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ROSENTHAL TYPE INEQUALITIES FOR FREE CHAOS

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Let $\mathcal{A}$ denote the reduced amalgamated free product of a family $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ of von Neumann algebras over a von Neumann subalgebra $\mathcal{B}$ with respect to normal faithful conditional expectations $\mathsf{E}_k : \mathsf{A}_k \to \mathcal{B}$. We investigate the norm in $L_p(\mathcal{A})$ of homogeneous polynomials of a given degree d. We first generalize Voiculescu's inequality to arbitrary degree $d \geq 1$ and indices $1 \leq p \leq \infty$. This can be regarded as a free analogue of the classical Rosenthal inequality. Our second result is a length-reduction formula from which we generalize recent results of Pisier, Ricard and the authors. All constants in our estimates are independent of n so that we may consider infinitely many free factors. As applications, we study square functions of free martingales. More precisely, we show that, in contrast with the Khintchine and Rosenthal inequalities, the free analogue of the Burkholder–Gundy inequalities does not hold in $L_\infty(\mathcal{A})$. At the end of the paper we also consider Khintchine type inequalities for Shlyakhtenko's generalized circular systems.

Introduction and main results. A strong interplay between harmonic analysis, probability theory and Banach space geometry can be found in the works of Burkholder, Gundy, Kwapień, Maurey, Pisier, Rosenthal and many others carried out mostly in the 1970s. Norm estimates for sums of independent random variables, as well as martingale inequalities, play a prominent role. Let us mention, for instance, the classical Khintchine and Rosenthal inequalities, Fefferman's duality theorem and the inequalities of Burkholder and Burkholder–Gundy for martingales. On the other hand, in the last two decades the noncommutative analogues of these aspects have been considerably developed. Important tools in this process come from free probability, operator space theory and theory of noncommutative martingales.

In this paper we continue this line of research by studying L_p-estimates for homogeneous polynomials of free random variables. Our results are motivated by the classical Rosenthal inequality [38]. That is, given a family $f_1, f_2, f_3, \ldots$ of

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independent, mean-zero random variables over a probability space Ω, we have

$$(\mathsf{R}_p) \qquad \left\| \sum_{k=1}^n f_k \right\|_{L_p(\Omega)} \sim_{c_p} \left(\sum_{k=1}^n \| f_k \|_2^2 \right)^{1/2} + \left(\sum_{k=1}^n \| f_k \|_p^p \right)^{1/p},$$

for $2 \le p < \infty$. We use $\mathsf{A} \sim_c \mathsf{B}$ for $c^{-1}\mathsf{A} \le \mathsf{B} \le c\mathsf{A}$ and $\mathsf{A} \lesssim_c \mathsf{B}$ for $\mathsf{A} \le c\mathsf{B}$. The growth rate for the constant c_p as $p \to \infty$ is $p/\log p$ (see [12]) and so the Rosenthal inequality fails on $L_\infty(\Omega)$. In sharp contrast are Voiculescu's inequality [44] and its operator-valued analogue [14] which are valid in L_∞. Let $\mathcal{A} = \mathsf{A}_1 * \mathsf{A}_2 * \cdots * \mathsf{A}_n$ denote the reduced free product of a family $\mathsf{A}_1, \mathsf{A}_2, \dots, \mathsf{A}_n$ of von Neumann algebras equipped with normal faithful (*n.f.* for short) states $\phi_1, \phi_2, \dots, \phi_n$, respectively. Then, given $a_1 \in \mathsf{A}_1, a_2 \in \mathsf{A}_2, \dots, a_n \in \mathsf{A}_n$ mean-zero random variables (i.e., freely independent) in $\mathcal{A}$ and a collection $b_1, b_2, \dots, b_n \in \mathcal{B}(\mathcal{H})$ of bounded linear operators on some Hilbert space $\mathcal{H}$, Voiculescu's inequality claims that

$$\begin{aligned} \left\| \sum_{k=1}^n a_k \otimes b_k \right\|_{\mathcal{A}\bar{\otimes}\mathcal{B}(\mathcal{H})} \sim_c & \sup_{1 \le k \le n} \| a_k \otimes b_k \|_{\mathsf{A}_k \bar{\otimes} \mathcal{B}(\mathcal{H})} \\ (\mathsf{V}_\infty) \qquad & + \left\| \left(\sum_{k=1}^n \phi_k(a_k^* a_k) b_k^* b_k \right)^{1/2} \right\|_{\mathcal{B}(\mathcal{H})} \\ & + \left\| \left(\sum_{k=1}^n \phi_k(a_k a_k^*) b_k b_k^* \right)^{1/2} \right\|_{\mathcal{B}(\mathcal{H})} \end{aligned}$$

for some universal positive constant c. The equivalence (V_∞) was proved by Voiculescu [44] in the tracial scalar-valued case. The general case as stated above (or, more generally, using amalgamated free product) can be found in [14]. This result can be regarded as the operator-valued free analogue of the Rosenthal inequality for homogeneous free polynomials of degree 1 and $p = \infty$. Quite surprisingly, the L_∞-estimates (which do not hold in the classical case) are easier to obtain in the free case by virtue of the Fock space representation. In contrast with the classical situation, the passage from L_∞ to L_p in the free setting is much more delicate. This is mainly because of the fact that a concrete Fock space representation does not seem available for $L_p(\mathcal{A})$.

Our first contribution in this paper consists of generalizing Voiculescu's inequality to homogeneous free polynomials of arbitrary degree d and to any index $1 \le p \le \infty$. Let us be more precise and fix some notation. Assume that $\mathcal{B}$ is a common von Neumann subalgebra of $\mathsf{A}_1, \mathsf{A}_2, \dots, \mathsf{A}_n$ such that there is a normal faithful conditional expectation $\mathsf{E}_k : \mathsf{A}_k \to \mathcal{B}$ for each k. Let $\mathcal{A}$ be the reduced amalgamated free product $*_{\mathcal{B}} \mathsf{A}_k$ of $\mathsf{A}_1, \mathsf{A}_2, \dots, \mathsf{A}_n$ over $\mathcal{B}$ with respect to the E_k. $\mathsf{E} : \mathcal{A} \to \mathcal{B}$ will denote the corresponding conditional expectation and $\mathbf{P}_{\mathcal{A}}(p, d)$ the subspace of $L_p(\mathcal{A})$ of homogeneous free polynomials of degree d. Then, given

$1 \le k \le n$, we consider the map $\mathcal{Q}_k$ on $\mathbf{P}_{\mathcal{A}}(p,d)$ which collects all reduced words starting and ending with a letter in A_k. Then we have the following result.

THEOREM A. *If $2 \le p \le \infty$ and $a_1, a_2, \ldots, a_n \in \mathbf{P}_{\mathcal{A}}(p,d)$, we have*

$$\Big\| \sum_{k=1}^n \mathcal{Q}_k(a_k) \Big\|_p \lesssim_{cd^7} \Big(\sum_{k=1}^n \|\mathcal{Q}_k(a_k)\|_p^p \Big)^{1/p} + \Big\| \Big(\sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(a_k)^* \mathcal{Q}_k(a_k)) \Big)^{1/2} \Big\|_p$$
$$+ \Big\| \Big(\sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(a_k) \mathcal{Q}_k(a_k)^*) \Big)^{1/2} \Big\|_p,$$

while the reverse inequality holds up to a constant less than or equal to cd.

We note that the operator-valued case is also contemplated in Theorem A since we are allowing amalgamation; see Remark 1.1 below for more details. On the other hand, we also point out that, since freeness implies noncommutative independence, the case of degree 1 polynomials for $2 \le p < \infty$ follows from the noncommutative analogue of Rosenthal's inequality [17, 18]. However, the constants obtained in this way are not uniformly bounded as $p \to \infty$; see Remark 2.13 for a more detailed discussion. Finally, we should also emphasize that Theorem A can be easily generalized to the case $1 \le p \le 2$ by duality; see Remark 3.7 for the details.

Our second major result is a length-reduction formula for homogeneous free polynomials in $L_p(\mathcal{A})$. Again, we need to fix some notation. In what follows, Λ will denote a finite index set and we shall keep the terminology for $\mathcal{A}$, $\mathcal{B}$ and $\mathsf{E}: \mathcal{A} \to \mathcal{B}$. Then, we use the following notation suggested by quantum mechanics

$$\Big\| \sum_{\alpha \in \Lambda} b(\alpha) \langle a(\alpha)| \Big\|_p = \Big\| \Big(\sum_{\alpha,\beta \in \Lambda} b(\alpha) \mathsf{E}(a(\alpha) a(\beta)^*) b(\beta)^* \Big)^{1/2} \Big\|_p,$$
$$\Big\| \sum_{\alpha \in \Lambda} |a(\alpha)\rangle b(\alpha) \Big\|_p = \Big\| \Big(\sum_{\alpha,\beta \in \Lambda} b(\alpha)^* \mathsf{E}(a(\alpha)^* a(\beta)) b(\beta) \Big)^{1/2} \Big\|_p,$$

where $a(\alpha) \in L_q(\mathcal{A})$ and $b(\alpha) \in L_r(\mathcal{A})$ with $1/q + 1/r = 1/p$. Finally, given $1 \le k \le n$, we consider the map $\mathcal{L}_k$ (resp. $\mathcal{R}_k$) on $\mathbf{P}_{\mathcal{A}}(p,d)$ which collects the reduced words starting (resp. ending) with a letter in A_k. Thus, we have

$$\mathcal{Q}_k = \mathcal{L}_k \mathcal{R}_k = \mathcal{R}_k \mathcal{L}_k.$$

We shall write $\mathbf{P}_{\mathcal{A}}(d)$ for $\mathbf{P}_{\mathcal{A}}(p,d)$ with $p = \infty$. Our second result is the following.

THEOREM B. *Let $2 \le p \le \infty$ and let $x_k(\alpha) \in L_p(\mathsf{A}_k)$ with $\mathsf{E}(x_k(\alpha)) = 0$ for each $1 \le k \le n$ and α running over a finite set Λ. Let $w_k(\alpha) \in \mathbf{P}_{\mathcal{A}}(d)$ for some*

$d \geq 0$ and satisfying $\mathcal{R}_k(w_k(\alpha)) = 0$ for all $1 \leq k \leq n$ and every $\alpha \in \Lambda$. Then, we have the equivalence

$$\Big\| \sum_{k,\alpha} w_k(\alpha) x_k(\alpha) \Big\|_{L_p(\mathcal{A})} \sim_{cd^2} \Big\| \sum_{k,\alpha} w_k(\alpha) \langle x_k(\alpha)| \Big\|_p + \Big\| \sum_{k,\alpha} |w_k(\alpha)\rangle x_k(\alpha) \Big\|_p .$$

Similarly, if $\mathcal{L}_k(w_k(\alpha)) = 0$, we have

$$\Big\| \sum_{k,\alpha} x_k(\alpha) w_k(\alpha) \Big\|_{L_p(\mathcal{A})} \sim_{cd^2} \Big\| \sum_{k,\alpha} |x_k(\alpha)\rangle w_k(\alpha) \Big\|_p + \Big\| \sum_{k,\alpha} x_k(\alpha) \langle w_k(\alpha)| \Big\|_p .$$

A large part of this paper will be devoted to the proofs of Theorems A and B. One of the key points in both proofs is the main complementation result in [37] (cf. Theorem 2.1 below) since it allows us to use interpolation starting from the case $p = \infty$, for which both results hold with constants independent of d. Our main application of Theorem B is a Khintchine type inequality. In the classical case, Khintchine's inequality is a particular case of Rosenthal's inequality with relevant constant $c_p \sim \sqrt{p}$ as $p \to \infty$. However, as in the Rosenthal/Voiculescu case, the free analogue of Khintchine's inequality holds in L_∞. Indeed, the first example of this phenomenon was found by Leinert [23], who replaced the Bernoulli random variables by the operators $\lambda(g_1), \lambda(g_2), \dots, \lambda(g_n)$ arising from the generators $g_1, g_2, \dots, g_n$ of a free group $\mathbb{F}_n$ via the left regular representation λ. More generally, if $\mathcal{W}_d$ denotes the subset of reduced words in $\mathbb{F}_n$ of length d and $C^*_\lambda(\mathbb{F}_n)$ stands for the reduced C*-algebra on $\mathbb{F}_n$, Haagerup [8] proved that

$$(\mathsf{H}_\infty) \qquad \Big\| \sum_{w \in \mathcal{W}_d} \alpha_w \lambda(w) \Big\|_{C^*_\lambda(\mathbb{F}_n)} \sim_{1+d} \Big(\sum_{w \in \mathcal{W}_d} |\alpha_w|^2 \Big)^{1/2} .$$

There are two ways to extend these inequalities. The *first step* consists of considering operator-valued coefficients. In the classical case, the operator-valued analogue is the so-called noncommutative Khintchine inequality by Lust-Piquard and Pisier [24, 25]. Leinert's result was extended to the operator-valued case by Haagerup and Pisier in [10], while Haagerup's inequality (H_∞) was generalized by Buchholz [3]. Finally, the result in [3] has been recently extended to arbitrary indices $1 \leq p \leq \infty$ by Pisier and Parcet in [27].

The *second step* consists in replacing the free generators by arbitrary free random variables and $C^*_\lambda(\mathbb{F}_n)$ by a reduced amalgamated free product von Neumann algebra $\mathcal{A}$. In this case we find the recent paper [37] by Ricard and the third-named author, where Buchholz's result was extended to arbitrary reduced amalgamated free products; see also [4] and [26] for the case of q-Gaussians.

In this paper we shall apply Theorem B to generalize the main results of [27, 37] and to do so we need to combine the brackets $|\ \rangle$ and $\langle\ |$; see Section 3 below for precise definitions. We obtain the following Khintchine-type inequality.

THEOREM C. *Let x be a d-homogeneous free polynomial*

$$x=\sum_{\alpha\in\Lambda}\sum_{j_1\neq j_2\neq\cdots\neq j_d} x_{j_1}(\alpha)\cdots x_{j_d}(\alpha)\in L_p(\mathcal{A})$$

for some $2\le p\le\infty$. Then we have

$$cd^{-5/2}(\Sigma_1+\Sigma_2)\le \|x\|_p\le c^d d!^2(\Sigma_1+\Sigma_2)$$

where Σ_1 is given by

$$\sum_{s=0}^{d}\Bigg\|\sum_{\alpha\in\Lambda}\sum_{j_1\neq\cdots\neq j_d}|x_{j_1}(\alpha)\cdots x_{j_s}(\alpha)\rangle\langle x_{j_{s+1}}(\alpha)\cdots x_{j_d}(\alpha)|\Bigg\|_p,$$

and Σ_2 has the form

$$\sum_{s=1}^{d}\Bigg(\sum_{j_s=1}^{n}\Bigg\|\sum_{\alpha\in\Lambda}\sum_{\substack{1\le j_1\neq\cdots\neq j_{s-1}\le n\\ 1\le j_{s+1}\neq\cdots\neq j_d\le n\\ j_{s-1}\neq j_s\neq j_{s+1}}}|x_{j_1}(\alpha)\cdots x_{j_{s-1}}(\alpha)\rangle x_{j_s}(\alpha)$$
$$\times\langle x_{j_{s+1}}(\alpha)\cdots x_{j_d}(\alpha)|\Bigg\|_p^p\Bigg)^{1/p}.$$

Our proof of Theorem C is an inductive application of Theorem B and provides a natural explanation of the norms Σ_1 and Σ_2. This leads naturally to three terms if $d=1$, 5 terms if $d=2$, and so on.... We refer to Section 3 below for a more detailed explanation of the norms Σ_1 and Σ_2. In the case of $p=\infty$, this result was obtained in [37] in a slightly different form but with an equivalence constant depending linearly on the degree d, which is essential for the applications there. The inductive nature of our arguments leads to worse constant; see Remark 3.8 for the better constant in the lower estimate.

In the last part of the paper we shall apply our techniques to studying square functions of free martingales and Khintchine type inequalities for generalized circular systems. More precisely, we first study the free analogue of the Burkholder–Gundy inequalities [5]. The noncommutative version of these inequalities was obtained by Pisier and the Xu in [34]. Thus, since any free martingale is a noncommutative martingale, the only interesting case seems to be $p=\infty$. In contrast with the free Khintchine and Rosenthal inequalities, we shall prove that the free analogue of the Burkholder–Gundy inequalities does not hold in $L_\infty(\mathcal{A})$. To be more precise, let us consider an infinite family $\mathsf{A}_1,\mathsf{A}_2,\mathsf{A}_3,\dots$ of von Neumann algebras equipped with distinguished normal faithful states and the associated reduced free product $\mathcal{A}=*_k\mathsf{A}_k$. We consider the natural filtration

$$\mathcal{A}_n=\mathsf{A}_1*\mathsf{A}_2*\cdots*\mathsf{A}_n\qquad\text{with conditional expectation } \mathsf{E}_n:\mathcal{A}\to\mathcal{A}_n.$$

Any martingale adapted to this filtration is called a *free martingale*. Now, let $\mathcal{K}_n$ be the best constant for which the lower estimate below holds for all free martingales $x_1, x_2, \dots$ in $L_\infty(\mathcal{A})$:

$$\max\left\{\left\|\left(\sum_{k=1}^{2n} dx_k\, dx_k^*\right)^{1/2}\right\|_\infty, \left\|\left(\sum_{k=1}^{2n} dx_k^*\, dx_k\right)^{1/2}\right\|_\infty\right\} \le \mathcal{K}_n \left\|\sum_{k=1}^{2n} dx_k\right\|_\infty.$$

Then we have the following result.

THEOREM D. *$\mathcal{K}_n$ satisfies $\mathcal{K}_n \ge c \log n$ for some absolute positive constant c.*

The last section is devoted to Khintchine-type inequalities for Shlyakhtenko's generalized circular variables [39] and Hiai's generalized q-Gaussians [11]. In these particular cases, the resulting inequalities are much nicer than those of Theorem C. The Khintchine inequalities for 1-homogeneous polynomials of generalized Gaussians were already proved in [47]; see Theorem 5.1 for an explicit formulation. We obtain here its natural extension for Hiai's generalized q-Gaussians. Namely, let us consider a system of q-generalized circular variables $gq_k = \lambda_k \ell_q(e_k) + \mu_k \ell_q^*(e_{-k})$ (see Section 5 for precise definitions) and let Γ_q denote the von Neumann algebra generated by these variables in the GNS-construction with respect to the vacuum state $\phi_q(\cdot) = \langle \Omega, \cdot\Omega\rangle_q$. Then, if d_{ϕ_q} denotes the density associated to the state ϕ_q, we have the following inequalities for the L_p-variables:

$$gq_{k,p} = d_{\phi_q}^{1/(2p)} gq_k d_{\phi_q}^{1/(2p)}.$$

THEOREM E. *Let $\mathcal{N}$ be a von Neumann algebra and $1 \le p \le \infty$. Let us consider a finite sequence $x_1, x_2, \dots, x_n$ in $L_p(\mathcal{N})$. Then, the following equivalences hold up to a constant c_q depending only on q:*

(i) *If $1 \le p \le 2$, then*

$$\left\|\sum_{k=1}^{n} x_k \otimes gq_{k,p}\right\|_p \sim_{c_q} \inf_{x_k = a_k + b_k} \left\|\left(\sum_{k=1}^{n} \lambda_k^{2/p} \mu_k^{2/p'} a_k a_k^*\right)^{1/2}\right\|_p + \left\|\left(\sum_{k=1}^{n} \lambda_k^{2/p'} \mu_k^{2/p} b_k^* b_k\right)^{1/2}\right\|_p.$$

(ii) *If $2 \le p \le \infty$, then*

$$\left\|\sum_{k=1}^{n} x_k \otimes gq_{k,p}\right\|_p \sim_{c_q} \max\left\{\left\|\left(\sum_{k=1}^{n} \lambda_k^{2/p} \mu_k^{2/p'} x_k x_k^*\right)^{1/2}\right\|_p, \left\|\left(\sum_{k=1}^{n} \lambda_k^{2/p'} \mu_k^{2/p} x_k^* x_k\right)^{1/2}\right\|_p\right\}.$$

Moreover, if $\mathcal{G}q_p$ denotes the closed subspace of $L_p(\Gamma_q)$ generated by the system of the generalized q-Gaussians $(gq_{k,p})_{k\geq 1}$, there exists a completely bounded projection $\gamma q_p : L_p(\Gamma_q) \to \mathcal{G}q_p$ satisfying

$$\|\gamma q_p\|_{cb} \leq \left(\frac{2}{\sqrt{1-|q|}}\right)^{|1-2/p|}.$$

In our last result we calculate the Khintchine inequalities for 2-homogeneous polynomials of generalized free gaussians $g_k = \lambda_k \ell(e_k) + \mu_k \ell^*(e_{-k})$ (corresponding to the case of $q = 0$). As we shall see, our method is also valid for d-homogeneous polynomials and the resulting inequalities can be regarded as asymmetric versions of the main inequalities in [27]. Let Γ denote the von Neumann algebra generated by the system of g_k's in the GNS-construction with respect to the vacuum state $\phi(\cdot) = \langle \Omega, \cdot \Omega \rangle$. Our result reads as follows:

THEOREM F. *Let $\mathcal{N}$ be a von Neumann algebra and $1 \leq p \leq \infty$. Let us consider a finite double indexed family $x = (x_{ij})_{i,j\geq 1}$ in $L_p(\mathcal{N})$ and define the following norms associated to x:*

$$\mathcal{M}_p(x) = \left\| \sum_{i\neq j} (\mu_i \lambda_j)^{1/p} (\lambda_i \mu_j)^{1/p'} x_{ij} \otimes e_{ij} \right\|_{S_p(L_p(\mathcal{N}))},$$

$$\mathcal{R}_p(x) = \left\| \left(\sum_{i\neq j} (\mu_i \mu_j)^{2/p'} (\lambda_i \lambda_j)^{2/p} x_{ij} x_{ij}^* \right)^{1/2} \right\|_{L_p(\mathcal{N})},$$

$$\mathcal{C}_p(x) = \left\| \left(\sum_{i\neq j} (\mu_i \mu_j)^{2/p} (\lambda_i \lambda_j)^{2/p'} x_{ij}^* x_{ij} \right)^{1/2} \right\|_{L_p(\mathcal{N})}.$$

Then, the following equivalences hold up to an absolute positive constant c:

(i) *If $1 \leq p \leq 2$, then*

$$\left\| \sum_{i\neq j} x_{ij} \otimes d_\phi^{1/(2p)} g_i g_j d_\phi^{1/(2p)} \right\|_p \sim_c \inf_{x=a+b+c} \mathcal{R}_p(a) + \mathcal{M}_p(b) + \mathcal{C}_p(c).$$

(ii) *If $2 \leq p \leq \infty$, then*

$$\left\| \sum_{i\neq j} x_{ij} \otimes d_\phi^{1/(2p)} g_i g_j d_\phi^{1/(2p)} \right\|_p \sim_c \max\{\mathcal{R}_p(x), \mathcal{M}_p(x), \mathcal{C}_p(x)\}.$$

Moreover, if $\mathcal{G}_{p,2}$ denotes the subspace of $L_p(\Gamma)$ generated by the system

$$\{d_\phi^{1/(2p)} g_i g_j d_\phi^{1/(2p)} \mid 1 \leq i \neq j < \infty\},$$

there exists a projection $\gamma_{p,2} : L_p(\Gamma) \to \mathcal{G}_{p,2}$ with cb-norm uniformly bounded on p.

We conclude the Introduction with some general remarks. We shall assume some familiarity with Voiculescu's free probability [43–45] and Pisier's vector-valued noncommutative integration [31]. In fact, we will be concerned only with the vector-valued Schatten classes and their column/row subspaces. On the other hand, since we are working over (amalgamated) free product von Neumann algebras, we shall need to use Haagerup noncommutative L_p-spaces [9, 41]. As is well known, Haagerup L_p-spaces have trivial intersection and thereby do not form an interpolation scale. However, the complex interpolation method will be a basic tool in this paper. This problem is solved by means of Kosaki's definition of L_p-spaces; see [20, 42]. We also refer the reader to Chapter 1 in [15] or to the survey [35] for a quick review of Haagerup's and Kosaki's definitions of noncommutative L_p-spaces and the compatibility between them. In particular, using such compatibility, in what follows we shall use the complex interpolation method without further details. At some specific points, we shall also need some basic notions from operator space theory [7, 32], Hilbert modules [22] and Tomita's modular theory [19, 30]. Along the paper, c will denote an absolute positive constant that may change from one instance to another.

1. Amalgamated free products. We begin by recalling the construction of the reduced amalgamated free product of a family of von Neumann algebras. Amalgamated free products of C*-algebras, which we also outline below, were introduced by Voiculescu [43]. Let $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ be a family of von Neumann algebras and let $\mathcal{B}$ be a common von Neumann subalgebra of all of them. We assume that there are normal faithful conditional expectations $\mathsf{E}_k : \mathsf{A}_k \to \mathcal{B}$. In addition, we also assume the existence of a von Neumann algebra $\mathcal{A}$ containing $\mathcal{B}$ with a normal faithful conditional expectation $\mathsf{E} : \mathcal{A} \to \mathcal{B}$ and the existence of $*$-homomorphisms $\pi_k : \mathsf{A}_k \to \mathcal{A}$ such that

$$\mathsf{E} \circ \pi_k = \mathsf{E}_k \quad \text{and} \quad \pi_{k|_{\mathcal{B}}} = id_{\mathcal{B}}.$$

The family $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ is called *freely independent* over E if

$$\mathsf{E}(\pi_{j_1}(a_1)\pi_{j_2}(a_2)\cdots\pi_{j_m}(a_m)) = 0$$

whenever $a_k \in \mathsf{A}_{j_k}$ are such that $\mathsf{E}(\pi_{j_k}(a_k)) = 0$ for all $1 \le k \le m$ and $j_1 \neq j_2 \neq \cdots \neq j_m$. In what follows we may identify A_k with the von Neumann subalgebra $\pi_k(\mathsf{A}_k)$ of $\mathcal{A}$ with no risk of confusion. In particular, we may use E or E_k indistinctively over A_k. Moreover, for notational convenience, we shall only use E almost all the time. In the scalar case, $\mathcal{B}$ is the complex field and the conditional expectations E and $\mathsf{E}_1, \mathsf{E}_2, \ldots, \mathsf{E}_n$ are replaced by normal faithful states.

As in the scalar-valued case, operator-valued freeness admits a natural Fock space representation. We first assume that $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ are C*-algebras having $\mathcal{B}$ as a common C*-subalgebra. Let us consider the mean-zero subspaces

$$\mathring{\mathsf{A}}_k = \{a_k \in \mathsf{A}_k \,|\, \mathsf{E}(a_k) = 0\}.$$

We define the Hilbert $\mathcal{B}$-module

$$\mathring{\mathsf{A}}_{j_1} \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_m}$$

equipped with the $\mathcal{B}$-valued inner product

$$\langle a_1 \otimes \cdots \otimes a_m, a_1' \otimes \cdots \otimes a_m' \rangle = \mathsf{E}_{j_m}(a_m^* \cdots \mathsf{E}_{j_2}(a_2^* \mathsf{E}_{j_1}(a_1^* a_1') a_2') \cdots a_m').$$

Then, the usual Fock space is replaced by the Hilbert $\mathcal{B}$-module

$$\mathcal{H}_{\mathcal{B}} = \mathcal{B} \oplus \bigoplus_{m \geq 1} \bigoplus_{j_1 \neq j_2 \neq \cdots \neq j_m} \mathring{\mathsf{A}}_{j_1} \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_m}.$$

The direct sums above are assumed to be $\mathcal{B}$-orthogonal. Let $\mathcal{L}(\mathcal{H}_{\mathcal{B}})$ stand for the algebra of adjointable maps on $\mathcal{H}_{\mathcal{B}}$. Recall that a linear right $\mathcal{B}$-module map $\mathrm{T} : \mathcal{H}_{\mathcal{B}} \to \mathcal{H}_{\mathcal{B}}$ is called *adjointable* if there exists $\mathrm{S} : \mathcal{H}_{\mathcal{B}} \to \mathcal{H}_{\mathcal{B}}$ such that

$$\langle x, \mathrm{T}y \rangle = \langle \mathrm{S}x, y \rangle \qquad \text{for all } x, y \in \mathcal{H}_{\mathcal{B}}.$$

Let us also recall how elements of A_k act on $\mathcal{H}_{\mathcal{B}}$. We decompose any $a_k \in \mathsf{A}_k$ as

$$a_k = \mathring{a}_k + \mathsf{E}_k(a_k).$$

An element in $\mathcal{B}$ acts on $\mathcal{H}_{\mathcal{B}}$ by left multiplication. Therefore, it suffices to define the action of mean-zero elements. The $*$-homomorphism $\pi_k : \mathsf{A}_k \to \mathcal{L}(\mathcal{H}_{\mathcal{B}})$ has the following form:

$$\begin{aligned} &\pi_k(\mathring{a}_k)(x_{j_1} \otimes \cdots \otimes x_{j_m}) \\ &\quad = \begin{cases} \mathring{a}_k \otimes x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_m}, & \text{if } k \neq j_1, \\ \mathsf{E}(\mathring{a}_k x_{j_1}) x_{j_2} \otimes \cdots \otimes x_{j_m} & \\ \qquad \oplus \big(\mathring{a}_k x_{j_1} - \mathsf{E}(\mathring{a}_k x_{j_1})\big) \otimes x_{j_2} \otimes \cdots \otimes x_{j_m}, & \text{if } k = j_1. \end{cases} \end{aligned}$$

This definition also applies for the empty word. Then, since the algebra $\mathcal{L}(\mathcal{H}_{\mathcal{B}})$ is a C^*-algebra [22], we can define the *reduced $\mathcal{B}$-amalgamated free product* $\mathrm{C}^*(*_{\mathcal{B}} \mathsf{A}_k)$ as the C^*-closure of linear combinations of operators of the form

$$\pi_{j_1}(a_1) \pi_{j_2}(a_2) \cdots \pi_{j_m}(a_m).$$

The C^*-algebra $\mathrm{C}^*(*_{\mathcal{B}} \mathsf{A}_k)$ is usually denoted in the literature by

$$*_k(\mathsf{A}_k, \mathsf{E}_k).$$

However, we shall use a more relaxed notation; see Remark 1.2 below.

Now we assume that $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ and $\mathcal{B}$ are von Neumann algebras and that $\mathcal{B}$ comes equipped with a normal faithful state $\varphi : \mathcal{B} \to \mathbb{C}$. This provides us with the induced *n.f.* states $\phi : \mathcal{A} \to \mathbb{C}$ and $\phi_k : \mathsf{A}_k \to \mathbb{C}$ given by

$$\phi = \varphi \circ \mathsf{E} \quad \text{and} \quad \phi_k = \varphi \circ \mathsf{E}_k.$$

The Hilbert space

$$L_2(\mathring{\mathsf{A}}_{j_1} \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_m}, \varphi)$$

is obtained from $\mathring{\mathsf{A}}_{j_1} \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_m}$ by considering the inner product

$$\langle a_1 \otimes \cdots \otimes a_m, a_1' \otimes \cdots \otimes a_m' \rangle_\varphi = \varphi(\langle a_1 \otimes \cdots \otimes a_m, a_1' \otimes \cdots \otimes a_m' \rangle).$$

Then we define the orthogonal direct sum

$$\mathcal{H}_\varphi = L_2(\mathcal{B}) \oplus \bigoplus_{m\geq 1} \bigoplus_{j_1 \neq j_2 \neq \cdots \neq j_m} L_2(\mathring{\mathsf{A}}_{j_1} \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_m}, \varphi).$$

Let us consider the $*$-representation $\lambda : \mathcal{L}(\mathcal{H}_{\mathcal{B}}) \to \mathcal{B}(\mathcal{H}_\varphi)$ defined by $(\lambda(\mathrm{T})x) = \mathrm{T}x$. The faithfulness of λ is implied by the fact that φ is also faithful. Indeed, assume that $\lambda(\mathrm{T}^*\mathrm{T}) = 0$, then we have

$$\langle \mathrm{T}^*\mathrm{T}x, x\rangle_\varphi = \varphi(\langle \mathrm{T}x, \mathrm{T}x\rangle) = 0 \qquad \text{for all } x \in \mathcal{H}_{\mathcal{B}}.$$

Since φ is faithful, $\mathrm{T}x = 0$ (as an element in $\mathcal{H}_{\mathcal{B}}$) for all $x \in \mathcal{H}_{\mathcal{B}}$, and so $\mathrm{T} = 0$. Then, the *$\mathcal{B}$-amalgamated reduced free product* $*_{\mathcal{B}}\mathsf{A}_k$ is defined as the weak* closure of $\mathrm{C}^*(*_{\mathcal{B}}\mathsf{A}_k)$ in $\mathcal{B}(\mathcal{H}_\varphi)$. After decomposing

$$a_k = \mathring{a}_k + \mathsf{E}(a_k)$$

and identifying $\mathring{\mathsf{A}}_k$ with $\lambda(\pi_k(\mathring{\mathsf{A}}_k))$, we can think of $*_{\mathcal{B}}\mathsf{A}_k$ as

$$*_{\mathcal{B}}\mathsf{A}_k = \Bigg(\mathcal{B} \oplus \bigoplus_{m\geq 1} \bigoplus_{j_1 \neq j_2 \neq \cdots \neq j_m} \mathring{\mathsf{A}}_{j_1}\mathring{\mathsf{A}}_{j_2} \cdots \mathring{\mathsf{A}}_{j_m}\Bigg)''.$$

Again, the usual notation for $*_{\mathcal{B}}\mathsf{A}_k$ is a bit more explicit one $\bar{*}_k(\mathsf{A}_k, \mathsf{E}_k)$.

Let us consider the orthogonal projections

$$\mathcal{Q}_\varnothing : \mathcal{H}_\varphi \to L_2(\mathcal{B}),$$
$$\mathcal{Q}_{j_1\cdots j_m} : \mathcal{H}_\varphi \to L_2(\mathring{\mathsf{A}}_{j_1} \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathring{\mathsf{A}}_{j_m}, \varphi).$$

Then $\mathsf{E} : *_{\mathcal{B}}\mathsf{A}_k \to \mathcal{B}$ is given by $\mathsf{E}(a) = \mathcal{Q}_\varnothing a \mathcal{Q}_\varnothing$ and the mappings

$$\mathcal{E}_{\mathsf{A}_k} : *_{\mathcal{B}}\mathsf{A}_k \ni a \mapsto \mathcal{Q}_{\mathsf{A}_k} a \mathcal{Q}_{\mathsf{A}_k} \in \mathsf{A}_k \qquad (\mathcal{Q}_{\mathsf{A}_k} = \mathcal{Q}_\varnothing + \mathcal{Q}_k),$$

are *n.f.* conditional expectations. In particular, it turns out that $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ are von Neumann subalgebras of $*_{\mathcal{B}}\mathsf{A}_k$ freely independent over E. Reciprocally, if $\mathsf{E} : \mathcal{A} \to \mathcal{B}$ is an *n.f.* conditional expectation and $\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n$ is a collection of von Neumann subalgebras of $\mathcal{A}$ freely independent over E and generating $\mathcal{A}$, then $\mathcal{A}$ is isomorphic to $*_{\mathcal{B}}\mathsf{A}_k$.

REMARK 1.1. Let $\mathcal{A} = \mathsf{A}_1 * \mathsf{A}_2 * \cdots * \mathsf{A}_n$ be a reduced free product of von Neumann algebras (i.e., $\mathcal{A}$ is amalgamated over the complex field) equipped with its natural *n.f.* state ϕ. Let $\mathcal{B}$ be a σ-finite von Neumann algebra, not necessarily included in $\mathcal{A}$. A relevant example of the construction outlined above is the following. Let us consider the conditional expectation $\mathsf{E} : \mathcal{A} \bar{\otimes} \mathcal{B} \to \mathcal{B}$ defined by $\mathsf{E}(a \otimes b) = \phi(a) 1_{\mathcal{A}} \otimes b$. Then $\mathsf{A}_1 \bar{\otimes} \mathcal{B}, \mathsf{A}_2 \bar{\otimes} \mathcal{B}, \ldots, \mathsf{A}_n \bar{\otimes} \mathcal{B}$ are freely independent subalgebras of $\mathcal{A} \bar{\otimes} \mathcal{B}$ over E. In particular, we obtain

$$\mathcal{M} = \mathcal{A} \bar{\otimes} \mathcal{B} = *_{\mathcal{B}} (\mathsf{A}_k \bar{\otimes} \mathcal{B}).$$

Therefore, taking $\mathcal{B}$ to be $\mathcal{B}(\ell_2)$, it turns out that the complete boundedness of a map $u : L_p(\mathcal{A}) \to L_p(\mathcal{A})$ is equivalent to the boundedness (with the same norm) of the map $u \otimes id_{\mathcal{B}} : L_p(\mathcal{M}) \to L_p(\mathcal{M})$. In other words, since our results are presented for general amalgamated free products, complete boundedness follows automatically and is instrumental in some of our arguments. This will be used below without any further reference.

REMARK 1.2. Let $\mathcal{A}$ be a von Neumann algebra equipped with an *n.f.* state ϕ and $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$. According to Takesaki [40], the existence and uniqueness of an *n.f.* conditional expectation $\mathsf{E} : \mathcal{A} \to \mathcal{B}$ is equivalent to the invariance of $\mathcal{B}$ under the action of the modular automorphism group σ_t^{ϕ} associated to $(\mathcal{A}, \phi)$. Moreover, in that case we have $\phi \circ \mathsf{E} = \phi$ and following Connes [6],

$$\mathsf{E} \circ \sigma_t^{\phi} = \sigma_t^{\phi} \circ \mathsf{E}.$$

In what follows we shall assume this invariance in all the von Neumann subalgebras considered. Hence, we may think of a *natural* conditional expectation $\mathsf{E} : \mathcal{A} \to \mathcal{B}$. This somehow justifies our relaxed notation for reduced amalgamated free products, where we do not make explicit the associated conditional expectations. This should not cause any confusion since only reduced free products are considered here.

2. Rosenthal/Voiculescu type inequalities. In this section we present a free analogue of Rosenthal's inequality (R_p). Let $\mathcal{A}$ be the amalgamated reduced free product $*_{\mathcal{B}} \mathsf{A}_k$ with $1 \le k \le n$ and $\mathcal{B}$ a common von Neumann subalgebra of the A_k's, equipped with an *n.f.* state φ. As we have already seen, the state φ induces an *n.f.* state ϕ on $\mathcal{A}$ by setting $\phi = \varphi \circ \mathsf{E}$. Given a nonnegative integer d, we shall write $\mathbf{P}_{\mathcal{A}}(d)$ for the closure of elements of the form

$$a = \sum_{\alpha \in \Lambda} \sum_{j_1 \neq j_2 \neq \cdots \neq j_d} a_{j_1}(\alpha) a_{j_2}(\alpha) \cdots a_{j_d}(\alpha), \tag{1}$$

with $a_{j_k}(\alpha) \in \mathring{\mathsf{A}}_{j_k}$ and α running over a finite set Λ. In other words, $\mathbf{P}_{\mathcal{A}}(d)$ is the subspace of $\mathcal{A}$ of homogeneous free polynomials of degree d. When d is 0, the expression (1) does not make sense. $\mathbf{P}_{\mathcal{A}}(0)$ is meant to be $\mathcal{B}$. Then we define the space $\mathbf{P}_{\mathcal{A}}(p,d)$ as the closure in $L_p(\mathcal{A})$ of

$$\mathbf{P}_{\mathcal{A}}(d)d_\phi^{1/p},$$

where d_ϕ denotes the density of the state ϕ. Note that, by using approximation with analytic elements, we might have well located the density d_ϕ on the left of $\mathbf{P}_{\mathcal{A}}(d)$ with no consequence in the definition of $\mathbf{P}_{\mathcal{A}}(p,d)$.

Similarly, $\mathbf{Q}_{\mathcal{A}}(d)$ denotes the subspace of polynomials of degree less than or equal to d in $\mathcal{A}$ and

$$\mathbf{Q}_{\mathcal{A}}(p,d) = \bigoplus_{k=0}^{d} \mathbf{P}_{\mathcal{A}}(p,k) \qquad \text{with } \mathbf{P}_{\mathcal{A}}(p,0) = L_p(\mathcal{B}).$$

The complementation result below from [37] is crucial for our further purposes. Indeed, it was proved there that $\mathbf{P}_{\mathcal{A}}(d)$ and $\mathbf{Q}_{\mathcal{A}}(d)$ are complemented in $\mathcal{A}$ with projection constants controlled by $4d$ and $2d+1$ respectively. Thus, transposition and complex interpolation yield the following result for $1 \le p \le \infty$.

THEOREM 2.1. *The following results hold*:

(a) $\mathbf{P}_{\mathcal{A}}(p,d)$ *is complemented in* $L_p(\mathcal{A})$ *with projection constant* $\le 4d$.
(b) $\mathbf{Q}_{\mathcal{A}}(p,d)$ *is complemented in* $L_p(\mathcal{A})$ *with projection constant* $\le 2d+1$.

REMARK 2.2. In what follows we shall write

$$\Pi_{\mathcal{A}}(p,d) : L_p(\mathcal{A}) \to \mathbf{P}_{\mathcal{A}}(p,d) \quad \text{and} \quad \Gamma_{\mathcal{A}}(p,d) : L_p(\mathcal{A}) \to \mathbf{Q}_{\mathcal{A}}(p,d)$$

for the natural projections determined by Theorem 2.1. It is worthy of mention that both projections above are completely determined by the natural projections $\Pi_{\mathcal{A}}(\infty,d)$ and $\Gamma_{\mathcal{A}}(\infty,d)$ from [37]. More precisely, given $x \in \mathcal{A}$, we have

$$\begin{aligned} \Pi_{\mathcal{A}}(p,d)(xd_\phi^{1/p}) &= \Pi_{\mathcal{A}}(\infty,d)(x)d_\phi^{1/p} \quad \text{and} \\ \Gamma_{\mathcal{A}}(p,d)(xd_\phi^{1/p}) &= \Gamma_{\mathcal{A}}(\infty,d)(x)d_\phi^{1/p}. \end{aligned} \tag{2}$$

In particular, by the density of the subspace $\mathcal{A}d_\phi^{1/p}$ in $L_p(\mathcal{A})$, the relations above completely determine the projections $\Pi_{\mathcal{A}}(p,d)$ and $\Gamma_{\mathcal{A}}(p,d)$. This will be essential in what follows for the interpolation of the spaces $\mathbf{P}_{\mathcal{A}}(p,d)$ and $\mathbf{Q}_{\mathcal{A}}(p,d)$ by the complex method. Another relevant fact implicitly used in the sequel is that both $\Pi_{\mathcal{A}}(\infty,d)$ and $\Gamma_{\mathcal{A}}(\infty,d)$ commute with the modular automorphism group of ϕ. Indeed, this follows from the action of the modular group σ_t^ϕ on reduced words

$$\sigma_t^\phi(a_{j_1}a_{j_2}\cdots a_{j_d}) = \sigma_t^\phi(a_{j_1})\sigma_t^\phi(a_{j_2})\cdots\sigma_t^\phi(a_{j_d}).$$

2.1. *The mappings $\mathcal{L}_k$ and $\mathcal{R}_k$.* Elements in $\mathring{\mathsf{A}}_k$ will be called mean-zero letters of A_k. Given $1 \le k \le n$, we consider the map $\mathcal{L}_k$ on $\mathbf{P}_{\mathcal{A}}(p,d)$ which collects all the reduced words starting with a mean-zero letter in A_k. Similarly, the map $\mathcal{R}_k$ collects all the reduced words ending with a mean-zero letter in A_k. That is, if a is given by the expression (1), we have

$$\mathcal{L}_k(a) = \sum_{\alpha \in \Lambda} \sum_{j_1 = k \neq j_2 \neq \cdots \neq j_d} a_{j_1}(\alpha) a_{j_2}(\alpha) \cdots a_{j_d}(\alpha),$$

$$\mathcal{R}_k(a) = \sum_{\alpha \in \Lambda} \sum_{j_1 \neq j_2 \neq \cdots \neq k = j_d} a_{j_1}(\alpha) a_{j_2}(\alpha) \cdots a_{j_d}(\alpha).$$

Of course, both $\mathcal{L}_k$ and $\mathcal{R}_k$ vanish on $\mathbf{P}_{\mathcal{A}}(p,0)$. The mappings $\mathcal{L}_k$ and $\mathcal{R}_k$ were introduced by Voiculescu [44] and clearly satisfy $\mathcal{L}_k(a^*) = \mathcal{R}_k(a)^*$. They are clearly $\mathcal{B}$-bimodule maps which commute with the modular automorphism group and with densities as in (2). Note also that $\mathcal{L}_k$ and $\mathcal{R}_k$ can also be regarded as orthogonal projections on $L_2(\mathcal{A})$. Thus, when $p = 2$, we need no restriction to the subspaces of homogeneous polynomials. In this particular case, we shall denote $\mathcal{L}_k$ and $\mathcal{R}_k$ respectively by L_k and R_k. Now we prove some fundamental freeness relations that will be used throughout the whole paper with no further reference.

LEMMA 2.3. *If $1 \le i, j \le n$ and $a_i, a_j \in \mathbf{P}_{\mathcal{A}}(d)$, we have*

$$\mathsf{L}_i [a_i - \mathcal{R}_i(a_i)]^* [a_j - \mathcal{R}_j(a_j)] \mathsf{L}_j = \delta_{ij} \mathsf{E}\big([a_i - \mathcal{R}_i(a_i)]^* [a_j - \mathcal{R}_j(a_j)]\big) \mathsf{L}_j, \tag{3}$$

$$(1 - \mathsf{L}_i)[\mathcal{R}_i(a_i)]^* [\mathcal{R}_j(a_j)](1 - \mathsf{L}_j) = \delta_{ij} \mathsf{E}\big([\mathcal{R}_i(a_i)]^* [\mathcal{R}_j(a_j)]\big)(1 - \mathsf{L}_j). \tag{4}$$

PROOF. By the GNS construction on $(\mathcal{A}, \phi)$, we know that $\mathcal{A}$ acts on $L_2(\mathcal{A})$ by left multiplication. Thus, we may regard the left-hand sides of (3) and (4) as mappings on $L_2(\mathcal{A})$. To prove (3), we first note that

$$[a_i - \mathcal{R}_i(a_i)]^* [a_j - \mathcal{R}_j(a_j)]$$

is a linear combination of words of the following form:

$$w_{xy} = x_{i_d}^* x_{i_{d-1}}^* \cdots x_{i_1}^* y_{j_1} \cdots y_{j_{d-1}} y_{j_d},$$

where $x_{i_s} \in \mathring{\mathsf{A}}_{i_s}$, $y_{j_s} \in \mathring{\mathsf{A}}_{j_s}$ and

$$i_1 \neq i_2 \neq \cdots \neq i_d \neq i,$$
$$j \neq j_d \neq \cdots \neq j_2 \neq j_1.$$

When $i_1 \neq j_1$, it turns out that w_{xy} is a reduced word and, since $j_d \neq j$, the map $w_{xy} \mathsf{L}_j$ can only act as a tensor so that the range of $w_{xy} \mathsf{L}_j$ lies in the orthocomplement of $\mathsf{L}_i(L_2(\mathcal{A}))$, since $i_d \neq i$. In other words, in that case we have

$$\mathsf{L}_i w_{xy} \mathsf{L}_j = 0 = \mathsf{L}_i \mathsf{E}(w_{xy}) \mathsf{L}_j.$$

When $i_1 = j_1$, we may write $w_{xy} = w'_{xy} + w''_{xy}$ with

$$w'_{xy} = x^*_{i_d} x^*_{i_{d-1}} \cdots x^*_{i_2} \mathsf{E}(x^*_{i_1} y_{j_1}) y_{j_2} \cdots y_{j_{d-1}} y_{j_d}.$$

If $d \geq 2$, the argument above implies again that $\mathsf{L}_i w''_{xy} \mathsf{L}_j = 0$ since w''_{xy} is a reduced word not starting with mean-zero letters in A_i nor ending with mean-zero letters in A_j. Then it is clear that we can iterate the same argument and obtain

$$\mathsf{L}_i w_{xy} \mathsf{L}_j = \mathsf{L}_i \mathsf{E}(x^*_{i_d} \cdots \mathsf{E}(x^*_{i_1} y_{j_1}) \cdots y_{j_d}) \mathsf{L}_j = \mathsf{L}_i \mathsf{E}(w_{xy}) \mathsf{L}_j = \delta_{ij} \mathsf{E}(w_{xy}) \mathsf{L}_j.$$

The second identity follows easily by freeness. Summing up, we obtain (3).

The proof of (4) is quite similar. Indeed, now we may write $[\mathcal{R}_i(a_i)]^*[\mathcal{R}_j(a_j)]$ as a linear combination of words w_{xy} with the form given above and satisfying

$$\begin{aligned} i_1 \neq i_2 \neq \cdots \neq i_d = i, \\ j = j_d \neq \cdots \neq j_2 \neq j_1. \end{aligned}$$

Then the arguments above lead to the following identity:

$$\begin{aligned} (1 - \mathsf{L}_i) w_{xy} (1 - \mathsf{L}_j) &= (1 - \mathsf{L}_i) \mathsf{E}(x^*_{i_d} \cdots \mathsf{E}(x^*_{i_1} y_{j_1}) \cdots y_{j_d})(1 - \mathsf{L}_j) \\ &= \delta_{ij} \mathsf{E}(x^*_{i_d} \cdots \mathsf{E}(x^*_{i_1} y_{j_1}) \cdots y_{j_d})(1 - \mathsf{L}_j) \\ &= \delta_{ij} \mathsf{E}(w_{xy})(1 - \mathsf{L}_j), \end{aligned}$$

where the second identity holds because the only way not to have

$$\mathsf{E}(x^*_{i_d} \cdots \mathsf{E}(x^*_{i_1} y_{j_1}) \cdots y_{j_d}) = 0$$

is the case where the indices i_s and j_s fit, that is, $i_s = j_s$ for $1 \leq s \leq d$. Therefore, since $i = i_d$ and $j_d = j$, the symbol δ_{ij} appears. Summing up one more time, we obtain the identity (4). This completes the proof. □

REMARK 2.4. The assumption that a_i and a_j are homogeneous and of the same degree is essential in Lemma 2.3. Indeed, the following counterexample was brought to our attention by Ken Dykema. Let $\mathbb{F}_2$ denote a free group on two generators g_1, g_2 and let $\lambda : \mathbb{F}_2 \to \mathcal{B}(\ell_2(\mathbb{F}_2))$ stand for the left regular representation. Let A_k be the von Neumann algebra generated by $\lambda(g_k)$ for $k = 1, 2$. In this case, $\mathcal{A} = \mathsf{A}_1 * \mathsf{A}_2$ is the von Neumann algebra generated by λ and the conditional expectation E is just $1_{\mathcal{A}} \tau$, where τ is the natural trace on $\mathcal{A}$. Then we consider the (nonhomogeneous) polynomial $a = \lambda(g_2) + \lambda(g_2 g_1 g_2)$. Clearly, we have $\mathcal{R}_1(a) = 0$ and

$$\begin{aligned} a^* a &= \big(\lambda(g_2)^* + \lambda(g_2 g_1 g_2)^*\big)\big(\lambda(g_2) + \lambda(g_2 g_1 g_2)\big) \\ &= \mathsf{E}(a^* a) + \lambda(g_1 g_2) + \lambda(g_1 g_2)^*. \end{aligned}$$

Taking, for instance, $h = \delta_{g_1 g_2}$, we see that

$$\mathsf{L}_1 a^* a \mathsf{L}_1(h) = \mathsf{L}_1 \mathsf{E}(a^* a) \mathsf{L}_1(h) + \delta_{g_1 g_2 g_1 g_2} \neq \mathsf{L}_1 \mathsf{E}(a^* a) \mathsf{L}_1(h).$$

Thus, identity (3) does not hold for a. A similar counterexample can be constructed for (4). In particular, since identities (3) and (4) are essential in most of our results below, this explains why this paper is written in terms of homogeneous polynomials.

LEMMA 2.5. *If $1 \le p \le \infty$ and $a_1, a_2, \dots, a_n \in \mathbf{P}_{\mathcal{A}}(p,d)$,*

$$\left\| \left(\sum_{k=1}^n \mathcal{R}_k(a_k)^* \mathcal{R}_k(a_k) \right)^{1/2} \right\|_p \le c d^2 \left\| \left(\sum_{k=1}^n a_k^* a_k \right)^{1/2} \right\|_p,$$
$$\left\| \left(\sum_{k=1}^n \mathcal{R}_k(a_k) \mathcal{R}_k(a_k)^* \right)^{1/2} \right\|_p \le c d^2 \left\| \left(\sum_{k=1}^n a_k a_k^* \right)^{1/2} \right\|_p.$$

Moreover, the same inequalities hold with the operator $\mathcal{L}_k$ instead of $\mathcal{R}_k$.

PROOF. Recall that we noted previously that $\mathcal{L}_k(a^*) = \mathcal{R}_k(a)^*$. Consequently, it suffices to prove the inequalities for the $\mathcal{R}_k$'s. On the other hand, in the row/column terminology (i.e., taking R_p^n and C_p^n to be the first row and column of the Schatten class S_p^n), the two terms on the right-hand side are the norms of $(a_1, a_2, \dots, a_n)$ in $R_p^n(L_p(\mathcal{A}))$ and $C_p^n(L_p(\mathcal{A}))$, respectively. According to [31], both spaces embed completely isometrically into and are completely contractively complemented in the space

$$S_p^n(L_p(\mathcal{A})) = L_p(\mathrm{M}_n \otimes \mathcal{A}) = L_p\big(*_{\mathrm{M}_n \otimes \mathcal{B}}(\mathrm{M}_n \otimes \mathsf{A}_k)\big) = L_p(\mathcal{A}_n).$$

Therefore, by means of Theorem 2.1 (applied to the amplified algebra $\mathcal{A}_n$), we know that $\mathbf{P}_{\mathcal{A}_n}(p,d)$ is complemented in $L_p(\mathcal{A}_n)$ with projection constant $4d$. Using the same projection restricted to $R_p^n(L_p(\mathcal{A}))$ and $C_p^n(L_p(\mathcal{A}))$, we conclude that the respective subspaces of homogeneous polynomials $R_p^n(\mathbf{P}_{\mathcal{A}}(p,d))$ and $C_p^n(\mathbf{P}_{\mathcal{A}}(p,d))$ form interpolation scales with equivalent norms up to a constant $4d$. By this observation, it suffices to show that the assertion holds for $p=1$ and $p=\infty$ with constant in both cases controlled by cd. In fact, in the latter case we shall even prove that the constant does not depend on d. This will be used sometimes in the paper without further reference. We prove the desired estimates in several steps.

Step 1. Let us prove the first inequality of $\mathcal{R}_k$'s for $p=\infty$. The GNS construction on $(\mathcal{A}, \phi)$ implies that $\mathcal{A}$ acts on $L_2(\mathcal{A})$ by left multiplication. Thus, we may regard $a_k \mathsf{L}_k$, $\mathcal{R}_k(a_k)(1-\mathsf{L}_k)$ and $(id_{\mathcal{A}} - \mathcal{R}_k)(a_k)\mathsf{L}_k$ as mappings on $L_2(\mathcal{A})$. Note that 1 is understood here as the identity map on $L_2(\mathcal{A})$, while $id_{\mathcal{A}}$ denotes the identity map on $\mathcal{A}$. In particular, since we have

$$\mathcal{R}_k(a_k) = a_k \mathsf{L}_k + \mathcal{R}_k(a_k)(1-\mathsf{L}_k) - (id_{\mathcal{A}} - \mathcal{R}_k)(a_k)\mathsf{L}_k,$$

we obtain, by the triangle inequality [with e_{ij} denoting the usual matrix units in $\mathcal{B}(\ell_2)$],

$$\begin{aligned}\left\| \left(\sum_{k=1}^n \mathcal{R}_k(a_k)^* \mathcal{R}_k(a_k) \right)^{1/2} \right\|_\infty &\le \left\| \sum_{k=1}^n e_{k1} \otimes a_k \mathsf{L}_k \right\|_\infty \\ &\quad + \left\| \sum_{k=1}^n e_{k1} \otimes \mathcal{R}_k(a_k)(1 - \mathsf{L}_k) \right\|_\infty \\ &\quad + \left\| \sum_{k=1}^n e_{k1} \otimes (id_{\mathcal{A}} - \mathcal{R}_k)(a_k) \mathsf{L}_k \right\|_\infty .\end{aligned}$$

If $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3$ denote respectively the terms on the right-hand side, we have

$$\mathrm{A}_1 = \left\| \left(\sum_{k=1}^n e_{kk} \otimes a_k \right) \left(\sum_{k=1}^n e_{k1} \otimes \mathsf{L}_k \right) \right\|_\infty \le \max_{1 \le k \le n} \|a_k\|_\infty \left\| \sum_{k=1}^n \mathsf{L}_k \right\|_\infty^{1/2} .$$

Thus, since $\sum_k \mathsf{L}_k = 1 - \mathsf{E}$, we find

$$\mathrm{A}_1 \le \left\| \left(\sum_{k=1}^n a_k^* a_k \right)^{1/2} \right\|_\infty .$$

On the other hand, by (4), we have

$$\begin{aligned}\mathrm{A}_2 &= \left\| \sum_{k=1}^n (1 - \mathsf{L}_k) \mathcal{R}_k(a_k)^* \mathcal{R}_k(a_k)(1 - \mathsf{L}_k) \right\|_\infty^{1/2} \\ &= \left\| \sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(a_k)^* \mathcal{R}_k(a_k))(1 - \mathsf{L}_k) \right\|_\infty^{1/2} .\end{aligned}$$

Now, since L_k commutes with $\mathcal{B}$, the last term is

$$\begin{aligned}&\sum_k \mathsf{E}(\mathcal{R}_k(a_k)^* \mathcal{R}_k(a_k))^{1/2} (1 - \mathsf{L}_k) \mathsf{E}(\mathcal{R}_k(a_k)^* \mathcal{R}_k(a_k))^{1/2} \\ &\qquad \le \sum_k \mathsf{E}(\mathcal{R}_k(a_k)^* \mathcal{R}_k(a_k)).\end{aligned}$$

Next, we observe that

$$\mathsf{E}(\mathcal{R}_k(a_k)^* \mathcal{R}_k(a_k)) \le \mathsf{E}(a_k^* a_k). \tag{5}$$

Indeed, since a_k is mean-zero,

$$a_k = \sum_j \mathcal{R}_j(a_k);$$

so, by freeness,

$$\begin{aligned}\mathsf{E}(a_k^*a_k) &= \sum_{i,j}\mathsf{E}(\mathcal{R}_i(a_k)^*\mathcal{R}_j(a_k)) = \sum_j \mathsf{E}(\mathcal{R}_j(a_k)^*\mathcal{R}_j(a_k)) \\ &\geq \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k)).\end{aligned}$$

This proves (5). Combining the estimates above, we find

$$\mathrm{A}_2 \leq \left\|\sum_{k=1}^n \mathsf{E}(a_k^*a_k)\right\|_\infty^{1/2} \leq \left\|\left(\sum_{k=1}^n a_k^*a_k\right)^{1/2}\right\|_\infty.$$

The estimate for A_3 is similar to the one for A_2:

$$\begin{aligned}\mathrm{A}_3 &= \left\|\sum_{k=1}^n \mathsf{L}_k[a_k-\mathcal{R}_k(a_k)]^*[a_k-\mathcal{R}_k(a_k)]\mathsf{L}_k\right\|_\infty^{1/2} \\ &= \left\|\sum_{k=1}^n \mathsf{E}\big([a_k-\mathcal{R}_k(a_k)]^*[a_k-\mathcal{R}_k(a_k)]\big)\mathsf{L}_k\right\|_\infty^{1/2} \\ &\leq \left\|\sum_{k=1}^n \mathsf{E}\big([a_k-\mathcal{R}_k(a_k)]^*[a_k-\mathcal{R}_k(a_k)]\big)\right\|_\infty^{1/2},\end{aligned}$$

where the last inequality follows once more from the fact that L_k commutes with $\mathcal{B}$. Now, using that $\mathsf{E}(a^*\mathcal{R}_k(a)) = \mathsf{E}(\mathcal{R}_k(a)^*\mathcal{R}_k(a))$ for any homogeneous polynomial a, we easily find

$$\mathsf{E}\big([a_k-\mathcal{R}_k(a_k)]^*[a_k-\mathcal{R}_k(a_k)]\big) = \mathsf{E}\big(a_k^*a_k - \mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k)\big) \leq \mathsf{E}(a_k^*a_k). \tag{6}$$

Step 2. Now we prove the second inequality for $\mathcal{R}_k$'s. As above, we have

$$\begin{aligned}\left\|\left(\sum_{k=1}^n \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*\right)^{1/2}\right\|_\infty &\leq \left\|\sum_{k=1}^n e_{1k}\otimes a_k\mathsf{L}_k\right\|_\infty \\ &\quad + \left\|\sum_{k=1}^n e_{1k}\otimes \mathcal{R}_k(a_k)(1-\mathsf{L}_k)\right\|_\infty \\ &\quad + \left\|\sum_{k=1}^n e_{1k}\otimes (id_{\mathcal{A}}-\mathcal{R}_k)(a_k)\mathsf{L}_k\right\|_\infty.\end{aligned}$$

We write $\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3$ for the terms on the right-hand side. The estimate of B_1 is trivial:

$$\mathrm{B}_1 = \left\|\left(\sum_{k=1}^n e_{1k}\otimes a_k\right)\left(\sum_{k=1}^n e_{kk}\otimes \mathsf{L}_k\right)\right\|_\infty \leq \left\|\left(\sum_{k=1}^n a_ka_k^*\right)^{1/2}\right\|_\infty.$$

On the other hand, by (4) and (5), we may write

$$\begin{aligned}
\mathrm{B}_2 &= \Big\| \sum_{i,j=1}^{n} e_{ij} \otimes (1-\mathsf{L}_i)\mathcal{R}_i(a_i)^* \mathcal{R}_j(a_j)(1-\mathsf{L}_j) \Big\|_\infty^{1/2} \\
&= \Big\| \sum_{k=1}^{n} e_{kk} \otimes \mathsf{E}(\mathcal{R}_k(a_k)^* \mathcal{R}_k(a_k))(1-\mathsf{L}_k) \Big\|_\infty^{1/2} \\
&\le \max_{1\le k\le n} \|\mathsf{E}(a_k^* a_k)\|_\infty^{1/2}.
\end{aligned}$$

Finally, to estimate B_3, we use (3) and (6):

$$\begin{aligned}
\mathrm{B}_3 &= \Big\| \sum_{i,j=1}^{n} e_{ij} \otimes \mathsf{L}_i [a_i - \mathcal{R}_i(a_i)]^* [a_j - \mathcal{R}_j(a_j)] \mathsf{L}_j \Big\|_\infty^{1/2} \\
&= \Big\| \sum_{k=1}^{n} e_{kk} \otimes \mathsf{E}\big([a_k - \mathcal{R}_k(a_k)]^* [a_k - \mathcal{R}_k(a_k)]\big) \mathsf{L}_k \Big\|_\infty^{1/2} \\
&\le \max_{1\le k\le n} \big\| \mathsf{E}\big([a_k - \mathcal{R}_k(a_k)]^* [a_k - \mathcal{R}_k(a_k)]\big) \big\|_\infty^{1/2} \\
&\le \max_{1\le k\le n} \|\mathsf{E}(a_k^* a_k)\|_\infty^{1/2}.
\end{aligned}$$

Step 3. Now we use a duality argument to prove the same estimates in $L_1(\mathcal{A})$. Recall d_ϕ denotes the density associated to the state ϕ of $\mathcal{A}$. Let $a \in \mathbf{P}_\mathcal{A}(d)$ and $x \in \mathcal{A}$ be a finite sum of reduced words. Then we have

$$\begin{aligned}
\langle x, \mathcal{R}_k(d_\phi a)\rangle &= \mathrm{tr}_\mathcal{A}(x^* \mathcal{R}_k(d_\phi a)) = \mathrm{tr}_\mathcal{A}(d_\phi \mathcal{R}_k(a) x^*) \\
&= \varphi(\mathsf{E}(\mathcal{R}_k(a)x^*)) = \varphi(\mathsf{E}(a\mathcal{L}_k(x^*))) \qquad (7)\\
&= \mathrm{tr}_\mathcal{A}(d_\phi a \mathcal{R}_k(x)^*) = \langle \mathcal{R}_k(x), d_\phi a\rangle.
\end{aligned}$$

Therefore, since we have

$$R_1^n(\mathbf{P}_\mathcal{A}(d,1))^* \simeq_{4d} R_\infty^n(\mathbf{P}_\mathcal{A}(d)),$$

from Theorem 2.1, we may use approximation and deduce

$$\begin{aligned}
&\Big\| \sum_k e_{1k} \otimes \mathcal{R}_k(a_k) \Big\|_1 \\
&\qquad \le 4d \sup_{\|x\|_{R_\infty^n(\mathbf{P}_\mathcal{A}(d))}\le 1} \sum_k \langle x_k, \mathcal{R}_k(a_k)\rangle \\
&\qquad = 4d \sup_{\|x\|_{R_\infty^n(\mathbf{P}_\mathcal{A}(d))}\le 1} \sum_k \langle \mathcal{R}_k(x_k), a_k\rangle
\end{aligned}$$

$$\leq 4d \sup_{\|x\|_{R_\infty^n(\mathbf{P}_{\mathcal{A}}(d))}\leq 1} \left\| \left(\sum_k \mathcal{R}_k(x_k)\mathcal{R}_k(x_k)^* \right)^{1/2} \right\|_\infty \left\| \left(\sum_k a_k a_k^* \right)^{1/2} \right\|_1 .$$

By step 2,

$$\left\| \left(\sum_k \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^* \right)^{1/2} \right\|_1 \leq 12d \left\| \left(\sum_k a_k a_k^* \right)^{1/2} \right\|_1 .$$

Using step 1 and columns instead of rows, we get the remaining estimate. □

REMARK 2.6. A detailed reading of the proof of Lemma 2.5 shows that the constant is controlled by cd^2 for $1 \leq p \leq 2$ and by cd for $2 \leq p \leq \infty$. Moreover, the same arguments are valid to show that Lemma 2.5 also holds replacing $\mathcal{L}_k$ or $\mathcal{R}_k$ by $\mathcal{Q}_k = \mathcal{L}_k\mathcal{R}_k = \mathcal{R}_k\mathcal{L}_k$ (to be used below). These generalizations of Lemma 2.5 will be used several times in the sequel.

REMARK 2.7. In Remark 2.4 we have partially justified why this paper is written in terms of homogeneous polynomials. On the other hand, Lemma 2.5 for $n = 1$ shows that $\mathcal{L}_k$ and $\mathcal{R}_k$ are bounded operators when acting on $\mathbf{P}_{\mathcal{A}}(p, d)$ for any $1 \leq p \leq \infty$ and $d \geq 1$. Another relevant fact which justifies the use of homogeneous polynomials is that $\mathcal{L}_k$ and $\mathcal{R}_k$ are not bounded on $L_\infty(\mathcal{A})$. The following simple counterexample was brought to our attention by Ana Maria Popa. Consider again the free group $\mathbb{F}_2$ with two generators g_1, g_2 and keep the terminology employed in Remark 2.4. Let H be the subgroup of $\mathbb{F}_2$ generated by $w = g_1 g_2$. Of course, it is clear that H is isomorphic to $\mathbb{Z}$ and that $\lambda(\mathsf{H})'' \simeq L_\infty(\mathbb{T})$. Moreover, we obviously have $\mathcal{L}_1(\lambda(w^k)) = \delta_{k>0}\lambda(w^k)$. In particular, if $\mathcal{A} = \lambda(\mathbb{F}_2)''$ denotes the reduced group von Neumann algebra, it turns out that the restriction of $\mathcal{L}_1 : \mathcal{A} \to \mathcal{A}$ to $\lambda(\mathsf{H})$ behaves as $\frac{1}{2}(id_{L_\infty(\mathbb{T})} + \mathbf{H})$, where $\mathbf{H}$ denotes the Hilbert transform on the circle. The claim follows since the Hilbert transform is known to be unbounded on $L_\infty(\mathbb{T})$. Moreover, the same example also shows that the map $\mathcal{Q}_k = \mathcal{R}_k\mathcal{L}_k$ (to be used below) is unbounded on $L_\infty(\mathcal{A})$. Indeed, $\mathcal{Q}_1$ is not bounded on the subspace $\lambda(\mathsf{H}g_1)''$ since

$$\mathcal{Q}_1(\lambda(w^k g_1)) = \delta_{k>0}\lambda(w^k g_1).$$

PROPOSITION 2.8. *If $a_1, a_2, \ldots, a_n \in \mathbf{P}_{\mathcal{A}}(d)$, we have*

$$\left\| \sum_{k=1}^n \mathcal{L}_k(a_k) \right\|_\infty \sim_c \max\left\{ \left\| \sum_{k=1}^n \mathcal{L}_k(a_k)^*\mathcal{L}_k(a_k) \right\|_\infty^{1/2}, \left\| \sum_{k=1}^n \mathsf{E}(\mathcal{L}_k(a_k)\mathcal{L}_k(a_k)^*) \right\|_\infty^{1/2} \right\},$$

$$\left\| \sum_{k=1}^n \mathcal{R}_k(a_k) \right\|_\infty \sim_c \max\left\{ \left\| \sum_{k=1}^n \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^* \right\|_\infty^{1/2}, \left\| \sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k)) \right\|_\infty^{1/2} \right\}.$$

PROOF. Once more, we only prove the assertion for $\mathcal{R}_k$. We have

$$\begin{aligned}\Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)\Big\|_\infty &\le \Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathsf{L}_k\Big\|_\infty + \Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)(1-\mathsf{L}_k)\Big\|_\infty \\ &= \Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathsf{L}_k\mathcal{R}_k(a_k)^*\Big\|_\infty^{1/2} \\ &\quad + \Big\|\sum_{i,j=1}^n (1-\mathsf{L}_i)\mathcal{R}_i(a_i)^*\mathcal{R}_j(a_j)(1-\mathsf{L}_j)\Big\|_\infty^{1/2} \\ &= \Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathsf{L}_k\mathcal{R}_k(a_k)^*\Big\|_\infty^{1/2} \\ &\quad + \Big\|\sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k))(1-\mathsf{L}_k)\Big\|_\infty^{1/2}.\end{aligned}$$

The first term is clearly bounded by $\sum_k \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*$. For the second term, we argue as in the proof of Lemma 2.5. That is, using that L_k commutes with $\mathcal{B}$, we can write $\sum_k \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k))(1-\mathsf{L}_k)$ as

$$\sum_k \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k))^{1/2}(1-\mathsf{L}_k)\mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k))^{1/2}.$$

Thus, we obtain the upper estimate

$$\Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)\Big\|_\infty \le \Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*\Big\|_\infty^{1/2} + \Big\|\sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k))\Big\|_\infty^{1/2}.$$

For the lower estimate, using freeness, we clearly have

$$\begin{aligned}&\Big\|\sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k))\Big\|_\infty \\ &\qquad = \Big\|\sum_{i,j=1}^n \mathsf{E}(\mathcal{R}_i(a_i)^*\mathcal{R}_j(a_j))\Big\|_\infty \\ &\qquad \le \Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)\Big\|_\infty^2.\end{aligned} \tag{8}$$

Thus, it remains to show that

$$\Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*\Big\|_\infty^{1/2} \le c\Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)\Big\|_\infty.$$

To that aim, we observe from (8) and the calculation above that

$$\begin{aligned}\Bigl\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathsf{L}_k\Bigr\|_\infty &\le \Bigl\|\sum_{k=1}^n \mathcal{R}_k(a_k)\Bigr\|_\infty + \Bigl\|\sum_{k=1}^n \mathcal{R}_k(a_k)(1-\mathsf{L}_k)\Bigr\|_\infty \\ &\le 2\Bigl\|\sum_{k=1}^n \mathcal{R}_k(a_k)\Bigr\|_\infty.\end{aligned}$$

Hence, since the term

$$\mathcal{S}(a,\varepsilon) = \Bigl\|\sum_{k=1}^n \varepsilon_k \mathcal{R}_k(a_k)\mathsf{L}_k\Bigr\|_\infty + \Bigl\|\sum_{k=1}^n \mathsf{E}((\varepsilon_k\mathcal{R}_k(a_k))^*(\varepsilon_k\mathcal{R}_k(a_k)))\Bigr\|_\infty^{1/2}$$

is independent of any choice of signs $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in \Omega = \{\pm 1\}^n$, we find

$$\Bigl\|\sum_{k=1}^n \varepsilon_k \mathcal{R}_k(a_k)\Bigr\|_\infty \le \mathcal{S}(a,\varepsilon) \le 3\Bigl\|\sum_{k=1}^n \mathcal{R}_k(a_k)\Bigr\|_\infty. \tag{9}$$

Therefore, we obtain

$$\begin{aligned}\Bigl\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*\Bigr\|_\infty &= \Bigl\|\int_\Omega \sum_{i,j=1}^n \varepsilon_i\mathcal{R}_i(a_i)\varepsilon_j\mathcal{R}_j(a_j)^*\,d\varepsilon\Bigr\|_\infty \\ &\le \int_\Omega \Bigl\|\sum_{i,j=1}^n \varepsilon_i\mathcal{R}_i(a_i)\varepsilon_j\mathcal{R}_j(a_j)^*\Bigr\|_\infty d\varepsilon \\ &\le \int_\Omega \Bigl\|\sum_{i=1}^n \varepsilon_i\mathcal{R}_i(a_i)\Bigr\|_\infty \Bigl\|\sum_{j=1}^n \varepsilon_j\mathcal{R}_j(a_j)^*\Bigr\|_\infty d\varepsilon \\ &\le 9\Bigl\|\sum_{k=1}^n \mathcal{R}_k(a_k)\Bigr\|_\infty^2.\end{aligned}$$

This is the remaining inequality to complete the proof of the lower estimate. □

COROLLARY 2.9. *If $2 \le p \le \infty$ and $a_1, a_2, \dots, a_n \in \mathbf{P}_{\mathcal{A}}(p,d)$, we have*

$$\Bigl\|\sum_{k=1}^n \mathcal{L}_k(a_k)\Bigr\|_p \le cd^2 \max\Bigl\{\Bigl\|\sum_{k=1}^n \mathcal{L}_k(a_k)\mathcal{L}_k(a_k)^*\Bigr\|_{p/2}^{1/2}, \Bigl\|\sum_{k=1}^n \mathcal{L}_k(a_k)^*\mathcal{L}_k(a_k)\Bigr\|_{p/2}^{1/2}\Bigr\},$$

$$\Bigl\|\sum_{k=1}^n \mathcal{R}_k(a_k)\Bigr\|_p \le cd^2 \max\Bigl\{\Bigl\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*\Bigr\|_{p/2}^{1/2}, \Bigl\|\sum_{k=1}^n \mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k)\Bigr\|_{p/2}^{1/2}\Bigr\}.$$

PROOF. We only prove the second inequality. According to Proposition 2.8, the case $p = \infty$ follows with constant 3, while the case $p = 2$ holds with constant 1 by orthogonality. Therefore, it suffices to show that we can interpolate. To that aim, we observe that the term of the right-hand side can be rewritten as

$$\max\left\{\left\|\sum_{k=1}^{n} e_{1k}\otimes\mathcal{R}_k(a_k)\right\|_{S_p^n(L_p(\mathcal{A}))},\left\|\sum_{k=1}^{n} e_{k1}\otimes\mathcal{R}_k(a_k)\right\|_{S_p^n(L_p(\mathcal{A}))}\right\}.$$

In other words, this is the norm of $(\mathcal{R}_k(a_k))$ in

$$RC_p^n(L_p(\mathcal{A}))=R_p^n(L_p(\mathcal{A}))\cap C_p^n(L_p(\mathcal{A})).$$

On the other hand, by Theorem 2.1, we know that $R_p^n(\mathbf{P}_{\mathcal{A}}(p,d))$ and $C_p^n(\mathbf{P}_{\mathcal{A}}(p,d))$ are complemented respectively in $R_p^n(L_p(\mathcal{A}))$ and $C_p^n(L_p(\mathcal{A}))$ with projection constant less than or equal to $4d$. Thus, taking the same projection on $RC_p^n(L_p(\mathcal{A}))$ and using that $RC_p^n(L_p(\mathcal{A}))$ is an interpolation scale (see [25, 31]), we conclude that $RC_p^n(\mathbf{P}_{\mathcal{A}}(p,d))$ is an interpolation scale with equivalent norms up to a constant controlled by cd. Then we need to consider the subspace of $RC_p^n(\mathbf{P}_{\mathcal{A}}(p,d))$ made up of elements for which the kth component is in $\mathcal{R}_k(L_p(\mathcal{A}))$. The associated projection is

$$\Pi_{\mathcal{R}}=\sum_k \delta_k\otimes\mathcal{R}_k,$$

where (δ_k) denotes the common basis of R and C when (R,C) is viewed as a compatible couple. According to Lemma 2.5 and Remark 2.6, the projection $\Pi_{\mathcal{R}}$ is bounded and of norm $\le cd$. Therefore, the family of spaces $\Pi_{\mathcal{R}}(RC_p^n(\mathbf{P}_{\mathcal{A}}(p,d)))$, $2\le p\le\infty$, forms an interpolation scale with equivalent norms up to a constant controlled by cd^2. This completes the proof. □

2.2. *Proof of Theorem* A *and applications.* We now study generalizations of Voiculescu's inequality [44], originally formulated for 1-homogeneous polynomials in a free product von Neumann algebra. Our main result is Theorem A (stated in the Introduction), which extends Voiculescu's inequality in three aspects: we allow amalgamation, homogeneous free polynomials of arbitrary degree and our inequalities hold in $L_p(\mathcal{A})$ for $2\le p\le\infty$. In particular, Theorem A can be regarded as a generalization of Rosenthal's inequality (R_p) in the free setting.

The notation

$$\mathcal{Q}_k=\mathcal{R}_k\mathcal{L}_k=\mathcal{L}_k\mathcal{R}_k$$

for the projection onto words starting and ending in $\mathring{\mathsf{A}}_k$ is crucial for our analysis.

LEMMA 2.10. *If* $a\in\mathbf{P}_{\mathcal{A}}(d)$, *we have*

$$\max_{1\le k\le n}\|\mathcal{Q}_k(a)\|_\infty+\left\|\sum_{k=1}^{n}\mathsf{E}(\mathcal{Q}_k(a)^*\mathcal{Q}_k(a))\right\|_\infty^{1/2}+\left\|\sum_{k=1}^{n}\mathsf{E}(\mathcal{Q}_k(a)\mathcal{Q}_k(a)^*)\right\|_\infty^{1/2}$$
$$\le c\|a\|_\infty.$$

Moreover, if $a_1, a_2, \dots, a_n \in \mathbf{P}_{\mathcal{A}}(d)$, *we have*

$$\left\| \sum_{k=1}^n \mathcal{Q}_k(a_k) \right\|_\infty \sim_c \max_{1 \le k \le n} \|\mathcal{Q}_k(a_k)\|_\infty + \left\| \left(\sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(a_k)^* \mathcal{Q}_k(a_k)) \right)^{1/2} \right\|_\infty + \left\| \left(\sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(a_k) \mathcal{Q}_k(a_k)^*) \right)^{1/2} \right\|_\infty .$$

PROOF. According to the proof of Lemma 2.5, we know that $\mathcal{L}_k$ and $\mathcal{R}_k$ are bounded maps on $\mathbf{P}_{\mathcal{A}}(d)$ with constant 3. In particular, we find $\|\mathcal{Q}_k(a)\|_\infty \le 9\|a\|_\infty$. On the other hand, using the identities

$$\mathsf{E}(a^* a) = \sum_k \mathsf{E}(\mathcal{L}_k(a)^* \mathcal{L}_k(a)) = \sum_k \mathsf{E}(\mathcal{R}_k(a)^* \mathcal{R}_k(a)),$$

for homogeneous polynomials [cf. the proof of (5)] we easily obtain the estimate

$$\begin{aligned} \left\| \sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(a)^* \mathcal{Q}_k(a)) \right\|_\infty &= \left\| \sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(\mathcal{L}_k(a))^* \mathcal{R}_k(\mathcal{L}_k(a))) \right\|_\infty \\ &\le \left\| \sum_{k=1}^n \mathsf{E}(\mathcal{L}_k(a)^* \mathcal{L}_k(a)) \right\|_\infty = \|\mathsf{E}(a^* a)\|_\infty \le \|a\|_\infty^2. \end{aligned}$$

Using this estimate for a^*, we deduce the first assertion.

To prove the second one, we note that $\mathcal{Q}_k(a_k) = \mathcal{Q}_k(a)$ for $a = \sum_k \mathcal{Q}_k(a_k)$. In particular, the lower estimate follows from the first assertion. For the upper estimate, we use

$$\sum_{k=1}^n \mathcal{Q}_k(a_k) = \sum_{k=1}^n \mathsf{L}_k \mathcal{Q}_k(a_k) \mathsf{L}_k + \sum_{k=1}^n \mathcal{Q}_k(a_k)(1 - \mathsf{L}_k) + \sum_{k=1}^n (1 - \mathsf{L}_k) \mathcal{Q}_k(a_k) \mathsf{L}_k.$$

The first term on the right-hand side clearly gives the maximum. The remaining two terms can be estimated by identity (4) in the same way as we did in Proposition 2.8. This completes the proof. □

LEMMA 2.11. *Let* $a_k \in \mathbf{P}_{\mathcal{A}}(p, d)$ *and signs* $\varepsilon_k = \pm 1$.

(i) *If* $1 \le p < 2$, *we have*

$$\left\| \sum_{k=1}^n \varepsilon_k \mathcal{Q}_k(a_k) \right\|_p \le c d^2 \left\| \sum_{k=1}^n \mathcal{Q}_k(a_k) \right\|_p .$$

(ii) *If* $2 \le p \le \infty$, *we have*

$$\left\| \sum_{k=1}^n \varepsilon_k \mathcal{Q}_k(a_k) \right\|_p \le c d \left\| \sum_{k=1}^n \mathcal{Q}_k(a_k) \right\|_p .$$

PROOF. If $a \in \mathbf{P}_{\mathcal{A}}(p,d)$, we claim that

$$(10)\qquad \begin{aligned} \Big\| \sum_{k=1}^n \varepsilon_k \mathcal{R}_k(a) \Big\|_p &\le c d^2 \Big\| \sum_{k=1}^n \mathcal{R}_k(a) \Big\|_p \qquad \text{for } 1 \le p < 2, \\ \Big\| \sum_{k=1}^n \varepsilon_k \mathcal{R}_k(a) \Big\|_p &\le c d \Big\| \sum_{k=1}^n \mathcal{R}_k(a) \Big\|_p \qquad \text{for } 2 \le p \le \infty. \end{aligned}$$

The second inequality clearly holds with constant 1 for $p = 2$. On the other hand, according to (9), it also holds for $p = \infty$ with constant 3. Therefore, since any $a \in \mathbf{P}_{\mathcal{A}}(p,d)$ satisfies $a = \sum_k \mathcal{R}_k(a)$, our claim follows for $2 \le p \le \infty$ by complex interpolation from Theorem 2.1.

Then a duality argument yields the first inequality in the claim. Indeed, by Theorem 2.1, one more time, we have $\mathbf{P}_{\mathcal{A}}(p,d)^* \simeq \mathbf{P}_{\mathcal{A}}(p',d)$ with equivalence constant controlled by $4d$. Therefore, given $1 \le p \le 2$, an element $a \in \mathbf{P}_{\mathcal{A}}(p,d)$ and signs $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, we choose $x \in \mathbf{P}_{\mathcal{A}}(p',d)$ of norm one such that

$$\begin{aligned} \Big\| \sum_{k=1}^n \varepsilon_k \mathcal{R}_k(a) \Big\|_p &\le 4d \, \mathrm{tr}_{\mathcal{A}} \Big(x^* \sum_{k=1}^n \varepsilon_k \mathcal{R}_k(a) \Big) \\ &= 4d \, \mathrm{tr}_{\mathcal{A}} \Big(\sum_{k=1}^n \varepsilon_k \mathcal{L}_k(x^*) a \Big) \\ &\le 4d \|a\|_p \Big\| \sum_{k=1}^n \varepsilon_k \mathcal{R}_k(x) \Big\|_{p'} \le c d^2 \|a\|_p. \end{aligned}$$

Taking $a = \sum_k \mathcal{R}_k(a_k)$, we see that (10) implies

$$(11)\qquad \begin{aligned} \Big\| \sum_{k=1}^n \varepsilon_k \mathcal{R}_k(a_k) \Big\|_p &\le c d^2 \Big\| \sum_{k=1}^n \mathcal{R}_k(a_k) \Big\|_p \qquad \text{for } 1 \le p < 2, \\ \Big\| \sum_{k=1}^n \varepsilon_k \mathcal{R}_k(a_k) \Big\|_p &\le c d \Big\| \sum_{k=1}^n \mathcal{R}_k(a_k) \Big\|_p \qquad \text{for } 2 \le p \le \infty. \end{aligned}$$

Therefore, the lemma immediately follows from (11) since $\mathcal{Q}_k = \mathcal{R}_k \mathcal{Q}_k$. $\square$

LEMMA 2.12. *If $1 \le p \le 2$ and $a_1, a_2, \dots, a_n \in \mathbf{Q}_{\mathcal{A}}(p,d)$, we have*

$$\Big\| \sum_{k=1}^n \mathcal{Q}_k(a_k) \Big\|_p \le c d^4 \Big(\sum_{k=1}^n \|a_k\|_p^p \Big)^{1/p}.$$

PROOF. Using the boundedness of the projection $\Gamma_{\mathcal{A}}(p,d)$ from Remark 2.2 and complex interpolation, it suffices to see that the inequalities associated to the extremal indices hold with constant controlled by cd^3. In the case $p = 2$, this follows by orthogonality with constant 1. When $p = 1$, we decompose the a_k's into their homogeneous parts and use the boundedness of

$$\mathcal{Q}_k \circ \Pi_{\mathcal{A}}(1, s) : L_1(\mathcal{A}) \to \mathbf{P}_{\mathcal{A}}(1, s).$$

Indeed, by step 3 in the proof of Lemma 2.5 and Remark 2.6, we have

$$\|\mathcal{Q}_k \circ \Pi_{\mathcal{A}}(1, s)\|_1 \le c(1 + s)\|\Pi_{\mathcal{A}}(1, s)\|_1.$$

Therefore, we find

$$\begin{aligned}\left\|\sum_{k=1}^n \mathcal{Q}_k(a_k)\right\|_1 &\le \sum_{k=1}^n \|\mathcal{Q}_k(a_k)\|_1 \le \sum_{k=1}^n \sum_{s=0}^d \|\mathcal{Q}_k(\Pi_{\mathcal{A}}(1, s)(a_k))\|_1 \\ &\le \sum_{k=1}^n \sum_{s=0}^d c(1 + s)\|\Pi_{\mathcal{A}}(1, s)(a_k)\|_1 \le c \sum_{k=1}^n \sum_{s=0}^d (1 + s)^2 \|a_k\|_1 \\ &= c\left(\sum_{s=0}^d (1 + s)^2\right)\left(\sum_{k=1}^n \|a_k\|_1\right) \le cd^3 \sum_{k=1}^n \|a_k\|_1.\end{aligned}$$

This proves the remaining estimate. The proof is complete. $\square$

PROOF OF THEOREM A. Lemma 2.10 implies the assertion for $p = \infty$. Thus, we may assume in what follows that $2 \le p < \infty$. Let us prove the lower estimate. First we observe that $L_p(\mathcal{A})$ has Rademacher cotype p for $2 \le p < \infty$. This, combined with Lemma 2.11, yields

$$\left(\sum_{k=1}^n \|\mathcal{Q}_k(a_k)\|_p^p\right)^{1/p} \le \int_\Omega \left\|\sum_{k=1}^n \varepsilon_k \mathcal{Q}_k(a_k)\right\|_p d\varepsilon \le cd \left\|\sum_{k=1}^n \mathcal{Q}_k(a_k)\right\|_p. \tag{12}$$

For the second term, we use

$$\sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(a_k)^* \mathcal{Q}_k(a_k)) = \sum_{i,j=1}^n \mathsf{E}(\mathcal{Q}_i(a_i)^* \mathcal{Q}_j(a_j)).$$

Hence, by the contractivity of E,

$$\left\|\sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(a_k)^* \mathcal{Q}_k(a_k))\right\|_{p/2} \le \left\|\sum_{k=1}^n \mathcal{Q}_k(a_k)\right\|_p^2.$$

The third term is estimated in the same way. Therefore, the lower estimate holds with constant cd. Now we prove the upper estimate. To that aim, we proceed in two steps. First we prove the case $2 \le p \le 4$ and after that we shall apply an induction argument.

Step 1. Since $\mathcal{R}_k(\mathcal{Q}_k(a_k)) = \mathcal{Q}_k(a_k)$, we may apply Corollary 2.9 and obtain

$$\begin{aligned}\Big\| \sum_{k=1}^n \mathcal{Q}_k(a_k) \Big\|_p &\le cd^2 \bigg(\Big\| \sum_{k=1}^n \mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^* \Big\|_{p/2}^{1/2} \\ &\quad + \Big\| \sum_{k=1}^n \mathcal{Q}_k(a_k)^*\mathcal{Q}_k(a_k) \Big\|_{p/2}^{1/2} \bigg).\end{aligned} \tag{13}$$

Then we observe that

$$\mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^* = \mathsf{E}(\mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^*) + \mathcal{Q}_k(\mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^*), \tag{14}$$

$$\mathcal{Q}_k(a_k)^*\mathcal{Q}_k(a_k) = \mathsf{E}(\mathcal{Q}_k(a_k)^*\mathcal{Q}_k(a_k)) + \mathcal{Q}_k(\mathcal{Q}_k(a_k)^*\mathcal{Q}_k(a_k)). \tag{15}$$

Let us first assume that $2 \le p \le 4$. Note that $\mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^*$ is not necessarily homogeneous. However, it is not difficult to see that it is a polynomial in $L_{p/2}(\mathcal{A})$ of degree $2d-1$. Therefore, it follows from Lemma 2.12 that

$$\begin{aligned}\Big\| \sum_{k=1}^n \mathcal{Q}_k(\mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^*) \Big\|_{p/2} &\le cd^4 \bigg(\sum_{k=1}^n \|\mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^*\|_{p/2}^{p/2} \bigg)^{2/p} \\ &= cd^4 \bigg(\sum_{k=1}^n \|\mathcal{Q}_k(a_k)\|_p^p \bigg)^{2/p}.\end{aligned}$$

By (14) and the triangle inequality, we deduce

$$\begin{aligned}\Big\| \sum_{k=1}^n \mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^* \Big\|_{p/2}^{1/2} &\le \Big\| \sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^*) \Big\|_{p/2}^{1/2} \\ &\quad + cd^2 \bigg(\sum_{k=1}^n \|\mathcal{Q}_k(a_k)\|_p^p \bigg)^{1/p}.\end{aligned}$$

Taking adjoints, we obtain a similar estimate for the last term of (13). Hence, given any index $2 \le p \le 4$, we have proved that the assertion holds with $\mathcal{C}_p(d) \le c_0 d^4$ for some absolute constant c_0.

Step 2. Now we proceed by induction and assume the assertion is proved in $L_{p/2}(\mathcal{A})$ with constant $\mathcal{C}_{p/2}(d)$ for some $4 < p < \infty$. Of course, we still have (13), (14) and (15) at our disposal. Thus, arguing as above, it suffices to estimate the term

$$\Big\| \sum_{k=1}^n \mathcal{Q}_k(\mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^*) \Big\|_{p/2}^{1/2}.$$

Let us write $x_k = \mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^*$. As observed above, we know that x_k is a polynomial of degree $2d-1$. Hence, we may use the projections $\Pi_{\mathcal{A}}(p,s)$ from Remark 2.2 and obtain the following inequality for $x_{ks} = \Pi_{\mathcal{A}}(p,s)(x_k)$:

$$\left\| \sum_{k=1}^n \mathcal{Q}_k(x_k) \right\|_{p/2} \le \sum_{s=1}^{2d-1} \left\| \sum_{k=1}^n \mathcal{Q}_k(x_{ks}) \right\|_{p/2}.$$

By the induction hypothesis, we have

$$\sum_{s=1}^{2d-1} \left\| \sum_{k=1}^n \mathcal{Q}_k(x_{ks}) \right\|_{p/2} \le \sum_{s=1}^{2d-1} \mathcal{C}_{p/2}(s)(\mathrm{A}_s + \mathrm{B}_s + \mathrm{C}_s).$$

By Remark 2.6, the first term on the right-hand side is estimated by

$$\begin{aligned} \mathrm{A}_s &= \left(\sum_{k=1}^n \|\mathcal{Q}_k(x_{ks})\|_{p/2}^{p/2} \right)^{2/p} \\ &\le cs \left(\sum_{k=1}^n \|\Pi_{\mathcal{A}}(p,s)(x_k)\|_{p/2}^{p/2} \right)^{2/p} \le cs^2 \left(\sum_{k=1}^n \|\mathcal{Q}_k(a_k)\|_p^p \right)^{2/p}. \end{aligned}$$

The second term is given by

$$\mathrm{B}_s = \left\| \sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(x_{ks})^* \mathcal{Q}_k(x_{ks})) \right\|_{p/4}^{1/2}.$$

Using $x_k = \sum_s x_{ks}$, freeness and (15), we have, for all $1 \le s \le 2d-1$,

$$\begin{aligned} \mathsf{E}(\mathcal{Q}_k(x_{ks})^* \mathcal{Q}_k(x_{ks})) &\le \sum_r \mathsf{E}(\mathcal{Q}_k(x_{kr})^* \mathcal{Q}_k(x_{kr})) \\ &= \sum_{q,r} \mathsf{E}(\mathcal{Q}_k(x_{kq})^* \mathcal{Q}_k(x_{kr})) \\ &= \mathsf{E}(\mathcal{Q}_k(x_k)^* \mathcal{Q}_k(x_k)) \\ &= \mathsf{E}\big((x_k - \mathsf{E}(x_k))^* (x_k - \mathsf{E}(x_k))\big) \\ &= \mathsf{E}(x_k^* x_k) - \mathsf{E}(x_k)^* \mathsf{E}(x_k) \le \mathsf{E}(x_k^* x_k). \end{aligned}$$

Then we apply Lemma 5.2 of [17] and then obtain

$$\begin{aligned} \mathrm{B}_s &\le \left\| \sum_{k=1}^n \mathsf{E}(x_k^* x_k) \right\|_{p/4}^{1/2} = \left\| \sum_{k=1}^n \mathsf{E}|\mathcal{Q}_k(a_k)^*|^4 \right\|_{p/4}^{1/2} \\ &\le \left\| \sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^*) \right\|_{p/2}^{(p-4)/(2p-4)} \left(\sum_{k=1}^n \|\mathcal{Q}_k(a_k)\|_p^p \right)^{2/(2p-4)}. \end{aligned}$$

The same estimate holds for C_s. Now, by homogeneity, we may assume that

$$\Big(\sum_{k=1}^n \|\mathcal{Q}_k(a_k)\|_p^p\Big)^{1/p} + \Big\|\sum_{k=1}^n \mathsf{E}[\mathcal{Q}_k(a_k)^*\mathcal{Q}_k(a_k)]\Big\|_{p/2}^{1/2} + \Big\|\sum_{k=1}^n \mathsf{E}[\mathcal{Q}_k(a_k)\mathcal{Q}_k(a_k)^*]\Big\|_{p/2}^{1/2} = 1.$$

Then combining the inequalities so far obtained, we deduce

$$\sum_{s=1}^{2d-1}\Big\|\sum_{k=1}^n \mathcal{Q}_k(x_{ks})\Big\|_{p/2} \leq \sum_{s=1}^{2d-1} \mathcal{C}_{p/2}(s)(2+cs^2).$$

Chasing through the inequalities above, we obtain the estimate

$$\mathcal{C}_p(d) \leq \sqrt{c}d^{7/2}\sqrt{\mathcal{C}_{p/2}(d)},$$

for some absolute constant c. Taking c big enough so that $c_0 \leq c$ and recalling that $\mathcal{C}_p(d) \leq c_0 d^4 \leq cd^7$ for $2 \leq p \leq 4$, it turns out that the growth of the constant $\mathcal{C}_p(d)$ as $d \to \infty$ is controlled by cd^7. This proves the assertion. □

REMARK 2.13. A noncommutative analogue of Rosenthal's inequality for general von Neumann algebras (nonnecessarily free products) was obtained in [17, 18]; see also [46] for the proof and the notion of noncommutative independence employed in it. As we have pointed out in the Introduction, recalling that freeness implies this notion of independence, Theorem A for $d = 1$ and $2 \leq p < \infty$ follows from the noncommutative Rosenthal inequality. However, the constants in [17, 18] are not uniformly bounded as $p \to \infty$, in sharp contrast with Theorem A. Similarly, one could try to derive Theorem A for $d \geq 1$ and $2 \leq p < \infty$ by proving that $\mathcal{Q}_1(a_1), \mathcal{Q}_2(a_2), \dots, \mathcal{Q}_n(a_n)$ are independent in the sense of [18]. Nevertheless, this alternative approach to Theorem A would provide constants depending on p, rather than on d.

Since any $a \in \mathbf{P}_{\mathcal{A}}(p,d)$ satisfies

$$a = \sum_{k=1}^n \mathcal{L}_k(a) = \sum_{k=1}^n \mathcal{R}_k(a),$$

the following result characterizes the L_p norm of all homogeneous free polynomials.

COROLLARY 2.14. *If $2 \leq p \leq \infty$ and $a_1, a_2, \dots, a_n \in \mathbf{P}_{\mathcal{A}}(p,d)$, we have*

$$\Big\|\sum_{k=1}^n \mathcal{L}_k(a_k)\Big\|_p \sim_{cd^7} \Big\|\sum_{k=1}^n \mathcal{L}_k(a_k)^*\mathcal{L}_k(a_k)\Big\|_{p/2}^{1/2} + \Big\|\sum_{k=1}^n \mathsf{E}(\mathcal{L}_k(a_k)\mathcal{L}_k(a_k)^*)\Big\|_{p/2}^{1/2}$$

and

$$\left\|\sum_{k=1}^n \mathcal{R}_k(a_k)\right\|_p \sim_{cd^7} \left\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*\right\|_{p/2}^{1/2} + \left\|\sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k))\right\|_{p/2}^{1/2}.$$

PROOF. By (10), we have

$$\begin{aligned}
&\left\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*\right\|_{p/2}^{1/2} \\
&\qquad = \left\|\int_\Omega \sum_{i,j=1}^n \varepsilon_i \mathcal{R}_i(a_i)\varepsilon_j \mathcal{R}_j(a_j)^*\, d\varepsilon\right\|_{p/2}^{1/2} \\
&\leq \left(\int_\Omega \left\|\sum_{i=1}^n \varepsilon_i \mathcal{R}_i(a_i)\right\|_p \left\|\sum_{j=1}^n \varepsilon_j \mathcal{R}_j(a_j)^*\right\|_p d\varepsilon\right)^{1/2} \\
&\leq cd\left\|\sum_{k=1}^n \mathcal{R}_k(a_k)\right\|_p.
\end{aligned}$$

On the other hand, by freeness,

$$\sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k)) = \mathsf{E}\left(\left(\sum_{i=1}^n \mathcal{R}_i(a_i)\right)^*\left(\sum_{j=1}^n \mathcal{R}_j(a_j)\right)\right).$$

Therefore, by the contractivity of E,

$$\left\|\sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k))\right\|_{p/2}^{1/2} \leq \left\|\sum_{k=1}^n \mathcal{R}_k(a_k)\right\|_p.$$

This gives the lower estimate.

For the upper estimate, we assume that $2 \leq p < \infty$, since the case $p = \infty$ was already proved in Proposition 2.8. Now we use the second inequality stated in Corollary 2.9:

$$\left\|\sum_{k=1}^n \mathcal{R}_k(a_k)\right\|_p \leq cd^2\left(\left\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*\right\|_{p/2}^{1/2} + \left\|\sum_{k=1}^n \mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k)\right\|_{p/2}^{1/2}\right).$$

On the other hand, it is clear that

$$\sum_{k=1}^n \mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k) = \sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k)) + \sum_{k=1}^n \mathcal{Q}_k(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k)). \tag{16}$$

Hence, it suffices to estimate the last term on the right-hand side. This part of the proof is similar to the corresponding one of the proof of Theorem A. Again, we observe that $x_k = \mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k)$ is no longer homogeneous but a polynomial of degree $\leq 2d$. Our argument for this term depends on the value of p.

Step 1. If $2 \leq p \leq 4$, we apply Lemma 2.12 and obtain

$$\begin{aligned}\Big\|\sum_{k=1}^n \mathcal{Q}_k(x_k)\Big\|_{p/2}^{1/2} &\leq cd^2\Big(\sum_{k=1}^n \|x_k\|_{p/2}^{p/2}\Big)^{1/p}\\ &= cd^2\Big(\sum_{k=1}^n \|\mathcal{R}_k(a_k)\|_p^p\Big)^{1/p} \leq cd^2\Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*\Big\|_{p/2}^{1/2},\end{aligned}$$

where the last inequality holds for $2 \leq p \leq \infty$ and follows by complex interpolation. Hence, in the case $2 \leq p \leq 4$, we have proved the upper estimate with constant cd^4.

Step 2. If $4 < p < \infty$, we take $x_{ks} = \Pi_{\mathcal{A}}(p,s)(x_k)$ and write

$$\Big\|\sum_{k=1}^n \mathcal{Q}_k(x_k)\Big\|_{p/2}^{1/2} \leq \Big(\sum_{s=1}^{2d}\Big\|\sum_{k=1}^n \mathcal{Q}_k(x_{ks})\Big\|_{p/2}\Big)^{1/2} \leq \sqrt{2d}\max_{1\leq s\leq 2d}\Big\|\sum_{k=1}^n \mathcal{Q}_k(x_{ks})\Big\|_{p/2}^{1/2}.$$

By Theorem A, we have

$$\begin{aligned}\Big\|\sum_{k=1}^n \mathcal{Q}_k(x_{ks})\Big\|_{p/2} \sim_{cs^7} &\Big(\sum_{k=1}^n \|\mathcal{Q}_k(x_{ks})\|_{p/2}^{p/2}\Big)^{2/p}\\ &+ \Big\|\sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(x_{ks})^*\mathcal{Q}_k(x_{ks}))\Big\|_{p/4}^{1/2}\\ &+ \Big\|\sum_{k=1}^n \mathsf{E}(\mathcal{Q}_k(x_{ks})\mathcal{Q}_k(x_{ks})^*)\Big\|_{p/4}^{1/2} = \mathrm{A}_s + \mathrm{B}_s + \mathrm{C}_s.\end{aligned}$$

These terms are estimated as in the proof of Theorem A (step 2):

$$\mathrm{A}_s \leq cs^2\Big(\sum_{k=1}^n \|\mathcal{R}_k(a_k)\|_p^p\Big)^{2/p}.$$

Similarly, we have

$$\max(\mathrm{B}_s,\mathrm{C}_s) \leq \Big\|\sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k))\Big\|_{p/2}^{(p-4)/(2p-4)}\Big(\sum_{k=1}^n \|\mathcal{R}_k(a_k)\|_p^p\Big)^{2/(2p-4)}.$$

On the other hand, by homogeneity, we may assume that

$$\Big\|\sum_{k=1}^n \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*\Big\|_{p/2}^{1/2} + \Big\|\sum_{k=1}^n \mathsf{E}(\mathcal{R}_k(a_k)^*\mathcal{R}_k(a_k))\Big\|_{p/2}^{1/2} = 1.$$

Using the estimates above and

$$\left(\sum_{k=1}^{n} \|\mathcal{R}_k(a_k)\|_p^p\right)^{1/p} \le \left\|\sum_{k=1}^{n} \mathcal{R}_k(a_k)\mathcal{R}_k(a_k)^*\right\|_{p/2}^{1/2},$$

we obtain

$$\left\|\sum_{k=1}^{n} \mathcal{Q}_k(x_k)\right\|_{p/2}^{1/2} \le \sqrt{2d} \max_{1\le s\le 2d} [cs^7(2+cs^2)]^{1/2},$$

for $4 < p < \infty$. Therefore, by Corollary 2.9 and (16), we find

$$\left\|\sum_{k=1}^{n} \mathcal{R}_k(a_k)\right\|_p \le cd^7.$$

This and step 1 yield the assertion for $\mathcal{R}_k$'s. For $\mathcal{L}_k$'s, we take adjoints. □

COROLLARY 2.15. *If $2 \le p \le \infty$ and $a \in \mathbf{P}_{\mathcal{A}}(p,d)$, we have*

$$\|a\|_p \sim_{cd^{14}} \left\|\sum_{i,j=1}^{n} e_{ij} \otimes \mathcal{L}_i\mathcal{R}_j(a)\right\|_{S_p^n(L_p(\mathcal{A}))} + \|\mathsf{E}(aa^*)^{1/2}\|_{L_p(\mathcal{B})} + \|\mathsf{E}(a^*a)^{1/2}\|_{L_p(\mathcal{B})}.$$

PROOF. We use $a = \sum_{k=1}^{n} \mathcal{R}_k(a)$ and Corollary 2.14:

$$\|a\|_p \sim_{cd^7} \left\|\sum_{k=1}^{n} \mathcal{R}_k(a)\mathcal{R}_k(a)^*\right\|_{p/2}^{1/2} + \left\|\sum_{k=1}^{n} \mathsf{E}(\mathcal{R}_k(a)^*\mathcal{R}_k(a))\right\|_{p/2}^{1/2} = \mathrm{A} + \mathrm{B}.$$

To estimate A, we use Corollary 2.14 for the $\mathcal{L}_k$'s

$$\begin{aligned}
&\left\|\sum_{k=1}^{n} \mathcal{R}_k(a)\mathcal{R}_k(a)^*\right\|_{p/2}^{1/2} \\
&\quad = \left\|\sum_{k=1}^{n} e_{1k} \otimes \mathcal{R}_k(a)\right\|_{S_p^n(L_p(\mathcal{A}))} \\
&\quad \sim_{cd^7} \left\|\sum_{i=1}^{n} e_{i1} \otimes \mathcal{L}_i\left(\sum_{j=1}^{n} e_{1j} \otimes \mathcal{R}_j(a)\right)\right\|_{S_p^{n^2}(L_p(\mathcal{A}))} \\
&\qquad + \left\|\sum_{i=1}^{n} \mathsf{E}\left(\left(\sum_{j=1}^{n} e_{1j} \otimes \mathcal{L}_i\mathcal{R}_j(a)\right)\left(\sum_{j=1}^{n} e_{1j} \otimes \mathcal{L}_i\mathcal{R}_j(a)\right)^*\right)\right\|_{S_{p/2}^n(L_{p/2}(\mathcal{B}))}^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \Big\| \sum_{i,j=1}^{n} e_{ij} \otimes \mathcal{L}_i \mathcal{R}_j(a) \Big\|_{S_p^n(L_p(\mathcal{A}))} \\
&\quad + \Big\| e_{11} \otimes \sum_{i,j=1}^{n} \mathsf{E}((\mathcal{L}_i \mathcal{R}_j(a))(\mathcal{L}_i \mathcal{R}_j(a))^*) \Big\|_{S_{p/2}^n(L_{p/2}(\mathcal{B}))}^{1/2} \\
&= \Big\| \sum_{i,j=1}^{n} e_{ij} \otimes \mathcal{L}_i \mathcal{R}_j(a) \Big\|_{S_p^n(L_p(\mathcal{A}))} + \|\mathsf{E}(aa^*)^{1/2}\|_{L_p(\mathcal{B})}.
\end{aligned}
$$

On the other hand, it is clear that

$$
\mathrm{B} = \Big\| \sum_{k=1}^{n} \mathsf{E}(\mathcal{R}_k(a)^* \mathcal{R}_k(a)) \Big\|_{p/2}^{1/2} = \|\mathsf{E}(a^*a)^{1/2}\|_{L_p(\mathcal{B})}.
$$

Thus, since we have used equivalences at each step, the proof is complete. $\square$

REMARK 2.16. By decomposing a free polynomial of degree d into its homogeneous parts, we automatically obtain trivial generalizations of Theorem A and Corollaries 2.14 and 2.15 for nonhomogeneous free polynomials of a fixed degree d. Most of the forthcoming results in this paper are susceptible of this kind of generalization.

3. A length-reduction formula. In this section we prove a length-reduction formula for polynomials in the free product. One more time, our standard assumptions are that $\mathcal{A} = *_{\mathcal{B}} \mathsf{A}_k$, where $1 \le k \le n$, $\mathcal{B}$ is equipped with a *n.f.* state φ which induces a *n.f.* state $\phi = \varphi \circ \mathsf{E}$ on $\mathcal{A}$ and $\mathsf{E} : \mathcal{A} \to \mathcal{B}$ is a *n.f.* conditional expectation. As usual, d_ϕ denotes the density of the state ϕ. We will need some preliminary facts on certain module maps. First, given $2 \le p \le \infty$, we define on $\mathcal{A} \otimes_{\mathcal{B}} L_p(\mathcal{A})$ the $L_{p/2}(\mathcal{A})$-valued inner product

$$
\langle\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle\rangle = y_1^* \mathsf{E}(x_1^* x_2) y_2.
$$

This allows us to define $L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$ and $L_p^r(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$ as the completion of the space $\mathcal{A} \otimes_{\mathcal{B}} L_p(\mathcal{A})$ with respect to the norms

$$
\begin{aligned}
\|z\|_{L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})} &= \| \langle\langle z, z \rangle\rangle \|_{L_{p/2}(\mathcal{A})}^{1/2}, \\
\|z\|_{L_p^r(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})} &= \| \langle\langle z^*, z^* \rangle\rangle \|_{L_{p/2}(\mathcal{A})}^{1/2}.
\end{aligned}
$$

Let $C_\infty(\mathcal{B})$ be the column subspace of the $\mathcal{B}$-valued Schatten class $S_\infty(\mathcal{B})$:

$$
C_\infty(\mathcal{B}) = \Big\{ \sum_k e_{k1} \otimes b_k \in \mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B} \Big\}.
$$

By [29], there exists a normal right $\mathcal{B}$-module map $u : \mathcal{A} \to C_\infty(\mathcal{B})$ satisfying

$$\mathsf{E}(x^* y) = \sum_{k=1}^{\infty} u_k(x)^* u_k(y) = u(x)^* u(y) \qquad \text{for all } x, y \in \mathcal{A}, \tag{17}$$

where u_k stands for the kth coordinate of u. More rigorously, to be able to apply [29], we need to assume $\mathcal{A}$ countably generated. However, for our purposes here and by a standard approximation argument, we can reduce the general case to this special one. Note that, according to [13, 16], this map canonically extends from $L_p(\mathcal{A})$ into $C_p(L_p(\mathcal{B}))$, still denoted by u. On the other hand, recalling that amalgamation gives $C_\infty(\mathcal{B}) \otimes_{\mathcal{B}} L_p(\mathcal{A}) = C_p(L_p(\mathcal{A}))$, we have an isometry

$$\widehat{u} = u \otimes id_{L_p(\mathcal{A})} : L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E}) \to C_p(L_p(\mathcal{A})). \tag{18}$$

Indeed, note that

$$\big(\widehat{u}(x_1 \otimes y_1)\big)^* \big(\widehat{u}(x_2 \otimes y_2)\big) = \sum_{k=1}^{\infty} y_1^* u_k(x_1)^* u_k(x_2) y_2 = y_1^* \mathsf{E}(x_1^* x_2) y_2.$$

Thus, linearity gives

$$\|\widehat{u}(z)\|_{C_p(L_p(\mathcal{A}))} = \|\langle\langle z, z\rangle\rangle\|_{L_{p/2}(\mathcal{A})}^{1/2} = \|z\|_{L_p^c(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{A}, \mathsf{E})}.$$

A similar argument holds in the row case and by Proposition 2.8 of [13] we deduce the following:

LEMMA 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be as above. Then,*

$$L_p^r(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E}) \quad \textit{and} \quad L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$$

are contractively complemented in the space $S_p(L_p(\mathcal{A}))$ for any $2 \le p \le \infty$.

In what follows, Λ will always denote a finite index set.

LEMMA 3.2. *Given $2 \le p \le \infty$, let us define*

$$\mathcal{W}_p = \left\{ \sum_{\alpha\in\Lambda} \sum_{k=1}^{n} x_k(\alpha) \otimes w_k(\alpha) \,|\, x_k(\alpha) \in \mathring{\mathsf{A}}_k, w_k(\alpha) \in L_p(\mathcal{A}) \right\}.$$

If we denote by $\mathcal{W}_p^r$ the closure of $\mathcal{W}_p$ with respect to the norm of $L_p^r(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$, then $\mathcal{W}_p^r$ is contractively complemented in $L_p^r(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$. Similarly, the same holds for the closure $\mathcal{W}_p^c$ of $\mathcal{W}_p$ in $L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$.

PROOF. By definition, $L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$ is the closure of

$$\sum_{\alpha\in\Lambda} x(\alpha) \otimes w(\alpha),$$

where $x(\alpha) \in \mathcal{A}$ and $w(\alpha) \in L_p(\mathcal{A})$. Let us recall the notation $\Pi_{\mathcal{A}}(p,d)$, introduced in Remark 2.2 for the projection from $L_p(\mathcal{A})$ onto the homogeneous polynomials of degree d. Then we clearly have

$$\begin{aligned} x(\alpha) &= \mathsf{E}(x(\alpha)) + \Pi_{\mathcal{A}}(p,1)(x(\alpha)) + \sum_{d\geq 2} \Pi_{\mathcal{A}}(p,d)(x(\alpha)) \\ &= x(\alpha,0) + x(\alpha,1) + x(\alpha,2). \end{aligned}$$

Now we define

$$\begin{aligned} \mathrm{A} &= \sum_{\alpha\in\Lambda} x(\alpha,1) \otimes w(\alpha), \\ \mathrm{B} &= \sum_{\alpha\in\Lambda} 1_{\mathcal{A}} \otimes x(\alpha,0)w(\alpha) + \sum_{\alpha\in\Lambda} x(\alpha,2) \otimes w(\alpha). \end{aligned}$$

Note that $\sum_\alpha x(\alpha) \otimes w(\alpha) = \mathrm{A} + \mathrm{B}$ and $\mathrm{A} \in \mathcal{W}_p$. On the other hand, by freeness,

$$\langle\langle \mathrm{A}+\mathrm{B}, \mathrm{A}+\mathrm{B}\rangle\rangle = \langle\langle \mathrm{A}, \mathrm{A}\rangle\rangle + \langle\langle \mathrm{B}, \mathrm{B}\rangle\rangle.$$

Therefore, by positivity,

$$\begin{aligned} \|\mathrm{A}\|^2_{L_p^c(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{A},\mathsf{E})} &= \|\langle\langle \mathrm{A}, \mathrm{A}\rangle\rangle\|_{p/2} \\ &\leq \|\langle\langle \mathrm{A}+\mathrm{B}, \mathrm{A}+\mathrm{B}\rangle\rangle\|_{p/2} \\ &= \left\| \sum_{\alpha\in\Lambda} x(\alpha) \otimes w(\alpha) \right\|^2_{L_p^c(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{A},\mathsf{E})}. \end{aligned}$$

By continuity, we find a contractive projection from $L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$ onto the space $\mathcal{W}_p^c$ for any given index $2 \leq p \leq \infty$. Obviously, the argument above also works for $L_p^r(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$. This completes the proof. $\square$

LEMMA 3.3. *If $2 \leq p \leq \infty$, the space*

$$\mathcal{Z}^r_{p,d} = \overline{\left\{ \sum_{\alpha\in\Lambda} \sum_{k=1}^n x_k(\alpha) \otimes w_k(\alpha) \in \mathcal{W}_p^r \,\middle|\, w_k(\alpha) \in \mathbf{P}_{\mathcal{A}}(p,d), \mathcal{R}_k(w_k(\alpha)) = 0 \right\}}$$

is complemented in $L_p^r(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$. Similarly, the space

$$\mathcal{Z}^c_{p,d} = \overline{\left\{ \sum_{\alpha\in\Lambda} \sum_{k=1}^n x_k(\alpha) \otimes w_k(\alpha) \in \mathcal{W}_p^c \,\middle|\, w_k(\alpha) \in \mathbf{P}_{\mathcal{A}}(p,d), \mathcal{L}_k(w_k(\alpha)) = 0 \right\}}$$

is complemented in $L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$. In both cases, the projection constant is $\leq cd^2$.

PROOF. Both complementation results can be proved using the same arguments. Thus, we only prove the second assertion. According to Lemma 3.2, it suffices to check that $\mathcal{Z}_{p,d}^c$ is complemented (with projection constant $\leq cd^2$) in $\mathcal{W}_p^c$. To that aim, we consider the intermediate space

$$\mathcal{W}_{p,d}^c = \overline{\left\{ \sum_{\alpha \in \Lambda} \sum_{k=1}^n x_k(\alpha) \otimes w_k(\alpha) \in \mathcal{W}_p^c \,\middle|\, w_k(\alpha) \in \mathbf{P}_{\mathcal{A}}(p,d) \right\}}.$$

$\mathcal{W}_{p,d}^c$ is complemented in $\mathcal{W}_p^c$ with constant $4d$. Indeed, using one more time the projection $\Pi_{\mathcal{A}}(p,d)$ onto the d-homogeneous polynomials, we write $w_{kd}(\alpha)$ for $\Pi_{\mathcal{A}}(p,d)(w_k(\alpha))$ and obtain from Lemma 3.1 and the discussion preceding it

$$\begin{aligned}
&\left\| \sum_{k,\alpha} x_k(\alpha) \otimes w_{kd}(\alpha) \right\|_{L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})} \\
&\quad = \left\| \sum_{j=1}^\infty e_{j1} \otimes \sum_{k,\alpha} u_j(x_k(\alpha)) w_{kd}(\alpha) \right\|_{C_p(L_p(\mathcal{A}))} \\
&\quad = \left\| \sum_{j=1}^\infty e_{j1} \otimes \Pi_{\mathcal{A}}(p,d) \left(\sum_{k,\alpha} u_j(x_k(\alpha)) w_k(\alpha) \right) \right\|_{C_p(L_p(\mathcal{A}))} \\
&\quad \leq \| id_{C_p} \otimes \Pi_{\mathcal{A}}(p,d) \|_{\mathcal{B}(C_p(L_p(\mathcal{A})))} \\
&\qquad \times \left\| \sum_{j=1}^\infty e_{j1} \otimes \sum_{k,\alpha} u_j(x_k(\alpha)) w_k(\alpha) \right\|_{C_p(L_p(\mathcal{A}))}.
\end{aligned}$$

On the other hand, combining Remarks 1.1 and 2.2, we deduce that $\Pi_{\mathcal{A}}(p,d)$ is a completely bounded map on $L_p(\mathcal{A})$ with cb-norm less than or equal to $4d$. Therefore, we deduce our claim

$$\left\| \sum_{k,\alpha} x_k(\alpha) \otimes w_{kd}(\alpha) \right\|_{L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})} \leq 4d \left\| \sum_{k,\alpha} x_k(\alpha) \otimes w_k(\alpha) \right\|_{L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})}.$$

It remains to see that $\mathcal{Z}_{p,d}^c$ is complemented (with projection constant less than or equal to cd) in $\mathcal{W}_{p,d}^c$. In other words, we are interested in proving the following inequality:

$$\begin{aligned}
&\left\| \sum_{k,\alpha} x_k(\alpha) \otimes (id_{\mathcal{A}} - \mathcal{L}_k)(w_{kd}(\alpha)) \right\|_{L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})} \\
&\qquad \leq cd \left\| \sum_{k,\alpha} x_k(\alpha) \otimes w_{kd}(\alpha) \right\|_{L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})}.
\end{aligned}$$

However, according to the triangle inequality, we may replace $id_{\mathcal{A}} - \mathcal{L}_k$ by $\mathcal{L}_k$ in the inequality above. Now we use the definition of the space $L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$ and freeness to obtain the following identity:

$$\begin{aligned}
&\Big\| \sum_{k,\alpha} x_k(\alpha) \otimes \mathcal{L}_k(w_{kd}(\alpha)) \Big\|_{L_p^c(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{A},\mathsf{E})} \\
&\quad = \Big\| \sum_{i,j,\alpha,\beta} \mathcal{L}_i(w_{id}(\alpha))^* \mathsf{E}(x_i(\alpha)^* x_j(\beta)) \mathcal{L}_j(w_{jd}(\beta)) \Big\|_{L_{p/2}(\mathcal{A})}^{1/2} \\
&\quad = \Big\| \sum_{k,\alpha,\beta} \mathcal{L}_k(w_{kd}(\alpha))^* \mathsf{E}(x_k(\alpha)^* x_k(\beta)) \mathcal{L}_k(w_{kd}(\beta)) \Big\|_{L_{p/2}(\mathcal{A})}^{1/2} \\
&\quad = \Big\| \sum_{k,\alpha,\beta} \mathcal{L}_k(u(x_k(\alpha)) w_{kd}(\alpha))^* \mathcal{L}_k(u(x_k(\beta)) w_{kd}(\beta)) \Big\|_{L_{p/2}(\mathcal{A})}^{1/2},
\end{aligned}$$

where the last identity uses (17) and the fact that $\mathcal{L}_k$ is a $\mathcal{B}$-module map. Recall that the $\mathcal{L}_k$'s in the last term on the right-hand side are acting on the amplified algebra $\mathcal{A} \bar{\otimes} \mathcal{B}(\ell_2)$ since the range of u is $C_\infty(\mathcal{B})$. Now the desired inequality follows after applying Lemma 2.5 and Remark 2.6 (acting on this amplified algebra) and undoing the identities used above. The proof is complete. $\square$

REMARK 3.4. In our definition of the spaces $L_p^r(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$ and $L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$, as well as in Lemmas 3.2 and 3.3, we have used tensors $x \otimes w$ with $x \in \mathcal{A}$ and $w \in L_p(\mathcal{A})$. Note that, according to the definition of the inner product $\langle\langle \cdot, \cdot \rangle\rangle$, it is relevant to distinguish between the first and second components of these tensors. However, in some forthcoming results (see, e.g., the proof of Lemma 3.5 below) we shall need to work with tensors $x \otimes w$, where $x \in L_p(\mathcal{A})$ and $w \in \mathcal{A}$. Thus, we have to understand which element of $L_p^r(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$ or $L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$ do we mean when writing $x \otimes w$. Let us consider a sequence $(x_n)_{n \ge 1}$ in $\mathcal{A}$ such that

$$x_n d_\phi^{1/p} \to x \qquad \text{as } n \to \infty$$

in $L_p(\mathcal{A})$. Then we set

$$x \otimes w = \lim_{n\to\infty} x_n \otimes d_\phi^{1/p} w.$$

To make sure our definition makes sense, we must see that the sequence on the right converges in the norms of $L_p^r(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$ and $L_p^c(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$. Let us see this for the first space, the other follows in the same way. By completeness, it suffices to show that we have a Cauchy sequence. This easily follows since

$$\begin{aligned}
&\|(x_n - x_m) \otimes d_\phi^{1/p} w\|_{L_p^r(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{A},\mathsf{E})} \\
&\quad = \|w^* d_\phi^{1/p} \mathsf{E}\big((x_n - x_m)^*(x_n - x_m)\big) d_\phi^{1/p} w\|_{L_{p/2}(\mathcal{A})}^{1/2} \\
&\quad \le \|w\|_{\mathcal{A}} \|d_\phi^{1/p} \mathsf{E}\big((x_n - x_m)^*(x_n - x_m)\big) d_\phi^{1/p}\|_{L_{p/2}(\mathcal{B})}^{1/2} \\
&\quad \le \|w\|_{\mathcal{A}} \|(x_n - x_m) d_\phi^{1/p}\|_{L_p(\mathcal{A})}
\end{aligned}$$

and the right-hand side converges to 0 as $n, m \to \infty$.

3.1. *Preliminary estimates.* This paragraph is devoted to some necessary estimates that will be used below. In the following we shall use the notation already defined in the Introduction:

$$\Big\| \sum_{k,\alpha} b_k(\alpha) \langle a_k(\alpha)| \Big\|_p = \Big\| \Big(\sum_{i,j,\alpha,\beta} b_i(\alpha) \mathsf{E}(a_i(\alpha) a_j(\beta)^*) b_j(\beta)^* \Big)^{1/2} \Big\|_p,$$

$$\Big\| \sum_{k,\alpha} |a_k(\alpha)\rangle b_k(\alpha) \Big\|_p = \Big\| \Big(\sum_{i,j,\alpha,\beta} b_i(\alpha)^* \mathsf{E}(a_i(\alpha)^* a_j(\beta)) b_j(\beta) \Big)^{1/2} \Big\|_p.$$

In other words,

$$\Big\| \sum_{k,\alpha} b_k(\alpha) \langle a_k(\alpha)| \Big\|_p = \Big\| \sum_{k,\alpha} a_k(\alpha) \otimes b_k(\alpha) \Big\|_{L_p^r(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{A},\mathsf{E})},$$

$$\Big\| \sum_{k,\alpha} |a_k(\alpha)\rangle b_k(\alpha) \Big\|_p = \Big\| \sum_{k,\alpha} a_k(\alpha) \otimes b_k(\alpha) \Big\|_{L_p^c(\mathcal{A}\otimes_{\mathcal{B}}\mathcal{A},\mathsf{E})}.$$

LEMMA 3.5. *Let $2 \le p, q \le \infty$ be two indices related by $1/2 = 1/p + 1/q$. Let $x_k(\alpha)$ be a mean-zero element in A_k for each $1 \le k \le n$ and α running over a finite set Λ. Let $w_k(\alpha) \in \mathbf{P}_{\mathcal{A}}(d)$ for some $d \ge 0$ and satisfying $\mathcal{R}_k(w_k(\alpha)) = 0$ for all $1 \le k \le n$ and every $\alpha \in \Lambda$. Then*

$$\Big\| \sum_{k,\alpha} w_k(\alpha) \mathsf{L}_k x_k(\alpha) d_\phi^{1/p} \Big\|_{\mathcal{B}(L_q(\mathcal{A}), L_2(\mathcal{A}))} \le \Big\| \sum_{k,\alpha} |w_k(\alpha)\rangle x_k(\alpha) d_\phi^{1/p} \Big\|_p,$$

$$\Big\| \sum_{k,\alpha} w_k(\alpha)(1 - \mathsf{L}_k) x_k(\alpha) d_\phi^{1/p} \Big\|_{\mathcal{B}(L_q(\mathcal{A}), L_2(\mathcal{A}))} \le c d^2 \Big\| \sum_{k,\alpha} w_k(\alpha) \langle x_k(\alpha) d_\phi^{1/p}| \Big\|_p.$$

PROOF. In what follows we use $x_k'(\alpha) = x_k(\alpha) d_\phi^{1/p}$. Given $z \in L_q(\mathcal{A})$, we have

$$h_k(\alpha) = x_k'(\alpha) z \in L_2(\mathcal{A})$$

and the vector $\mathsf{L}_k h_k(\alpha)$ is a linear combination of reduced words in $L_2(\mathcal{A})$ starting with a mean-zero letter in A_k. Therefore, since $\mathcal{R}_k(w_k(\alpha)) = 0$, the operator $w_k(\alpha)$ acts on $\mathsf{L}_k h_k(\alpha)$ by tensoring from the left. In particular, the $(d+1)$-th letter in the words of $w_k(\alpha)\mathsf{L}_k h_k(\alpha)$ is always in A_k and the inequality below follows from freeness by using (3), (17) and that L_k commutes with $\mathcal{B}$ like in step 1, Lemma 2.5:

$$\begin{aligned}
\left\| \left(\sum_{k,\alpha} w_k(\alpha)\mathsf{L}_k x'_k(\alpha) \right)(z) \right\|_2^2 &= \sum_{i,j,\alpha,\beta} \mathrm{tr}_{\mathcal{A}}(h_i(\alpha)^* \mathsf{L}_i w_i(\alpha)^* w_j(\beta)\mathsf{L}_j h_j(\beta)) \\
&= \sum_{k,\alpha,\beta} \mathrm{tr}_{\mathcal{A}}(h_k(\alpha)^* \mathsf{L}_k \mathsf{E}(w_k(\alpha)^* w_k(\beta))\mathsf{L}_k h_k(\beta)) \\
&\le \sum_{k,\alpha,\beta} \mathrm{tr}_{\mathcal{A}}(h_k(\alpha)^* \mathsf{E}(w_k(\alpha)^* w_k(\beta)) h_k(\beta)) \\
&= \mathrm{tr}_{\mathcal{A}}\left(z^* \sum_{i,j,\alpha,\beta} x'_i(\alpha)^* \mathsf{E}(w_i(\alpha)^* w_j(\beta)) x'_j(\beta) z \right) \\
&\le \|z\|_q^2 \left\| \sum_{i,j,\alpha,\beta} x'_i(\alpha)^* \mathsf{E}(w_i(\alpha)^* w_j(\beta)) x'_j(\beta) \right\|_{p/2}.
\end{aligned}$$

This proves the first inequality.

Let us prove the second one. According to Lemma 3.1, we know that the spaces $L_p^r(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \mathsf{E})$ form an interpolation scale for $2 \le p \le \infty$. Moreover, it follows from Lemma 3.3 that the spaces $\mathcal{Z}_{p,d}^r$ also form (up to a constant cd^2) an interpolation scale for $2 \le p \le \infty$. Therefore, since (see Remark 3.4)

$$\left\| \sum_{k,\alpha} w_k(\alpha) \langle x_k(\alpha) d_\phi^{1/p} | \right\|_p = \left\| \sum_{k,\alpha} x_k(\alpha) d_\phi^{1/p} \otimes w_k(\alpha) \right\|_{\mathcal{Z}_{p,d}^r},$$

it suffices to see (by complex interpolation) that the assertion holds when $p = 2$ and $p = \infty$ with some constant not depending on d. Let us use the same terminology for $x'_k(\alpha)$ as above. If $p = 2$, we have $q = \infty$ and the triangle inequality gives

$$\begin{aligned}
&\left\| \sum_{k,\alpha} w_k(\alpha)(1 - \mathsf{L}_k) x'_k(\alpha) \right\|_{\mathcal{B}(L_\infty(\mathcal{A}), L_2(\mathcal{A}))} \\
&\quad \le \left\| \sum_{k,\alpha} w_k(\alpha) x'_k(\alpha) \right\|_{\mathcal{B}(L_\infty(\mathcal{A}), L_2(\mathcal{A}))} \\
&\qquad + \left\| \sum_{k,\alpha} w_k(\alpha) \mathsf{L}_k x'_k(\alpha) \right\|_{\mathcal{B}(L_\infty(\mathcal{A}), L_2(\mathcal{A}))}.
\end{aligned}$$

The first term equals

$$\begin{aligned}\Big\|\sum_{k,\alpha} w_k(\alpha)x_k'(\alpha)\Big\|_2 &= \Big(\sum_{i,j,\alpha,\beta} \mathrm{tr}_{\mathcal{A}}[w_i(\alpha)x_i'(\alpha)x_j'(\beta)^* w_j(\beta)^*]\Big)^{1/2} \\ &= \Big(\sum_{i,j,\alpha,\beta} \mathrm{tr}_{\mathcal{A}}[w_i(\alpha)\mathsf{E}(x_i'(\alpha)x_j'(\beta)^*)w_j(\beta)^*]\Big)^{1/2} \\ &= \Big\|\sum_{k,\alpha} w_k(\alpha)\langle x_k'(\alpha)|\Big\|_2.\end{aligned}$$

Indeed, since we have $\mathcal{R}_k(w_k(\alpha)) = 0$, the first trace above vanishes for $i \neq j$ since in that case $w_i(\alpha)x_i'(\alpha)x_j'(\beta)^* w_j(\beta)^*$ is a homogeneous free polynomial of degree $(2d+2)$. On the other hand, if $i \neq j$, the second trace also vanishes since $\mathsf{E}(x_i'(\alpha)x_j'(\beta)^*)$ does. Thus, to justify the second inequality above, it remains to consider the case $i = j$. However, using again the hypothesis $\mathcal{R}_k(w_k(\alpha)) = 0$, we find by freeness

$$\mathrm{tr}_{\mathcal{A}}\big[w_k(\alpha)\big(x_k'(\alpha)x_k'(\beta)^* - \mathsf{E}(x_k'(\alpha)x_k'(\beta)^*)\big)w_k(\beta)^*\big] = 0.$$

To estimate the second term, we use the first inequality proved in this lemma:

$$\begin{aligned}\Big\|\sum_{k,\alpha} w_k(\alpha)\mathsf{L}_k x_k'(\alpha)\Big\|_{\mathcal{B}(L_\infty(\mathcal{A}),L_2(\mathcal{A}))}^2 &\le \Big\|\sum_{k,\alpha} |w_k(\alpha)\rangle x_k'(\alpha)\Big\|_2^2 \\ &= \sum_{i,j,\alpha,\beta} \mathrm{tr}_{\mathcal{A}}(x_i'(\alpha)^*\mathsf{E}(w_i(\alpha)^* w_j(\beta))x_j'(\beta)) \\ &= \sum_{i,j,\alpha,\beta} \mathrm{tr}_{\mathcal{A}}(x_i'(\alpha)^* w_i(\alpha)^* w_j(\beta)x_j'(\beta)) \\ &= \sum_{i,j,\alpha,\beta} \mathrm{tr}_{\mathcal{A}}(w_j(\beta)x_j'(\beta)x_i'(\alpha)^* w_i(\alpha)^*) \\ &= \sum_{i,j,\alpha,\beta} \mathrm{tr}_{\mathcal{A}}(w_j(\beta)\mathsf{E}(x_j'(\beta)x_i'(\alpha)^*)w_i(\alpha)^*) \\ &= \Big\|\sum_{k,\alpha} w_k(\alpha)\langle x_k'(\alpha)|\Big\|_2^2.\end{aligned}$$

The second and fourth identities above can be justified using again our hypothesis $\mathcal{R}_k(w_k(\alpha)) = 0$ and freeness in the same way we did above to deal with the first term. Therefore, we have proved that

$$\Big\|\sum_{k,\alpha} w_k(\alpha)(1-\mathsf{L}_k)x_k'(\alpha)\Big\|_{\mathcal{B}(L_\infty(\mathcal{A}),L_2(\mathcal{A}))} \le 2\Big\|\sum_{k,\alpha} w_k(\alpha)\langle x_k'(\alpha)|\Big\|_2.$$

To prove the assertion for $p = \infty$ and $q = 2$, we first note that

$$(1 - \mathsf{L}_k)x_k(\alpha) = (1 - \mathsf{L}_k)x_k(\alpha)\mathsf{L}_k.$$

This implies

$$\begin{aligned}
&\Big\| \sum_{k,\alpha} w_k(\alpha)(1 - \mathsf{L}_k)x_k(\alpha) \Big\|^2_{\mathcal{B}(L_2(\mathcal{A}),L_2(\mathcal{A}))} \\
&\quad = \Big\| \sum_{k,\alpha} w_k(\alpha)(1 - \mathsf{L}_k)x_k(\alpha)\mathsf{L}_k \Big\|^2_\infty \\
&\quad = \Big\| \sum_{k,\alpha,\beta} w_k(\alpha)(1 - \mathsf{L}_k)x_k(\alpha)x_k(\beta)^*(1 - \mathsf{L}_k)w_k(\beta)^* \Big\|_\infty \\
&\quad = \Big\| \sum_{k,\alpha,\beta} w_k(\alpha)(1 - \mathsf{L}_k)\mathsf{E}(x_k(\alpha)x_k(\beta)^*)(1 - \mathsf{L}_k)w_k(\beta)^* \Big\|_\infty \\
&\quad \le \Big\| \sum_{k,\alpha,\beta} w_k(\alpha)\mathsf{E}(x_k(\alpha)x_k(\beta)^*)w_k(\beta)^* \Big\|_\infty ,
\end{aligned}$$

as L_k commutes with $\mathcal{B}$. Hence, we have seen that

$$\Big\| \sum_{k,\alpha} w_k(\alpha)(1 - \mathsf{L}_k)x_k(\alpha) \Big\|_{\mathcal{B}(L_2(\mathcal{A}),L_2(\mathcal{A}))} \le \Big\| \sum_{k,\alpha} w_k(\alpha)\langle x_k(\alpha)| \Big\|_\infty .$$

This proves the assertion for $p = \infty$. The general case follows by interpolation. □

3.2. *Proofs of Theorems* B *and* C. Now we prove the second major result of this paper, a length-reduction formula for homogeneous polynomials on free random variables. As a consequence, we extend the main results in [27, 37].

PROOF OF THEOREM B. The second reduction formula clearly follows from the first one by taking adjoints. Thus, it suffices to prove the first reduction formula. We begin by proving the upper estimate. If $1/p + 1/q = 1/2$, we have

$$\begin{aligned}
\Big\| \sum_{k,\alpha} w_k(\alpha)x_k(\alpha) \Big\|_{L_p(\mathcal{A})} &= \Big\| \sum_{k,\alpha} w_k(\alpha)x_k(\alpha) \Big\|_{\mathcal{B}(L_q(\mathcal{A}),L_2(\mathcal{A}))} \\
&\le \Big\| \sum_{k,\alpha} w_k(\alpha)\mathsf{L}_k x_k(\alpha) \Big\|_{\mathcal{B}(L_q(\mathcal{A}),L_2(\mathcal{A}))} \\
&\quad + \Big\| \sum_{k,\alpha} w_k(\alpha)(1 - \mathsf{L}_k)x_k(\alpha) \Big\|_{\mathcal{B}(L_q(\mathcal{A}),L_2(\mathcal{A}))} .
\end{aligned}$$

If we approximate $x_k(\alpha)$ by elements of the form

$$z_k(\alpha) d_\phi^{1/p} \qquad \text{with } z_k(\alpha) \in \mathring{\mathsf{A}}_k,$$

the upper estimate follows from the inequalities in Lemma 3.5:

$$\left\| \sum_{k,\alpha} w_k(\alpha) x_k(\alpha) \right\|_{L_p(\mathcal{A})} \le \left\| \sum_{k,\alpha} |w_k(\alpha)\rangle x_k(\alpha) \right\|_p + c d^2 \left\| \sum_{k,\alpha} w_k(\alpha) \langle x_k(\alpha)| \right\|_p.$$

To prove the lower estimate, we use the projection

$$\Gamma_{\mathcal{A}}(p,d) : L_p(\mathcal{A}) \to \mathbf{Q}_{\mathcal{A}}(p,d),$$

which, according to Remark 2.2, is bounded by $2d+1$. Then we observe

$$\sum_{i,j,\alpha,\beta} x_i(\alpha)^* \mathsf{E}(w_i(\alpha)^* w_j(\beta)) x_j(\beta) = \Gamma_{\mathcal{A}}(p/2,2)(a^* a) \tag{19}$$

for $a = \sum_{k,\alpha} w_k(\alpha) x_k(\alpha) \in L_p(\mathcal{A})$. Indeed, the proof of this fact goes essentially as the proof of Lemma 2.3, recalling that $\mathcal{R}_k(w_k(\alpha)) = 0$. In particular, we deduce

$$\left\| \sum_{k,\alpha} |w_k(\alpha)\rangle x_k(\alpha) \right\|_p = \|\Gamma_{\mathcal{A}}(p/2,2)(a^* a)\|_{p/2}^{1/2} \le \sqrt{5} \left\| \sum_{k,\alpha} w_k(\alpha) x_k(\alpha) \right\|_p.$$

Thus, it remains to prove the estimate

$$\left\| \sum_{k,\alpha} w_k(\alpha) \langle x_k(\alpha)| \right\|_p \le \sqrt{4d+1} \left\| \sum_{k,\alpha} w_k(\alpha) x_k(\alpha) \right\|_p.$$

To that aim, we use again the projection $\Gamma_{\mathcal{A}}(p/2,2d)$ and Remark 2.2:

$$\begin{aligned} &\left\| \sum_{k,\alpha} w_k(\alpha) \langle x_k(\alpha)| \right\|_p^2 \\ &\qquad = \left\| \sum_{i,j,\alpha,\beta} w_i(\alpha) \mathsf{E}(x_i(\alpha) x_j(\beta)^*) w_j(\beta)^* \right\|_{p/2} \\ &\qquad = \left\| \Gamma_{\mathcal{A}}(p/2,2d) \left[\left(\sum_{k,\alpha} w_k(\alpha) x_k(\alpha) \right) \left(\sum_{k,\alpha} w_k(\alpha) x_k(\alpha) \right)^* \right] \right\|_{p/2}, \end{aligned}$$

where the last identity follows again by the same argument outline above for (19). Therefore, the assertion follows from the estimate $\|\Gamma_{\mathcal{A}}(p/2,2d)\| \le 4d+1$. $\square$

Our aim now is to iterate Theorem B to obtain a Khintchine type inequality, stated as Theorem C in the Introduction, which generalizes the main results of [3, 27, 37]. Before that, we analyze in more detail the meaning of the brackets $|\,\rangle$ and $\langle\,|$. That is, according to the mapping $u : \mathcal{A} \to C_\infty(\mathcal{B})$, we can always write

$$|a\rangle = u(a) \quad \text{and} \quad \langle a| = u(a^*)^*. \tag{20}$$

This remark allows us to combine the brackets $|\,\rangle$ and $\langle\,|$. In particular, our expressions for the norms Σ_1 and Σ_2 in the statement of Theorem C (cf. the Introduction) are explained by (18) and (20). Moreover, we can iterate them as we will often do in the sequel. That is, we shall write $\|x\rangle z\rangle$ for $u(u(x)z)$, where the first u acts on the amplified algebra $\mathcal{A}\bar{\otimes}\mathcal{B}(\ell_2)$ due to the fact that $u(x) \in C_\infty(\mathcal{B})$; see the last part of the proof of Lemma 3.3 for a similar situation. Of course, taking adjoints, we also have a meaning for iterated bracket $\langle z\langle x\|$.

LEMMA 3.6. *Let $2 \le p \le \infty$ and let $x_k(\alpha)$, $z_k(\alpha)$ and $w_k(\alpha)$ be homogeneous free polynomials of degree d_1, d_2 and d_3 respectively for all $1 \le k \le n$ and α running over a finite set Λ. Assume that $\sum_{k,\alpha} x_k(\alpha)z_k(\alpha)w_k(\alpha) \in L_p(\mathcal{A})$. Then, if $\mathcal{R}_k(x_k(\alpha)) = x_k(\alpha)$ and $\mathcal{L}_k(z_k(\alpha)) = 0$ for all (k,α), we have*

$$\Big\| \sum_{k,\alpha} \big| |x_k(\alpha)\rangle z_k(\alpha) \big\rangle w_k(\alpha) \Big\|_{C_p(C_p(L_p(\mathcal{A})))} = \Big\| \sum_{k,\alpha} |x_k(\alpha)z_k(\alpha)\rangle w_k(\alpha) \Big\|_{C_p(L_p(\mathcal{A}))}.$$

Similarly, we have the following:

- *if $\mathcal{L}_k(x_k(\alpha)) = x_k(\alpha)$ and $\mathcal{R}_k(z_k(\alpha)) = 0$,*
$$\Big\| \sum_{k,\alpha} w_k(\alpha) \big\langle z_k(\alpha) \langle x_k(\alpha)| \big| \Big\|_p = \Big\| \sum_{k,\alpha} w_k(\alpha) \langle z_k(\alpha)x_k(\alpha)| \Big\|_p;$$
- *if $\mathcal{R}_k(x_k(\alpha)) = 0$ and $\mathcal{L}_k(z_k(\alpha)) = z_k(\alpha)$,*
$$\Big\| \sum_{k,\alpha} \big| |x_k(\alpha)\rangle z_k(\alpha) \big\rangle w_k(\alpha) \Big\|_p = \Big\| \sum_{k,\alpha} |x_k(\alpha)z_k(\alpha)\rangle w_k(\alpha) \Big\|_p;$$
- *if $\mathcal{L}_k(x_k(\alpha)) = 0$ and $\mathcal{R}_k(z_k(\alpha)) = z_k(\alpha)$,*
$$\Big\| \sum_{k,\alpha} w_k(\alpha) \big\langle z_k(\alpha) \langle x_k(\alpha)| \big| \Big\|_p = \Big\| \sum_{k,\alpha} w_k(\alpha) \langle z_k(\alpha)x_k(\alpha)| \Big\|_p.$$

PROOF. By freeness, we have

$$\begin{aligned}
&\Big\| \sum_{k,\alpha} |x_k(\alpha)z_k(\alpha)\rangle w_k(\alpha) \Big\|_p \\
&\quad = \Big\| \Big(\sum_{i,j,\alpha,\beta} w_i(\alpha)^* \mathsf{E}(z_i(\alpha)^* x_i(\alpha)^* x_j(\beta) z_j(\beta)) w_j(\beta) \Big)^{1/2} \Big\|_p \\
&\quad = \Big\| \Big(\sum_{i,j,\alpha,\beta} w_i(\alpha)^* \mathsf{E}(z_i(\alpha)^* \mathsf{E}(x_i(\alpha)^* x_j(\beta)) z_j(\beta)) w_j(\beta) \Big)^{1/2} \Big\|_p.
\end{aligned}$$

Thus, using the defining property of $u : \mathcal{A} \to C_\infty(\mathcal{B})$, we obtain

$$\Big\| \sum_{k,\alpha} |x_k(\alpha) z_k(\alpha)\rangle w_k(\alpha) \Big\|_p = \Big\| \sum_{k,\alpha} u(u(x_k(\alpha)) z_k(\alpha)) w_k(\alpha) \Big\|_p$$
$$= \Big\| \sum_{k,\alpha} \big| |x_k(\alpha)\rangle z_k(\alpha) \big\rangle w_k(\alpha) \Big\|_p .$$

The three remaining identities follow similarly. This completes the proof. $\square$

In the proof of Theorem C below, we shall use a shorter notation to write sums like those appearing in the term Σ_2 (see the statement of Theorem C in the Introduction) as follows. For a fixed value k of j_s in $\{1, 2, \dots, n\}$, we shall write

$$\sum_{\substack{1 \le j_1 \ne \cdots \ne j_{s-1} \le n \\ 1 \le j_{s+1} \ne \cdots \ne j_d \le n \\ j_{s-1} \ne j_s = k \ne j_{s+1}}} \qquad \text{as} \qquad \sum_{\substack{j_1 \ne \cdots \ne j_d \\ [j_s = k]}} .$$

PROOF OF THEOREM C. The case of degree 1 follows automatically from Theorem A. Now we proceed by induction on d. Assume the assertion is true for degree $d-1$ with relevant constant $\mathcal{C}_p(d-1)$. Then we apply Theorem B and obtain

$$\|x\|_p \sim_{cd^2} \Big\| \sum_{\alpha \in \Lambda} \sum_{j_1 \ne \cdots \ne j_d} x_{j_1}(\alpha) \langle x_{j_2}(\alpha) \cdots x_{j_d}(\alpha)| \Big\|_p$$
$$+ \Big\| \sum_{\alpha \in \Lambda} \sum_{j_1 \ne \cdots \ne j_d} |x_{j_1}(\alpha)\rangle x_{j_2}(\alpha) \cdots x_{j_d}(\alpha) \Big\|_p = \mathrm{A} + \mathrm{B}.$$

The resulting terms are homogeneous polynomials of degree 1 and $d-1$ respectively. The first one belongs to $R_p(L_p(\mathcal{A}))$, while the second one lives in $C_p(L_p(\mathcal{A}))$. We estimate the first term by applying Theorem A (with $d = 1$) one more time on the amplified space $S_p(L_p(\mathcal{A}))$:

$$\mathrm{A} \sim_c \Big\| \sum_{\alpha \in \Lambda} \sum_{j_1 \ne \cdots \ne j_d} \big\langle x_{j_1}(\alpha) \langle x_{j_2}(\alpha) \cdots x_{j_d}(\alpha)| \big| \Big\|_p$$
$$+ \Big\| \sum_{\alpha \in \Lambda} \sum_{j_1 \ne \cdots \ne j_d} \big| x_{j_1}(\alpha) \langle x_{j_2}(\alpha) \cdots x_{j_d}(\alpha)| \big\rangle \Big\|_p$$
$$+ \bigg(\sum_{k=1}^n \Big\| \sum_{\alpha \in \Lambda} \sum_{\substack{j_1 \ne \cdots \ne j_d \\ [j_1 = k]}} x_k(\alpha) \langle x_{j_2}(\alpha) \cdots x_{j_d}(\alpha)| \Big\|_p^p \bigg)^{1/p} .$$

According to Lemma 3.6 and the fact that $u : \mathcal{A} \to C_\infty(\mathcal{B})$ is a right $\mathcal{B}$-module map, we easily obtain

$$
\begin{aligned}
\mathrm{A} \sim_c & \Big\| \sum_{\alpha\in\Lambda} \sum_{j_1\neq\cdots\neq j_d} \langle x_{j_1}(\alpha) x_{j_2}(\alpha)\cdots x_{j_d}(\alpha)| \Big\|_p \\
& + \Big\| \sum_{\alpha\in\Lambda} \sum_{j_1\neq\cdots\neq j_d} |x_{j_1}(\alpha)\rangle\langle x_{j_2}(\alpha)\cdots x_{j_d}(\alpha)| \Big\|_p \\
& + \bigg(\sum_{k=1}^n \Big\| \sum_{\alpha\in\Lambda} \sum_{\substack{j_1\neq\cdots\neq j_d \\ [j_1=k]}} x_k(\alpha)\langle x_{j_2}(\alpha)\cdots x_{j_d}(\alpha)| \Big\|_p^p \bigg)^{1/p}.
\end{aligned} \tag{21}
$$

On the other hand, the induction hypothesis gives $\mathrm{B} \sim_{\mathcal{C}_p(d-1)} \mathrm{B}_1 + \mathrm{B}_2$ with

$$
\mathrm{B}_1 = \sum_{s=1}^d \Big\| \sum_{\alpha, j_1\neq\cdots\neq j_d} \big| |x_{j_1}(\alpha)\rangle \cdots x_{j_s}(\alpha)\big\rangle \langle x_{j_{s+1}}(\alpha)\cdots x_{j_d}(\alpha)| \Big\|_p,
$$

and B_2 given by

$$
\sum_{s=2}^d \bigg(\sum_{k=1}^n \Big\| \sum_{\alpha\in\Lambda} \sum_{\substack{j_1\neq\cdots\neq j_d \\ [j_s=k]}} \big| |x_{j_1}(\alpha)\rangle \cdots x_{j_{s-1}}(\alpha)\big\rangle x_{j_s}(\alpha) \langle x_{j_{s+1}}(\alpha)\cdots x_{j_d}(\alpha)| \Big\|_p^p \bigg)^{1/p}.
$$

Moreover, the expressions above are simplified by means of Lemma 3.6 as follows:

$$
\begin{aligned}
\mathrm{B}_1 &= \sum_{s=1}^d \Big\| \sum_{\alpha, j_1\neq\cdots\neq j_d} |x_{j_1}(\alpha)\cdots x_{j_s}(\alpha)\rangle\langle x_{j_{s+1}}(\alpha)\cdots x_{j_d}(\alpha)| \Big\|_p, \\
\mathrm{B}_2 &= \sum_{s=2}^d \bigg(\sum_{k=1}^n \Big\| \sum_{\alpha\in\Lambda} \sum_{\substack{j_1\neq\cdots\neq j_d \\ [j_s=k]}} |x_{j_1}(\alpha)\cdots\rangle x_{j_s}(\alpha)\langle\cdots x_{j_d}(\alpha)| \Big\|_p^p \bigg)^{1/p}.
\end{aligned}
$$

Then we note that the first and third terms in (21) are the ones which are missing in B_1 and B_2 respectively to obtain $\Sigma_1 + \Sigma_2$, while the middle term in (21) already appears in B_1. Thus, we conclude that

$$
\|x\|_p \sim_{\mathcal{C}_p(d)} \Sigma_1 + \Sigma_2,
$$

where, after keeping track of the constants, we see that $\mathcal{C}_p(d)$ is controlled by

$$
\mathcal{C}_p(d) \leq c d^2 \mathcal{C}_p(d-1).
$$

Therefore, the bound $\mathcal{C}_p(d) \leq c^d d!^2$ follows from the recurrence above. $\square$

REMARK 3.7. From a more functional analytical point of view, the right-hand side of Theorem C can be regarded as the norm of x in an operator space which is the result of intersecting $2d+1$ operator spaces; see [3, 27, 37] for more explicit descriptions of these constructions. We do not state this result in detail since the notation becomes considerably more complicated. However, equipped with the description given in [37] and with Theorem C, it is not difficult to rephrase Theorem C as a complete isomorphism between $\mathbf{P}_{\mathcal{A}}(p,d)$ and certain p-direct sum of Haagerup tensor products of (subspaces of) L_p-spaces. Moreover, arguing as in [27], we could extend Theorem C to $1 \le p \le 2$ just replacing intersections by sums of operator spaces. The same observation is valid for Theorem A.

REMARK 3.8. The constant $c^d d!^2$ is far from being optimal. Nevertheless, we can improve the constant in the lower estimate of Theorem C. To that aim, we use the projection $\Gamma_{\mathcal{A}}(p/2, 2s) : L_{p/2}(\mathcal{A}) \to \mathbf{Q}_{\mathcal{A}}(p/2, 2s)$ so that $\Gamma_{\mathcal{A}}(p/2, 2s)(xx^*)$ has the form

$$\sum_{i_k, j_k, \alpha, \beta} x_{i_1}(\alpha) \cdots x_{i_s}(\alpha) \mathsf{E}(\cdots x_{i_d}(\alpha) x_{j_d}(\beta)^* \cdots) x_{j_s}(\beta)^* \cdots x_{j_1}(\beta)^*.$$

A similar expression holds for $\Gamma_{\mathcal{A}}(p/2, 2(d-s))(x^*x)$:

$$\sum_{i_k, j_k, \alpha, \beta} x_{i_d}(\alpha)^* \cdots x_{i_{s+1}}(\alpha)^* \mathsf{E}(\cdots x_{i_1}(\alpha)^* x_{j_1}(\beta) \cdots) x_{j_{s+1}}(\beta) \cdots x_{j_d}(\beta).$$

Therefore, since $\Gamma_{\mathcal{A}}(p/2, 2d)$ is bounded with constant $4d+1$, we find

$$\Bigg\| \sum_{\alpha \in \Lambda} \sum_{j_1 \neq \cdots \neq j_d} x_{j_1}(\alpha) \cdots x_{j_s}(\alpha) \langle x_{j_{s+1}}(\alpha) \cdots x_{j_d}(\alpha)| \Bigg\|_p \le \sqrt{4s+1} \|x\|_p,$$

$$\Bigg\| \sum_{\alpha \in \Lambda} \sum_{j_1 \neq \cdots \neq j_d} |x_{j_1}(\alpha) \cdots x_{j_s}(\alpha)\rangle x_{j_{s+1}}(\alpha) \cdots x_{j_d}(\alpha) \Bigg\|_p \le \sqrt{4(d-s)+1} \|x\|_p.$$

In particular, since $\min(s, d-s) \le d/2$, we deduce

$$\Bigg\| \sum_{\alpha \in \Lambda} \sum_{j_1 \neq \cdots \neq j_d} |x_{j_1}(\alpha) \cdots x_{j_s}(\alpha)\rangle \langle x_{j_{s+1}}(\alpha) \cdots x_{j_d}(\alpha)| \Bigg\|_p \le \sqrt{2d+1} \|x\|_p.$$

Therefore, we have proved the estimate

$$\Sigma_1 \le (d+1)\sqrt{2d+1} \|x\|_p.$$

Similarly, using $\Gamma_{\mathcal{A}}(p/2, 2)$ as in the proof of Theorem B, we obtain

$$\Bigg\| \sum_{j_s=1}^{n} \sum_{\alpha \in \Lambda} \sum_{j_1 \neq \cdots \neq j_d} |x_{j_1}(\alpha) \cdots x_{j_{s-1}}(\alpha)\rangle x_{j_s}(\alpha) \langle x_{j_{s+1}}(\alpha) \cdots x_{j_d}(\alpha)| \Bigg\|_p$$

$$\leq \sqrt{5}\left\| \sum_{j_s=1}^{n} \sum_{\alpha\in\Lambda} \sum_{j_1\neq\cdots\neq j_d} x_{j_1}(\alpha)\cdots x_{j_s}(\alpha)\langle x_{j_{s+1}}(\alpha)\cdots x_{j_d}(\alpha)| \right\|_p$$
$$\leq \sqrt{10d+5}\|x\|_p.$$

Hence, according to (12), we deduce

$$\Sigma_2 \leq 12d^2\sqrt{10d+5}\|x\|_p.$$

Motivated by the results in [37], we conjecture that the growth of the constant in the upper estimate of Theorem C should also be polynomial on d. However, at the time of this writing we cannot prove this.

REMARK 3.9. Theorem C also generalizes the main results in [3, 27]. Indeed, note that Theorem C uses $2d+1$ terms in contrast with the $d+1$ terms in [27]. However, in the particular case of free generators, it is easily seen that the terms associated to Σ_1 (exactly the $d+1$ terms appearing in [27]) dominate the terms in Σ_2. We refer the reader to the proofs of Lemma 4.1 and Theorem F below for computations very similar to the ones we are omitting here. Given $2\leq p\leq\infty$ and as a consequence of Theorem C and Remark 3.8, we can rephrase the Khintchine inequality in [27] as the following equivalence for any operator valued d-homogeneous polynomial x on the free generators $\lambda(g_1),\lambda(g_2),\dots,\lambda(g_n)$:

$$cd^{-3/2}\Sigma_1 \leq \|x\|_p \leq c^d d!^2\Sigma_1.$$

4. Square functions. Now we apply our length-reduction formula to study square functions associated to free martingales. All martingales in this section are adapted to the free filtration, the natural filtration of a reduced product already defined in the Introduction. More precisely, according to the Khintchine and Rosenthal inequalities for free random variables, it is natural to ask whether or not the Burkholder–Gundy inequality [34] holds in the free setting for $p=\infty$; see also [17] for the nonsemifinite case and [28, 36] for the weak type $(1,1)$ inequality associated to it. In this section we find a counterexample to this question. The following is the key step.

LEMMA 4.1. *Let* $\mathsf{A}_k = L_\infty(-2,2)$ *for* $k=0,1,2,\dots$ *equipped with the Wigner measure, and let* $\mathcal{A}=\mathsf{A}_0 * \mathsf{A}_1 * \mathsf{A}_2\cdots$ *be the associated reduced free product equipped with the n.f. tracial state* ϕ. *Consider a free family of semicircular elements* $w_k\in\mathsf{A}_{2k-1}$ *and* $w_k'\in\mathsf{A}_{2k}$ *for* $k\geq 1$. *Given an integer* n, *fix a mean-zero element* f *in* A_0 *such that*

$$\|f\|_{L_2(\mathsf{A}_0)} = 1/\sqrt{n} \quad \textit{and} \quad \|f\|_{L_\infty(\mathsf{A}_0)} = 1.$$

Let $a_{ij}\in\mathcal{B}(\ell_2)$ *and*

$$x_{2n} = \sum_{1\leq i,j\leq n} a_{ij}\otimes w_i f w_j' \in \mathcal{B}(\ell_2)\otimes\mathcal{A}.$$

Then

$$\|x_{2n}\|_{\mathcal{B}(\ell_2)\bar{\otimes}\mathcal{A}} \sim_c \Big\| \sum_{1\le i,j\le n} a_{ij} \otimes e_{ij} \Big\|_{\mathcal{B}(\ell_2)\otimes_{\min}\mathcal{B}(\ell_2)}.$$

PROOF. By Remark 1.1, we have

$$\mathcal{B}(\ell_2)\,\bar{\otimes}\,\mathcal{A} = \big(\mathcal{B}(\ell_2)\,\bar{\otimes}\,\mathsf{A}_0\big) *_{\mathcal{B}(\ell_2)} \big(\mathcal{B}(\ell_2)\,\bar{\otimes}\,\mathsf{A}_1\big) *_{\mathcal{B}(\ell_2)} \big(\mathcal{B}(\ell_2)\,\bar{\otimes}\,\mathsf{A}_2\big) *_{\mathcal{B}(\ell_2)} \cdots.$$

According to this isometry, we rewrite x_{2n} as follows:

$$x_{2n} = \sum_{i,j} a_{ij} \otimes w_i f w'_j = \sum_{i,j} (a_{ij} \otimes w_i)(1 \otimes f)(1 \otimes w'_j) = \sum_{i,j} x_{ij} y z_j.$$

In particular, Theorem C gives the following equivalence for $\mathsf{E} = \phi \otimes id_{\mathcal{B}(\ell_2)}$:

$$\begin{aligned}
\|x_{2n}\|_{\mathcal{B}(\ell_2)\bar{\otimes}\mathcal{A}} \sim_c\; & \|\mathsf{E}(x_{2n}x_{2n}^*)\|_\infty^{1/2} + \|\mathsf{E}(x_{2n}^*x_{2n})\|_\infty^{1/2} \\
& + \Big\| \sum_{i,j=1}^n x_{ij} \langle y z_j | \Big\|_\infty + \Big\| \sum_{i,j=1}^n |x_{ij} y\rangle z_j \Big\|_\infty \\
& + \Big\| \sum_{i,j=1}^n |x_{ij} y\rangle\langle z_j| \Big\|_\infty + \Big\| \sum_{i,j=1}^n |x_{ij}\rangle\langle y z_j| \Big\|_\infty \\
& + \Big\| \sum_{i,j=1}^n |x_{ij}\rangle y \langle z_j| \Big\|_\infty \\
=\; & \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F} + \mathrm{G}.
\end{aligned}$$

It is clear that

$$\mathrm{A} = \Big\| \sum_{ijkl} a_{ij} a_{kl}^* \phi(w_i f w'_j w'_l f^* w_k) \Big\|_\infty^{1/2} = \Big\| \sum_{ij} a_{ij} a_{ij}^* \phi(w_i f w_j'^2 f^* w_i) \Big\|_\infty^{1/2}.$$

Since $\phi(w_k^2) = \phi(w_k'^2) = 1$, this gives

$$\mathrm{A} = \|f\|_2 \Big\| \sum_{i,j=1}^n a_{ij} a_{ij}^* \Big\|_\infty^{1/2} = \frac{1}{\sqrt{n}} \Big\| \sum_{i,j=1}^n a_{ij} \otimes e_{1,ij} \Big\|_{\mathcal{B}(\ell_2)\otimes_{\min}\mathcal{B}(\ell_2)}.$$

The same argument gives rise to the identity

$$\mathrm{B} = \frac{1}{\sqrt{n}} \Big\| \sum_{i,j=1}^n a_{ij} \otimes e_{ij,1} \Big\|_{\mathcal{B}(\ell_2)\otimes_{\min}\mathcal{B}(\ell_2)}.$$

Let us estimate the term C:

$$\begin{aligned}
\mathrm{C} &= \Big\| \sum_{ijkl} x_{ij} \mathsf{E}(y z_j z_l^* y^*) x_{kl}^* \Big\|_\infty^{1/2} \\
&= \Big\| \sum_{ijkl} a_{ij} a_{kl}^* \otimes w_i \phi(f w_j' w_l' f^*) w_k \Big\|_\infty^{1/2} \\
&= \|f\|_2 \Big\| \sum_{j=1}^n e_{1j} \otimes \Big(\sum_{i=1}^n a_{ij} \otimes w_i \Big) \Big\|_\infty \\
&= \|f\|_2 \Big\| \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij} \otimes e_{1j} \Big) \otimes w_i \Big\|_\infty .
\end{aligned}$$

Now, applying the Khintchine inequality for free random variables [10],

$$\begin{aligned}
\mathrm{C} &\sim \|f\|_2 \max \Big\{ \Big\| \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij} \otimes e_{1j} \Big) \otimes e_{1i} \Big\|_\infty , \Big\| \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij} \otimes e_{1j} \Big) \otimes e_{i1} \Big\|_\infty \Big\} \\
&= \|f\|_2 \max \Big\{ \Big\| \sum_{i,j=1}^n a_{ij} \otimes e_{1,ij} \Big\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)} , \Big\| \sum_{i,j=1}^n a_{ij} \otimes e_{ij} \Big\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)} \Big\}.
\end{aligned}$$

Again, the same argument gives

$$\mathrm{D} \sim \|f\|_2 \max \Big\{ \Big\| \sum_{i,j=1}^n a_{ij} \otimes e_{ij,1} \Big\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)} , \Big\| \sum_{i,j=1}^n a_{ij} \otimes e_{ij} \Big\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)} \Big\}.$$

The term E is calculated as follows:

$$\begin{aligned}
\mathrm{E} &= \Big\| \sum_{ijkl} u(x_{ij} y) \mathsf{E}(z_j z_l^*) u(x_{kl} y)^* \Big\|_\infty^{1/2} \\
&= \Big\| \sum_{j=1}^n \Big(\sum_{i=1}^n u(a_{ij} \otimes w_i f) \Big) \Big(\sum_{k=1}^n u(a_{kj} \otimes w_k f) \Big)^* \Big\|_\infty^{1/2} \\
&= \Big\| \sum_{j=1}^n e_{1j} \otimes \Big(\sum_{i=1}^n u(a_{ij} \otimes w_i f) \Big) \Big\|_\infty \\
&= \Big\| \sum_{i,j=1}^n e_{ij} \otimes \Big(\sum_{r,s=1}^n a_{ri}^* \phi(f^* w_r w_s f) a_{sj} \Big) \Big\|_\infty^{1/2} \\
&= \|f\|_2 \Big\| \sum_{i,j=1}^n a_{ij} \otimes e_{ij} \Big\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)} = \frac{1}{\sqrt{n}} \Big\| \sum_{i,j=1}^n a_{ij} \otimes e_{ij} \Big\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)} .
\end{aligned}$$

The same identity holds for F:

$$\mathrm{F} = \frac{1}{\sqrt{n}} \left\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij} \right\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)}.$$

The calculation of G is very similar:

$$\begin{aligned}
\mathrm{G} &= \left\| \sum_{ijkl} u(x_{ij}) y \mathsf{E}(z_j z_l^*) y^* u(x_{kl})^* \right\|_\infty^{1/2} \\
&= \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{n} u(a_{ij} \otimes w_i) y \right) \left(\sum_{k=1}^{n} u(a_{kj} \otimes w_k) y \right)^* \right\|_\infty^{1/2} \\
&= \left\| \sum_{j=1}^{n} e_{1j} \otimes \left(\sum_{i=1}^{n} u(a_{ij} \otimes w_i) y \right) \right\|_\infty \\
&= \left\| \sum_{i,j=1}^{n} e_{ij} \otimes \left(\sum_{r,s=1}^{n} a_{ri}^* a_{sj} \otimes f^* \phi(w_r w_s) f \right) \right\|_\infty^{1/2} \\
&= \|f\|_\infty \left\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij} \right\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)} = \left\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij} \right\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)}.
\end{aligned}$$

On the other hand, we observe that the maps on $\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)$,

$$\begin{aligned}
\sum_{i,j=1}^{n} a_{ij} \otimes e_{i1} \otimes e_{1j} &\mapsto \sum_{i,j=1}^{n} a_{ij} \otimes e_{1i} \otimes e_{1j}, \\
\sum_{i,j=1}^{n} a_{ij} \otimes e_{i1} \otimes e_{1j} &\mapsto \sum_{i,j=1}^{n} a_{ij} \otimes e_{i1} \otimes e_{j1},
\end{aligned}$$

have norm $\sqrt{n}$. Indeed, this follows automatically from the well-known fact that the natural mappings $R_n \to C_n$ and $C_n \to R_n$ between n-dimensional row and column Hilbert spaces are completely bounded with cb-norm $\sqrt{n}$; see, for example, [7] or [32] for the proof. Thus, we deduce

$$\begin{aligned}
\left\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{1,ij} \right\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)} &\leq \sqrt{n} \left\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij} \right\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)}, \\
\left\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij,1} \right\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)} &\leq \sqrt{n} \left\| \sum_{i,j=1}^{n} a_{ij} \otimes e_{ij} \right\|_{\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)}.
\end{aligned}$$

The assertion follows easily from these inequalities and the estimates above. $\square$

The idea to find our counterexample follows an argument from [34]. We consider a suitable martingale for which the Burkholder–Gundy inequality implies an upper estimate for the triangular projection on $\mathcal{B}(\ell_2^n)$. This gives the logarithmic growth stated in Theorem D. After the proof of our counterexample or Theorem D, we shall study the reverse estimate for free martingales whose martingale differences are polynomials of a bounded degree.

PROOF OF THEOREM D. Let us define

$$x_{2n} = \sum_{1 \le i,j \le n} a_{ij} w_i f w_j' \qquad \text{with } a_{ij} \in \mathbb{C}.$$

Here w_i, f and w_j' are defined as in Lemma 4.1. Moreover, the enumeration given in the statement of Lemma 4.1 for the algebras $\mathsf{A}_0, \mathsf{A}_1, \mathsf{A}_2, \dots$ provides a natural martingale structure for the x_{2n}'s, that is, with respect to the natural filtration $(\mathcal{A}_k)$ defined by $\mathcal{A}_k = \mathsf{A}_0 * \mathsf{A}_1 * \mathsf{A}_2 * \cdots * \mathsf{A}_k$. An easy inspection gives the following expressions valid for all $k \ge 0$:

$$dx_{2k} = \sum_{1 \le i \le k} a_{ik} w_i f w_k' \quad \text{and} \quad dx_{2k-1} = \sum_{1 \le j < k} a_{kj} w_k f w_j'. \tag{22}$$

We are interested in the best constant $\mathcal{K}_n$ for

$$\max\left\{ \left\| \left(\sum_{k=1}^{2n} dx_k dx_k^* \right)^{1/2} \right\|_\infty, \left\| \left(\sum_{k=1}^{2n} dx_k^* dx_k \right)^{1/2} \right\|_\infty \right\} \le \mathcal{K}_n \left\| \sum_{k=1}^{2n} dx_k \right\|_\infty.$$

According to Lemma 4.1, we have

$$\left\| \sum_{k=1}^{2n} dx_k \right\|_\infty \sim_c \left\| \sum_{i,j=1}^{n} a_{ij} e_{ij} \right\|_{\mathcal{B}(\ell_2)}.$$

On the other hand, we observe that

$$\left\| \left(\sum_{k=1}^{n} dx_{2k} dx_{2k}^* \right)^{1/2} \right\|_\infty = \left\| \sum_{k=1}^{n} e_{1k} \otimes dx_{2k} \right\|_\infty = \left\| \sum_{i \le k} a_{ik} e_{1k} \otimes w_i f w_k' \right\|_\infty,$$

$$\left\| \left(\sum_{k=1}^{n} dx_{2k-1}^* dx_{2k-1} \right)^{1/2} \right\|_\infty = \left\| \sum_{k=1}^{n} e_{k1} \otimes dx_{2k-1} \right\|_\infty = \left\| \sum_{k > j} a_{kj} e_{k1} \otimes w_k f w_j' \right\|_\infty.$$

Thus, we may apply Lemma 4.1 one more time and obtain

$$\left\| \left(\sum_{k=1}^{n} dx_{2k} dx_{2k}^* \right)^{1/2} \right\|_\infty \sim_c \left\| \sum_{i \le j} a_{ij} e_{1j} \otimes e_{ij} \right\|_{\mathcal{B}(\ell_2 \otimes \ell_2)} = \left\| \sum_{\substack{i,j=1 \\ i \le j}}^{n} a_{ij} e_{ij} \right\|_{\mathcal{B}(\ell_2)},$$

$$\left\| \left(\sum_{k=1}^{n} dx_{2k-1}^* dx_{2k-1} \right)^{1/2} \right\|_\infty \sim_c \left\| \sum_{i > j} a_{ij} e_{i1} \otimes e_{ij} \right\|_{\mathcal{B}(\ell_2 \otimes \ell_2)} = \left\| \sum_{\substack{i,j=1 \\ i > j}}^{n} a_{ij} e_{ij} \right\|_{\mathcal{B}(\ell_2)}.$$

That is, $\mathcal{K}_n$ is bounded from below by c times the norm of the triangular projection on $\mathcal{B}(\ell_2^n)$. However, it is well known that the norm of the triangular projection grows like $\log n$; see, for example, [21]. This completes the proof. $\square$

After Theorem D, it remains open to see whether or not the reverse estimate in the Burkholder–Gundy inequalities holds for free martingales in $L_\infty(\mathcal{A})$. In the following result we give a partial solution to this problem. We will work with free martingales (i.e., adapted to the free filtration) with martingale differences

$$dx_k = \sum_{\alpha\in\Lambda} \sum_{j_1\neq\cdots\neq j_d} a_{j_1}^k(\alpha)\cdots a_{j_d}^k(\alpha) \quad \text{and} \quad a_{j_s}^k(\alpha) \in \mathring{\mathsf{A}}_{j_s}, \tag{23}$$

where $1 \le j_1, j_2, \dots, j_d \le k$. Note that since dx_k is a martingale difference with respect to the free filtration, at least one of the j_l's has to be k. That is, we assume that all the martingale differences are d-homogeneous free polynomials. We shall refer to these kind of martingales as *d-homogeneous free martingales*. Moreover, if x is a free martingale with dx_k being a (not necessarily homogeneous) free polynomial of degree d, we shall simply say that x is a *d-polynomial free martingale*. We shall also use the following notation:

$$\mathcal{S}_\infty(x,n) = \max\left\{ \left\| \left(\sum_{k=1}^n dx_k dx_k^* \right)^{1/2} \right\|_\infty, \left\| \left(\sum_{k=1}^n dx_k^* dx_k \right)^{1/2} \right\|_\infty \right\}.$$

Our main tools in the following result are again Theorems A and B.

PROPOSITION 4.2. *If x is a d-polynomial free martingale*,

$$\left\| \sum_{k=1}^n dx_k \right\|_\infty \le c^d d^2 \sqrt{d!}\, \mathcal{S}_\infty(x,n).$$

PROOF. Let us consider the inequality

$$\left\| \sum_{k=1}^n dx_k \right\|_\infty \le \mathcal{C}(d)\, \mathcal{S}_\infty(x,n) \tag{24}$$

valid for any d-homogeneous free martingale x with $d \ge 0$. To prove (24) and estimate $\mathcal{C}(d)$, we proceed by induction on d. Namely, for $d = 0$, we have $dx_1 = \mathsf{E}(x)$ and $dx_k = 0$ for $k = 2, 3, \dots$. In particular,

$$\left\| \sum_{k=1}^n dx_k \right\|_\infty = \|dx_1\|_\infty \le \mathcal{S}_\infty(x,n).$$

Therefore, (24) holds for $d = 0$ with $\mathcal{C}(0) = 1$. If $d = 1$, we observe that

$$\sum_{k=1}^n dx_k = \sum_{k=1}^n \mathcal{L}_k(dx_k).$$

Thus, Proposition 2.8 gives

$$\Big\|\sum_{k=1}^n dx_k\Big\|_\infty \leq 3\max\bigg\{\Big\|\sum_{k=1}^n \mathcal{L}_k(dx_k)\mathcal{L}_k(dx_k)^*\Big\|_\infty^{1/2}, \Big\|\sum_{k=1}^n \mathcal{L}_k(dx_k)^*\mathcal{L}_k(dx_k)\Big\|_\infty^{1/2}\bigg\}.$$

This, combined with the proof of Lemma 2.5, gives rise to

$$\Big\|\sum_{k=1}^n dx_k\Big\|_\infty \leq 9\mathcal{S}_\infty(x,n).$$

In particular, (24) holds for $d=1$ with $\mathcal{C}(1)\leq 9$. Now we assume that (24) holds for $(d-1)$-homogeneous free martingales with some constant $\mathcal{C}(d-1)$. To prove (24) for a d-homogeneous free martingale x, we decompose the martingale differences by means of the mappings $\mathcal{L}_k$ as follows:

$$\Big\|\sum_{k=1}^n dx_k\Big\|_\infty \leq \Big\|\sum_{k=1}^n \mathcal{L}_k(dx_k)\Big\|_\infty + \Big\|\sum_{k=1}^n (id_{\mathcal{A}}-\mathcal{L}_k)(dx_k)\Big\|_\infty = \mathrm{A}+\mathrm{B}.$$

The estimate

$$\mathrm{A}\leq 9\mathcal{S}_\infty(x,n), \tag{25}$$

follows as the inequality $\mathcal{C}(1)\leq 9$ above. On the other hand, we have

$$(id_{\mathcal{A}}-\mathcal{L}_k)(dx_k) = \sum_{\alpha\in\Lambda}\sum_{j=1}^{k-1} x_j^k(\alpha)w_j^k(\alpha),$$

with $x_j^k(\alpha)\in\mathring{\mathsf{A}}_j$ and $w_j^k(\alpha)\in\mathbf{P}_{\mathcal{A}}(d-1)$ satisfying $\mathcal{L}_j(w_j^k(\alpha))=0$. Indeed, this follows from the fact that no word in $(id_{\mathcal{A}}-\mathcal{L}_k)(dx_k)$ starts with a mean-zero letter in A_k and that $dx_k\in\mathcal{A}_k$. Thus, we may write B in the form

$$\mathrm{B} = \Big\|\sum_{(\alpha,k)\in\Delta}\sum_{j=1}^{n-1} x_j(\alpha,k)w_j(\alpha,k)\Big\|_\infty,$$

with $\Delta=\Lambda\times\{1,2,\ldots,n\}$ and

$$x_j(\alpha,k)w_j(\alpha,k) = \begin{cases} 0, & \text{if } j\geq k,\\ x_j^k(\alpha)w_j^k(\alpha), & \text{if } j<k.\end{cases}$$

According to Theorem B, we obtain

$$\Big\|\sum_{(\alpha,k),j} x_j(\alpha,k)w_j(\alpha,k)\Big\|_\infty \leq \Big\|\sum_{(\alpha,k),j} x_j(\alpha,k)\langle w_j(\alpha,k)|\Big\|_\infty + \Big\|\sum_{(\alpha,k),j} |x_j(\alpha,k)\rangle w_j(\alpha,k)\Big\|_\infty = \mathrm{B}_1+\mathrm{B}_2.$$

Note that the constant 1 in the inequality above holds since we are only considering the case $(p,q) = (\infty, 2)$ in the proof of Theorem B. Let us start by estimating the first term B_1. We claim that

$$\mathrm{B}_1^2 = \left\| \sum_{k=1}^n \sum_{\alpha,\beta\in\Lambda} \sum_{j_1,j_2=1}^{n-1} x_{j_1}(\alpha,k) \mathsf{E}(w_{j_1}(\alpha,k) w_{j_2}(\beta,k)^*) x_{j_2}(\beta,k)^* \right\|_\infty.$$

To see this, it suffices to show that

$$\mathsf{E}(w_{j_1}(\alpha,k_1) w_{j_2}(\beta,k_2)^*) = 0$$

for $k_1 \neq k_2$. Indeed, let us assume without lost of generality that $k_1 < k_2$. Then we know by construction that $w_{j_1}(\alpha,k_1) \in \mathcal{A}_{k_1}$ and that $w_{j_2}(\beta,k_2)$ contains a mean-zero letter in A_{k_2} with $k_2 > k_1$. Thus, our claim follows easily by freeness. Hence, we may write the identity above as follows:

$$\mathrm{B}_1 = \left\| \sum_{k=1}^n e_{1k} \otimes \sum_{\alpha\in\Lambda} \sum_{j=1}^{k-1} x_j^k(\alpha) \langle w_j^k(\alpha)| \right\|_\infty.$$

Arguing as in the proof of Theorem B, we obtain

$$\begin{aligned}
\mathrm{B}_1 &\leq \sqrt{5} \left\| \sum_{k=1}^n e_{1k} \otimes \sum_{\alpha\in\Lambda} \sum_{j=1}^{k-1} x_j^k(\alpha) w_j^k(\alpha) \right\|_\infty \\
&= \sqrt{5} \left\| \sum_{k=1}^n e_{1k} \otimes (id_{\mathcal{A}} - \mathcal{L}_k)(dx_k) \right\|_\infty \qquad (26)\\
&\leq \sqrt{5} \left[\left\| \sum_{k=1}^n e_{1k} \otimes dx_k \right\|_\infty + \left\| \sum_{k=1}^n e_{1k} \otimes \mathcal{L}_k(dx_k) \right\|_\infty \right] \leq 4\sqrt{5}\, \mathcal{S}_\infty(x,n),
\end{aligned}$$

where the last inequality follows from Lemma 2.5 one more time.

To estimate B_2, we observe that

$$\sum_{k=1}^n \sum_{\alpha\in\Lambda} \sum_{j=1}^{k-1} |x_j^k(\alpha)\rangle w_j^k(\alpha) \tag{27}$$

can be regarded as a sum of martingale differences on the von Neumann algebra $\mathcal{B}(\ell_2) \bar{\otimes} \mathcal{A}$ with respect to the index k and the filtration $\mathcal{B}(\ell_2) \bar{\otimes} \mathcal{A}_1, \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{A}_2, \dots$. Indeed, we have

$$id_{\mathcal{B}(\ell_2)} \otimes \mathsf{E}_{k-1} \left(\sum_{\alpha\in\Lambda} \sum_{j=1}^{k-1} |x_j^k(\alpha)\rangle w_j^k(\alpha) \right) = \sum_{\alpha\in\Lambda} \sum_{j=1}^{k-1} |x_j^k(\alpha)\rangle \mathsf{E}_{k-1}(w_j^k(\alpha)) = 0.$$

Then, since (27) forms a $(d-1)$-homogeneous free martingale, we may apply the induction hypothesis and obtain in this way the following upper bound for B_2:

$$\mathcal{C}(d-1) \max \left\{ \left\| \sum_{k=1}^n e_{1k} \otimes \sum_{\alpha,j} |x_j^k(\alpha)\rangle w_j^k(\alpha) \right\|_\infty, \left\| \sum_{k=1}^n e_{k1} \otimes \sum_{\alpha,j} |x_j^k(\alpha)\rangle w_j^k(\alpha) \right\|_\infty \right\}.$$

Then, arguing as in the proof of Theorem B $(2(2(d-1))+1=4d-3)$, we deduce

$$\mathrm{B}_2 \leq \sqrt{4d-3}\,\mathcal{C}(d-1) \times \max\left\{ \left\| \sum_{k=1}^n e_{1k} \otimes (id_{\mathcal{A}} - \mathcal{L}_k)(dx_k) \right\|_\infty, \left\| \sum_{k=1}^n e_{k1} \otimes (id_{\mathcal{A}} - \mathcal{L}_k)(dx_k) \right\|_\infty \right\}.$$

The triangle inequality and Lemma 2.5 produce

$$\mathrm{B}_2 \leq 4\sqrt{4d-3}\,\mathcal{C}(d-1)\,\mathcal{S}_\infty(x,n). \tag{28}$$

Now (25), (26), (28) give

$$\mathcal{C}(d) \leq \big(9+4\sqrt{5}\big) + 4\sqrt{4d-3}\,\mathcal{C}(d-1) \leq c\sqrt{d}\,\mathcal{C}(d-1).$$

Iterating the recurrence and using $\mathcal{C}_0 = 1$, we find $\mathcal{C}(d) \leq c^d\sqrt{d!}$. Therefore,

$$\left\| \sum_{k=1}^n dx_k \right\|_\infty \leq c^d \sqrt{d!}\,\mathcal{S}_\infty(x,n) \tag{29}$$

for d-homogeneous free martingales.

Now let x be any d-polynomial free martingale x. We may decompose x into its homogeneous parts $dx_k = \sum_s dx_k^s$ with $0 \leq s \leq d$. It is clear that $dx_1^s, dx_2^s, dx_3^s, \ldots$ are the martingale differences of an s-homogeneous free martingale x^s. Therefore, applying (29), we deduce

$$\left\| \sum_{k=1}^n dx_k \right\|_\infty \leq \sum_{s=0}^d \left\| \sum_{k=1}^n dx_k^s \right\|_\infty \leq \|\mathsf{E}(x)\|_\infty + \sum_{s=1}^d c^s \sqrt{s!}\,\mathcal{S}_\infty(x^s,n).$$

For the first term, we have

$$\|\mathsf{E}(x)\|_\infty = \|\mathsf{E}(\mathsf{E}_1(x))\|_\infty \leq \|\mathsf{E}_1(x)\|_\infty = \|dx_1\|_\infty \leq \mathcal{S}_\infty(x,n).$$

The rest of the terms are estimated by Theorem 2.1:

$$\begin{aligned}
\mathcal{S}_\infty(x^s,n) &\sim \left\| \sum_{k=1}^n e_{1k} \otimes dx_k^s \right\|_\infty + \left\| \sum_{k=1}^n e_{k1} \otimes dx_k^s \right\|_\infty \\
&= \left\| \big(id_{\mathcal{B}(\ell_2)} \otimes \Pi_{\mathcal{A}}(\infty,s)\big)\left(\sum_{k=1}^n e_{1k} \otimes dx_k \right) \right\|_\infty \\
&\quad + \left\| \big(id_{\mathcal{B}(\ell_2)} \otimes \Pi_{\mathcal{A}}(\infty,s)\big)\left(\sum_{k=1}^n e_{k1} \otimes dx_k \right) \right\|_\infty \leq 4s\,\mathcal{S}_\infty(x,n).
\end{aligned}$$

Our estimates give rise to

$$\left\| \sum_{k=1}^n dx_k \right\|_\infty \leq \left(1 + 4\sum_{s=1}^d c^s s\sqrt{s!} \right) \mathcal{S}_\infty(x,n) \leq c^d d^2 \sqrt{d!}\,\mathcal{S}_\infty(x,n).$$

This is the desired estimate. The proof is complete. □

REMARK 4.3. Proposition 4.2 extends to the case $2 \le p \le \infty$. Indeed, we just need to replace Proposition 2.8 by Corollary 2.14 and apply Theorem B in full generality. Of course, this would provide a worse constant. The relevance of Proposition 4.2 lies, however, in the fact that the resulting constants are uniformly bounded as $p \to \infty$, in contrast with the nonfree setting [34].

5. Generalized circular systems. In this last section we illustrate our results by investigating Khintchine type inequalities for Shlyakhtenko's generalized circular systems and Hiai's generalized q-Gaussians. Given an infinite dimensional and separable Hilbert space $\mathcal{H}$ equipped with a distinguished unit vector or vacuum Ω, we denote by $\mathcal{F}(\mathcal{H})$ the associated Fock space

$$\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n\ge 1} \mathcal{H}^{\otimes n}.$$

Given any vector $e \in \mathcal{H}$, we denote by $\ell(e)$ the left creation operator on $\mathcal{F}(\mathcal{H})$ associated with e, which acts by tensoring from the left. The adjoint map $\ell^*(e)$ is called the annihilation operator on $\mathcal{F}(\mathcal{H})$; see [45] for more details. Let us fix an orthonormal basis $(e_{\pm k})_{k\ge 1}$ in $\mathcal{H}$ and two sequences $(\lambda_k)_{k\ge 1}$ and $(\mu_k)_{k\ge 1}$ of positive numbers. Set

$$g_k = \lambda_k \ell(e_k) + \mu_k \ell^*(e_{-k}).$$

The g_k's are *generalized circular random variables* studied by Shlyakhtenko [39]. Let Γ denote the von Neumann algebra generated by the generalized circular system $(g_k)_{k\ge 1}$. Γ is equipped with the vacuum state ϕ given by $\phi(x) = \langle \Omega, x\Omega\rangle$. According to [39], ϕ is faithful and the g_k's are free with respect to ϕ. In fact, if Γ_k is the von Neumann subalgebra of Γ generated by g_k, then $(\Gamma, \phi) = *_{k\ge 1}(\Gamma_k, \phi_{\Gamma_k})$. Shlyakhtenko also calculated in [39] the modular group and showed that $\sigma_t(g_k) = (\lambda_k^{-1}\mu_k)^{2it} g_k$. In particular, the g_k's are analytic elements of Γ and eigenvectors of the modular automorphism group σ. Let us write d_ϕ for the density associated to the state ϕ on Γ. We shall also need the elements

$$g_{k,p} = d_\phi^{1/(2p)} g_k d_\phi^{1/(2p)} = (\lambda_k^{-1}\mu_k)^{1/p} g_k d_\phi^{1/p} = (\lambda_k \mu_k^{-1})^{1/p} d_\phi^{1/p} g_k. \tag{30}$$

The following is the Khintchine type inequality for 1-homogeneous polynomials on generalized circular random variables. Its proof can be found in [47], where the third-named author used Theorem A to obtain constants independent of p. When $\lambda_k = \mu_k$ for $k \ge 1$, the g_k's become a usual circular system in Voiculescu's sense and the result below reduces to Theorem 8.6.5 in [31]. On the other hand, the case $p = \infty$ was already proved by Pisier and Shlyakhtenko in [33].

THEOREM 5.1. *Let $\mathcal{N}$ be a von Neumann algebra and $1 \le p \le \infty$. Let us consider a finite sequence $x_1, x_2, \ldots, x_n$ in $L_p(\mathcal{N})$. Then, the following equivalences hold up to an absolute constant c independent of n:*

(i) *If* $1 \le p \le 2$, *then*

$$\left\| \sum_{k=1}^n x_k \otimes g_{k,p} \right\|_p \sim_c \inf_{x_k = a_k + b_k} \left\| \left(\sum_{k=1}^n \lambda_k^{2/p} \mu_k^{2/p'} a_k a_k^* \right)^{1/2} \right\|_p + \left\| \left(\sum_{k=1}^n \lambda_k^{2/p'} \mu_k^{2/p} b_k^* b_k \right)^{1/2} \right\|_p .$$

(ii) *If* $2 \le p \le \infty$, *then*

$$\left\| \sum_{k=1}^n x_k \otimes g_{k,p} \right\|_p \sim_c \max \left\{ \left\| \left(\sum_{k=1}^n \lambda_k^{2/p} \mu_k^{2/p'} x_k x_k^* \right)^{1/2} \right\|_p , \left\| \left(\sum_{k=1}^n \lambda_k^{2/p'} \mu_k^{2/p} x_k^* x_k \right)^{1/2} \right\|_p \right\}.$$

Moreover, let us write $\mathcal{G}_p$ *for the closed subspace of* $L_p(\Gamma)$ *generated by the system of generalized circular variables* $(g_{k,p})_{k \ge 1}$. *Then, there exists a completely bounded projection* $\gamma_p : L_p(\Gamma) \to \mathcal{G}_p$ *satisfying*

$$\|\gamma_p\|_{cb} \le 2^{|1-2/p|}.$$

REMARK 5.2. It is worthy of mention that Theorem 5.1 improves Theorem C in the case of generalized circular systems. Indeed, we have only used two terms while Theorem C needs three terms in the general case of 1-homogeneous polynomials. This phenomenon will also occur in the case of degree 2; see below.

As application, we collect some interpolation identities that arise from Theorem 5.1. Indeed, we consider the spaces $\mathcal{J}_p$ and $\mathcal{K}_p$, respectively defined as the closure of finite sequences in $L_p(\mathcal{N})$ with respect to the following norms:

$$\|(x_k)\|_{\mathcal{K}_p} = \inf_{x_k = a_k + b_k} \left\| \left(\sum_{k=1}^n \lambda_k^{2/p} \mu_k^{2/p'} a_k a_k^* \right)^{1/2} \right\|_p + \left\| \left(\sum_{k=1}^n \lambda_k^{2/p'} \mu_k^{2/p} b_k^* b_k \right)^{1/2} \right\|_p ,$$

$$\|(z_k)\|_{\mathcal{J}_p} = \max \left\{ \left\| \left(\sum_k \lambda_k^{2/p} \mu_k^{2/p'} z_k z_k^* \right)^{1/2} \right\|_p , \left\| \left(\sum_k \lambda_k^{2/p'} \mu_k^{2/p} z_k^* z_k \right)^{1/2} \right\|_p \right\}.$$

Given $1 \le p \le \infty$, we define the spaces

$$L_p(\mathcal{N}; RC_p(\lambda, \mu)) = \begin{cases} \mathcal{K}_p, & \text{for } 1 \le p \le 2, \\ \mathcal{J}_p, & \text{for } 2 \le p \le \infty, \end{cases}$$

and the maps

$$u_p : (x_k) \in L_p(\mathcal{N}; RC_p(\lambda, \mu)) \mapsto \sum_k x_k \otimes g_{k,p} \in L_p(\mathcal{N} \bar{\otimes} \Gamma).$$

COROLLARY 5.3. *If* $1 \le p_0, p_1 \le \infty$, $0 < \theta < 1$ *and* $1/p = (1-\theta)/p_0 + \theta/p_1$, *then*

$$[L_{p_0}(\mathcal{N}; RC_{p_0}(\lambda,\mu)), L_{p_1}(\mathcal{N}; RC_{p_1}(\lambda,\mu))]_\theta \simeq L_p(\mathcal{N}; RC_p(\lambda,\mu)).$$

Moreover, the relevant constants are majorized by a universal constant.

PROOF. Let us recall Kosaki's theorem [20]:

$$[L_{p_0}(\mathcal{N}\bar{\otimes}\Gamma), L_{p_1}(\mathcal{N}\bar{\otimes}\Gamma)]_\theta = L_p(\mathcal{N}\bar{\otimes}\Gamma).$$

More precisely, if the von Neumann algebra $\mathcal{N}$ is equipped with the *n.f.* state ψ and $d_{\psi\otimes\phi}$ denotes the density associated to $\psi\otimes\phi$, we use in the interpolation isometry above the symmetric inclusions

$$d_{\psi\otimes\phi}^{1/2p_0'} L_{p_0}(\mathcal{N}\bar{\otimes}\Gamma) d_{\psi\otimes\phi}^{1/2p_0'} \subset L_1(\mathcal{N}\bar{\otimes}\Gamma),$$

$$d_{\psi\otimes\phi}^{1/2p_1'} L_{p_1}(\mathcal{N}\bar{\otimes}\Gamma) d_{\psi\otimes\phi}^{1/2p_1'} \subset L_1(\mathcal{N}\bar{\otimes}\Gamma).$$

Then we recall from Theorem 5.1 that the maps u_p defined above are isomorphic embeddings. Using in addition the projection γ_p introduced in Theorem 5.1, we deduce the assertion. The proof is complete. $\square$

Corollary 5.3 provides interesting applications in the theory of operator spaces. Given two sequences $(\xi_k)_{k\ge1}$ and $(\rho_k)_{k\ge1}$ of positive numbers, we introduce the operator space $R_p(\xi)\cap C_p(\rho)$ as the span of the sequence $f_k = \xi_k e_{1k} + \rho_k e_{k1}$ in the Schatten class S_p. Note that

$$\begin{aligned}\Big\|\sum_k x_k\otimes f_k\Big\|_{L_p(\mathcal{N}; R_p(\xi)\cap C_p(\rho))} \\ \sim \max\left\{\left\|\left(\sum_k \xi_k^2 x_k x_k^*\right)^{1/2}\right\|_p, \left\|\left(\sum_k \rho_k^2 x_k^* x_k\right)^{1/2}\right\|_p\right\}.\end{aligned}$$

By duality, we understand the sum $R_p(\xi)+C_p(\rho)$ as a quotient space. Indeed, we consider the subspace $R_p\oplus C_p$ of S_p as the span of $(e_{1k}; e_{k1})_{k\ge1}$ in S_p. Then we have

$$R_p(\xi)+C_p(\rho) = R_p\oplus C_p/\Delta(\xi,\rho),$$

where Δ is the weighted diagonal $\Delta(\xi,\rho) = \mathrm{span}\{\xi_k e_{1k} - \rho_k e_{k1} \mid k\ge1\}$. Let π be the natural quotient map and let us consider the sequence $f_k = \pi(\xi_k e_{1k}) = \pi(\rho_k e_{k1})$ in $R_p(\xi)+C_p(\rho)$. Then we find

$$\begin{aligned}\Big\|\sum_k x_k\otimes f_k\Big\|_{L_p(\mathcal{N}; R_p(\xi)+C_p(\rho))} \\ \sim \inf_{x_k=a_k+b_k}\left\|\left(\sum_k \xi_k^2 a_k a_k^*\right)^{1/2}\right\|_p + \left\|\left(\sum_k \rho_k^2 b_k^* b_k\right)^{1/2}\right\|_p.\end{aligned}$$

COROLLARY 5.4. *Let* $(\lambda_k)_{k\geq 1}$ *and* $(\mu_k)_{k\geq 1}$ *be two sequences in* $\mathbb{R}_+$ *and* $1 \leq p \leq \infty$. *Then, the following cb-isomorphisms hold according to the value of* $\theta = 1/p$:

$$\begin{aligned}&[R(\lambda) \cap C(\mu), R(\lambda) + C(\mu)]_\theta \\ &\quad \simeq_{cb} \begin{cases} R_p(\lambda^\theta \mu^{1-\theta}) + C_p(\lambda^{1-\theta}\mu^\theta), & \text{if } 1 \leq p \leq 2, \\ R_p(\lambda^\theta \mu^{1-\theta}) \cap C_p(\lambda^{1-\theta}\mu^\theta), & \text{if } 2 \leq p \leq \infty. \end{cases}\end{aligned}$$

The relevant constants are majorized by an absolute constant.

PROOF. This is a reformulation of Corollary 5.3 in operator space terms. □

We now discuss the analogue of Theorem 5.1 for q-Gaussians. We refer to [1] for the basic definitions on q-deformation and to Hiai's paper [11] for the quasi-free q-deformation. Given an infinite dimensional separable Hilbert space $\mathcal{H}$ equipped with an orthonormal basis $(e_{\pm k})_{k\geq 1}$ and given $-1 < q < 1$, we denote by $\mathcal{F}_q(\mathcal{H})$ the associated q-Fock space

$$\mathcal{F}_q(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n\geq 1} \mathcal{H}^{\otimes n}$$

equipped with the q-scalar product induced by

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_q = \delta_{nm} \sum_{\pi \in \mathcal{S}_n} q^{i(\pi)} \langle f_1, g_{\pi(1)} \rangle \cdots \langle f_n, g_{\pi(n)} \rangle,$$

where $\mathcal{S}_n$ denotes the symmetric group of permutations of n elements and $i(\pi)$ stands for the number of inversions of π. Given a vector $e \in \mathcal{H}$, we write $\ell_q(e)$ for the left creation operator and $\ell_q^*(e)$ for the left annihilation; see [1] for the precise definitions. As in the free case, we define

$$gq_k = \lambda_k \ell_q(e_k) + \mu_k \ell_q^*(e_{-k})$$

after having fixed two sequences $(\lambda_k)_{k\geq 1}$ and $(\mu_k)_{k\geq 1}$ of positive numbers. The gq_k's are *q-generalized circular variables*. The von Neumann algebra generated by these variables in the GNS-construction with respect to the vacuum state $\phi_q(\cdot) = \langle \Omega, \cdot\Omega \rangle_q$ will be denoted by Γ_q. A discussion of the modular group of ϕ_q and important properties of these von Neumann algebras can be found in Hiai's paper. Indeed, we still have

$$\sigma_t(gq_k) = (\lambda_k^{-1}\mu_k)^{2it} gq_k.$$

Therefore, gq_k is an analytic element and we find as above

$$gq_{k,p} = d_{\phi_q}^{1/(2p)} gq_k d_{\phi_q}^{1/(2p)} = (\lambda_k^{-1}\mu_k)^{1/p} gq_k d_{\phi_q}^{1/p} = (\lambda_k \mu_k^{-1})^{1/p} d_{\phi_q}^{1/p} gq_k. \tag{31}$$

PROOF OF THEOREM E. Let us first see that the map

$$u_p : (x_k) \in \mathcal{K}_p \mapsto \sum_k x_k \otimes gq_{k,p} \in L_p(\mathcal{N} \bar{\otimes} \Gamma_q) \tag{32}$$

is a contraction for $1 \le p \le 2$. According to [17], we have

$$\|x\|_p \le \min\{\|\mathsf{E}(xx^*)^{1/2}\|_p, \|\mathsf{E}(x^*x)^{1/2}\|_p\} \qquad \text{for } 1 \le p \le 2.$$

Taking $x = \sum_k x_k \otimes gq_{k,p}$, the L_p-norm of x is bounded above by

$$\min\left\{\left\|\left(\sum_{i,j} \mathsf{E}(x_i x_j^* \otimes gq_{i,p} gq_{j,p}^*)\right)^{1/2}\right\|_p, \left\|\left(\sum_{i,j} \mathsf{E}(x_i^* x_j \otimes gq_{i,p}^* gq_{j,p})\right)^{1/2}\right\|_p\right\},$$

where $\mathsf{E} = id_{\mathcal{N}} \otimes \phi_q$ in our case. Therefore, recalling from (31) that

$$gq_{i,p} gq_{j,p}^* = d_{\phi_q}^{1/(2p)} gq_i d_{\phi_q}^{1/p} gq_j^* d_{\phi_q}^{1/(2p)} = (\lambda_i \mu_i^{-1} \lambda_j \mu_j^{-1})^{1/p} d_{\phi_q}^{1/p} gq_i gq_j^* d_{\phi_q}^{1/p},$$

$$gq_{i,p}^* gq_{j,p} = d_{\phi_q}^{1/(2p)} gq_i^* d_{\phi_q}^{1/p} gq_j d_{\phi_q}^{1/(2p)} = (\lambda_i^{-1} \mu_i \lambda_j^{-1} \mu_j)^{1/p} d_{\phi_q}^{1/p} gq_i^* gq_j d_{\phi_q}^{1/p},$$

and using the identities $\phi_q(gq_i gq_j^*) = \delta_{ij}\mu_i^2$ and $\phi_q(gq_i^* gq_j) = \delta_{ij}\lambda_i^2$, we deduce

$$\mathsf{E}(x_i x_j^* \otimes gq_{i,p} gq_{j,p}^*) = \delta_{ij} \lambda_i^{2/p} \mu_i^{2/p'} x_i x_i^*,$$

$$\mathsf{E}(x_i^* x_j \otimes gq_{i,p}^* gq_{j,p}) = \delta_{ij} \lambda_i^{2/p'} \mu_i^{2/p} x_i^* x_i.$$

Therefore, the triangle inequality yields

$$\begin{aligned} &\left\|\sum_{k=1}^n x_k \otimes gq_{k,p}\right\|_p \\ &\qquad \le \min\left\{\left\|\left(\sum_{k=1}^n \lambda_k^{2/p} \mu_k^{2/p'} x_k x_k^*\right)^{1/2}\right\|_p, \left\|\left(\sum_{k=1}^n \lambda_k^{2/p'} \mu_k^{2/p} x_k^* x_k\right)^{1/2}\right\|_p\right\}. \end{aligned}$$

This proves the contractivity of (32) for $1 \le p \le 2$. Now we show that

$$u_p : (x_k) \in \mathcal{J}_p \mapsto \sum_k x_k \otimes gq_{k,p} \in L_p(\mathcal{N} \bar{\otimes} \Gamma_q) \tag{33}$$

is bounded for $2 \le p \le \infty$ with a constant c_q depending only on q. If $p = 2$, the result follows by the orthogonality of the $gq_{k,2}$'s in $L_2(\Gamma_q)$. Therefore, according to Corollary 5.3, it suffices to estimate the norm of $u_\infty : \mathcal{J}_\infty \to \mathcal{N} \bar{\otimes} \Gamma_q$ and apply complex interpolation. By the definition of gq_k, we have

$$\sum_k x_k \otimes gq_k = \sum_k \lambda_k x_k \otimes \ell_q(e_k) + \sum_k \mu_k x_k \otimes \ell_q^*(e_{-k}).$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}\Big\|\sum_k \lambda_k x_k \otimes \ell_q(e_k)\Big\|_\infty &\leq \Big\|\Big(\sum_k \lambda_k^2 x_k^* x_k\Big)^{1/2}\Big\|_\infty \Big\|\Big(\sum_k \ell_q(e_k)\ell_q(e_k)^*\Big)^{1/2}\Big\|_\infty \\ &\leq \frac{1}{\sqrt{1-|q|}}\Big\|\Big(\sum_k \lambda_k^2 x_k^* x_k\Big)^{1/2}\Big\|_\infty,\end{aligned}$$

where the last inequality follows from [2]. Similarly, we have

$$\Big\|\sum_k \mu_k x_k \otimes \ell_q^*(e_{-k})\Big\|_\infty \leq \frac{1}{\sqrt{1-|q|}}\Big\|\Big(\sum_k \mu_k^2 x_k x_k^*\Big)^{1/2}\Big\|_\infty.$$

Thus, we obtain $\|u_\infty\| \leq 2/\sqrt{1-|q|}$ and

$$\|u_p : \mathcal{J}_p \to L_p(\mathcal{N} \bar{\otimes} \Gamma_q)\| \leq \left(\frac{2}{\sqrt{1-|q|}}\right)^{1-2/p} \qquad \text{for } 2 \leq p \leq \infty.$$

The crucial observation here is that

$$\begin{aligned}&\langle u_p((x_k)), u_{p'}((z_k))\rangle \\ &\quad = \sum_{i,j} \mathrm{tr}_{\mathcal{N}}(x_i^* z_j)\mathrm{tr}_{\Gamma_q}(g q_{i,p}^* g q_{j,p'}) \\ &\quad = \sum_{i,j} \mathrm{tr}_{\mathcal{N}}(x_i^* z_j)(\lambda_i^{-1}\mu_i)^{1/p}(\lambda_j^{-1}\mu_j)^{1/p'}\,\mathrm{tr}_{\Gamma_q}(d_\phi^{1/p} g q_i^* g q_j d_\phi^{1/p'}) \qquad (34) \\ &\quad = \sum_{i,j} \mathrm{tr}_{\mathcal{N}}(x_i^* z_j)(\lambda_i^{-1}\mu_i)^{1/p}(\lambda_j^{-1}\mu_j)^{1/p'}\phi_q(g q_i^* g q_j) \\ &\quad = \sum_k \lambda_k\mu_k\,\mathrm{tr}_{\mathcal{N}}(x_k^* y_k) = \langle (x_k), (z_k)\rangle.\end{aligned}$$

This relation and the boundedness of the maps (32) and (33) immediately imply the inequalities stated in (i) and (ii).

On the other hand, according to (34), we know that $u_{p'}^* u_p$ is the identity map and we may construct the following projection for every index $1 \leq p \leq \infty$:

$$u_p u_{p'}^* = id_{L_p(\mathcal{N})} \otimes \gamma q_p : L_p(\mathcal{N}\bar{\otimes}\Gamma_q) \to L_p(\mathcal{N}; \mathcal{G}q_p).$$

By elementary properties from [31] of vector-valued noncommutative L_p spaces, it suffices to prove that the maps above are bounded with the following constants for $1 \leq p \leq 2 \leq p' \leq \infty$:

$$\max\{\|\gamma q_p\|_{cb}, \|\gamma q_{p'}\|_{cb}\} = \max\{\|u_p u_{p'}^*\|, \|u_{p'} u_p^*\|\} \leq \left(\frac{2}{\sqrt{1-|q|}}\right)^{2/p-1}.$$

Recalling that the second estimate follows from the first by taking adjoints and that the estimate for $p = 2$ is trivial, it suffices to prove the estimate for $u_1 u_\infty^*$ and apply complex interpolation. However, according to our previous estimates, we find $\|u_1 u_\infty^*\| \le 2/\sqrt{1-|q|}$, as desired. This completes the proof. □

After this intermezzo on q-Gaussians, we conclude by illustrating our inequalities for 2-homogeneous polynomials on generalized circular variables. Again, our result in this particular case improves Theorem C since we obtain only three terms out of the five given there.

SKETCH OF THE PROOF OF THEOREM F. Following the arguments in Theorem E, it suffices to prove the assertion for $2 \le p \le \infty$ since the case $1 \le p \le 2$ and the complementation result follow from the same duality arguments. In order to prove the assertion for $2 \le p \le \infty$, we first consider a finite index set Λ to factorize

$$\sum_{i\neq j} x_{ij} \otimes d_\phi^{1/(2p)} g_i g_j d_\phi^{1/(2p)} = \sum_{i\neq j} \big(x_{ij} \otimes d_\phi^{1/(2p)} g_i\big)\big(1 \otimes g_j d^{1/(2p)}\big) = \sum_{i\neq j} \alpha_{ij}\beta_j.$$

According to Theorem B, we have, for $2 \le p \le \infty$,

$$\Big\| \sum_{i\neq j} \alpha_{ij}\beta_j \Big\|_p \sim_c \Big\| \sum_{i\neq j} |\alpha_{ij}\rangle \beta_j \Big\|_p + \Big\| \sum_{i\neq j} \alpha_{ij} \langle \beta_j| \Big\|_p .$$

Let us denote the terms on the right-hand side by A and B respectively. To simplify the expressions for A and B, we need to calculate $\mathsf{E}(\alpha_{ij}^*\alpha_{kl})$ and $\mathsf{E}(\beta_j\beta_l^*)$. According to (30), we easily find

$$\mathsf{E}(\alpha_{ij}^*\alpha_{kl}) = \delta_{ik}(x_{ij}^* x_{il} \otimes \lambda_i^{2/p'} \mu_i^{2/p} d_\phi^{1/p}),$$

$$\mathsf{E}(b_j b_l^*) = \delta_{jl}(1 \otimes \lambda_j^{2/p} \mu_j^{2/p'} d_\phi^{1/p}).$$

Using these relations and recalling the factorization above of x_{ij}, we obtain

$$\mathrm{A} = \Big\| \Big(\sum_i \lambda_i^{2/p'} \mu_i^{2/p} \sum_{j_1,j_2} x_{ij_1}^* x_{ij_2} \otimes g_{j_1,p}^* g_{j_2,p} \Big)^{1/2} \Big\|_p ,$$

$$\mathrm{B} = \Big\| \Big(\sum_j \lambda_j^{2/p} \mu_j^{2/p'} \sum_{i_1,i_2} x_{i_1 j} x_{i_2 j}^* \otimes g_{i_1,p} g_{i_2,p}^* \Big)^{1/2} \Big\|_p .$$

Equivalently, we have

$$\mathrm{A} = \Big\| \sum_i \lambda_i^{1/p'} \mu_i^{1/p} \Big(\sum_j x_{ij} \otimes g_{j,p} \Big) \otimes e_{i1} \Big\|_p = \Big\| \sum_k a_k \otimes g_{k,p} \Big\|_p ,$$

$$\mathrm{B} = \Big\| \sum_j \lambda_j^{1/p} \mu_j^{1/p'} \Big(\sum_i x_{ij} \otimes g_{i,p} \Big) \otimes e_{1j} \Big\|_p = \Big\| \sum_k b_k \otimes g_{k,p} \Big\|_p ,$$

where a_k and b_k are respectively given by

$$a_k = \sum_i \lambda_i^{1/p'} \mu_i^{1/p} x_{ik} \otimes e_{i1} \quad \text{and} \quad b_k = \sum_j \lambda_j^{1/p} \mu_j^{1/p'} x_{kj} \otimes e_{1j}.$$

According to Theorem 5.1, we obtain

$$\mathrm{A} \sim_c \left\| \left(\sum_{k=1}^n \lambda_k^{2/p} \mu_k^{2/p'} a_k a_k^* \right)^{1/2} \right\|_p + \left\| \left(\sum_{k=1}^n \lambda_k^{2/p'} \mu_k^{2/p} a_k^* a_k \right)^{1/2} \right\|_p = \mathrm{A}_1 + \mathrm{A}_2,$$

$$\mathrm{B} \sim_c \left\| \left(\sum_{k=1}^n \lambda_k^{2/p} \mu_k^{2/p'} b_k b_k^* \right)^{1/2} \right\|_p + \left\| \left(\sum_{k=1}^n \lambda_k^{2/p'} \mu_k^{2/p} b_k^* b_k \right)^{1/2} \right\|_p = \mathrm{B}_1 + \mathrm{B}_2.$$

Finally, using the terminology introduced in the statement of Theorem F, we have

$$\mathrm{B}_1 = \mathcal{R}_p(x), \qquad \mathrm{A}_1 = \mathcal{M}_p(x) = \mathrm{B}_2, \qquad \mathrm{A}_2 = \mathcal{C}_p(x).$$

Details of the identities above are left to the reader. This completes the proof. □

REMARK 5.5. Although it is out of the scope of this paper, the methods used in the proof of Theorem F are also valid for any degree $d \geq 1$. In this way, the L_p norm of an operator-valued d-homogeneous polynomial on generalized circular random variables behaves as the *asymmetric* version of the main result in [27]; see Theorem 2.6 there. Thus, we obtain $d+1$ terms instead of the $2d+1$ given by Theorem C.

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