L^1(\mu).$$

Intuitively, the condition $\min\{L, R\} \geq 2$ and $\max\{L, R\} \geq 3$ ensures that a finite box $[a, b]$ contains distinct paths with jumps in $\Lambda \setminus \{0\}$, crossing the box in opposite directions and with disjoint supports. For example:



The case $L = R = 2$ is critical (as appears in Theorem 3.15), since such paths still exist (in contrary to the situation $\min\{L, R\} = 1$) but must be exclusively composed of jumps of size two. In a related way, the criticality of the $(2, 2)$-model was also transparent in the rather striking properties of conjugation with non-negative matrices of the matrix M in this case (see [5]). Heuristics were given in [7] that such a property was specific to the case $L = R = 2$.

Let us explain the strategy for understanding the geometrical constraints imposed to V_R. To perform such an analysis, recall that V_R is seen as spanning the intersection of two subspaces of $\mathbb{R}^d$. We then explicitly describe the geometrical constraints on these subspaces, represented by the limits, as $a \to -\infty$ and $b \to +\infty$, of $Dir(\mathcal{R}_{-1}(a, 0))$ and $Dir(\mathcal{L}_{-1}(-1, b))$. We then split the problem in two independent parts, since the previous decomposable vectors involve *disjoint* halfs of the environment. In order to get the *exact* constraints on V_R, we need to determine the *exact* geometrical properties of $\mathcal{R}_{-1}(a, 0)$ and $\mathcal{L}_{-1}(-1, b)$. In other words, we shall determine the *minimal* stable convex cones for $(-1)^{R-1} \bigwedge^R M$ and $(-1)^{L-1} \bigwedge^L M^{-1}$. A subtlety is that this study cannot be deduced from the one in [7] on minimal stable cones for the matrices $(-1)^{R-1} \bigwedge^R ({}^tM)$ and $(-1)^{L-1} \bigwedge^L ({}^tM)^{-1}$. The latter gave, by duality, stable cones for $(-1)^{R-1} \bigwedge^R M$ and $(-1)^{L-1} \bigwedge^L M^{-1}$, but these will be seen not to be minimal as soon as $\min\{L, R\} \geq 2$.

We proceed symmetrically to the investigation of the exact geometrical constraints on W_R, defined as the central eigenvector of ${}^tM$. In contrast to V_R, the components of W_R always have the same fixed sign. In fact we completely determine the structure of the vectors V_R and W_R. In this analysis, the mechanism of the model is highlighted and appears to be intimately related to "extremal" finite boxes (in the sense explained above) and to "exit games" defined with such boxes. As a result, M provides a rather remarkable example of a random matrix where the geometrical features of some central Lyapunov eigenvectors can be described with a high level of precision.

Next, the families of minimal stable cones of $(-1)^{R-1} \bigwedge^R M$ and $(-1)^{L-1} \bigwedge^L M^{-1}$ and that of $(-1)^{R-1} \bigwedge^R ({}^tM)$ and $(-1)^{L-1} \bigwedge^L ({}^tM)^{-1}$ are both used to understand the geometrical link between V_R and W_R and related non-singularity results. Equation (IM) can then be studied precisely.

We also reformulate the criterion for (IM) in the case of transience to the right ($\gamma_R(M,T)>0$) via the auxiliary matrix D of size R presented in [7], associated to the subsequence of best right records. It was defined by:

$$D=\begin{pmatrix} 0 & 1 & \cdots \\ \cdots & \cdots & 1 \\ P_0\{-\infty,1,R\} & \cdots & P_0\{-\infty,1,1\} \end{pmatrix}. \tag{5}$$

Since $\gamma_R(M,T)>0$, D is strictly sub-stochastic. More precisely $\gamma_1(D,T^{-1})<0$, by Theorem 6.3 and Lemma 7.1 of [7]. Introduce the unique random (bounded) vector $W\in(\mathbb{R}_+)^R$ with $\langle W,e_R\rangle=1$ and the unique positive ρ (with $\log\rho$ bounded) satisfying $DTW=\rho W$. Then:

Proposition 1.9. *Let $\gamma_R(M,T)>0$. Then $(IM)\Leftrightarrow(\sum_{k=0}^{d-1}|1-T^k\rho|)^{-1}\in L^1(\mu)$.*

The above sum involves only d terms. When $L=R=1$, the criterion is $1/P_0(-\infty,1,-)\in L^1(\mu)$. Via for instance Proposition 2.2 of [6], one recovers the usual result established in [8].

We finally classically deduce a characterization of the LLN with positive speed, when combining Theorem 1.8 with Proposition 9.1 from [7]. For integers $a<b$, with at least a or b finite, denote by $\tau(a,b)$ the exit time of the interval $[a+1,b-1]$.

Theorem 1.10. (i) *The following assertions are equivalent*:

1. *There exists a constant $c>0$ such that: $\frac{\xi_n(\omega)}{n}\underset{n\to+\infty}{\longrightarrow}c$, $\mathcal{P}_0^\omega$-a.s, μ-a.s.*
2. *$\gamma_R(M,T)<0$ and (IM) holds.*
3. *$\gamma_R(M,T)<0$ and $\|\sum_{n\geq 0}(\lambda_R\cdots T^{n-1}\lambda_R)T^nV_R\|\in L^1(\mu)$.*
4. *$\int_\Omega E_0(\tau(-\infty,1))\,\mathrm{d}\mu<+\infty$.*

(ii) *The following assertions are equivalent*:

1. *There exists a constant $c<0$ such that: $\frac{\xi_n(\omega)}{n}\underset{n\to+\infty}{\longrightarrow}c$, $\mathcal{P}_0^\omega$-a.s, μ-a.s.*
2. *$\gamma_R(M,T)>0$ and (IM) holds.*
3. *$\gamma_R(M,T)>0$ and $\|\sum_{n\geq 1}(T^{-1}\lambda_R\cdots T^{-n}\lambda_R)^{-1}T^{-n}V_R\|\in L^1(\mu)$.*
4. *$\int_\Omega E_0(\tau(-1,+\infty))\,\mathrm{d}\mu<+\infty$.*

(iii) *In all remaining cases: $\frac{\xi_n(\omega)}{n}\underset{n\to+\infty}{\longrightarrow}0$, $\mathcal{P}_0^\omega$-a.s, μ-a.s.*

Using exit times and when the random walk satisfies the LLN with non-zero speed, then the invariant measure ν with $\mathrm{d}\nu=\pi\,\mathrm{d}\mu$, π satisfying (IM), can be simply expressed (see (38)) as in [1]. A formula for the average speed is given in Proposition 9.1 of [7], but an expression for quantities such as $E_0(\tau(-\infty,1))$ is not available, in contrary to the strip case (cf. [24]).

Plan of the article: Section 2 concerns preliminaries, Section 3 details the geometry of the Lyapunov eigenvectors relevant for the analysis and Section 4 focuses on the invariant measure equation and the Law of Large Numbers.

2. Preliminary part

2.1. *Algebraic conventions*

We fix notations and remind a few basic facts regarding exterior algebra. On this topic, one may consult Arnold [2] (p. 118–121), Federer [10] (Chapter 1) or Karoubi–Leruste [13] (Chapter 1).

- Consider $\mathbb{R}^d$ with canonical basis $(e_i)_{1\leq i\leq d}$. Convene that $e_i=0$ if $i\notin\{1,\dots,d\}$. The space $\mathbb{R}^d$ is endowed with its Euclidean structure, to which $\perp$ refers to.
- For any $0\leq n\leq d$, $\bigwedge^n\mathbb{R}^d$ denotes the exterior power of $\mathbb{R}^d$ of order n, which can be identified with the set of asymmetric n-linear forms on the dual of $\mathbb{R}^d$. Elements of $\bigwedge^n\mathbb{R}^d$ are called *n-vectors*. Those of the form

$u_1 \wedge \cdots \wedge u_n$, where $u_i \in \mathbb{R}^d$ and $\wedge$ denotes the *wedge product* (see the definition in [13]), are called *decomposable n-vectors*. Recall that any n-vector can be represented (not uniquely) as a finite linear combination of decomposable n-vectors. In particular, the canonical basis of $\bigwedge^n \mathbb{R}^d$ is:

$$\{e_{i,n} = e_{i_1} \wedge \cdots \wedge e_{i_n} \mid i \in I_n\}, \quad \text{where } I_n = \big\{i = (i_1, \ldots, i_n) \mid 1 \le i_1 < \cdots < i_n \le d\big\}.$$

– A decomposable n-vector $u_1 \wedge \cdots \wedge u_n \in \bigwedge^n \mathbb{R}^d$, where $u_i \in \mathbb{R}^d$, is written as $\bigwedge_{i=1}^n u_i$.
– If a decomposable n-vector appears in the form $(\wedge \cdots \bigwedge_k z \wedge \cdots) \in \bigwedge^n \mathbb{R}^d$, the *below* subscript k means that z is at place k.
– In the sequel $\bigwedge^n \mathbb{R}^d$ is endowed with its Euclidean structure inherited from $\mathbb{R}^d$ (see for instance Theorem 10.3, p. 28 of [13]). We also use the symbol $\perp$. For any decomposable n-vectors $\bigwedge_{i=1}^n u_i$ and $\bigwedge_{i=1}^n v_i$ in $\bigwedge^n \mathbb{R}^d$, recall that:

$$\left\langle \bigwedge_{i=1}^n u_i, \bigwedge_{i=1}^n v_i \right\rangle = \det\big(\langle u_i, v_j\rangle\big).$$

The expression for any couple of n-vectors is obtained by bilinearity.
– Vectorial subspaces of $\mathbb{R}^d$ can be identified with directions of decomposable vectors. If $\bigwedge_{i=1}^n u_i$ is a non-zero decomposable n-vector of $\bigwedge^n \mathbb{R}^d$, write $S(\bigwedge_{i=1}^n u_i)$ for the subspace $\mathrm{Vect}(u_1, \ldots, u_n) \subset \mathbb{R}^d$.

- For $0 \le n \le d$, the n-th exterior power $\bigwedge^n A$ of a matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ is a matrix in $\mathrm{Mat}_{C_n^d}(\mathbb{R})$ acting on $\bigwedge^n \mathbb{R}^d$, whose value is defined by:

$$\bigwedge^n A\left(\bigwedge_{i=1}^n u_i\right) = \bigwedge_{i=1}^n Au_i,$$

where $\bigwedge_{i=1}^n u_i$ is a decomposable n-vector in $\bigwedge^n \mathbb{R}^d$. By linearity, this definition extends to indecomposable elements of $\bigwedge^n \mathbb{R}^d$.
- For the sake of simplicity, any n-tuple $i = (i_1, \ldots, i_n) \in I_n$ is also considered as a set. Write $z \in i$ if $z = i_j$ for some $1 \le j \le n$. Also $i^c = (1, \ldots, d) \setminus i$. If $i \in I_n$ and $j \in I_m$, $i \cap j$ stands for the ordered set of elements both in i and j. The same holds for $i \cup j$.
- A *cone* is here always a convex cone, that is a non-empty subset stable under non-negative linear combinations. We say that a cone is *minimal* with respect to a certain property if no strict subcone except $\{0\}$ has this property. If $n \ge 1$ and $B \subset \mathbb{R}^n$, write $\mathrm{Vect}(B) \subset \mathbb{R}^n$ for the subspace generated by B and $\mathrm{Vect}_+(B) \subset \mathbb{R}^n$ for the cone generated by B.

2.2. *Lyapunov spectrum and Lyapunov eigenvectors*

An exposition on Lyapunov exponents and Oseledec's theorem [20] can be found in [2,16] or [23].

As a first observation, condition (1) implies that $\log \|M\|$ and $\log \|M^{-1}\|$ are bounded quantities. The Lyapunov exponents of (M, T) are then well defined and all finite. More precisely, recalling cocycle notations introduced in (4):

Proposition 2.1 (See [16]). (i) *The Lyapunov exponents $\gamma_1(M, T) \ge \cdots \ge \gamma_d(M, T)$ of (M, T) can be recursively defined by the equalities, for $1 \le i \le d$:*

$$\gamma_1(M, T) + \cdots + \gamma_i(M, T) = \lim_{n\to+\infty} \frac{1}{n} \int \log\left\| \bigwedge^i M_n \right\| \mathrm{d}\mu. \tag{6}$$

(ii) *We have: $\gamma_i(M, T) = \gamma_i({}^tM, T^{-1})$ and $\gamma_i(M^{-1}, T^{-1}) = -\gamma_{d+1-i}(M, T)$, $1 \le i \le d$.*

Given $Y \in \mathbb{R}^d$, its Lyapunov exponent with respect to (M, T) is defined as:

$$\gamma(Y, M, T) = \limsup_{n\to+\infty} \frac{1}{n} \log \|M_n Y\|. \tag{7}$$

Oseledec's theorem [20] describes the Lyapunov exponent of vectors in terms of the Lyapunov exponents $(\gamma_i(M,T))_{1\le i\le d}$ and expresses the result using a random filtration of $\mathbb{R}^d$ in subspaces. A corollary in the invertible case is the existence of random bases of $\mathbb{R}^d$ of the following form.

Theorem 2.2 (See [16]). (i) *There exists a measurable basis* $(V_i)_{1\le i\le d}$ *of* $\mathbb{R}^d$ *such that* $\|V_i\|=1$ *and satisfying*:

$$\lim_{n\to\pm\infty}\frac{1}{|n|}\log\|M_n V_i\| = \pm\gamma_i(M,T), \quad \forall 1\le i\le d.$$

(ii) *There exists a measurable basis* $(W_i)_{1\le i\le d}$ *in* $\mathbb{R}^d$ *such that* $\|W_i\|=1$ *and satisfying*:

$$\lim_{n\to\pm\infty}\frac{1}{|n|}\log\left\|\left({}^tM^{-1}\right)_{-n}W_i\right\| = \pm\gamma_i(M,T), \quad \forall 1\le i\le d.$$

We call such a basis a *Lyapunov basis* and its elements, *Lyapunov eigenvectors*. We next denote by $(V_i)_{1\le i\le d}$ and $(W_i)_{1\le i\le d}$ choices as above of Lyapunov bases. As recalled in the Introduction, the simplicity of some $\gamma_i(M,T)$ $(\gamma_{i-1}(M,T)>\gamma_i(M,T)>\gamma_{i+1}(M,T))$ implies the uniqueness in direction of both V_i and W_i (see [16] and Proposition 2.6 of the present text). Theorem 1.4, point (ii), thus implies that $Dir(V_R)$ and $Dir(W_R)$ are unique.

We shall in fact show that there are natural definitions for V_R and W_R, each one as a special unit vector spanning the one-dimensional intersection of two subspaces.

2.3. *Algebraic preliminaries*

We first develop calculations for finding explicitly $E\cap F$, when two subspaces E and F of $\mathbb{R}^d$ verifying $\mathrm{Dim}(E\cap F)=1$ are represented by non-zero decomposable vectors.

Definition 2.3. (i) *Let* $0\le n\le d$. *For* $i\in I_n$, *denote by* $\varepsilon_n(i)$ *the signature of the permutation of* $[1,d]$ *mapping* i *on* $[1,n]$ *and* i^c *on* $[n+1,d]$, *preserving intrinsic orders of* i *and* i^c.

(ii)) *Let* $0\le n\le d$ *and* $A\in \mathrm{Mat}_d(\mathbb{R})$. *Define a matrix* $Com_n(A)$ *acting on* $\bigwedge^n\mathbb{R}^d$ *by*:

$$Com_n(A)(i,j)=\varepsilon_n(i)\varepsilon_n(j)\left(\wedge^{d-n}A\right)\left(i^c,j^c\right), \quad \forall (i,j)\in I_n\times I_n.$$

One easily checks that if $A\in GL_d(\mathbb{R})$, *then* $\bigwedge^n A^{-1}=(\det A)^{-1t}Com_n(A)$.

(iii) *For all* $0\le n\le d$, *define a map* $Ort_n:\bigwedge^n\mathbb{R}^d\longrightarrow\bigwedge^{d-n}\mathbb{R}^d$ *by* $Ort_n(e_{i,n})=\varepsilon_n(i)e_{i^c,d-n}, \forall i\in I_n$, *and then extended by linearity to* $\bigwedge^n\mathbb{R}^d$. *If* $x\in\bigwedge^n\mathbb{R}^d$, *we write* $x^{\perp *}$ *for* $Ort_n(x)$.

(iv) *Define a bilinear map* $\mathrm{Int}:\bigwedge^R\mathbb{R}^d\times\bigwedge^L\mathbb{R}^d\longrightarrow\mathbb{R}^d$ *by* $\mathrm{Int}(x,y)=(x^{\perp *}\wedge y^{\perp *})^{\perp *}$.

The properties of Ort_n and Int used in the sequel are detailed in the following lemma.

Lemma 2.4. (i) *For* $0\le n\le d$, Ort_n *is an isometry from* $\bigwedge^n\mathbb{R}^d$ *on* $\bigwedge^{d-n}\mathbb{R}^d$ *for their Euclidean structures. If* $A\in GL_d(\mathbb{R})$ *and* $x\in\bigwedge^n\mathbb{R}^d$, *then*:

$$\left(\bigwedge^n Ax\right)^{\perp *}=\det A\left(\bigwedge^{d-n}{}^tA^{-1}\right)\left(x^{\perp *}\right). \tag{8}$$

(ii) *If* $x\in\bigwedge^n\mathbb{R}^d$ *is a non-zero decomposable* n*-vector, then* $x^{\perp *}\in\bigwedge^{d-n}\mathbb{R}^d$ *is a non-zero decomposable* $(d-n)$*-vector satisfying* $S(x^{\perp *})=S(x)^{\perp}$. *If* $x\in\bigwedge^R\mathbb{R}^d$ *and* $y\in\bigwedge^L\mathbb{R}^d$ *are non-zero decomposable vectors, then* $\mathrm{Int}(x,y)\in\mathbb{R}^d$ *spans* $S(x)\cap S(y)$, *if* $\mathrm{Dim}(S(x)\cap S(y))=1$, *and equals* 0 *otherwise*.

Proof. (i) The linear application Ort_n maps the canonical orthogonal basis of $\bigwedge^n \mathbb{R}^d$ onto that of $\bigwedge^{d-n} \mathbb{R}^d$, up to signs. Thus Ort_n is an isometry. Next, by linearity we check (8) for any $e_{i,n}$, $i \in I_n$:

$$\begin{aligned}
\left(\bigwedge^n Ae_{i,n}\right)^{\perp *} &= \sum_{j\in I_n} \left(\wedge^n A\right)(j,i)\varepsilon_n(j)e_{j^c,d-n} \\
&= \det A \sum_{j\in I_n} \left(\bigwedge^{d-n} {}^tA^{-1}\right)\left(j^c, i^c\right)\varepsilon_n(i)e_{j^c,d-n} \\
&= \det A\varepsilon_n(i)\left(\bigwedge^{d-n} {}^tA^{-1}\right)e_{i^c,d-n} = \det A\left(\bigwedge^{d-n} {}^tA^{-1}\right)\left(e_{i,n}^{\perp *}\right).
\end{aligned} \tag{9}$$

(ii) If $E \subset \mathbb{R}^d$ is a n-dimensional subspace, choose $A \in GL_d(\mathbb{R})$ in such a way that its columns from 1 to n form an orthonormal basis of E and those from $n+1$ to d an orthonormal basis of $E^\perp$. Since $A^{-1} = {}^tA$, applying (8) proves the first claim. The second one then follows from the remark that for subspaces E and F, one has $E \cap F = (E^\perp + F^\perp)^\perp$. □

2.4. *Choice for V_R and W_R*

Using lemma (2.4), we now make explicit choices for the Lyapunov eigenvectors V_R and W_R. Concerning for instance V_R, we show that it can be defined in such a way that there is a $\lambda_R > 0$ verifying $MV_R = \lambda_R TV_R$. The possibility of choosing $\lambda_R > 0$ is non-obvious, as even a random scalar not necessarily admits a non-negative element in its multiplicative coboundary class. When $\gamma_R(M,T) \neq 0$, this result is also a consequence of Proposition 8.4 in [7].

Introduce the matrices $(-1)^{R-1}\bigwedge^R M$ and $(-1)^{L-1}\bigwedge^L M^{-1}$. Summing up the results of [7]:

Proposition 2.5 (See [7]). (i) *The exponent $\gamma_1(\bigwedge^R M, T)$ is simple. Let $\mathcal{V}_R \in \bigwedge^R \mathbb{R}^d$ and $\alpha_R \in \mathbb{R}_+$ be defined by*:

$$\mathcal{V}_R = \lim_{n\to+\infty} \frac{\mathcal{R}_{-1}(-n,0)}{P_{-1}(-n,0,-)} \quad \text{and} \quad \alpha_R = \frac{1}{P_0(-\infty,1,R)} \lim_{n\to+\infty} \frac{P_0(-n,1,-)}{P_{-1}(-n,0,-)}.$$

Then $(-1)^{R-1}\bigwedge^R M\mathcal{V}_R = \alpha_R T\mathcal{V}_R$ and $\mathcal{V}_R$ has maximal Lyapunov exponent for $(\bigwedge^R M, T)$.

(ii) *The exponent $\gamma_1(\bigwedge^L M^{-1}, T^{-1})$ is simple. Let $\mathcal{V}_L \in \bigwedge^L \mathbb{R}^d$ and $\alpha_L \in \mathbb{R}_+$ be defined by*:

$$\mathcal{V}_L = \lim_{n\to+\infty} \frac{\mathcal{L}_{-1}(-1,n)}{P_0(-1,n,+)}, \qquad \alpha_L = \frac{1}{P_0(-1,+\infty,-L)} \lim_{n\to+\infty} \frac{P_0(-1,n,+)}{P_1(0,n,+)}.$$

Then $(-1)^{L-1}\bigwedge^L M^{-1}T\mathcal{V}_L = \alpha_L\mathcal{V}_L$ and $T\mathcal{V}_L$ has maximal Lyapunov exponent for $(\bigwedge^L M^{-1}, T^{-1})$.

(iii) *Let random vectors $\mathcal{W}_R \in \bigwedge^R \mathbb{R}^d$ with $\|\mathcal{W}_R\| = 1$, $\mathcal{W}_L \in \bigwedge^L \mathbb{R}^d$ with $\|\mathcal{W}_L\| = 1$ and random scalars $\beta_R > 0$, $\beta_L > 0$ be such that*:

$$\begin{cases} (-1)^{R-1}\bigwedge^R\left({}^tM\right)T\mathcal{W}_R = \beta_R\mathcal{W}_R, \\ (-1)^{L-1}\bigwedge^L\left({}^tM\right)^{-1}\mathcal{W}_L = \beta_L T\mathcal{W}_L, \end{cases}$$

and $\mathcal{W}_R$ and $\mathcal{W}_L$ have maximal exponent for $(\bigwedge^R({}^tM), T^{-1})$ and $(\bigwedge^L({}^tM)^{-1}, T)$, respectively.

As mentioned in the introduction, $\mathcal{V}_R$, $\mathcal{V}_L$, $\mathcal{W}_R$ and $\mathcal{W}_L$ can be concretely handled, using respectively the cone contraction properties of the matrices $(-1)^{R-1}\bigwedge^R M$, $(-1)^{L-1}\bigwedge^L M^{-1}$, $(-1)^{R-1}\bigwedge^R({}^tM)$, $(-1)^{L-1}\bigwedge^L({}^tM)^{-1}$, as detailed in [7]. We have the following proposition.

Proposition 2.6. (i) *We have* $S(\mathcal{V}_R) = \mathrm{Vect}(V_1, \dots, V_R)$, $S(\mathcal{V}_L) = \mathrm{Vect}(V_R, \dots, V_d)$, $\mathrm{Vect}(V_R) = S(\mathcal{V}_R) \cap S(\mathcal{V}_L)$ *and* V_R *is uniquely defined in direction. We then define*:

$$V_R = \frac{\mathrm{Int}(\mathcal{V}_R, \mathcal{V}_L)}{\|\mathrm{Int}(\mathcal{V}_R, \mathcal{V}_L)\|} \quad \textit{and} \quad \lambda_R = \frac{p_R}{p_{-L}} \times \frac{\alpha_R}{\alpha_L} \times \frac{\|\mathrm{Int}(T\mathcal{V}_R, T\mathcal{V}_L)\|}{\|\mathrm{Int}(\mathcal{V}_R, \mathcal{V}_L)\|} > 0.$$

Then $MV_R = \lambda_R T V_R$ *and* λ_R *is bounded away from* 0 *and* $+\infty$, *verifying* $\int \log \lambda_R \, \mathrm{d}\mu = \gamma_R(M, T)$.

(ii) *Similarly* $S(\mathcal{W}_R) = \mathrm{Vect}(W_1, \dots, W_R)$, $S(\mathcal{W}_L) = \mathrm{Vect}(W_R, \dots, W_d)$, $\mathrm{Vect}(W_R) = S(\mathcal{W}_R) \cap S(\mathcal{W}_L)$ *and* W_R *is uniquely defined in direction. We then define*:

$$W_R = \frac{\mathrm{Int}(\mathcal{W}_R, \mathcal{W}_L)}{\|\mathrm{Int}(\mathcal{W}_R, \mathcal{W}_L)\|} \quad \textit{and} \quad \rho_R = \frac{p_R}{p_{-L}} \times \frac{\beta_R}{\beta_L} \times \frac{\|\mathrm{Int}(\mathcal{W}_R, \mathcal{W}_L)\|}{\|\mathrm{Int}(T\mathcal{W}_R, T\mathcal{W}_L)\|} > 0.$$

Then $MTW_R = \rho_R W_R$ *and* ρ_R *is bounded away from* 0 *and* $+\infty$, *verifying* $\int \log \rho_R \, \mathrm{d}\mu = \gamma_R(M, T)$.

(iii) *We have* $S(\mathcal{V}_R)^\perp = \mathrm{Vect}(W_{R+1}, \dots, W_d)$, $S(\mathcal{V}_L)^\perp = \mathrm{Vect}(W_1, \dots, W_{R-1})$ *and* $S(\mathcal{W}_R)^\perp = \mathrm{Vect}(V_{R+1}, \dots, V_d)$, $S(\mathcal{W}_L)^\perp = \mathrm{Vect}(V_1, \dots, V_{R-1})$.

Proof. For (i), Proposition 2.5 gives that $\gamma_1(\bigwedge^R M, T)$ is simple. Therefore the direction of $\mathcal{V}_R$ is unique in $\bigwedge^R \mathbb{R}^d$ and the same is true for $\mathcal{V}_L$ in $\bigwedge^L \mathbb{R}^d$. Also (see [16], page 325) we have $S(\mathcal{V}_R) = \mathrm{Vect}(V_1, \dots, V_R)$ and $S(\mathcal{V}_L) = \mathrm{Vect}(V_R, \dots, V_d)$. Therefore $\mathrm{Vect}(V_R) = S(\mathcal{V}_R) \cap S(\mathcal{V}_L)$ and the direction of V_R is then uniquely determined. Next, using repeatedly (8) and Proposition 2.5:

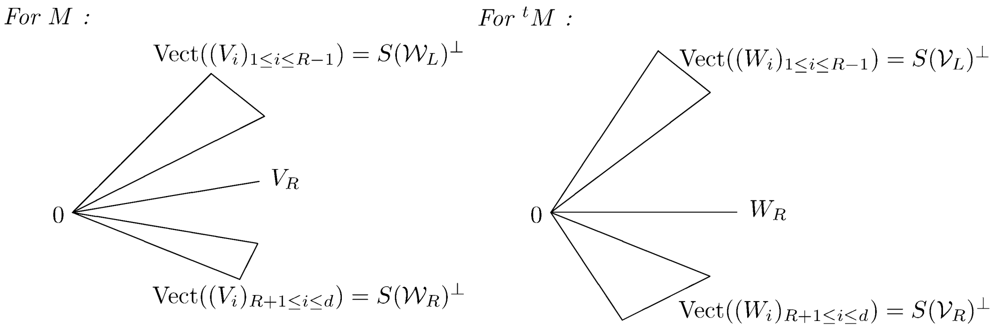
$$\begin{aligned} M\,\mathrm{Int}(\mathcal{V}_R, \mathcal{V}_L) &= M\big[\mathcal{V}_R^{\perp *} \wedge \mathcal{V}_L^{\perp *}\big]^{\perp *} = \det M \Big[\bigwedge^{d-1}\big({}^t M^{-1}\big)\big(\mathcal{V}_R^{\perp *} \wedge \mathcal{V}_L^{\perp *}\big)\Big]^{\perp *} \\ &= \det M \Big[\Big(\bigwedge^{L-1}\big({}^t M^{-1}\big)\mathcal{V}_R^{\perp *}\Big) \wedge \Big(\bigwedge^{R-1}\big({}^t M^{-1}\big)\mathcal{V}_L^{\perp *}\Big)\Big]^{\perp *} \\ &= \frac{1}{\det M}\Big[\Big(\bigwedge^{R} M\mathcal{V}_R\Big)^{\perp *} \wedge \Big(\bigwedge^{L} M\mathcal{V}_L\Big)^{\perp *}\Big]^{\perp *} \\ &= (-1)^{d-1}\frac{p_R}{p_{-L}}(-1)^{(R-1)+(L-1)}\frac{\alpha_R}{\alpha_L}\,\mathrm{Int}(T\mathcal{V}_R, T\mathcal{V}_L). \end{aligned}$$

Since $(-1)^{d-1+R-1+L-1} = 1$, this completes (i). Point (ii) is similar. Next, (iii) is standard, but we include the proof for completeness. Let $1 \le i < j \le d$, with $i \le R \le j$. We show that $V_i \perp W_j$ and $V_j \perp W_i$. Since for $n \ge 0$, one has $I = (T^{-n} M_n) M_{-n}$, we get:

$$\begin{aligned} \langle V_i, W_j \rangle &= \big\langle \big(T^{-n} M_n\big)(M_{-n}) V_i, W_j \big\rangle \\ &= \big\langle M_{-n} V_i, \big({}^t M^{-1}\big)_{-n} W_j \big\rangle = \mathrm{O}\big(\exp\big(-n\big[\gamma_i(M, T) - \gamma_j(M, T) - \eta\big]\big)\big), \quad \text{for all } \eta > 0. \end{aligned}$$

As $\gamma_i(M, T) > \gamma_j(M, T)$, the conclusion follows. The reverse case is similar. □

As a summary and using point (iii) of the previous proposition, we get the following picture for the Lyapunov eigenvectors in $\mathbb{R}^d$:



3. Geometrical properties of Lyapunov eigenvectors

Recall that V_R is seen as spanning the intersection of the subspaces $S(\mathcal{V}_R)$ and $S(\mathcal{V}_L)$ and that W_R is seen similarly. We first compute the minimal cones in their respective vector spaces where lie $\mathcal{W}_R$, $\mathcal{W}_L$, W_R and next $\mathcal{V}_R$, $\mathcal{V}_L$, V_R. In a last part we show non-singularity results.

In the analysis, we need to introduce the class $\mathcal{M}$ of matrices having the same form as M.

Definition 3.1. *Introduce, if $R \geq 2$:*

$$\mathcal{M} = \big\{M(\delta,\eta) \in \mathrm{Mat}_d(\mathbb{R}) \mid \delta = (\delta_i)_{1\le i\le R-1}, \eta = (\eta_j)_{1\le j\le L},\ \textit{with}\ \delta_i \geq 0, \eta_j \geq 0\big\},$$

where $M(\delta,\eta)_{i,j} = 1_{i=j+1}$, for $2 \le i \le d$, and:

$$M(\delta,\eta)_{1,j} = \begin{cases} -(1+\delta_1+\cdots+\delta_j), & 1 \le j \le R-1,\\ \eta_1+\cdots+\eta_{L+R-j}, & R \le j \le d.\end{cases}$$

If $R = 1$, the class $\mathcal{M}$ reduces to $\mathcal{M} = \{M(\eta) \mid \eta = (\eta_j)_{1\le j\le L},\ \textit{with}\ \eta_j \geq 0\}$.

3.1. *Minimal stable geometrical cones for $\mathcal{W}_R$, $\mathcal{W}_L$ and W_R*

The story concerning $\mathcal{W}_R$ and $\mathcal{W}_L$ is contained in [7], Section 3. Changing a little the notations, we have:

Theorem 3.2 (See [7]). *Introduce a set of indices, a set of edges and a cone in $\bigwedge^R \mathbb{R}^d$:*

$$\begin{cases} I_{t,+} = \big\{k = (k_1,\dots,k_R) \mid 0 \le k_j \le L-1, i+k_i \neq j+k_j,\ \textit{if}\ i \neq j\big\}\\ \mathcal{E}_{t,+} = \big\{\zeta_k = \bigwedge_{j=1}^R \big(\sum_{j\le s\le j+k_j} e_s\big) \mid k \in I_{t,+}\big\}\\ \mathcal{C}_{t,+} = \mathrm{Vect}_+(\mathcal{E}_{t,+}).\end{cases}$$

Similarly, define a set of indices, a set of edges and a cone in $\bigwedge^L \mathbb{R}^d$:

$$\begin{cases} I_{t,-} = \big\{l = (l_R,\dots,l_d) \mid 0 \le l_j \le R-1, i-l_i \neq j-l_j, \textit{if}\ i \neq j\big\}\\ \mathcal{E}_{t,-} = \big\{\chi_l = \bigwedge_{j=R}^d \big(\sum_{j-l_j\le s\le j} e_s\big) \mid l \in I_{t,-}\big\}\\ \mathcal{C}_{t,-} = \mathrm{Vect}_+(\mathcal{E}_{t,-}).\end{cases}$$

Then, the set of edges of $\mathcal{C}_{t,+}$ is $\mathcal{E}_{t,+}$. This cone has non-empty interior, is stable under the class $(-1)^{R-1}\bigwedge^R({}^t\mathcal{M})$ and is minimal with respect to this property. Also $\mathcal{W}_R \in \mathcal{C}_{t,+}$ and for some constant $C > 0$, $dist(\mathcal{W}_R, \partial\mathcal{C}_{t,+}) \geq C$. Moreover, denoting by $S_{\bigwedge^R \mathbb{R}^d}$ the unit sphere of $\bigwedge^R \mathbb{R}^d$:

$$\mathcal{W}_R = \lim_{n\to+\infty} (-1)^{n(R-1)} \bigwedge^R \big({}^tM_n\big)\big(\mathcal{C}_{t,+}\big) \cap S_{\bigwedge^R \mathbb{R}^d}, \quad \textit{non-increasingly}. \tag{10}$$

Moreover, the limit is uniform on Ω.

A by-product of the proof of Proposition 3.7 of [7] is the next result:

Proposition 3.3. *There exist iid environments where the direction of $\mathcal{W}_R$ is arbitrary close to that of any element of $\mathcal{E}_{t,+}$ with positive μ-probability, taking $\varepsilon > 0$ small enough (where ε is defined in condition (1)). The same properties hold for $\mathcal{C}_{t,-}$, the class $(-1)^{L-1} \bigwedge^L ({}^t\mathcal{M})^{-1}$ and $\mathcal{W}_L$ in $\bigwedge^L \mathbb{R}^d$.*

Proof. From the proof of point (iii) of Proposition 3.7 of [7], matrices $M^1, \dots, M^R$ in $\mathcal{M}$ satisfying condition (1) can be taken in such a way that uniformly in $U \in \mathcal{C}_{t,+}$ the vector

$$\left((-1)^{R-1} \bigwedge^R ({}^tM^1)\right) \cdots \left((-1)^{R-1} \bigwedge^R ({}^tM^R)\right) U$$

is arbitrary close in direction to that of any edge of $\mathcal{E}_{t,+}$, taking $\varepsilon > 0$ small enough. Using (10) and the fact that the limit is non-increasing, simply choose an independent medium where M is close to each M^i, $1 \le i \le R$, with positive probability. With at least the product of these R probabilities, $\mathcal{W}_R$ is close to the desired edge. □

Considering now W_R, the aim of this section is to prove the following positivity result:

Theorem 3.4. *There exist a constant $C > 0$ and positive random coefficients $(c_{i,j})_{i \le R \le j}$ such that:*

$$W_R = (-1)^{d-1} \sum_{i \le R \le j} c_{i,j} \left(\sum_{i \le s \le j} e_s \right), \quad \text{with } \frac{1}{C} \le c_{i,j} \le C. \tag{11}$$

Moreover, there are iid environments where the random vector W_R is arbitrary close in direction to any $(-1)^{d-1} \sum_{i \le s \le j} e_s$ with positive probability, taking $\varepsilon > 0$ (defined in (1)) small enough.

In view of Proposition 2.6, we need to compute $\operatorname{Int}(\mathcal{W}_R, \mathcal{W}_L)$ and by bilinearity, $\operatorname{Int}(\zeta_k, \chi_l)$, for $(k,l) \in I_{t,+} \times I_{t,-}$. The statement of the result requires the introduction of finite algorithms.

Definition 3.5. (i) *Let $k \in I_{t,+}$. Fixing $1 \le j \le R$, let $j_0 = j$. For $n \ge 0$ and if $j_n + k_{j_n} < R$, set $j_{n+1} = j_n + k_{j_n} + 1$. Define $t_k(j) = j_n + k_{j_n}$, where n is the first index with $j_n + k_{j_n} \ge R$. This defines $t_k : [1, R] \to [R, d]$.*

(ii) *Let $l \in I_{t,-}$. Fixing $R \le j \le d$, let $j_0 = j$. For $n \ge 0$ and if $j_n - l_{j_n} > R$, set $j_{n+1} = j_n - l_{j_n} - 1$. Define $s_l(j) = j_n - l_{j_n}$, where n is the first index with $j_n - l_{j_n} \le R$. This defines $s_l : [R, d] \longmapsto [1, R]$.*

(iii) *For $(k,l) \in I_{t,+} \times I_{t,-}$, set $\varphi_{k,l} = s_l \circ t_k : [1, R] \longmapsto [1, R]$ and $\psi_{k,l} = t_k \circ s_l : [R, d] \longmapsto [R, d]$.*

As a transformation of $[1, R]$, $\varphi_{k,l}$ admits attracting limit cycles in $[1, R]$: any $1 \le i \le R$ ends in a limit cycle under iterations of $\varphi_{k,l}$. The same holds for $\psi_{k,l}$ in $[R, d]$ and the limit cycles of $\varphi_{k,l}$ and $\psi_{k,l}$ are in bijection via t_k and s_l. Let $m_{k,l}$ be the number of attracting limit cycles for $\varphi_{k,l}$ (it is also that of $\psi_{k,l}$). The limit cycle to which R is attracted under iteration of $\varphi_{k,l}$ is denoted by $C_{k,l}$ and its length by $n_{k,l}$.

Remark. *We illustrate via examples various possibilities for $\varphi_{k,l}$ and its limit cycles:*

1. Let $R = 2$, $L = 2$ ($d = 3$), with $k = (1, 1)$ and $l = (1, 0)$. Then $\varphi_{k,l}(1) = 1$, $\varphi_{k,l}(2) = 1$. There is only one cycle, that is $m_{k,l} = 1$.
2. Let $R = 5$, $L = 4$ ($d = 8$), with $k = (2, 3, 1, 2, 2)$ and $l = (2, 4, 2, 4)$. Then $\varphi_{k,l}(1) = 2$, $\varphi_{k,l}(2) = 3$, $\varphi_{k,l}(3) = 5$, $\varphi_{k,l}(4) = 2$, $\varphi_{k,l}(5) = 5$. Then $m_{k,l} = 1$.
3. Let $R = 7$, $L = 5$ ($d = 11$), with $k = (4, 1, 1, 4, 3, 3, 0)$ and $l = (0, 3, 6, 2, 5)$. Then $\varphi_{k,l}(1) = 3$, $\varphi_{k,l}(2) = 5$, $\varphi_{k,l}(3) = 5$, $\varphi_{k,l}(4) = 5$, $\varphi_{k,l}(5) = 5$, $\varphi_{k,l}(6) = 3$, $\varphi_{k,l}(7) = 7$. Then $m_{k,l} = 2$.

We have the following result:

Theorem 3.6. (i) *Let $(k,l) \in I_{t,+} \times I_{t,-}$. Then the coordinates of* $\operatorname{Int}(\zeta_k, \chi_l)$ *are*:

$$\operatorname{Int}(\zeta_k, \chi_l) = (-1)^{d-1} 1_{m_{k,l}=1} \times \begin{pmatrix} [\#(C_{k,l} \cap [1, i])]_{1 \le i \le R} \\ [\#(t_k(C_{k,l}) \cap [i, d])]_{R+1 \le i \le d} \end{pmatrix} \in \mathbb{R}^d.$$

(ii) *The edges of* $\mathrm{Vect}_+\{\mathrm{Int}(\zeta_k,\chi_l) \mid (k,l) \in I_{t,+} \times I_{t,-}\}$ *are* $(-1)^{d-1}\{\sum_{s=i}^{j} e_s \mid i \le R \le j\}$.

Proof. *Step* 1: Let $(k,l) \in I_{t,+} \times I_{t,-}$ and define P and Q in $GL_d(\mathbb{R})$ by:

$$P_{i,j} = \begin{cases} 1_{j \le i \le j+k_j}, & 1 \le j \le R, \\ 1_{i=j}, & R+1 \le j \le d, \end{cases} \quad \text{and} \quad Q_{i,j} = \begin{cases} 1_{i=j}, & 1 \le j \le R-1, \\ 1_{j-l_j \le i \le j}, & R \le j \le d. \end{cases}$$

From (8) and $\varepsilon_L((R,\dots,d)) = (-1)^{(R-1)(d-1)} = (-1)^{(R-1)L}$, we get:

$$\zeta_k^{\perp *} = \left(\bigwedge^{L-1} {}^tP^{-1}\right)\left(\bigwedge_{i=R+1}^{d} e_i\right) \quad \text{and} \quad \chi_l^{\perp *} = (-1)^{(R-1)L}\left(\bigwedge^{R-1} {}^tQ^{-1}\right)\left(\bigwedge_{i=1}^{R-1} e_i\right).$$

Introduce next:

$$\begin{aligned} \mathcal{O} &= \left(\left[{}^tP^{-1}\right]_{\mathrm{col}:R+1,\dots,d}, \left[{}^tQ^{-1}\right]_{\mathrm{col}:1,\dots,R-1}\right) \\ &= \begin{pmatrix} A_1 & I_{R-1} \\ A_2 & B_2 \\ I_{L-1} & B_1 \end{pmatrix} \in \mathrm{Mat}_{d,d-1}(\mathbb{R}), \end{aligned}$$

with $A_1 \in \mathrm{Mat}_{R-1,L-1}(\mathbb{R})$, $A_2 \in \mathrm{Mat}_{1,L-1}(\mathbb{R})$ and $B_2 \in \mathrm{Mat}_{1,R-1}(\mathbb{R})$, $B_1 \in \mathrm{Mat}_{L-1,R-1}(\mathbb{R})$. Denote by $\mathcal{O}_{*w}$ the submatrix without line w. Then:

$$\zeta_k^{\perp *} \wedge \chi_l^{\perp *} = (-1)^{(R-1)L} \sum_{w=1}^{d} \det(\mathcal{O}_{*w})\left(\bigwedge_{i \in [1,d]\setminus\{w\}} e_i\right).$$

Finally:

$$\mathrm{Int}(\zeta_k,\chi_l) = (-1)^{(R-1)L+d} \sum_{w=1}^{d} (-1)^w \det(\mathcal{O}_{*w}) e_w. \tag{12}$$

Step 2: We precise P^{-1} and Q^{-1}. Let $R+1 \le i \le d$ and $1 \le j \le R$. Then, obviously:

$$\sum_{s=j}^{(j+k_j)\wedge R} \left(P^{-1}\right)_{i,s} + 1_{i \le j+k_j} = 0. \tag{13}$$

Adding repeatedly terms like (13), Definition 3.5 provides $\sum_{s=j}^{R} (P^{-1})_{i,s} = -1_{i \le t_k(j)}$. Therefore:

$$\left(P^{-1}\right)_{i,j} = \begin{cases} -1_{i \le t_k(j)} + 1_{i \le t_k(j+1)}, & 1 \le j < R, \\ -1_{i \le t_k(R)}, & j = R. \end{cases}$$

Similarly, for $1 \le i \le R-1$ and $R \le j \le d$:

$$\left(Q^{-1}\right)_{i,j} = \begin{cases} -1_{i \ge s_l(j)} + 1_{i \ge s_l(j-1)}, & R < j \le d \\ -1_{i \ge s_l(R)}, & j = R. \end{cases}$$

Step 3: Fixing $1 \le w \le R$, we compute $\det(\mathcal{O}_{*w})$. First, column operations give:

$$\det(\mathcal{O}_{*w}) = (-1)^{d(L-1)} \det\begin{pmatrix} I_{R-1} - A_1B_1 \\ B_2 - A_2B_1 \end{pmatrix}_{*w}. \tag{14}$$

Next, for $1 \le i, j \le R-1$:

$$
\begin{aligned}
(A_1 B_1)_{i,j} &= \sum_{u=R+1}^{d} \left({}^t P^{-1}\right)_{i,u} \left({}^t Q^{-1}\right)_{u,j} \\
&= -\sum_{u=R+1}^{t_k(i)} (-1_{j \ge s_l(u)} + 1_{j \ge s_l(u-1)}) + \sum_{u=R+1}^{t_k(i+1)} (-1_{j \ge s_l(u)} + 1_{j \ge s_l(u-1)}) \\
&= 1_{j \ge \varphi_{k,l}(i)} - 1_{j \ge \varphi_{k,l}(i+1)}.
\end{aligned}
$$

Similarly, for $1 \le j \le R-1$: $(B_2 - A_2 B_1)(j) = ({}^t P^{-1})_{R,j} - \sum_{u=R+1}^{d} ({}^t Q^{-1})_{u,j} = -1_{j \ge \varphi_{k,l(R)}}$. Introduce $X_j \in \mathbb{R}^R$, with $X_j(i) = 1_{i \ge j}$. Thus $X_{R+1} = 0$. Set also $X_{\varphi_{k,l}(R+1)} = 0$. Transposing in (14), we obtain:

$$
\begin{aligned}
\det(\mathcal{O}_{*w}) &= (-1)^{d(L-1)+R+w} \bigwedge_{j=1, j \ne w}^{R} \left(X_j - X_{j+1} - (X_{\varphi_{k,l}(j)} - X_{\varphi_{k,l}(j+1)})\right) \bigwedge_{w} e_R \\
&= (-1)^{d(L-1)+R+w} \sum_{1 \le j \le w < j' \le R+1} (-1)^{w-j+R+1-j'} Y\left(j, j'\right),
\end{aligned}
\tag{15}
$$

with $Y(j, j') = \bigwedge_{s=1, s \ne j}^{w} (X_s - X_{\varphi_{k,l}(s)}) \bigwedge_w e_R \bigwedge_{s=w+1, s \ne j'}^{R+1} (X_s - X_{\varphi_{k,l}(s)})$. Observe that $Y(j, j') = 0$, if $j' < R+1$. Then:

$$
\begin{aligned}
Y(j, R+1) &= \bigwedge_{s=1, s \ne j}^{w} (X_s - X_{\varphi_{k,l}(s)}) \bigwedge_{w} e_R \bigwedge_{s=w+1}^{R} (X_s - X_{\varphi_{k,l}(s)}) \\
&= \sum_{t=0}^{R-1} (-1)^t \sum_{u=[1,R] \setminus v \setminus \{j\}\, v \in I_t, v \subset [1,R] \setminus \{j\}} (-1)^{w-j} \bigwedge_{s \in v} X_{\varphi_{k,l}(s)} \bigwedge_{s' \in u} X_{s'} \bigwedge_{j} X_R.
\end{aligned}
$$

Non-zero contributing subsets v in the right-hand side check $R \in v \cup \{j\}$ and $\varphi_{k,l}(v) = (v \cup \{j\}) \setminus \{R\}$. In particular $\varphi_{k,l}$ is injective on v. If $j = R$, then $\varphi_{k,l}$ is a bijection of v and thus v is any union of limit cycles for $\varphi_{k,l}$ that do not contain j. If $j < R$, then $R \in v$ and v is the union of a sequence $\{R, \varphi_{k,l}(R), \ldots, (\varphi_{k,l})^{s-1}(R)\}$, with $j = (\varphi_{k,l})^s(R)$, for a smallest $s \ge 1$, and any collection of limit cycles that do not contain j. Let m_j be the number of limit cycles that do not contain j and $(C_{s,j})_{1 \le s \le m_j}$ be these cycles. Write $\mathrm{Orb}(R)$ for the orbit of R under $\varphi_{k,l}$. Then:

$$
\begin{aligned}
Y(j, R+1) &= (-1)^{w-j} \left(\sum_{q=0}^{m_j} \sum_{(C_{i_h,j})_{1 \le h \le q}} (-1)^{(\sum_{1 \le h \le q} \# C_{i_h,j}) + (\sum_{1 \le h \le q} (\# C_{i_h,j} - 1))} \right) \\
&\quad \times \left(\sum_{s \ge 0} 1_{j = \varphi_{k,l}^s(R);\, j \ne \varphi_{k,l}^p(R), p < s} (-1)^s (-1)^s \right), \\
&= (-1)^{w-j} 1_{m_j = 0} 1_{j \in \mathrm{Orb}(R)} = (-1)^{w-j} 1_{m_{k,l} = 1, j \in C_{k,l}}.
\end{aligned}
$$

Using (15), we obtain:

$$
\det(\mathcal{O}_{*w}) = (-1)^{d(L-1)+R+w} \sum_{1 \le j \le w} 1_{m_{k,l} = 1, j \in C_{k,l}}.
$$

Since $(-1)^{d(L-1)+R+w}(-1)^{(R-1)L+d+w} = (-1)^{d-1}$, the coefficient of $\mathrm{Int}(\zeta_k, \chi_l)$ in (12) with respect to e_w is $(-1)^{d-1} 1_{m_{k,l}=1} \#(C_{k,l} \cap [1, w])$. The conclusion therefore holds for $1 \le w \le R$. The result for $R \le w \le d$ is proved similarly, using the limit cycle defined by $\psi_{k,l}$, namely $t_k(C_{k,l})$. This concludes the proof of (i).

Step 4: We prove (ii). Consider $(k,l) \in I_{t,+} \times I_{t,-}$. For $1 \le i \le n_{k,l}$, let:

$$\begin{cases} \alpha_i = \min\big\{1 \le s \le R \mid \#\big([1,s] \cap C_{k,l}\big) = i\big\}, \\ \beta_i = \max\big\{R \le s \le d \mid \#\big([s,d] \cap t_k(C_{k,l})\big) = i\big\}. \end{cases}$$

Then:

$$\mathrm{Int}(\zeta_k, \chi_l) = (-1)^{d-1} 1_{m_{k,l}=1} \sum_{i=1}^{n_{k,l}} \left(\sum_{j=\alpha_i}^{\beta_i} e_j \right) \in (-1)^{d-1} \mathrm{Vect}_+ \left(\sum_{i \le s \le j} e_s \,\Big|\, i \le R \le j \right).$$

Next, observe that all $\sum_{s=i}^{j} e_s$, with $i \le R \le j$, are extremal in the cone they generate. We now verify that such a vector is some $(-1)^{d-1} \mathrm{Int}(\zeta_k, \chi_l)$. Indeed, take $k = (0, \dots, 0, j-R)$ and $l = (R-i, 0, \dots, 0)$. Then $\varphi_{k,l}(z) = i$, for $1 \le z \le R$. Thus $t_k(i) = j$, $m_{k,l} = 1$ and $(-1)^{d-1} \mathrm{Int}(\zeta_k, \chi_l) = \sum_{i \le s \le j} e_s$. This concludes the proof of the theorem. □

We can now prove Theorem 3.4 on W_R.

Proof of Theorem 3.4. From Theorem 3.2, there is a constant $C > 0$ such that $\mathcal{W}_R$ and $\mathcal{W}_L$ can be written as:

$$\mathcal{W}_R = \sum_{k \in I_{t,+}} \alpha_k \zeta_k \quad \text{and} \quad \mathcal{W}_L = \sum_{l \in I_{t,-}} \beta_l \chi_l, \quad \text{with } \frac{1}{C} \le \alpha_k,\ \beta_l \le C.$$

Bilinearity of Int provides $\mathrm{Int}(\mathcal{W}_R, \mathcal{W}_L) = \sum_{k \in I_{t,+}, l \in I_{t,-}} \alpha_k \beta_l \,\mathrm{Int}(\zeta_k, \chi_l)$. The definition of W_R (Proposition 2.6) and Theorem 3.6 give (11).

Finally, recall from Theorem 3.2 that $\mathcal{W}_R$ is built only via the matrices $(T^k M)_{k \ge 0}$ and $\mathcal{W}_L$ using only $(T^k M)_{k \le -1}$. Since taking $\varepsilon > 0$ small enough, there exist iid environments where $\mathcal{W}_R$ and $\mathcal{W}_L$ are close in direction respectively to any ζ_k and any χ_l with positive probability, there also exists an iid environment where $\mathrm{Int}(\mathcal{W}_R, \mathcal{W}_L)$ is arbitrary close in direction to any non-zero $\mathrm{Int}(\zeta_k, \chi_l)$, with positive probability. Since any $(-1)^{d-1} \sum_{i \le s \le j} e_s$ is of this form, this concludes the proof of the theorem. □

3.2. *Minimal stable geometrical cones for $\mathcal{V}_R$ and $\mathcal{V}_L$*

We next turn to $\mathcal{V}_R$ and $\mathcal{V}_L$ and determine the minimal stable cones where respectively lie these decomposable vectors, focusing on $\mathcal{V}_R$.

It was shown in [7], Proposition 6.2, that $\mathcal{V}_R$ belongs to the algebraic dual cone $(\mathcal{C}_{t,+})^*$ of $\mathcal{C}_{t,+}$ and that this cone is stable under the linear action of the class $(-1)^{R-1} \wedge^R \mathcal{M}$. However the following study reveals that $(\mathcal{C}_{t,+})^*$ is not minimal for this property, for instance as soon as $\min\{R, L\} \ge 2$. We exhibit below the minimal stable cone, which is intimately related to the mechanism of the random walk. The description of such a cone is required when studying the geometrical properties of V_R. Mention that the key point in this section is Lemma 3.10.

We first make a change of basis for the matrix M which eases the study of $(-1)^{R-1} \bigwedge^R \mathcal{M}$. Mention that it is dissymmetric in L and R and that another one is natural when considering the matrix $(-1)^{L-1} \bigwedge^L \mathcal{M}^{-1}$.

Definition 3.7. (i) *Define $U \in GL_d(\mathbb{R})$ by $U_{i,j} = 1_{i \le j \le R-1}$, for $1 \le i \le R-1$ and $U_{i,j} = 1_{i \ge j \ge R}$, for $R \le i \le d$. If $R = 1$, only the second part remains. Define then the class $\mathcal{M}' = U \mathcal{M} U^{-1}$. If $M(\delta, \eta) \in \mathcal{M}$, the matrix $M'(\delta, \eta) = U M(\delta, \eta) U^{-1}$ is*:

$$M'(\delta, \eta)_{i,j} = \begin{cases} -\delta_j, & i = 1, 1 \le j \le R-2, \\ -(1 + \delta_{R-1}), & i = 1, j = R-1, \\ \eta_{R+L-j}, & i = 1, R \le j \le d, \\ 1_{i=j+1}, & 2 \le i \le d, j \ne R-1, \\ 1_{i \ge R} - 1_{i \le R-1}, & 2 \le i \le d, j = R-1. \end{cases}$$

(ii) *For integers $a < k < b$, set $\mathcal{R}'_k(a, b) = (-1)^R \bigwedge^R U \mathcal{R}_k(a, b)$.*

An essential remark, already pointed out in the introduction, is that for $a \le k$ and $b \ge k+1$, then the direction of $\mathcal{R}'_k(a,b)$ is independent on b (see Lemma 5.2 in [7]). We shall then focus on $\mathcal{R}'_0(a,1)$, $a \le 0$. Observe that, due to the change of basis, $\mathcal{R}'_0(a,1)$ can be written as:

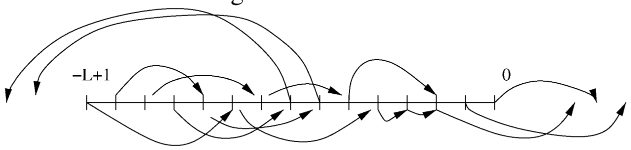
$$\mathcal{R}'_0(a,1) = \left(\bigwedge_{j=1}^{R-1} \left[e_j - \sum_{i=R}^{d} e_i \, p_{-R-i}(a,1,R+1-j) \right] \right) \wedge \left(\sum_{i=R}^{d} e_i \, p_{-R-i}(a,1,-) \right). \tag{16}$$

When $a \le 0$, we will show that $\mathcal{R}'_0(a,1)$ belongs to an explicit polyhedral minimal cone, whose edges are indexed by "left-extremal boxes" of length L, namely graphs in $[-L+1,\dots,0]$ built with deterministic transitions at each site of $[-L+1,0]$. We give the definition below.

Definition 3.8. (i) *A left-extremal box B is a graph obtained by choosing at each site of $[-L+1,0]$ a transition among $\{-L\} \cup \{+1,\dots,+R\}$. Each path leaves $[-L+1,0]$ in $(-\infty,-L]$ or in $[1,+\infty)$. Let $I_j(B) \subset [-L+1,0]$ be the subset of sites i, such that starting at i and following the graph, the exit is at j, $1 \le j \le R$, and $I_-(B)$ be the subset of sites where the exit is on the left-hand side. The set of left-extremal boxes is denoted by $\mathcal{B}_L$.*

(ii) *A right-extremal box B is defined similarly as a graph in $[1,R]$ resulting from the choice at each site of a transition among $\{-L,\dots,-1\} \cup \{+R\}$. Denote by $J_j(B)$, $0 \le j \le L-1$, and $J_+(B)$ the exit sets. The set of right-extremal boxes is written as $\mathcal{B}_R$.*

An example of left extremal box is the following one:



We next introduce families of edges and cones.

Definition 3.9. (i) *Let $\mathcal{P}_+ = \{(R - I_R(B),\dots,R - I_2(B), R - I_-(B)) \mid B \in \mathcal{B}_L, I_-(B) \ne \emptyset\}$ and define $\mathcal{E}'_+ \subset \mathcal{E}'_{+,1} \subset \mathcal{E}'_{+,2}$ as:*

$$\left\{ \bigwedge_{j=1}^{R-1} \left(e_j - \sum_{i \in I_j} e_i \right) \wedge \sum_{i \in I_-} e_i \;\middle|\; (*) \right\},$$

where $(*)$ is:

$$\begin{cases} (I_1,\dots,I_{R-1},I_-) \in \mathcal{P}_+, & \text{for } \mathcal{E}'_{+,1} \text{ and } \mathcal{E}'_+ \text{ if } R=1, \\ (I_1,\dots,I_{R-1},I_-) \in \mathcal{P}_+, \#I_- = 1, & \text{for } \mathcal{E}'_+, \text{ if } R \ge 2, \\ (I_1,\dots,I_{R-1},I_-) \text{ disjoint subsets of } [R,d], I_- \ne \emptyset, & \text{for } \mathcal{E}'_{+,2}. \end{cases}$$

Define then $\mathcal{C}'_+ = \mathrm{Vect}_+(\mathcal{E}'_+)$, $\mathcal{C}'_{+,1} = \mathrm{Vect}_+(\mathcal{E}'_{+,1})$ and $\mathcal{C}'_{+,2} = \mathrm{Vect}_+(\mathcal{E}'_{+,2})$. Finally, let $\mathcal{E}_+ = \bigwedge^R U^{-1} \mathcal{E}'_+$ and $\mathcal{C}_+ = \mathrm{Vect}_+(\mathcal{E}_+)$. Precisely, if $R \ge 2$ (omitting the last condition if $R=1$):

$$\mathcal{E}_+ = \left\{ \bigwedge_{j=1}^{R-1} \left(e_j - e_{j-1} - \sum_{i \in I_j} (e_i - e_{i+1}) \right) \wedge \sum_{i \in I_-} (e_i - e_{i+1}) \mid (I_1,\dots,I_{R-1},I_-) \in \mathcal{P}_+, \#I_- = 1 \right\}.$$

(ii) *Set $\mathcal{P}_- = \{(J_+(B), J_{-1}(B),\dots,J_{-L+1}(B)) \mid B \in \mathcal{B}_R, J_+(B) \ne \emptyset\}$ and $\mathcal{C}_- = \mathrm{Vect}_+(\mathcal{E}_-)$, where if $L \ge 2$ (omitting the last condition if $L=1$):*

$$\mathcal{E}_- = \left\{ \left(\sum_{i \in J_+} (e_i - e_{i-1}) \right) \bigwedge_{j=R+1}^{d} \left(e_j - e_{j+1} - \sum_{i \in J_j} (e_i - e_{i-1}) \right) \;\middle|\; (J_+, J_{R+1},\dots,J_d) \in \mathcal{P}_-, \#J_+ = 1 \right\}.$$

The following lemma details the linear action of a matrix in $(-1)^{R-1}\bigwedge^R \mathcal{M}'$ on decomposable R-vectors of $\bigwedge^R \mathbb{R}^d$ having "the same form" as $\mathcal{R}'_0(a,1)$.

Lemma 3.10. *Let* $\mathcal{A}=\{A=(\alpha_{i,j})_{R\le i\le d, j\in\{1,\dots,R-1,-\}}\mid \alpha_{i,j}\ge 0, \sum_{j\in\{1,\dots,R-1,-\}}\alpha_{i,j}\le 1, \textit{for } R\le i\le d\}$. *Set*:

$$Z(A)=\left(\bigwedge_{j=1}^{R-1}\left[e_j-\sum_{l=R}^{d}e_l\alpha_{l,j}\right]\right)\wedge\left(\sum_{l=R}^{d}e_l\alpha_{l,-}\right),\quad A\in\mathcal{A}.$$

Let $A\in\mathcal{A}$ *and* $M'(\delta,\eta)\in\mathcal{M}'$, *with* $\delta=(\delta_i)_{1\le i\le R-1}$ *and* $\eta=(\eta_j)_{1\le j\le L}$. *Then*:

$$(-1)^{R-1}\bigwedge^R M'(\delta,\eta)Z(A)=\sum_{j=1}^{R-1}\left(\delta_j+\sum_{l=R}^{d}\eta_{L+R-l}\alpha_{l,j}\right)Z(A_j)+\left(\sum_{l=R}^{d}\eta_{R+L-l}\alpha_{l,-}\right)Z(A_-)+Z(A_0),$$

where A_- *and* $(A_j)_{0\le j\le R-1}$ *are defined by*:

(i) *For* $1\le j\le R-2$, *where the singular column is at place* $j+1$:

$$A_j=\begin{pmatrix}0&0&0&\cdots&1&\cdots&0&0\\ \vdots&\alpha_{R,1}&\alpha_{R,2}&\cdots&(1-\sum_{s\neq j}\alpha_{R,s})&\cdots&\alpha_{R,R-2}&\alpha_{R,-}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 0&\alpha_{d-1,1}&\alpha_{d-1,2}&\cdots&(1-\sum_{s\neq j}\alpha_{d-1,s})&\cdots&\alpha_{d-1,R-2}&\alpha_{d-1,-}\end{pmatrix}.$$

(ii)

$$A_{R-1}=\begin{pmatrix}0&0&\cdots&0&0\\ \vdots&\alpha_{R,1}&\cdots&\alpha_{R,R-2}&\alpha_{R,-}\\ \vdots&\vdots&\vdots&\vdots&\vdots\\ 0&\alpha_{d-1,1}&\cdots&\alpha_{d-1,R-2}&\alpha_{d-1,-}\end{pmatrix}.$$

(iii)

$$A_-=\begin{pmatrix}0&0&\cdots&0&1\\ \vdots&\alpha_{R,1}&\cdots&\alpha_{R,R-2}&(1-\sum_{s\neq -}\alpha_{R,s})\\ \vdots&\vdots&\vdots&\vdots&\vdots\\ 0&\alpha_{d-1,1}&\cdots&\alpha_{d-1,R-2}&(1-\sum_{s\neq -}\alpha_{d-1,s})\end{pmatrix}.$$

(iv)

$$A_0=\begin{pmatrix}1&0&\cdots&0&0\\ (1-\sum_s\alpha_{R,s})&\alpha_{R,1}&\cdots&\alpha_{R,R-2}&\alpha_{R,-}\\ \vdots&\vdots&\vdots&\vdots&\vdots\\ (1-\sum_s\alpha_{d-1,s})&\alpha_{d-1,1}&\cdots&\alpha_{d-1,R-2}&\alpha_{d-1,-}\end{pmatrix}.$$

Proof. Recall the expression for $M'(\delta,\eta)$ (Definition 3.7). With $A=(\alpha_{i,j})_{R\le i\le d, j\in\{1,\dots,R-1,-\}}$, we get:

$$\begin{aligned}\bigwedge^R M'(\delta,\eta)Z(A)&=\bigwedge^R M'(\delta,\eta)\left(\bigwedge_{j=1}^{R-1}\left[e_j-\sum_{l=R}^{d}e_l\alpha_{l,j}\right]\right)\wedge\left(\sum_{l=R}^{d}e_l\alpha_{l,-}\right)\\ &=\left(\bigwedge_{j=1}^{R-2}\left[-\delta_je_1+e_{j+1}-\sum_{l=R}^{d}(\eta_{R+L-l}e_1+e_{l+1})\alpha_{l,j}\right]\right)\end{aligned}$$

$$\wedge\left(-(\delta_{R-1}+1)e_1-\sum_{k=2}^{R-1}e_k+\sum_{l=R}^{d}e_l-\sum_{l=R}^{d}(\eta_{R+L-l}e_1+e_{l+1})\alpha_{l,R-1}\right)$$
$$\wedge\left(\sum_{l=R}^{d}(\eta_{R+L-l}e_1+e_{l+1})\alpha_{l,-}\right).$$

Therefore:

$$\bigwedge^{R}M'(\delta,\eta)Z(A)=\sum_{j=1}^{R-2}\left(-\delta_j-\sum_{l=R}^{d}\eta_{L+R-l}\alpha_{l,j}\right)F_j+\left(-\delta_{R-1}-\sum_{l=R}^{d}\eta_{L+R-l}\alpha_{l,R-1}\right)F_{R-1}$$
$$+\left(\sum_{l=R}^{d}\eta_{R+L-l}\alpha_{l,-}\right)F_-+F_0,$$

where we now detail each F_j, F_{R-1}, F_0 and F_-:

1. For $1\le j\le R-2$:

$$F_j=\left(\bigwedge_{i=1}^{j-1}\left[e_{i+1}-\sum_{l=R}^{d}\alpha_{l,i}e_{l+1}\right]\right)\wedge e_1\left(\bigwedge_{i=j+1}^{R-2}\left[e_{i+1}-\sum_{l=R}^{d}\alpha_{l,i}e_{l+1}\right]\right)$$
$$\wedge\left(-\sum_{k=1}^{R-1}e_k+\sum_{l=R}^{d}e_l-\sum_{l=R}^{d}\alpha_{l,R-1}e_{l+1}\right)\wedge\left(\sum_{l=R}^{d}\alpha_{l,-}e_{l+1}\right)$$
$$=(-1)^Re_1\left(\bigwedge_{i=1}^{j-1}\left[e_{i+1}-\sum_{l=R}^{d}\alpha_{l,i}e_{l+1}\right]\right)\wedge\left(e_{j+1}-\sum_{l=R}^{d}e_l+\sum_{l=R}^{d}\left(\sum_{s\neq j}\alpha_{l,s}\right)e_{l+1}\right)$$
$$\wedge\left(\bigwedge_{i=j+1}^{R-2}\left[e_{i+1}-\sum_{l=R}^{d}\alpha_{l,i}e_{l+1}\right]\right)\wedge\left(\sum_{l=R}^{d}\alpha_{l,-}e_{l+1}\right).$$

2.

$$F_{R-1}=\left(\bigwedge_{i=1}^{R-2}\left[e_{i+1}-\sum_{l=R}^{d}\alpha_{l,i}e_{l+1}\right]\right)\wedge e_1\wedge\left(\sum_{l=R}^{d}\alpha_{l,-}e_{l+1}\right)$$
$$=(-1)^{R-2}e_1\left(\bigwedge_{i=1}^{R-2}\left[e_{i+1}-\sum_{l=R}^{d}\alpha_{l,i}e_{l+1}\right]\right)\wedge\left(\sum_{l=R}^{d}\alpha_{l,-}e_{l+1}\right).$$

3.

$$F_-=\left(\bigwedge_{i=1}^{R-2}\left[e_{i+1}-\sum_{l=R}^{d}\alpha_{l,i}e_{l+1}\right]\right)\wedge\left(-\sum_{k=2}^{R-1}e_k+\sum_{l=R}^{d}e_l-\sum_{l=R}^{d}\alpha_{l,R-1}e_{l+1}\right)\wedge e_1$$
$$=(-1)^{R-1}e_1\left(\bigwedge_{i=1}^{R-2}\left[e_{i+1}-\sum_{l=R}^{d}\alpha_{l,i}e_{l+1}\right]\right)\wedge\left(\sum_{l=R}^{d}e_l-\sum_{l=R}^{d}\left(\sum_{s\neq -}\alpha_{l,s}\right)e_{l+1}\right).$$

4.

$$F_0=\left(\bigwedge_{j=1}^{R-2}\left[e_{j+1}-\sum_{l=R}^{d}e_{l+1}\alpha_{l,j}\right]\right)\wedge\left(-e_1-\sum_{k=2}^{R-1}e_k+\sum_{l=R}^{d}e_l-\sum_{l=R}^{d}e_{l+1}\alpha_{l,R-1}\right)\wedge\left(\sum_{l=R}^{d}e_{l+1}\alpha_{l,-}\right)$$

$$= (-1)^{R-1}\left(e_1 - \sum_{l=R}^{d} e_l + \sum_{l=R}^{d}\left(\sum_{s}\alpha_{l,s}\right)e_{l+1}\right)\left(\bigwedge_{j=1}^{R-2}\left[e_{j+1} - \sum_{l=R}^{d} e_{l+1}\alpha_{l,j}\right]\right) \wedge \left(\sum_{l=R}^{d} e_{l+1}\alpha_{l,-}\right).$$

This concludes the proof of the lemma. □

We next detail the geometrical properties of the cones introduced in Definition 3.9.

Proposition 3.11. (i) *Let $A \in \mathcal{A}$ and $(M'^i)_{1\le i\le L}$ be in $\mathcal{M}'$. Then $(-1)^{(R-1)L} \bigwedge^R (M'^L \cdots M'^1) Z(A) \in \mathcal{C}'_{+,1}$. In particular, $(-1)^{(R-1)L} \bigwedge^R (M'^L \cdots M'^1)(\mathcal{C}'_{+,2}) \subset \mathcal{C}'_{+,1}$.*

(ii) *The cone $\mathcal{C}_{+,1}$ is stable under $(-1)^{R-1} \bigwedge^R \mathcal{M}$ and is minimal for this property.*

(iii) *The cone $\mathcal{C}_{+,1}$ has non-empty interior. Any cone in $\bigwedge^R \mathbb{R}^d$ stable under $(-1)^{R-1} \bigwedge^R \mathcal{M}$ and with non-empty interior contains either $\mathcal{C}_{+,1}$ or $(-\mathcal{C}_{+,1})$.*

(iv) *One has $\mathcal{C}'_+ = \mathcal{C}'_{+,1}$. The edges of $\mathcal{C}_+$ are the elements of $\mathcal{E}_+$.*

(v) *One has $\mathcal{C}'_+ \subset \mathcal{C}'_{+,2}$, with equality if and only if $L = 1$.*

(vi) *Introduce the cone*:

$$\mathcal{D} = \mathrm{Vect}_+\left\{ \bigwedge_{1\le j\le R}\left(\sum_{j\le i\le j+k_j} e_i\right) \,\middle|\, k_j \ge 0, i + k_i \ne j + k_j, \quad \text{for } i \ne j\right\} \subset \bigwedge^R \mathbb{R}^d.$$

Then $\mathcal{C}_+ \subset (\mathcal{D})^*$. In particular $\mathcal{C}_+ \subset (\mathcal{C}_{t,+})^*$ and equality holds if and only if $L = 1$.

Proof. (i) Let $A \in \mathcal{A}$ and $M'(\delta, \eta) \in \mathcal{M}'$, with $\delta = (\delta_i)_{1\le i\le R-1}$ and $\eta = (\eta_j)_{1\le j\le L}$. As a first step, we interpret the multiplication $(-1)^{R-1} \bigwedge^R M'(\delta, \eta) \times Z(A)$ in terms of an evolution on a graph. Define, as in Definition 3.8 and for each $1 \le l \le L$, an extremal box in $[-l+1, 0]$ as a graph resulting from the choice at each site of a transition among $\{-L\} \cup \{+1, \ldots, +R\}$. Recall also Definitions 1.2 and 3.7, and expression (16) for $\mathcal{R}'_0(a, 1)$, $a \le 0$.

We shall prove by induction on $1 \le l \le L$: *"The lines from R to $R + l - 1$ of any decomposable R-vector obtained from $Z(A)$ (when written as a matrix as in Lemma 3.10) by successive applications of Lemma 3.10 with $(-1)^{R-1} \bigwedge^R M'^1$, …, $(-1)^{R-1} \bigwedge^R M'^l$ are the ones of some $\mathcal{R}'_0(-L, 1)$, for some extremal box in $[-l+1, 0]$."*

Recall that the exit sets of a left-extremal box intervening in Definition 3.9 are indexed by $\{2, \ldots, R, -\}$. Then:

1. Let $l = 1$. Considering line R of $Z(A_j)$, it corresponds to an extremal box in $\{0\}$, if choosing the transition at 0 to be to $R - j$, if $0 \le j \le R$, and to $-L$, if $j = -$. One then completes arbitrarily the medium in $[-L+1, 0]$, so that $Z(A_j)$ is some $\mathcal{R}'_0(-L, 1)$.
2. Passage from l to $l + 1$. Start with some $Z(A)$, whose lines from R to $R + l - 1$ correspond to an extremal box in $[-l+1, 0]$, and apply $(-1)^{R-1} \bigwedge^R M'^{l+1}$. Using Lemma 3.10, one gets a positive linear sum of $Z(A_j)$. Fixing j, let us check that lines from R to $R + l$ correspond to an extremal box in $[-l, 0]$, choosing at 0 the same transitions as in the case $l = 1$ and shifting the medium in $[-l+1, 0]$ to $[-l, -1]$. The case of line R is clear (as above). Next:

- If starting from a departure point in $[-l+1, 0]$ and following the graph, the exit was at u, with $3 \le u \le R$, then in the new medium in $[-l, 0]$ the exit is at $2 \le u - 1 \le R - 1$. This corresponds to a down-shift and a right-shift in A_j with respect to A. The case when the path ended with a final jump of $-L$ is treated in the same way.
- If the exit was at 2, the new exit is at 1. Such a departure site does not appear in $Z(A_j)$ and it corresponds to the fact that the column $(\alpha_{i,R-1})_i$ disappears.
- If the exit was at 1, one then now passes to 0 and the exit is the same as that of 0.

This proves the first assertion of (i). Next, any element in $\mathcal{E}'_{+,2}$ is some $Z(A)$, with $A \in \mathcal{A}$, giving the second claim. Remark also that any element in $\mathcal{E}'_{+,1}$ is some $Z(A)$, with $A \in \mathcal{A}$ corresponding to a left-extremal box. The second part of the proof gives that $\mathcal{C}'_{+,1}$ is stable under $(-1)^{R-1} \bigwedge^R \mathcal{M}'$.

(ii) Stability was proved above, using $\mathcal{C}'_{+,1}$ and $(-1)^{R-1} \bigwedge^R \mathcal{M}'$. Concerning minimality, observe first that $\bigwedge_{1\le i\le R-1} e_i \wedge (e_R + \cdots + e_d) \in \mathcal{C}'_+$, for the left-extremal box in $[-L+1, 0]$ that consists in jumping of $-L$ at each site (see (16)). Fix an element in $\mathcal{E}'_{+,1}$, defined by a left-extremal box B. Apply successively Lemma 3.10 to

$\bigwedge_{1\le i\le R-1} e_i \wedge (e_R+\dots+e_d)$, with the matrices $(-1)^{R-1}\bigwedge^R M'(\delta^{L-1},\eta^{L-1}),\dots,(-1)^{R-1}\bigwedge^R M'(\delta^0,\eta^0)$, where for $0\le l\le L-1$:

$$\begin{cases} \delta^l=(H1_{i=R-j})_{1\le i\le R-1},\ \eta^l=(0)_{1\le i\le L}, & \text{if the jump at } l \text{ in } B \text{ is } +j,\ 1\le j\le R-1,\\ \delta^l=(0)_{1\le i\le R-1},\ \eta^l=(0)_{1\le i\le L}, & \text{if the jump at } l \text{ in } B \text{ is } +R,\\ \delta^l=(0)_{1\le i\le R-1},\ \eta^l=(H1_{i=1})_{1\le i\le L}, & \text{if the jump at } l \text{ in } B \text{ is } -L.\end{cases}.$$

Observe that until applying the last step, the coefficient $\alpha_{d,-}$ is always 1. Thus, if the third case occurs, Lemma 3.10 only gives $HZ(A_-)+Z(A_0)$. Making $H\to+\infty$, the direction of the result tends to that of the given element of $\mathcal{E}'_{+,1}$.

Consider now a cone $\emptyset\ne\tilde{\mathcal{C}}\subset\mathcal{C}'_{+,1}$ stable under $(-1)^{R-1}\bigwedge^R\mathcal{M}'$. Take $0\ne x\in\tilde{\mathcal{C}}$ and write it as $x=\sum_{f\in F}c_f f$, where $\emptyset\ne F\subset\mathcal{E}'_{+,1}$ and $c_f>0$, $f\in F$. With $\delta=(H1_{i=R-1})_{1\le i\le R-1}$ and $\eta=(0)_{1\le i\le L}$, apply enough times Lemma 3.10 with $(-1)^{R-1}\bigwedge^R M'(\delta,\eta)$ to the previous equality, so that any term $Z(A)$, $A=(\alpha_{i,j})$, appearing in the sum with a coefficient H^n (with $n\ge1$) in front of it, checks $\alpha_{d,-}=1$ and $\alpha_{i,-}=0$, for $R\le i\le d-1$. Taking then H large, the remaining terms are negligible.

Remarking that in the dominating terms, $\alpha_{d,j}=0$, $1\le j\le R-1$, set $\delta=(0)_{1\le i\le R-1}$ and $\eta=(H1_{i=1})_{1\le i\le L}$. Apply next $(-1)^{R-1}\bigwedge^R M'(\delta,\eta)$ to the equality. The dominating R-vectors $Z(A)$, with $A=(\alpha_{i,j})$, then verify $\alpha_{R,j}=0$, $1\le j\le R-1$, and $\alpha_{R,-}=1$. Finally, apply $L-1$ times $(-1)^{R-1}\bigwedge^R M'(\delta,\eta)$ with $\delta=(0)_{1\le i\le R-1}$ and $\eta=(H1_{i=L})_{1\le i\le L}$. Taking H large, the direction of the sum is arbitrary close to that of $\bigwedge_{1\le i\le R-1}e_i\wedge(e_R+\dots+e_d)$. Since the sum belongs to $\tilde{\mathcal{C}}$, the first part of the study implies that $\mathcal{C}_{+,1}\subset\tilde{\mathcal{C}}$.

(iii) We prove the statement for the cone $\mathcal{C}'_{+,1}$ and the class $\mathcal{M}'$. In view of point (i), with invertible matrices in $\mathcal{M}'$, and since an invertible linear map is open, it is enough to show that $\mathcal{C}'_{+,2}$ has non empty-interior. Let then $w\in\bigwedge^R\mathbb{R}^d$ verify $w\perp\mathcal{C}'_{+,2}$ and fix $i=(i_1,\dots,i_R)\in I_R$.

If $i=(1,\dots,R)$, then $e_{i,R}\in\mathcal{E}'_{+,2}$ and $w\perp e_{i,R}$, otherwise let u be such that $i_u<R\le i_{u+1}$. Choose $A\in\mathcal{A}$ so that lines in A with index $i\notin\{i_v\mid u+1\le v\le R\}$ are zero and for each $u+1\le v\le R$, then line i_v is zero except one element equal to 1 and placed at a column with index in $(1,\dots,R)\setminus(i_1,\dots,i_u)$. Then $Z(A)\in\mathcal{E}'_{+,2}$. Note that the quantity $e_{q(R)}-e_{i_R}$ appears in the wedge product expression of $Z(A)$, for some $1\le q(R)\le R$. Define next $Z(A')\in\mathcal{E}'_{+,2}$, replacing $e_{q(R)}-e_{i_R}$ by $e_{q(R)}$ in the expression of $Z(A)$. Then $w\perp(Z(A)-Z(A'))$. The last quantity contains only e_{i_R} at place $q(R)$ in its expression. Recursively, we get $w\perp e_{i,R}$. As this holds for all $i\in I_R$, finally $w=0$.

Take next a cone $\tilde{\mathcal{C}}\subset\bigwedge^R\mathbb{R}^d$ stable under $(-1)^{R-1}\bigwedge^R\mathcal{M}'$ and with non-empty interior. Let x be interior to $\tilde{\mathcal{C}}$. Since $\mathcal{C}'_{+,1}$ has non-empty interior, write $x=\sum_{f\in\mathcal{E}'_{+,1}}c_f f$. Up to adding some $\eta e_1\wedge\dots\wedge e_R$, suppose that $C:=\sum_{f\in\mathcal{E}'_{+,1},R\in I^f_-}c_f\ne0$, where $(I^f_j)\in\mathcal{P}_+$ are associated to each $f\in\mathcal{E}_{+,1}$. Using the stability of $\tilde{\mathcal{C}}$ under $(-1)^{R-1}\bigwedge^R\mathcal{M}'$ and Lemma 3.10, apply $(L-1)$ times to the above equality the matrix $(-1)^{R-1}\bigwedge^R M'(\delta,\eta)$, with $\delta=(H1_{i=R-1})_{1\le i\le R-1}$ and $\eta=(0)_{1\le i\le L}$. Taking $H>0$ large, the deduced sum is equivalent to $CH^{L-1}\times\bigwedge_{j=1}^{R-1}e_j\wedge e_d\in\tilde{\mathcal{C}}$. From point (ii) and since $\bigwedge_{j=1}^{R-1}e_j\wedge e_d\in\tilde{\mathcal{C}}_{+,1}$, we deduce that $\mathcal{C}'_{+,1}\subset\operatorname{sign}(C)\tilde{\mathcal{C}}$.

(iv) If $R=1$, then $\mathcal{C}'_+=\mathcal{C}'_{+,1}$, by Definition 3.9. In this case, one clearly has $\mathcal{E}'_+=\{e_i+\dots+e_d\mid1\le i\le L\}\subset\mathbb{R}^L$, which is also the set of edges of $\mathcal{C}'_+$. Suppose then that $R\ge2$. We show below that $\mathcal{E}'_+$ is the set of edges of $\mathcal{C}'_+$ and that any element of $\mathcal{E}'_{+,1}$ is a non-negative linear combination of elements of $\mathcal{E}'_+$. In view of Definition 3.9, write any element $f\in\mathcal{E}'_{+,1}$ in the form:

$$f=\bigwedge_{j=1}^{R-1}\Big(e_j-\sum_{i\in I^f_j}e_i\Big)\wedge\sum_{i\in I^f_-}e_i. \tag{17}$$

Observe that:

$$f=\sum_{u=0}^{R-1}\sum_{k\in I_u\cap[1,R-1]}(-1)^{R-1-u}\Big(\bigwedge_{j\in k}e_j\bigwedge_{j\in[1,R-1]\setminus k}\sum_{i\in I^f_j}e_i\Big)\wedge\sum_{i\in I^f_-}e_i.$$

Suppose then that there exist $f_0 \in \mathcal{E}'_+$ and a subset $\emptyset \neq F \subset \mathcal{E}'_+$, with $f_0 = \sum_{f\in F} c_f f$, where each c_f is > 0. Since $I^f_j \subset [R, d]$, the above decomposition implies that this equality is equivalent to the fact that, for all $0 \le u \le R-1$ and all $1 \le i_1 < \cdots < i_u \le R-1$:

$$\left[\bigwedge_{j=1}^{u}\Bigl(\sum_{i\in I^{f_0}_{i_j}} e_i\Bigr)\right] \wedge \Bigl(\sum_{i\in I^{f_0}_-} e_i\Bigr) = \sum_{f\in F} c_f \left[\bigwedge_{j=1}^{u}\Bigl(\sum_{i\in I^{f}_{i_j}} e_i\Bigr)\right] \wedge \Bigl(\sum_{i\in I^{f}_-} e_i\Bigr). \tag{18}$$

In particular, $\sum_{i\in I^{f_0}_-} e_i = \sum_{f\in F} c_f(\sum_{i\in I^f_-} e_i)$. As $\#I^{f_0}_- = 1$ and $c_f > 0$ for all $f \in F$, there exist $R \le w \le d$ such that $I^{f_0}_- = I^f_- = \{e_w\}$, $f \in F$. Another case of (18) gives, for $2 \le j \le R$:

$$\Bigl(\sum_{i\in I^{f_0}_j} e_i\Bigr) \wedge e_w = \sum_{f\in F} c_f \Bigl(\sum_{i\in I^{f}_j} e_i\Bigr) \wedge e_w \quad \text{and thus} \quad \sum_{i\in I^{f_0}_-} e_i = \sum_{f\in F} c_f \Bigl(\sum_{i\in I^{f}_-} e_i\Bigr),$$

as w does not belong to any I^f_j. Since $c_f > 0$ for all $f \in F$, it is necessary that $I^f_j = I^{f_0}_j$, $f \in F$. Thus $\mathcal{E}'_+$ is the set of edges of $\mathcal{C}'_+$. We next check that the elements of $\mathcal{E}'_{+,1}$ are positive linear combinations of elements in $\mathcal{E}'_+$. Take $f \in \mathcal{E}'_{+,1}$, with $\#\mathrm{I}^f_- \ge 2$, and recall that f corresponds to some left-extremal box in $[-L+1, 0]$ with exit sets $I_1, \dots, I_R$ and I_-, as in Definition 3.8. Then f can be written as in (17), with $I^f_j = R - I_{R+1-j}$ and $I^f_- = R - I_-$. Let $u > v$ be the two greatest indices of I_- and w be the smallest index of $\bigcup_{1\le i\le R} I_i$. Suppose that $w \in I_x$, $x \in \{1, \dots, R\}$.

There is then a path in $[w, 0]$ made of jumps among $\{+1, \dots, +R\}$ leaving $[-L+1, 0]$ at x and the whole path belongs to I_x. Suppose that $x = 1$ and write:

$$f = \bigwedge_{j=1}^{R-1}\Bigl(e_j - \sum_{i\in I_{R+1-j}} e_{R-i}\Bigr) \wedge e_{R-u} + \bigwedge_{j=1}^{R-1}\Bigl(e_j - \sum_{i\in I_{R+1-j}} e_{R-i}\Bigr) \wedge \sum_{i\in I_-\setminus\{u\}} e_{R-i}. \tag{19}$$

We claim that the above two R-vectors belong to $\mathcal{E}'_{+,1}$. Considering the first term, it is obtained by adding $I_- \setminus \{u\}$ to $I_x = I_1$. Indeed, there is at least one element z of the path defined by w in I_1, including $+1$, that verifies $v < z \le v + R$. Remark we use that $R \ge 2$. Link then v to z by a jump of size $\le R$. More generally, the ordered sequence defined by $I_- \setminus \{u\}$ decomposes into blocks of consecutive elements. The top element of each block is such that some point of the path defined by w that is at distance $\le R$. Connect these two elements by a positive jump of size $\le R$ and make a jump of $+1$ at each non-top element of a block. This connects $I_- \setminus \{u\}$ on I_1 and the first term is an element of $\mathcal{E}'_{+,1}$. Similarly, the second term in (19) is treated by adding a connection from u to I_1.

If $x > 1$, the same reasoning holds, using the following decomposition instead of (19):

$$f = \left[\bigwedge_{j=1}^{R-1}\Bigl(e_j - \sum_{i\in J_{R+1-j}} e_{R-i}\Bigr)\right] \wedge \sum_{i\in I_-\setminus\{u\}} e_{R-i} + \left[\bigwedge_{j=1}^{R-1}\Bigl(e_j - \sum_{i\in K_{R+1-j}} e_{R-i}\Bigr)\right] \wedge e_{R-u}, \tag{20}$$

where $J_j = K_j = I_j$ for $j \in \{2, \dots, R\} \setminus \{x\}$ and $J_x = I_x \cup \{u\}$, $K_x = I_x \cup (I_- \setminus \{u\})$. Finally, either with (19) or (20), the cardinal of I_- decreases at each step of at least of one unity, so the desired decomposition follows recursively.

(v) If $L = 1$, we have $\mathcal{C}'_+ = \mathcal{C}'_{+,2} = \mathbb{R}_+(\bigwedge_{1\le i\le R} e_i)$. If $L \ge 2$ and $R = 1$, then $\mathcal{C}'_+ = \mathrm{Vect}_+(e_i + \cdots + e_d \mid 1 \le i \le d) \neq \mathcal{C}'_{+,2} = \mathrm{Vect}_+(e_i \mid 1 \le i \le d)$. Let then $\min\{L, R\} \ge 2$ and denote by $\mathcal{E}'_{+,3}$ the subset of $\mathcal{E}'_{+,2}$ corresponding to elements defined with $\#I_- = 1$. A simple corollary of decompositions (19), (20) and of the first part of the proof of (iv) is that $\mathcal{E}'_{+,3}$ is the set of edges of $\mathcal{C}'_{+,2}$. To show that $\mathcal{C}'_+ \neq \mathcal{C}'_{+,2}$, we exhibit an element in $\mathcal{E}'_{+,3} \setminus \mathcal{E}'_+$. Let us check that $(e_1 - e_d)\bigwedge_{j=2}^{R} e_j \in \mathcal{E}'_{+,3}$ convenes.

If some $\zeta \in \mathcal{E}'_{+,3}$ were colinear to $(e_1 - e_d)\bigwedge_{j=2}^{R} e_j$, since the first part in the proof of (iv) implies uniqueness of the representation, there would be equality and there would be a left-extremal box in $[-L+1, 0]$, where the jump at 0 is $-L$, all $-L+2 \le i \le -1$ exit at 1 and $-L+1$ exits at R. This requires a jump from $-L+1$ to R, but $R + L - 1 > R$, since $L \ge 2$, which is impossible. Thus $(e_1 - e_d)\bigwedge_{j=2}^{R} e_j \notin \mathcal{E}'_+$.

(vi) Observe first that $\mathcal{C}_{t,+} \subset \mathcal{D}$. If $L = 1$, then $\mathcal{C}_+ = \mathcal{C}_{t,+} = \mathbb{R}^+$ and thus $\mathcal{C}_+ = (\mathcal{C}_{t,+})^* = \mathbb{R}^+$. Suppose next $L \geq 2$. Point (v) gives $\mathcal{C}'_+ \subset \mathcal{C}'_{+,2}$, with strict inclusion. We show $\bigwedge^R U^{-1}(\mathcal{C}'_{+,2}) \subset (\mathcal{D})^*$, giving $\mathcal{C}_+ = \bigwedge^R U^{-1}(\mathcal{C}'_+) \subset \bigwedge^R U^{-1}(\mathcal{C}'_{+,2}) \subset (\mathcal{D})^*$.

Let $\zeta = \bigwedge_{j=1}^R (\sum_{j \leq s \leq j+k_j} e_s)$ be a generator of $\mathcal{D}$ and ζ' be an edge of $\bigwedge^R U^{-1}(\mathcal{C}'_{+,2})$, written as follows (see Definition 3.9), with disjoint sets $(I_j)_{1 \leq j \leq R-1}$ and I_- in $[R, d]$:

$$\zeta' = \bigwedge_{j=1}^{R-1} \left(e_j - e_{j-1} - \sum_{i \in I_j} (e_i - e_{i+1}) \right) \wedge \sum_{i \in I_-} (e_i - e_{i+1}).$$

Set $A_1 = I_1$, $A_j = \{j-1\} \cup I_j$, for $2 \leq j \leq R-1$, and $A_R = I_-$. The (A_j) are all disjoint. One checks that $\langle \zeta, \zeta' \rangle_{\bigwedge^R \mathbb{R}^d} = \det[(1_{i=j} - 1_{i+k_i \in A_j})_{1 \leq i \leq R, 1 \leq j \leq R-1} (1_{i+k_i \in A_R})_{1 \leq i \leq R}]$. Developing with respect to the last column:

$$\langle \zeta, \zeta' \rangle_{\bigwedge^R \mathbb{R}^d} = \sum_{l=1}^R u_l, \quad \text{with } u_l = \det\big[(1_{i=j} - 1_{i+k_i \in A_j})_{1 \leq i \leq R, 1 \leq j \leq R-1} (1_{i+k_i \in A_R, i=l})_{1 \leq i \leq R}\big].$$

First, $u_R = 1_{R+k_R} \det[(1_{i=j} - 1_{i+k_i \in A_j})_{1 \leq i \leq R-1, 1 \leq j \leq R-1}] \geq 0$, since the involved matrix has a positive dominating diagonal. If $l < R$ and $l + k_l \notin A_R$, then $u_l = 0$. If $l + k_l \in A_R$, then after manipulations on columns, $u_l = \det[(1_{i=j} - 1_{i+k_i \in A_j})_{1 \leq i \leq R, 1 \leq j \leq R-1} (1_{i+k_i \in A_l})_{1 \leq i \leq R}]_{*l,*l}$, where "$*l, *l$" means suppressing line l and column l. This determinant is of the same type as the original expression for $\langle \zeta, \zeta' \rangle_{\bigwedge^R \mathbb{R}^d}$. The result follows by recurrence on the dimension. □

We next detail consequences of Proposition 3.11 for the decomposable vectors $\mathcal{R}_k(a,b)$ and $\mathcal{V}_R$. We obtain the following result.

Proposition 3.12. *For integers $a < k < b$, $(-1)^{(b-k-1)(R-1)} \mathcal{R}_k(a,b) \in (-1)^R \mathcal{C}_+$. There is a constant $C > 0$ such that $\mathcal{V}_R$ can be written as*:

$$\mathcal{V}_R = (-1)^R \sum_{f \in \mathcal{E}_+} c_f f, \quad \textit{with } \frac{1}{C} \leq c_f \leq C, \textit{ for all } f \in \mathcal{E}_+. \tag{21}$$

Moreover there are iid environments where the direction of $\mathcal{V}_R$ is arbitrary close to that of any element of $\mathcal{E}_+$ with positive μ-probability, taking $\varepsilon > 0$ small enough (*see* (1)).

Proof. If $a < k < b$, Lemma 5.2 of [7] gives $\mathcal{R}_k(a,b) = (-1)^{(R-1)(b-1-k)} (d_{k+1} \cdots d_{b-1}) \mathcal{R}_k(a, k+1)$, with $d_s = P_s(a, s+1, s+R)$. This proves the first claim.

Since $\mathcal{V}_R = \lim_{n \to +\infty} \mathcal{R}_{-1}(-n, 0)/P_{-1}(-n, 0, -)$ and $(-1)^R \bigwedge^R U \mathcal{R}_{-1}(-n, 0)$ is some $Z(A)$, with $A \in \mathcal{A}$, we get that $\mathcal{V}_R \in (-1)^R \mathcal{C}_+$. Next, for $s \geq 0$:

$$\mathcal{V}_R = (-1)^{(s+1)(R-1)} \big(T^{-1}\alpha_R \cdots T^{-s-1}\alpha_R\big)^{-1} \bigwedge^R \big(T^{-1}M \cdots T^{-s-1}M\big) T^{-s-1} \mathcal{V}_R. \tag{22}$$

Recall that α_R is bounded away from zero and $+\infty$. Remark next that, when writing $\mathcal{V}_R = \lim_{n \to +\infty} \mathcal{R}_{-1}(-n, 0)/P_{-1}(-n, 0, -)$, the last vector also has components bounded away from 0 and $+\infty$ (see Proposition 6.2, p. 329 of [7]). Taking $s = 2L - 1$ and using Lemma 3.10 in (22) (L times point (ii) and L times point (iii)), observe that there is a constant $C > 0$ such that $\mathcal{V}_R = (-1)^R \sum_{f \in \mathcal{E}_+} c'_f f$, with $0 \leq c'_f \leq C$ and $c'_{f_0} \geq 1/C$, where $f_0 = \bigwedge_{1 \leq j \leq R-1} e_j \wedge (e_R + \cdots + e_d)$. Take then $s = L - 1$ in (22). Applying next the first part of the construction in point (ii) in Proposition 3.11, one gets (21).

Finally, the last claim is proved as in Theorem 3.2, as a consequence of the minimality of $\mathcal{C}_+$. □

3.3. *Geometrical constraints on V_R*

We now use the previous analysis on $\mathcal{V}_R$ and $\mathcal{V}_L$ to determine the geometrical conditions imposed to V_R, exactly in the same way as what was done for W_R, using properties of $\mathcal{W}_R$ and $\mathcal{W}_L$.

First, as a consequence of Proposition 3.12 on $\mathcal{V}_R$ and its analogue for $\mathcal{V}_L$, we have the following result, whose proof is the same as that of Theorem 3.4.

Theorem 3.13. (i) *There exist a constant $C>0$ and random coefficients $(c_{\zeta,\chi})_{\zeta\in\mathcal{E}_+,\chi\in\mathcal{E}_-}$ satisfying*:

$$V_R=(-1)^{d-1}\sum_{\zeta\in\mathcal{E}_+,\chi\in\mathcal{E}_-}c_{\zeta,\chi}\,\mathrm{Int}(\zeta,\chi),\quad \textit{with}\ \frac{1}{C}\le c_{\zeta,\chi}\le C. \tag{23}$$

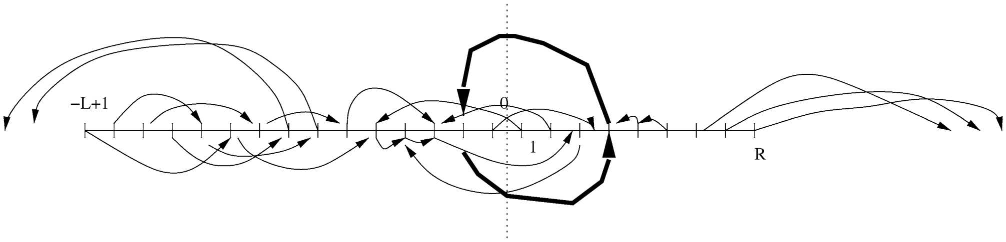
(ii) *There exist iid environments where V_R is arbitrary close in direction to that of any vector $(-1)^{d-1}\,\mathrm{Int}(\zeta,\chi)\neq 0$ with positive μ-probability, for all $(\zeta,\chi)\in\mathcal{E}_+\times\mathcal{E}_-$, taking $\varepsilon>0$ (defined in (1)) small enough.*

We shall now determine $\mathrm{Int}(\zeta,\chi)$, for $\zeta\in\mathcal{E}_+$, $\chi\in\mathcal{E}_-$. Recall Definition 3.8 on left and right-extremal boxes. We now glue such boxes.

Definition 3.14. (i) *An extremal box $B_{L,R}=B_L\cup B_R$ is the graph in $[-L+1,R]$ obtained by taking a left-extremal box B_L and a right-extremal box B_R: a transition is chosen among $\{-L\}\cup\{1,\dots,R\}$ at each site in $[-L+1,0]$ and a transition among $\{-L,\dots,-1\}\cup\{+R\}$ is chosen at each site in $[1,R]$.*

(ii) *Any path in an extremal box (when following the graph) either exits $[-L+1,R]$ or ends on a cycle. Let us call "Cycle-free" an extremal box with no cycle. In this case, the path starting at $i\in[-L+1,R]$ finally leaves $[-L+1,R]$ on the right-hand side or on the left-hand side. Write then respectively $ex(i)=+$ and $ex(i)=-$.*

An example of extremal box with at least one cycle (in thick) is the following one:



It is important to notice that if $B_{L,R}=B_L\cup B_R$, then the property of being Cycle-free and the exit function only depend on the exit sets $(I_i(B_L))_{i\in\{1,\dots,R,-\}}$ and $(J_j(B_R))_{j\in\{-L+1,\dots,0,+\}}$. Also, any $\zeta\in\mathcal{E}_+$ is uniquely associated to the $(I_i(B_L))_{i\in\{1,\dots,R,-\}}$, where B_L is any left-extremal box used to represent ζ. The same holds for $\chi\in\mathcal{E}_-$, with right-extremal boxes. Let us next say that (ζ,χ) is Cycle-free if any associated extremal box $B_L\cup B_R$ is Cycle-free. We then denote by $\mathrm{ex}_{\zeta,\chi}$ the exit function. We have the following result:

Theorem 3.15. (i) *Let $\zeta\in\mathcal{E}_+$ and $\chi\in\mathcal{E}_-$. Then*:

$$\mathrm{Int}(\zeta,\chi)=(-1)^{d-1}1_{(\zeta,\chi)\text{ is Cycle-free}}\left(1_{\mathrm{ex}_{\zeta,\chi}(R+1-i)=+}-1_{\mathrm{ex}_{\zeta,\chi}(R-i)=+}\right)_{1\le i\le d}\in\mathbb{R}^d. \tag{24}$$

(ii) *The cone $(-1)^{d-1}\,\mathrm{Vect}_+\{\mathrm{Int}(\zeta,\chi),\zeta\in\mathcal{E}_+,\chi\in\mathcal{E}_-\}$ is*:

1. *If $L=1$: $(\mathbb{R}_+)^R$.*
2. *If $R=1$: $(\mathbb{R}_+)^L$.*
3. *If $L=R=2$: $\mathbb{R}_+e_1+\mathbb{R}_+e_3+\mathbb{R}(e_1-e_2+e_3)\subset\mathbb{R}^3$.*
4. *If $\min\{L,R\}\ge 2$ and $\max\{L,R\}\ge 3$: $\mathbb{R}^d$.*

As a corollary of Theorems 3.13 and 3.15, δV_R does not always lie in the non-negative cone of $\mathbb{R}^d$ for a constant $\delta \in \{\pm 1\}$, since as soon as $\min\{L, R\} \geq 2$, some $\mathrm{Int}(\zeta, \chi)$ does not verify this. This contrasts with Theorem 3.4 about W_R.

It confirms that the statement of Lemma 5 p. 192 of [17] is incorrect and that condition $(C3)$ of [19] is not valid in general. Restrictive hypotheses on the support of μ may however ensure that the Lyapunov eigenvector V_R lies in the non-negative cone of $\mathbb{R}^d$. Indeed, it is not hard to check that this property is true when the environment is constant.

Proof of Theorem 3.15. (i) *Step* 1: Define $U_1 \in GL_d(\mathbb{R})$ by $(U_1)_{ij} = 1_{i \leq j \leq R-1}$, for $1 \leq i \leq R-1$, and $(U_1)_{ij} = 1_{i \geq j \geq R}$, for $R \leq i \leq d$. Similarly, let $U_2 \in GL_d(\mathbb{R})$ be such that $(U_2)_{ij} = 1_{i \leq j \leq R}$, for $1 \leq i \leq R$, and $(U_2)_{ij} = 1_{i \geq j \geq R+1}$, for $R+1 \leq i \leq d$. Then, generic edges $\zeta \in \mathcal{E}_+$ and $\chi \in \mathcal{E}_-$, as introduced in Definition 3.9, can respectively be written as $\zeta = (\bigwedge^R U_1^{-1})\tilde{\zeta}$ and $\chi = (\bigwedge^L U_2^{-1})\tilde{\chi}$, where:

$$\tilde{\zeta} = \bigwedge_{j=1}^{R-1} \Big(e_j - \sum_{i \in I_j} e_i\Big) \wedge \sum_{i \in I_-} e_i \quad \text{and} \quad \tilde{\chi} = \sum_{i \in J_+} e_i \bigwedge_{j=R+1}^{d} \Big(e_j - \sum_{i \in J_j} e_i\Big), \tag{25}$$

with partitioning sets $(I_i)_{i \in \{1,\dots,R,-\}}$ in $\{R, \dots, d\}$ and $(J_j)_{j \in \{-L+1,\dots,0,+\}}$ in $\{1, \dots, R\}$, with $I_- \neq \emptyset$ and $J_+ \neq \emptyset$. As $\zeta^{\perp *} = (\bigwedge^{L-1} {}^tU_1)(\tilde{\zeta}^{\perp *})$ and $\chi^{\perp *} = (\bigwedge^{R-1} {}^tU_2)(\tilde{\chi}^{\perp *})$, we get:

$$\mathrm{Int}(\zeta, \chi) = \bigg[\bigwedge^{d-1} ({}^tU_1)\Big(\tilde{\zeta}^{\perp *} \wedge \Big(\bigwedge^{R-1} ({}^tU_1^{-1}\, {}^tU_2)\tilde{\chi}^{\perp *}\Big)\Big)\bigg]^{\perp *} = U_1^{-1}\Big(\tilde{\zeta}^{\perp *} \wedge \Big(\bigwedge^{R-1} H\Big)\tilde{\chi}^{\perp *}\Big)^{\perp *}, \tag{26}$$

with $H = {}^tU_1^{-1}\, {}^tU_2$ satisfying $H_{ij} = 1_{i=j}$, if $i \neq R$, and $H_{Rj} = 1_{j \leq R} - 1_{j \geq R+1}$.

Let us first treat the case $L = 1$. Then, $I_+ = \{R\}$ and $\zeta = \bigwedge_{1 \leq i \leq R} e_i$. Also $J_+ = [1, a]$, for some $1 \leq a \leq R$ and $\chi = e_a$. Therefore, $\mathrm{Int}(\bigwedge_{1 \leq k \leq R} e_k, e_a) = (-1)^{a-1}(\bigwedge_{1 \leq k \leq R, k \neq a} e_k)^{\perp *} = (-1)^{R+1} e_a$. As the associated extremal box in $[0, R]$ is such that points $R - a + 1, \dots, R$ leave the box on the right-hand side, whereas the other ones leave it on the left side, (ζ, χ) is Cycle-free and the right-hand side of (24) equals $(-1)^{R+1} e_a$. This concludes the case $L = 1$. The situation when $R = 1$ is similar.

Suppose next that $\min\{L, R\} \geq 2$. Then I_- and J_+ are singletons, written as $I_- = \{u\}$ and $J_+ = \{v\}$. First, associate matrices P to $\tilde{\zeta}$ and Q to $\tilde{\chi}$, respectively.

- Let $P = \begin{pmatrix} I_{R-1} & 0 \\ A & K \end{pmatrix} \in GL_d(\mathbb{R})$, with $A = (-1_{i \in I_j})_{R \leq i \leq d, 1 \leq j \leq R-1}$, $K = (1_{i = \sigma(j)})_{R \leq i, j \leq d}$, where σ is the permutation of $\{R, \dots, d\}$ equal to the identity if $u = R$, and to the transposition (u, R), if $u \neq R$. Set $\varepsilon_u = 1$, if $u = R$, and $\varepsilon_u = -1$, if $u \neq R$. Observe that:

$$ {}^tP^{-1} = \begin{pmatrix} I_{R-1} & (1_{\sigma(j) \in I_i})_{1 \leq i \leq R-1, R \leq j \leq d} \\ 0 & (1_{\sigma(j) = i})_{R \leq i \leq d, R \leq j \leq d} \end{pmatrix}.$$

  We have $\tilde{\zeta} = (\bigwedge^R P)(\bigwedge_{i=1}^R e_i)$ and $\tilde{\zeta}^{\perp *} = \varepsilon_u \bigwedge^{L-1}({}^tP^{-1})(\bigwedge_{i=R+1}^d e_i)$, using (8).

- Let $Q = \begin{pmatrix} L & B \\ 0 & I_{L-1} \end{pmatrix} \in GL_d(\mathbb{R})$, with $B = (-1_{i \in J_j})_{1 \leq i \leq R, R+1 \leq j \leq d}$ and $L = (1_{i = \tau(j)})_{1 \leq i, j \leq R}$, where τ is the permutation of $\{1, \dots, R\}$ equal to the identity if $v = R$, and to the transposition (v, R), if $v \neq R$. Set $\varepsilon_v = 1$, if $v = R$, and $\varepsilon_v = -1$, if $v \neq R$. Then:

$$ {}^tQ^{-1} = \begin{pmatrix} (1_{\tau(j) = i})_{1 \leq i \leq R, 1 \leq j \leq R} & 0 \\ (1_{\tau(j) \in J_i})_{R+1 \leq i \leq d, 1 \leq j \leq R} & I_{L-1} \end{pmatrix}.$$

  Also, $\tilde{\chi} = (\bigwedge^R Q)(\bigwedge_{i=R}^d e_i)$ and $\tilde{\chi}^{\perp *} = \varepsilon_v (-1)^{(R-1)L} \bigwedge^{R-1}({}^tQ^{-1})(\bigwedge_{i=1}^{R-1} e_i)$. Therefore, using that $\tau(j) \in J_- \Leftrightarrow j = R$:

$$\Big(\bigwedge^{R-1} H\Big)(\tilde{\chi}^{\perp *}) = \varepsilon_v (-1)^{(R-1)L} \bigwedge_{j=1}^{R-1} \begin{pmatrix} (1_{\tau(j) = i})_{1 \leq i \leq R-1} \\ 1_{\tau(j) \in J_R} \\ (1_{\tau(j) \in J_i})_{R+1 \leq i \leq d} \end{pmatrix}.$$

Via (26), $U_1(\mathrm{Int}(\zeta,\chi)) = \varepsilon_u\varepsilon_v(-1)^{(R-1)L+d(L-1)+d+R}\sum_{w=1}^d e_w\Delta_w = \varepsilon_u\varepsilon_v(-1)^{d-1}\sum_{w=1}^d e_w\Delta_w$, proceeding as in *Step* 1 of Theorem 3.6, where:

$$\Delta_w = \left| \begin{pmatrix} (1_{\tau(j)=i})_{1\le i\le R-1} \\ 1_{\tau(j)\in J_R} \\ (1_{\tau(j)\in J_i})_{R+1\le i\le d} \end{pmatrix}_{1\le j\le R-1},\ (1_{i=w})_{j=R},\ \begin{pmatrix} (1_{\sigma(j)\in I_i})_{1\le i\le R-1} \\ 1_{\sigma(j)=R} \\ (1_{\sigma(j)=i})_{R+1\le i\le d} \end{pmatrix}_{R+1\le j\le d} \right|.$$

Step 2: Define $\varphi:[R,d]\longrightarrow[1,R]\cup\{-\}$, by $\varphi(i)=j\Leftrightarrow i\in I_j$, and $\psi:[1,R]\longrightarrow[R,d]\cup\{+\}$, by $\psi(i)=j\Leftrightarrow i\in J_j$. To distinguish R in $[1,R]$ of R in $[R,d]$, we write it as R'. Set $\eta=\varphi\circ\psi$, when it is defined and denote by m_η the number of limit cycles of η in $[1,R]\setminus(J_u\cup\{v\})$. An orbit under iterations of η is written as Orb_η. Considering $[R,d]\cup[1,R]$ as $(d,\dots,R,R',\dots,1)$, if (ζ,χ) is Cycle-free, write $\mathrm{exit}(w)=+$, if starting from $w\in(d,\dots,R)$ and iterating successively φ and ψ, the exit is on the right. A similar definition holds for $w\in(R',\dots,1)$.

We now suppose that $R\le w\le d$ and compute Δ_w. First and via column operations:

$$\begin{aligned}\Delta_w &= \varepsilon_u \left| \begin{pmatrix} (1_{i=j,j\ne v})_{1\le i\le R-1} \\ (1_{i=\psi(j),j\ne v}+1_{i=\psi(R),j=v})_{R\le i\le d}\end{pmatrix}_{1\le j\le R-1},\ \begin{pmatrix} (1_{i=\varphi(j),j\ne u})_{1\le i\le R-1} \\ (1_{i=j,j\ne u}+1_{i=w,j=u})_{R\le i\le d}\end{pmatrix}_{R\le j\le d} \right| \\ &= \begin{cases} \varepsilon_u\varepsilon_v\bigwedge_{j=1,j\ne v}^R(e_j-e_{\eta(j)}1_{\psi(j)\ne u})\bigwedge_v e_R, & \text{if } w=u,\\ -\varepsilon_u\varepsilon_v\sum_{j\in J_u}\alpha_j, \quad \text{with } \alpha_j=\bigwedge_{s=1,s\notin\{j,v\}}^R(e_s-e_{\eta(s)}1_{\psi(s)\ne u})\bigwedge_j e_R\bigwedge_v e_{\varphi(w)}, & \text{if } w\ne u.\end{cases}\end{aligned}$$

Assume next that $w\ne u$. Then:

$$\alpha_j = \sum_{\substack{A:\ A\cap(J_u\cup\{v\})=\emptyset \\ B=[1,R']\setminus(A\cup J_u\cup\{v\})}} (-1)^{\#A}\bigwedge_{s\in A}e_{\eta(s)}\bigwedge_{t\in B}e_t\bigwedge_j e_R\bigwedge_v e_{\varphi(w)}.$$

Above, non-zero contributing subsets A must check $\eta(A)=(A\cup\{j,v\})\setminus\{R',\varphi(w)\}$. Distinguish the following cases:

- If $\{j,v\}=\{R',\varphi(w)\}$, a non-zero contributing A verifies $\eta(A)=A$ and is a union of limit cycles for η in $[1,R']\setminus\{v\}\setminus J_u$. As in the proof of Theorem 3.6: $\alpha_j=1_{m_\eta=0}(1_{j=R'}-1_{j=\varphi(w)})$.
- If $v\notin\{R',\varphi(w)\}$ and $j=R'$, a non-zero contributing A is a union of limit cycles for η in $[1,R']\setminus\{v\}\setminus J_u$ and a sequence of the form $(\varphi(w),\eta(\varphi(w)),\dots,\eta^p(\varphi(w)))$, with $p\ge 0$, and $\eta^{p+1}(\varphi(w))=v$. Then $\alpha_j=1_{m_{\eta=0},v\in\mathrm{Orb}_\eta(\varphi(w))}$. A similar reasoning provides:

$$\alpha_j=\begin{cases} -1_{m_{\eta=0}}1_{v\in\mathrm{Orb}_\eta(R')}, & \text{if } v\notin\{R',\varphi(w)\},\ j=\varphi(w),\\ -1_{m_{\eta=0}}1_{j\in\mathrm{Orb}_\eta(\varphi(w))}, & \text{if } j\notin\{R',\varphi(w)\},\ v=R',\\ -1_{m_{\eta=0}}1_{j\in\mathrm{Orb}_\eta(R')}, & \text{if } j\notin\{R',\varphi(w)\},\ v=\varphi(w).\end{cases}$$

- If $\{j,v\}\cap\{R',\varphi(w)\}=\emptyset$, a non-zero contributing A is a union of limit cycles for η in $[1,R']\setminus\{v\}\setminus J_u$ and of two sequences of the form $(\varphi(w),\eta(\varphi(w)),\dots,\eta^p(\varphi(w)))$, with $p\ge0$, and $(R',\eta(R'),\dots,\eta^q(R'))$, with $q\ge0$, satisfying $\{\eta^{p+1}(\varphi(w)),\eta^{q+1}(R')\}=\{j,v\}$. Then:

$$\alpha_j=1_{m_{\eta=0}}(1_{v\in\mathrm{Orb}_\eta(\varphi(w)),j\in\mathrm{Orb}_\eta(R')}-1_{j\in\mathrm{Orb}_\eta(\varphi(w)),v\in\mathrm{Orb}_\eta(R')}).$$

Thus the above formula is valid in all cases. Finally, if $R\le w\le d$, $w\ne u$:

$$\begin{aligned}\varepsilon_u\varepsilon_v\Delta_w &= 1_{m_\eta=0}(-1_{\mathrm{exit}(w)=+,\mathrm{exit}(R')=-}+1_{\mathrm{exit}(w)=-,\mathrm{exit}(R')=+})\\ &= 1_{m_\eta=0}(1_{\mathrm{exit}(R')=+}-1_{\mathrm{exit}(w)=+}).\end{aligned}$$

In the same way, $\varepsilon_u\varepsilon_v\Delta_u=1_{m_\eta=0}1_{\mathrm{exit}(R')=+}$. Hence, $(\mathrm{Int}(\zeta,\chi))_R=1_{m_\eta=0}(1_{\mathrm{exit}(R')=+}-1_{\mathrm{exit}(R)=+})$ and $(\mathrm{Int}(\zeta,\chi))_w=1_{m_\eta=0}(1_{\mathrm{exit}(w-1)=+}-1_{\mathrm{exit}(w)=+})$, $R+1\le w\le d$. Similarly, $(\mathrm{Int}(\zeta,\chi))_w=1_{m_\eta=0}(1_{\mathrm{exit}(w+1)=-}-1_{\mathrm{exit}(w)=-})=1_{m_\eta=0}(1_{\mathrm{exit}(w)=+}-1_{\mathrm{exit}(w+1)=+})$, $1\le w\le R-1$. This concludes point (i).

(ii) If $L=1$, then $\operatorname{Int}(\zeta,\chi)$ has the form $(-1)^{R+1}e_a$ and every $1\le a\le R$ can be taken. The case when $R=1$ is similar. Suppose next that $L=R=2$. We list below, according to left exit points $a\in[-1,0]$ and right exit points $b\in[1,2]$, the vector given by point (i):

- Let $a=0$. If $b=1$, we get ${}^t(-1,1,-1)$. If $b=2$, then ${}^t(1,0,0)$.
- Let $a=-1$. If $b=1$, then ${}^t(0,0,1)$. If $b=2$ and $0\to 1$, $1\to -1$, then ${}^t(1,0,0)$. If $b=2$ and $0\to 2$, $1\to -1$, then ${}^t(1,-1,1)$. If $b=2$ and $0\to 2$, $1\to 0$, then ${}^t(0,0,1)$.

This concludes the case $L=R=2$. Suppose next that $\min\{L,R\}\ge 2$ and $\max\{L,R\}\ge 3$ and for instance $L\ge 3$. We show that the dual cone of $(-1)^{d-1}\operatorname{Vect}_+\{\operatorname{Int}(\zeta,\chi),\zeta\in\mathcal{E}_+,\chi\in\mathcal{E}_-\}$ is $\{0\}$. Let then $X={}^t(x_1,\dots,x_d)\in\mathbb{R}^d$ be such that $(-1)^{d-1}\langle X,\operatorname{Int}(\zeta,\chi)\rangle\ge 0$ for all $(\zeta,\chi)\in\mathcal{E}_+\times\mathcal{E}_-$. We prove that $X=0$ by choosing adequate extremal boxes in $[-L+1,\dots,-2,-1,0]\cup[1,2,\dots,R]$. As above, let a and b be respectively the left and right exit points.

- Take $a=0$, $2\le b\le R$ and the graph $(R\to R-1\to\cdots\to b+1\to b-1\to\cdots\to 1\to 0;\ -L+1\to -L+2\to\cdots\to -1\to 1)$. If $b=1$, take $a=-1$ and the graph $(R\to R-1\to\cdots\to 2\to -1;\ -L+1\to -L+2\to\cdots\to -2\to 0\to 2)$. This provides $(-e_{R-b}+e_{R-b+1})$, for all $1\le b\le R$. Thus $x_R\ge x_{R-1}\ge\cdots\ge x_1\ge 0$. Let now $a\le -1$, $b=R$ and the graph $(R-1\to\cdots\to 1\to 0;\ -L+1\to -L+2\to\cdots\to a-1\to a+1\to\cdots\to -1\to 0\to R)$. If $a=0$, take $b=1$ and the graph $(R\to\cdots\to 2\to -1;\ -L+1\to -L+2\to\cdots\to -1\to 1)$. These cases give $(e_{R-a}-e_{R-a+1})$, for all $-L+1\le a\le 0$. Thus $x_R\ge x_{R+1}\ge\cdots\ge x_d\ge 0$.
- Let $a=-L+1$, $b=R$ and the graph $(R-1\to\cdots\to 1\to -L+1;\ -L+2\to\cdots\to 0\to R)$. We get $e_1-e_R+e_d$, giving $x_1+x_d\ge x_R$.
- Take $a=0$, $b=1$ and $(R\to\cdots\to 2\to 0;\ -L+1\to -L+2\to\cdots\to -1\to 1)$. Thus $-e_{R-1}+e_R-e_{R+1}$ and $x_R\ge x_{R-1}+x_{R+1}$. Hence, this already provides $x_1=\cdots=x_{R-1}\ge 0$, $x_{R+1}=\cdots=x_d\ge 0$ and $x_R=x_1+x_d$.
- If $R\ge 3$, take $a=-1$, $b=2$ and $(R\to\cdots\to 3\to 0;\ 1\to -1;\ -L+1\to -L+2\to\cdots\to -2\to 0\to 2)$. If $R=2$, take $(1\to -1;\ -L+1\to -L+2\to\cdots\to -2\to 0\to 2)$. This gives $e_{R-1}-e_R+e_{R+1}-e_{R+2}$. Thus $x_{R+1}=\cdots=x_d=0$ and $x_R=x_1$.
- If $R\ge 3$, take $a=-1$, $b=2$ and the graph $(R\to\cdots\to 3\to 1\to -1;\ -L+1\to -L+2\to\cdots\to -2\to 0\to 2)$. This provides $-e_{R-2}+e_{R-1}-e_R+e_{R+1}-e_{R+2}$, giving $x_R=0$ and thus $X=0$. If $R=2$, take $a=-1$, $b=1$ and the graph $(2\to -1;\ -L+1\to -L+2\to\cdots\to -2\to 0\to 1)$. This gives $-e_1+e_3-e_4$. Thus $x_1=0$ and then $X=0$.

This concludes the proof of point (ii) of the theorem. □

3.4. *Non-singularity results*

We finish this section by proving non-singularity results for $\mathcal{W}_R$, $\mathcal{W}_L$, W_R and $\mathcal{V}_R$, $\mathcal{V}_L$, V_R. These are crucial for the sequel.

Proposition 3.16. (i) *For $(\zeta_1,\zeta_2)\in\mathcal{E}_{t,+}\times\mathcal{E}_+$, $\langle\zeta_1,\zeta_2\rangle\ge 0$. For each $\zeta_1\in\mathcal{E}_{t,+}$, there is $\zeta_2\in\mathcal{E}_+$ with $\langle\zeta_1,\zeta_2\rangle\ge 1$, and for each $\zeta_2\in\mathcal{E}_+$, there is $\zeta_1\in\mathcal{E}_{t,+}$ with $\langle\zeta_1,\zeta_2\rangle\ge 1$. The same holds for $\mathcal{E}_{t,-}$ and $\mathcal{E}_-$. Thus, there is a constant $C>0$ such that*:

$$(-1)^R\langle\mathcal{V}_R,\mathcal{W}_R\rangle=(-1)^R\big\langle\mathcal{V}_R^{\perp *},\mathcal{W}_R^{\perp *}\big\rangle\ge C\quad\textit{and}\quad(-1)^L\langle\mathcal{V}_L,\mathcal{W}_L\rangle=(-1)^L\big\langle\mathcal{V}_L^{\perp *},\mathcal{W}_L^{\perp *}\big\rangle\ge C.$$

(ii) *For some constant $C>0$, $(-1)^{d-1}\langle V_R,W_R\rangle\|\operatorname{Int}(\mathcal{W}_R,\mathcal{W}_L)\|\|\operatorname{Int}(\mathcal{V}_R,\mathcal{V}_L)\|\ge C$. In particular, for another constant $C'>0$: $(-1)^{d-1}\langle V_R,W_R\rangle\ge C'$. We also have the equality $\lambda_R=\rho_R\langle V_R,W_R\rangle/T\langle V_R,W_R\rangle$.*

(iii) *We have $|\mathcal{W}_L^{\perp *}\wedge\mathcal{W}_R^{\perp *}\wedge V_R|=|\langle W_R,V_R\rangle|\|\operatorname{Int}(\mathcal{W}_R,\mathcal{W}_L)\|$, as well as $|\mathcal{V}_L^{\perp *}\wedge\mathcal{V}_R^{\perp *}\wedge W_R|=|\langle V_R,W_R\rangle|\times\|\operatorname{Int}(\mathcal{V}_R,\mathcal{V}_L)\|$. In particular, using* (ii), *each of the configurations $(\mathcal{W}_L^{\perp *},\mathcal{W}_R^{\perp *},V_R)$ and $(\mathcal{V}_L^{\perp *},\mathcal{V}_R^{\perp *},W_R)$ in $\mathbb{R}^d$ are non-singular.*

Proof. (i) Since $\mathcal{C}_+\subset(\mathcal{C}_{t,+})^*$ (Proposition 3.11), we have $\langle\zeta_1,\zeta_2\rangle\ge 0$, for all $(\zeta_1,\zeta_2)\in\mathcal{E}_{t,+}\times\mathcal{E}_+$. Fixing ζ_1, if this quantity were always equal to 0, then $\zeta_1=0$, since $\mathcal{C}_+$ has non-empty interior. Finally, remark that $\langle\zeta_1,\zeta_2\rangle$ is an integer. The last point follows from Theorem 3.2 and Proposition 3.12.

(ii) The last point follows from $\langle V_R, W_R\rangle = (1/\rho_R)\langle V_R, {}^tMTW_R\rangle = (\lambda_R/\rho_R)\langle TV_R, TW_R\rangle$. Next, using that Ort_n is an isometry for all $0 \le n \le d$:

$$\begin{aligned}\langle W_R, V_R\rangle \|\mathrm{Int}(\mathcal{W}_R, \mathcal{W}_L)\| \|\mathrm{Int}(\mathcal{V}_R, \mathcal{V}_L)\| &= \langle (\mathcal{W}_R^{\perp *} \wedge \mathcal{W}_L^{\perp *})^{\perp *}, (\mathcal{V}_R^{\perp *} \wedge \mathcal{V}_L^{\perp *})^{\perp *}\rangle \\ &= \langle \mathcal{W}_R^{\perp *} \wedge \mathcal{W}_L^{\perp *}, \mathcal{V}_R^{\perp *} \wedge \mathcal{V}_L^{\perp *}\rangle \\ &= \langle \mathcal{W}_R^{\perp *}, \mathcal{V}_R^{\perp *}\rangle\langle \mathcal{W}_L^{\perp *}, \mathcal{V}_L^{\perp *}\rangle,\end{aligned}$$

since $S(\mathcal{V}_R^{\perp *}) \perp S(\mathcal{W}_L^{\perp *})$, as well as $S(\mathcal{W}_R^{\perp *}) \perp S(\mathcal{V}_L^{\perp *})$. We conclude with point (i).

(iii) The equalities $|\mathcal{W}_L^{\perp *} \wedge \mathcal{W}_R^{\perp *} \wedge V_R| = |\langle (\mathcal{W}_L^{\perp *} \wedge \mathcal{W}_R^{\perp *})^{\perp *}, V_R\rangle| = |\langle W_R, V_R\rangle| \|\mathrm{Int}(\mathcal{W}_R, \mathcal{W}_L)\|$ treat the first case. The second one is similar. □

We finally study the behaviour of quantities like $(M_nX)_{n\ge 0}$, when $X \in S(\mathcal{W}_R^{\perp *})$. By definition of $S(\mathcal{W}_R^{\perp *})$, such a quantity tends exponentially fast towards 0, as $n \to +\infty$. We show that the convergence is uniformly exponential.

Proposition 3.17. *There exist constants* $0 < c < 1$ *and* $C > 0$ *such that*:

$$\begin{cases} \forall n \ge 0, \forall X \in S(\mathcal{W}_R^{\perp *}), & \|M_nX\| \le Cc^n\|X\|, \\ \forall n \ge 0, \forall Y \in S(\mathcal{W}_L^{\perp *}), & \|M_{-n}Y\| \le Cc^n\|Y\|, \end{cases} \tag{27}$$

and:

$$\begin{cases} \forall n \ge 0, \forall X \in S(T^n\mathcal{V}_R^{\perp *}), & \|{}^t(M_n)X\| \le Cc^n\|X\| \\ \forall n \ge 0, \forall Y \in S(T^{-n}\mathcal{V}_L^{\perp *}), & \|{}^t(M_{-n})Y\| \le Cc^n\|Y\|. \end{cases} \tag{28}$$

Proof. *Step* 1: We first make reductions, using the matrix $K_r = \mathrm{diag}(1, r, \dots, r^{d-1})$, as in [6]. Let $M(\delta, \eta) \in \mathcal{M}$, satisfying condition (1). Recalling Definition 3.1, introduce $A_j = 1 + \delta_1 + \cdots + \delta_j$, $1 \le j \le R-1$, and $B_{L+R-j} = \eta_1 + \cdots + \eta_{L+R-j}$, $R \le j \le d$. Then $K_rM(\delta, \eta)K_r^{-1} = rM(\delta', \eta')$, with (δ', η') associated to $A'_j = A_j/r^j$, $1 \le j \le R-1$, and $B'_{L+R-j} = B_{L+R-j}/r^{R+L-j}$, $R \le j \le d$. Condition (1) thus implies that for r close enough to 1, $M(\delta', \eta') \in \mathcal{M}$ and a condition similar to (1) holds with another constant.

Setting $M' = r^{-1}(K_rMK_r^{-1})$, we get $K_rM_nK_r^{-1} = r^n(M')_n$, $n \in \mathbb{Z}$. Also, the subspaces related to M' and defined by Oseledec's theorem are the images by K_r of those related to M. We thus only need to show the proposition with $c = 1$. Since the Lyapunov exponents of (M', T) verify $\gamma_i(M', T) = \gamma_i(M, T) - \log r$, $1 \le i \le d$, we also suppose that $\gamma_R(M, T) \ne 0$, up to perturbing.

Step 2: We show that the first inequalities in (27) and in (28) are equivalent. For instance, denote by $p : \mathbb{R}^d \to \mathbb{R}^d$ the orthogonal projection on $S(\mathcal{W}_R^{\perp *})$. Then, for some constant C and a.s.:

$$\forall X \in S(\mathcal{V}_R^{\perp *}), \quad \|X\| \le C\|p(X)\|, \tag{29}$$

since if this were not true, it easily contradicts point (i) of Proposition 3.16. Suppose next that the first inequality of (27) holds and take $n \ge 0$ and $X \in S(T^n\mathcal{V}_R^{\perp *})$. As ${}^t(M_n)X \in S(\mathcal{V}_R^{\perp *})$, denoting by $(f_1, \dots, f_{L-1})$ an orthonormal basis of $S(\mathcal{W}_R^{\perp *})$, we get:

$$\|{}^t(M_n)X\| \le C\|p({}^t(M_n)X)\| \le C\sum_{i=1}^{n} |\langle X, M_nf_i\rangle| \le C'\|X\|.$$

The proof of the other direction is similar, as well as that of the equivalence between the second inequalities in (27) and (28).

Step 3: We prove (27). By symmetry, we only consider the first inequality with $c = 1$, as discussed above. As in [6], let $V^j = V_{-1}(-1, +\infty, -j)$, $1 \le j \le L$. Fixing $1 \le j \le L$, (3) gives $M_nV^j = V_{n-1}(-1, +\infty, -j)$, $n \ge 0$. Remark that each component of $V_{n-1}(-1, +\infty, -j)$ is the difference of two probabilities and thus is bounded by one. According to the previous discussion, we distinguish two cases:

– If $\gamma_R(M,T) > 0$, then (see [6]), $V^1 + \cdots + V^L = 0$ and $(V^1, \ldots, V^{L-1})$ span $S(\mathcal{W}_R^{\perp *})$ and $\| \bigwedge_{1\le j\le L-1} V^j \| \ge 1$. Therefore, the result follows from the previous remark.

– If $\gamma_R(M,T) < 0$, observe that the $(V^1, \ldots, V^L)$ are linearly independent. We consider the cone $\mathcal{D} \subset \wedge^R \mathbb{R}^d$ of point (vi) in Proposition 3.11. Recall that $\mathcal{C}_{t,+} \subset \mathcal{D}$ and remark that point (i) of Proposition 3.16 still holds, when replacing $\mathcal{C}_{t,+}$ by $\mathcal{D}$, as $\mathcal{C}_+ \subset (\mathcal{D})^*$. We will show that:

$$(-1)^{L-1+(d-1)R} \left(\bigwedge_{j=1}^{L-1} V^j \right)^{\perp *} \in \mathcal{D}. \tag{30}$$

Let us show that this proves the result. First, $\|(\bigwedge_{1\le j\le L-1} V^j)^{\perp *}\| = \| \bigwedge_{1\le j\le L-1} V^j \| \ge 1$ and this quantity is clearly bounded. Next:

$$(-1)^{L-1+(d-1)R+R} \left\langle \bigwedge_{1\le j\le L-1} V^j, \mathcal{V}_R^{\perp *} \right\rangle = (-1)^{L-1+(d-1)R+R} \left\langle \left(\bigwedge_{1\le j\le L-1} V^j \right)^{\perp *}, \mathcal{V}_R \right\rangle \ge C, \tag{31}$$

for some constant $C > 0$. Indeed, $\|(\bigwedge_{1\le j\le L-1} V^j)^{\perp *}\| \ge 1$ and at least one component in the decomposition of $(-1)^{L-1+(d-1)R}(\bigwedge_{1\le j\le L-1} V^j)^{\perp *}$ according to the elements ζ defining $\mathcal{D}$ is not small. Since all components of $(-1)^R \mathcal{V}_R$ are greater than some constant $c' > 0$ (Proposition 3.12), we get (31) via point (i) of Proposition 3.16. As a result, (29) is valid, when replacing $S(\mathcal{W}_R^{\perp *})$ by $S(\bigwedge_{1\le j\le L-1} V^j)$. Since the sequence $(M_n V^j)_{n\ge 0}$ is bounded, for $1 \le j \le L-1$, this proves the first inequality in (28). We finally show (30). Observe first that:

$$(-1)^{L-1} \bigwedge_{1\le j\le L-1} V^j = \bigwedge_{1\le j\le L-1} \left(\sum_{1\le i\le R} \big(P'_{R-i}(j) - P'_{R-i-1}(j)\big) e_i + e_{R+j} \right),$$

setting $P'_i(j) = \sum_{l=1}^{j} P_i(-1, +\infty, -l)$. Using Lemma 2.4:

$$\begin{aligned}
(-1)^{L-1} \left(\bigwedge_{j=1}^{L-1} V^j \right)^{\perp *} &= (-1)^{(d-1)R} \bigwedge_{1\le j\le R} \left(e_j - \sum_{i=R+1}^{d} e_i \big(P'_{R-j}(i-R) - P'_{R-j-1}(i-R) \big) \right) \\
&= (-1)^{(d-1)R} \bigwedge_{1\le j\le R} \left(e_j - \sum_{i=R+1}^{d} e_i \big(Q_{j+1}(i) - Q_j(i) \big) \right),
\end{aligned}$$

with $Q_j(i) = 1 - P'_{R-j}(i-R)$. Remark that $Q_{R+1}(i) = 0$, $R+1 \le i \le d$. Consequently:

$$(-1)^{L-1+(d-1)R} \left(\bigwedge_{j=1}^{L-1} V^j \right)^{\perp *} = \bigwedge_{1\le j\le R} \left(e_j + \cdots + e_R + \sum_{i=R+1}^{d} e_i Q_j(i) \right). \tag{32}$$

Fixing $1 \le j \le R$, $i \longmapsto Q_j(i)$ is a non-increasing function, verifying $0 \le Q_j(i) \le 1$. Thus, for $1 \le j \le R$, we get $e_j + \cdots + e_R + \sum_{i=R+1}^{d} e_i Q_j(i) \in \mathrm{Vect}_+\{\sum_{j\le s\le m} e_s \mid R \le m \le d\}$.

In (32), multilinearity gives that $(-1)^{L-1+(d-1)R}(\bigwedge_{j=1}^{L-1} V^j)^{\perp *}$ is a non-negative linear combination of elements of the form $\bigwedge_{1\le j\le R}(e_j + \cdots + e_{m_j})$, with $m_j \ge R$ for all $1 \le j \le R$. Such an element can clearly be written as an element generating $\mathcal{D}$. This proves (30) and concludes the proof of the proposition. $\square$

4. Invariant measure equation and Law of Large Numbers

4.1. *Characterization of* (IM)

We consider condition (IM), described in Definition 1.7 and show how the previous algebraic study clarifies the analysis. We discuss the invariant measure equation according to the sign of $\gamma_R(M,T)$. We follow the strategy of [6] and begin with a reformulation of the equation $P^*\pi=\pi$. Recall that $P^*f(\omega)=\sum_{z\in\Lambda}p_z(T^{-z}\omega)f(T^{-z}\omega)$.

Proposition 4.1. *The equation $\pi=P^*\pi$ is equivalent to the equality $Z=TZ$, where*:

$$Z=-\sum_{r=1}^{R}T^{-r}\left(\frac{p_r+\cdots+p_R}{p_R}\right)T^{-r}x+\sum_{l=1}^{L}T^{l-1}\left(\frac{p_{-l}+\cdots+p_{-L}}{p_R}\right)T^{l-1}x,\quad \textit{with } x=p_R\pi. \tag{33}$$

Ergodicity of $(\Omega,\mathcal{F},\mu,T)$ then implies that the equation $\pi=P^*\pi$ is equivalent to the equality $Z=-c$ for some constant c. In this case $c=\sum_{z\in\Lambda}\int zp_z\pi\,\mathrm{d}\mu$.

Proof. Observe that the equality $\pi=P^*\pi$ can be written as:

$$\pi=p_0\pi+\sum_{r=1}^{R}T^{-r}p_rT^{-r}\pi+\sum_{l=1}^{L}T^{l}p_{-l}T^{l}\pi,$$

or, equivalently, with $x=p_R\pi$:

$$\frac{1-p_0}{p_R}x=\sum_{r=1}^{R}T^{-r}\left(\frac{p_r}{p_R}\right)T^{-r}x+\sum_{l=1}^{L}T^{l}\left(\frac{p_{-l}}{p_R}\right)T^{l}x,$$

that is:

$$\begin{aligned}&\frac{1-p_0}{p_R}x-\sum_{r=1}^{R}T^{-r}\left(\frac{p_r+\cdots+p_R}{p_R}\right)T^{-r}x+\sum_{l=2}^{L}T^{l-1}\left(\frac{p_{-l}+\cdots+p_{-L}}{p_R}\right)T^{l-1}x\\&\quad=-\sum_{r=2}^{R}T^{-r+1}\left(\frac{p_r+\cdots+p_R}{p_R}\right)T^{-r+1}x+\sum_{l=1}^{L}T^{l}\left(\frac{p_{-l}+\cdots+p_{-L}}{p_R}\right)T^{l}x.\end{aligned}$$

Since

$$-\left(\frac{p_1+\cdots+p_R}{p_R}\right)x-\left(\frac{p_{-1}+\cdots+p_{-L}}{p_R}\right)x+\left(\frac{1-p_0}{p_R}\right)x=0,$$

we get $Z-TZ=0$.

As all steps proceeded by equivalence, this proves the first claim. The formula for the speed follows by taking expectation in (33), using the definition of π. □

We next rewrite (33), using some conjugate of ${}^tM$. Introduce the following auxiliary matrices.

Definition 4.2. (i) *Let $N\in GL_d(\mathbb{R})$ be the random matrix defined as* (*suppressing the first case, if $R=1$*):

$$T^{R-1}N_{i,j}=\begin{cases}-T^{j-1}a_j, & i{=}1,\ 1\le j\le R-1,\\ T^{j-1}b_{L+R-j}, & i=1,\ R\le j\le d,\\ 1_{i=j+1}, & 2\le i\le d.\end{cases}$$

(ii) *Define* $(c_i)_{2\le i\le d}$ *as* $c_i = -a_i$ *for* $2 \le i \le R-1$ (*if* $R \ge 3$) *and* $c_i = b_{L+R-i}$ *for* $R \le i \le d$. *Define* $\Phi \in GL_d(\mathbb{R})$ *by*:

$$\Phi(i,j) = \begin{cases} 1_{j=1}, & i=1, \\ 0, & i \ge 2,\ j \notin [2, d+2-i], \\ T^{i-2}c_{j+i-2}, & i\ge 2,\ j \in [2, d+2-i]. \end{cases}$$

Remark. *Notice for the sequel that* Φ *and* Φ^{-1} *are bounded maps. Also* ${}^t\Phi e_1 = e_1$. *The next proposition directly follows from Proposition* 4.1 *and Definition* 4.2.

Proposition 4.3. (i) *One has* $M = (T\Phi)^{-1}(T^{R-1\,t}N)\Phi$.

(ii) *With the notations of Proposition* 4.1, *the equation* $\pi = P^*\pi$ *is equivalent to the equality*:

$$T^{-1}X = NX + ce_1, \quad \text{with } X = {}^t(T^{-R+1}x, \dots, x, \dots, T^{L-1}x).$$

(iii) *Set* $X = T^{-R+2\,t}\Phi^{-1} \sim T^{-R+2}Y$. *Then* (IM) *is equivalent to the two conditions*:

- *There exists* Y *and a constant* c' *such that* $Y = {}^tMTY + c'e_1$.
- *We have* $\langle Y, e_1\rangle > 0$, μ*-a.s, and* $\|Y\| \in L^1(\mu)$.

Also, up to a positive multiplicative constant, c' *is the average speed of the random walk.*

We next characterize (IM), proving Theorem 1.8:

Proof of Theorem 1.8. (i) Suppose that $\gamma_R(M,T) = 0$. Since the Law of Large Numbers holds, the average speed is 0. If (IM) holds, then for some $Y \in \mathbb{R}^d$ with $\|Y\| \in L^1(\mu)$, we have $Y = {}^tMTY$ and $\langle Y, e_1\rangle > 0$, μ-a.s. Therefore, the Lyapunov exponent of Y with respect to $({}^tM, T^{-1})$ (cf. definition (7)) is ≤ 0, and similarly with respect to $({}^tM^{-1}, T)$. This property is only shared by vectors colinear to W_R. Thus, for some γ, we have $Y = \gamma W_R$. As $\|W_R\| = 1$, we deduce that $\gamma \in L^1(\mu)$. One also checks that $\gamma = T\gamma\rho_R$, where ρ_R is defined in Proposition 2.6, (ii).

Consider next the first coordinate in the equality $Y = \gamma W_R$. As $(-1)^{d-1}\langle W_R, e_1\rangle > 0$, μ-a.s. (Corollary 3.4), we get that $(-1)^{d-1}\gamma > 0$, μ-a.s. Proposition 3.16 then provides $\lambda_R = \gamma\langle V_R, W_R\rangle/(T\gamma T\langle V_R, W_R\rangle)$. Setting $\varphi = (-1)^{d-1}\gamma\langle V_R, W_R\rangle$ gives the result. Reciprocally, if $\lambda_R = \varphi/T\varphi$ with $\varphi \in L^1(\mu)$ and $\varphi > 0$, μ-a.s., then ${}^tMTW_R = \rho_R W_R$ can be rewritten as:

$$ {}^tMT\left(W_R\frac{\varphi}{\langle V_R, W_R\rangle}\right) = \left(W_R\frac{\varphi}{\langle V_R, W_R\rangle}\right) \tag{34}$$

and $W_R\varphi/\langle V_R, W_R\rangle$ has the desired qualities. So (IM) is verified.

(ii) Suppose that $\gamma_R(M,T) < 0$. Decompose first Y and e_1 with respect to suitable subspaces:

$$\begin{cases} Y = H + K + \gamma W_R, & \text{with } H \in S(\mathcal{V}_L^{\perp *}), K \in S(\mathcal{V}_R^{\perp *}), \gamma \in \mathbb{R}, \\ e_1 = H_0 + K_0 + \gamma_0 W_R, & \text{with } H_0 \in S(\mathcal{V}_L^{\perp *}), K_0 \in S(\mathcal{V}_R^{\perp *}), \gamma_0 \in \mathbb{R}. \end{cases}$$

Using Oseledec's theorem (see [16]), equation $Y = {}^tMTY + c'e_1$ is equivalent to:

$$H = {}^tMTH + c'H_0, \qquad K = {}^tMTK + c'K_0, \qquad \gamma = \rho_R T\gamma + c'\gamma_0. \tag{35}$$

Proposition 3.16 implies that H_0, K_0 and γ_0 are bounded quantities. Let us check that the solution of the previous system is given by:

$$\begin{cases} H = -c'\sum_{n\ge 1}\left(T^{-1}\left({}^tM^{-1}\right)\cdots T^{-n}\left({}^tM^{-1}\right)\right)T^{-n}H_0, \\ K = c'\sum_{n\ge 0}\left({}^tM \cdots T^{n-1}({}^tM)\right)T^nK_0, \\ \gamma = c'\sum_{n\ge 0}T^n\gamma_0\left(\rho_R\cdots T^{n-1}\rho_R\right). \end{cases}$$

Considering K for instance, the expression follows by iterations. Indeed, $T^n K$ is bounded along a subsequence, by Poincaré recurrence theorem, so $({}^t M \cdots T^{n-1}({}^t M))T^n K$ tends to zero along this subsequence, via Proposition 3.17. As $\sum_{n\geq 0}({}^t M \cdots T^{n-1}({}^t M))T^n K_0$ converges (Proposition 3.17), this gives the result. Proposition 3.17 also implies that H and K are bounded quantities. To conclude this preliminary analysis, remark that $\gamma_0 = \langle e_1, V_R\rangle / \langle V_R, W_R\rangle$. Since $\rho_R = \lambda_R \langle T V_R, T W_R\rangle / \langle V_R, W_R\rangle$, we get that:

$$\gamma = \frac{c'}{\langle V_R, W_R\rangle} Z, \quad \text{with } Z = \sum_{n\geq 0} T^n \langle V_R, e_1\rangle \big(\lambda_R \cdots T^{n-1}\lambda_R\big). \tag{36}$$

We next have the following discussion:

- If the integrability condition is verified, one can then solve $Y = {}^t M T Y + e_1$ with $\|Y\| \in L^1(\mu)$. Back to (IM), this provides $\pi \in L^1(\mu)$ with $\pi = P^*\pi$ and $\mu(\pi \neq 0) > 0$. To get a non-negative solution, observe that $|\pi| \leq P^*|\pi|$, μ-a.s. But this leads to a sub-invariant quantity in Proposition 4.1. As ergodicity ensures that a sub-invariant quantity is invariant, we deduce that $|\pi| = P^*|\pi|$, μ-a.s. Therefore (see [15]) $|\pi| > 0$, μ-a.s., and the quantity $|\pi| / \int |\pi| \, \mathrm{d}\mu$ checks (IM).
- Suppose that (IM) is verified. The average speed c in the Law of Large Numbers is ≥ 0, as the random walk is transient to the right. If $c = 0$, the argument of [6] about the recurrence of the ergodic sums is still valid and implies the recurrence of the random walk. Thus $c > 0$. If Y verifies $Y = {}^t M T Y + c e_1$, then Proposition 3.16 implies that the corresponding quantities H, K and γ are integrable (recall that H and K are bounded). Thus $Z \in L^1(\mu)$, as $c > 0$, meaning that the first component of $\sum_{n\geq 0}(\lambda_R \cdots T^{n-1}\lambda_R) T^n V_R$ is in $L^1(\mu)$. The case of the other components is deduced from the equality $M V_R = \lambda_R T V_R$, as the quantities λ_R and $1/\lambda_R$ are bounded.

This ends the proof of point (ii). The proof of (iii) is symmetric. □

Remark. *Let us focus on the transient case $\gamma_R(M,T) < 0$. If $\min\{L,R\} = 1$, as explained in the Introduction, then the condition for (IM) reduces to $\sum_{n\geq 0}(\lambda_R \cdots T^{n-1}\lambda_R) \in L^1(\mu)$. This is also the case if $L = R = 2$, since Theorems 3.13 and 3.15 say that $\langle V_2, {}^t(1,2,1)\rangle$ is uniformly positive. Such a remark cannot be made if $\min\{L,R\} \geq 2$ and $\max\{L,R\} \geq 3$, as the algebraic dual cone of the natural cone where any V_R lies is reduced to $\{0\}$.*

Proof of Proposition 1.9. Recall the definition of D given in (5) and the fact that the hypothesis $\gamma_R(M,T) > 0$ implies $\gamma_1(D, T^{-1}) < 0$. Let W and ρ be as in the proposition. As in [7], let also:

$$\tilde{V}_R = T^{-L+1t}\big(\big(1/T\rho \cdots T^{d-2}\rho\big)\big(1 - 1/T^{d-1}\rho\big), \ldots, (1 - 1/T\rho), (\rho - 1)\big) \quad \text{and} \quad \tilde{\lambda}_R = 1/T^{-L+2}\rho.$$

It was shown that $M\tilde{V}_R = \tilde{\lambda}_R T \tilde{V}_R$ and $\tilde{V}_R$ is colinear to V_R. At this point of the discussion, we make an apology for the incorrect corollary mentioned at the end of the statement of Proposition 8.4 of [7] on the boundedness of $\log \eta$. Indeed ergodicity implies that η is necessarily a constant multiple of $\|\tilde{V}_R\|$, but this quantity can be close to 0. It is in fact the heart of the problem.

It is now plain that $\lambda_R = \tilde{\lambda}_R \|T\tilde{V}_R\| / \|\tilde{V}_R\|$ and $V_R = \delta \tilde{V}_R / \|\tilde{V}_R\|$ for some random variable $\delta \in \{\pm 1\}$. However it is easily seen that δ is T-invariant. By ergodicity, δ is constant and we now suppose that $V_R = \tilde{V}_R / \|\tilde{V}_R\|$. Considering next the condition for (IM) in Theorem 1.8 when $\gamma_R(M,T) > 0$:

$$\sum_{n\geq 1}\big(T^{-1}\lambda_R \cdots T^{-n}\lambda_R\big)^{-1} T^{-n} V_R = \frac{1}{\|\tilde{V}_R\|} \sum_{n\geq 1}\big(T^{-1}\tilde{\lambda}_R \cdots T^{-n}\tilde{\lambda}_R\big)^{-1} T^{-n}\tilde{V}_R. \tag{37}$$

As mentioned in the course of the proof of Theorem 1.8, the integrability condition of the quantity appearing in (37) is equivalent to that of any of its component. Since the last component of $\tilde{V}_R$ checks $(\tilde{V}_R)_d = 1/T^{-1}\tilde{\lambda}_R - 1$, the last component of the right-hand side of (37) is a telescopic sum simply equal to $-1/(T^{-1}\tilde{\lambda}_R \|\tilde{V}_R\|)$. Using the expression for $\tilde{V}_R$ and the fact that $\log \rho$ is bounded, we get the result. □

Notice that the previous arguments also give:

Corollary 4.4. *If $\gamma_R(M,T) > 0$, then the components of $\sum_{n\geq 1}(T^{-1}\lambda_R \cdots T^{-n}\lambda_R)^{-1} T^{-n} V_R$ are all bounded away from 0 and have the same fixed sign. A similar statement holds in the case when $\gamma_R(M,T) < 0$.*

4.2. *Classification with respect to speed*

Recall that the quenched LLN always holds (Corollary 9.2 of [7]). We now show Theorem 1.10, providing a criterion for the non-zero speed of the random walk. Recall that $\tau(a,b)$ denotes the exit time of the maybe half-infinite interval $[a+1,b-1]$.

Proof of the Theorem 1.10. Consider point (i), the case of (ii) being symmetric. Then $4 \Leftrightarrow 1$ is Proposition 9.1 of [7] and $2 \Leftrightarrow 3$ is Theorem 1.8. This also gives $2 \Rightarrow 1$, by the argument of recurrence of the ergodic sums given in [6] and mentioned at the end of the proof of Theorem 1.8. We finally prove that $4 \Rightarrow 2$.

As 1 holds, the recurrence criterion (Theorem 1.4) gives $\gamma_R(M,T) < 0$. Set $\tau = \tau(-\infty, 1)$ and let π_1 be the *bounded* positive invariant density defined in Proposition 9.1 of [7]. We define a finite measure ν on $(\Omega, \mathcal{F})$ for all $B \in \mathcal{F}$ by:

$$\nu(B) = \int \left[\int \sum_{k=0}^{\tau-1} 1_B(\omega_k)\, \mathrm{d}\mathcal{P}_0^{\omega} \right] \pi_1(\omega)\, \mathrm{d}\mu(\omega), \tag{38}$$

where $(\omega_k)_{k\ge 0}$ is the sequence of the environments seen from the particle. Using the invariance properties of π_1 (see Proposition 9.1 of [7] and Proposition 3.6 of [6]) and following the proof of Theorem 3.1 of [1], we deduce that $P\nu = \nu$ and that ν is absolutely continuous with respect to μ. This concludes the proof of the theorem. □

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