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PRODUCT-LIMIT ESTIMATORS OF THE SURVIVAL FUNCTION WITH TWICE CENSORED DATA

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A model for competing (resp. complementary) risks survival data where the failure time can be left (resp. right) censored is proposed. Product-limit estimators for the survival functions of the individual risks are derived. We deduce the strong convergence of our estimators on the whole real half-line without any additional assumptions and their asymptotic normality under conditions concerning only the observed distribution. When the observations are generated according to the double censoring model introduced by Turnbull, the product-limit estimators represent upper and lower bounds for Turnbull's estimator.

1. Introduction. Consider the problem of nonparametric inference with competing risks survival data. The novelty we propose is that the failure time can be left-censored, for instance, at the time the study starts. For simplicity, we consider two distinct competing risks of failure, the extension to more than two competing risks being straightforward. Let T and V_1 denote the latent independent lifetimes for each cause of failure. The failure time is $\min(T, V_1)$ and it can be censored from the left by a censoring time U_1. The observations are independent copies of a lifetime Y, a finite nonnegative random variable and a discrete random variable A with values in $\{0, 1, 2\}$, where 2 indicates a left-censored failure time, while 0 and 1 correspond to an observation equal to T and V_1, respectively. If T is the lifetime of interest, we say that Y is a *twice censored* observation of T. Associated with the problem of competing risks is the dual problem of complementary risks where the observed failure time is the maximum of the lifetimes for each cause of failure (e.g., [1]). The extension we consider here is that the failure time can be right-censored, for instance, at the time the experience ends.

By the plug-in (or substitution) principle applied for the empirical distribution, the nonparametric estimation of the distribution of a latent lifetime of interest is straightforward as soon as this distribution can be expressed as an explicit function of the distribution of the observed variables. The two models we propose in this paper allow for explicit inversion formulae, that is, the latent distributions of interest are explicit functionals of the distribution of the observations.

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In Section 2 we introduce our latent models, while in Section 3 we provide the inversion formulae. In Section 4 we compare our model with the doubly censored data latent model proposed by Turnbull [14]. We show that the inversion formulae provide lower and upper bounds for the distribution of interest identified by Turnbull's model. Applying the inversion formulae to the empirical distribution, we deduce in Section 5 the product-limit estimators. In Sections 6 and 7 we deduce the almost sure uniform convergence and the asymptotic normality for our functionals.

2. Latent variables models. The random variables we consider take values in $\overline{\mathbb{R}}^+ = [0, \infty]$ endowed with $\overline{\mathcal{B}}^+$ the Borel σ-field. If X is such a variable, F_X denotes its distribution.

For the first latent model considered (call it Model I), let T and V_1 be two lifetimes and let U_1 be a left-censoring time. Assume that T, V_1 and U_1 are independent. Suppose that Y and A are observed, where $Y = \max[\min(T, V_1), U_1]$ and

$$A = \begin{cases} 0, & \text{if } U_1 < T \le V_1, \\ 1, & \text{if } U_1 < V_1 < T, \\ 2, & \text{if } \min(T, V_1) \le U_1. \end{cases}$$

Define the observed subdistributions of Y as

$$H_k(B) = P[Y \in B, A = k], \qquad k = 0, 1, 2,$$

where B is a Borel set in $[0, \infty]$; the distribution of Y is $H = H_0 + H_1 + H_2$. In Model I, the subdistributions of Y can be expressed in terms of the distributions of the latent variables as follows:

$$\begin{aligned} H_0(dt) &= F_{U_1}([0, t))F_{V_1}([t, \infty])F_T(dt), \\ H_1(dt) &= F_{U_1}([0, t))F_T((t, \infty])F_{V_1}(dt), \\ H_2(dt) &= \{1 - F_T((t, \infty])F_{V_1}((t, \infty])\}F_{U_1}(dt) \end{aligned} \tag{1}$$

[necessarily $H_0(\{0\}) = H_1(\{0\}) = 0$]. If $S_1 = \min(T, V_1)$ and $H_{01} = H_0 + H_1$, the three equations imply

$$H_{01}(dt) = F_{U_1}([0, t))F_{S_1}(dt), \qquad H_2(dt) = F_{S_1}([0, t])F_{U_1}(dt). \tag{2}$$

This indicates that the problem of inverting the model, that is, expressing the distributions of the latent variables in terms of the subdistributions of Y, can be solved in two steps. First, determine the distributions of U_1 and S_1 as in an independent left-censoring model. Next, use these distributions and the first equation in (1) to recover the distribution of T.

As an application of Model I, consider a reliability system which consists of three components U_1, T and V_1, with T and V_1 in series and U_1 in parallel with this series system (see, e.g., [8], Chapter 15). The lifetimes of U_1, T and V_1 are independent and when the system fails we are able to determine which component failed at the same time as the system. Morales, Pardo and Quesada [9] propose the application of this model to study a certain cause of death for trees on a farm.

For our second latent model (call it Model *II*), let U_2 and T be two lifetimes and let V_2 be a right-censoring time. Suppose T, U_2 and V_2 are independent. The observed variables are Y and A, where $Y = \min[\max(T, U_2), V_2]$ and

$$A = \begin{cases} 0, & \text{if } U_2 < T \le V_2, \\ 1, & \text{if } V_2 < \max(U_2, T), \\ 2, & \text{if } T \le U_2 \le V_2. \end{cases}$$

In Model *II*, the relationship between the subdistributions of Y and the distributions of the latent variables is described by the equations

$$\begin{aligned} H_0(dt) &= F_{U_2}([0,t))F_{V_2}([t,\infty])F_T(dt), \\ H_1(dt) &= \{1 - F_T([0,t])F_{U_2}([0,t])\}F_{V_2}(dt), \\ H_2(dt) &= F_T([0,t])F_{V_2}([t,\infty])F_{U_2}(dt) \end{aligned} \tag{3}$$

[necessarily $H_0(\{0\}) = 0$]. If $S_2 = \max(U_2, T)$ and $H_{02} = H_0 + H_2$, we obtain

$$H_{02}(dt) = F_{V_2}([t,\infty])F_{S_2}(dt), \qquad H_1(dt) = F_{S_2}((t,\infty])F_{V_2}(dt). \tag{4}$$

These relations show that Model *II* can be inverted in two steps. First, as in an independent right-censoring model, recover the distributions of V_2 and S_2 from H_{02} and H_1. Second, use the distributions of V_2 and S_2 and the first equation in (3) to determine the distribution of T.

Model *II* can be interpreted as follows: consider a system consisting of three components U_2, T and V_2 with independent lifetimes. Put T and U_2 in parallel and V_2 in series with this parallel system (see also [2], page 767). Again, assume that we are able to determine which component failed at the same time as the system.

3. Inversion formulae. Recall that if F is a probability distribution on $(\overline{\mathbb{R}}^+, \overline{\mathcal{B}}^+)$, the associated hazard measure is $L([0,t]) = -\ln F((t,\infty])$. Two more hazard measures can be defined,

$$L^-(dt) = \frac{F(dt)}{F([t,\infty])} \quad \text{and} \quad L^+(dt) = \frac{F(dt)}{F((t,\infty])},$$

which we call the predictable and the unpredictable hazard measure, respectively. The three hazard measures have the same continuous parts. Moreover, their point masses are in bijection: $L(\{t\}) = -\ln[1 - L^-(\{t\})] = \ln[1 + L^+(\{t\})]$. The probability distribution F can be expressed as

$$F((t,\infty]) = \exp\{-L([0,t])\} = \prod_{[0,t]}(1 - L^-(ds)) = \left[\prod_{[0,t]}(1 + L^+(ds))\right]^{-1},$$

where $\prod$ is the product-integral (e.g., [6]). The mass of L at infinity is irrelevant for F and $F(\{\infty\}) = \exp\{-L([0,\infty))\}$.

Similarly, by reversing time, the reverse hazard measure associated to F is $M((t,\infty]) = -\ln F([0,t])$. Moreover, the predictable and unpredictable reverse hazard measures are defined as

$$M^-(dt) = \frac{F(dt)}{F([0,t])} \quad \text{and} \quad M^+(dt) = \frac{F(dt)}{F([0,t))},$$

respectively. The three reverse hazard measures have the same continuous parts and their point masses satisfy $M(\{t\}) = -\ln[1 - M^-(\{t\})] = \ln[1 + M^+(\{t\})]$. We have

$$F([0,t]) = \exp\{-M((t,\infty])\} = \prod_{(t,\infty]}(1 - M^-(ds)) = \left[\prod_{(t,\infty]}(1 + M^+(ds))\right]^{-1}.$$

The mass $M(\{0\})$ is irrelevant for F. Moreover, $F(\{0\}) = \exp\{-M((0,\infty])\}$.

Given a nonnegative measure on $(\overline{\mathbb{R}}^+, \overline{\mathcal{B}}^+)$, we can always define a probability distribution on the same space by considering this measure as being one of L, L^- or L^+ (resp. M, M^- or M^+) and using the relations above. For instance, in the independent right-censoring model, one defines $L^-(dt) = H_0(dt)/H([t,\infty])$, with H_0 the subdistribution of the uncensored data. Then, by the equations of the model, the distribution corresponding to this L^- is nothing else than the distribution of the lifetime of interest. The reverse hazard measures M, M^- and M^+ are the counterparts of L, L^- and L^+ to be used in left-censoring models.

We can invert our models using the hazard measures above. Since, apart from mild conditions at the origin, the inversion formulae below apply to *any* subdistributions (H_0, H_1, H_2), we deduce them without any reference to the latent variables.

For inverting Model I, assume $H_0(\{0\}) = H_1(\{0\}) = 0$. In view of (2), proceed as for inverting a left-censoring model and define the predictable reverse hazard measures

$$M_2^-(dt) = \frac{H_2(dt)}{H([0,t])}, \qquad M_{01}^-(dt) = \frac{H_{01}(dt)}{H([0,t)) + H_{01}(\{t\})} \tag{5}$$

and let F_2^I and F_{01}^I be the corresponding distributions. By this definition, we have $H([0,t]) = F_2^I([0,t])F_{01}^I([0,t])$. In the second step of the inversion, note that the first equation in (1) and the definition of S_1 imply $H_0(dt)/F_{U_1}([0,t))F_{S_1}([t,\infty]) = F_T(dt)/F_T([t,\infty])$. This suggests defining the predictable hazard measure

$$L_T^{I-}(dt) = \frac{H_0(dt)}{F_2^I([0,t))F_{01}^I([t,\infty])}. \tag{6}$$

Let F_T^I be its associated distribution.

For Model II, assume $H_0(\{0\}) = 0$. Look at the relation (4) and, exactly as in a right-censoring model, define the predictable hazard measures

$$L_{02}^{-}(dt) = \frac{H_{02}(dt)}{H([t,\infty])}, \qquad L_1^{-}(dt) = \frac{H_1(dt)}{H((t,\infty]) + H_1(\{t\})}.$$

Let F_{02}^{II} and F_1^{II} denote the corresponding distributions. Clearly, $H((t,\infty]) = F_1^{II}((t,\infty])F_{02}^{II}((t,\infty])$. In the second step of the inversion, by the first equation in (3) and the definition of S_2, $H_0(dt)/\{F_{V_2}([t,\infty])F_{S_2}([0,t)) + H_0(\{t\})\} = F_T(dt)/F_T([0,t])$. Consequently, define the predictable reverse hazard measure

$$M_T^{II-}(dt) = \frac{H_0(dt)}{F_1^{II}([t,\infty])F_{02}^{II}([0,t)) + H_0(\{t\})} \tag{7}$$

and let F_T^{II} be its associated distribution.

Now, consider the identification problem. If Model I is correct, we look for conditions ensuring that $F_T^I = F_T$ on $\overline{\mathbb{R}}^+$. Define the support of μ, a nonnegative measure on $[0,\infty]$, as $supp(\mu) = \{t : \mu([0,t])\mu([t,\infty]) > 0\}$. Let $B_1 = \{t : F_{U_1}([0,t))F_{V_1}([t,\infty]) > 0\}$. Deduce from (6) that the support of L_T^{I-} is equal to the support of H_0. As $supp(H_0) = B_1 \cap supp(F_T)$,

$$F_T^I = F_T \qquad \text{on } \overline{\mathbb{R}}^+ \quad \Longleftrightarrow \quad supp(F_T) \subset B_1.$$

By similar arguments, if Model II is correct, $F_T^{II} = F_T$ on $\overline{\mathbb{R}}^+$ if and only if $supp(F_T) \subset \{t : F_{U_2}([0,t))F_{V_2}([t,\infty]) > 0\}$.

4. Comparisons with the doubly censored data model. The models we propose are closely related to the model for doubly (left and right) censored observations introduced by Turnbull [14]. In Turnbull's model the lifetime T is independent of the censoring variables (L, R) and $L \le R$. The observations are independent copies of Y and A, where

$$Y = \max[\min(T, R), L] = \min[\max(T, L), R],$$

$$A = \begin{cases} 0, & \text{if } L < T \le R \text{ (no censoring)}, \\ 1, & \text{if } (L \le)R < T \text{ (right censoring)}, \\ 2, & \text{if } T \le L(\le R) \text{ (left censoring)}. \end{cases}$$

If $H_k(dt) = P(Y \in dt, A = k)$, $k = 0, 1, 2$, the equations of the model are

$$\begin{aligned} H_0(dt) &= \{F_L([0,t)) - F_R([0,t))\}F_T(dt), \\ H_1(dt) &= F_T((t,\infty])F_R(dt), \\ H_2(dt) &= F_T([0,t])F_L(dt). \end{aligned} \tag{8}$$

Note that the assumptions of the model imply

$$H([0,t]) = F_L([0,t])F_T([0,t]) + F_R([0,t])F_T((t,\infty]). \tag{9}$$

In Turnbull's model T is censored from the left by L and from the right by R and the observation Y is always the variable in the middle. This is different from the censoring mechanisms we consider: in Model I the variable $\min(T, V_1)$ is left-censored, while in Model II the variable $\max(U_2, T)$ is right-censored.

Turnbull [14] proposed a nonparametric maximum likelihood estimator that can be obtained as the implicit solution of the equations (8). The implicit definition of Turnbull's estimator makes its asymptotic properties quite difficult (see [7]). Moreover, a numerical algorithm is needed for the applications.

We are interested in the relationship between our F_T^I, F_T^{II} and F_T identified by Turnbull's model. In fact, for any subdistributions H_0, H_1 and H_2 with $H_0(\{0\}) = H_1(\{0\}) = 0$,

$$F_T^I([0,t]) \le F_T([0,t]) \le F_T^{II}([0,t]) \qquad \forall t \ge 0,$$

where F_T is the distribution of T identified by Turnbull's model. Indeed, in Model I use definition (6) and $H([0,t]) = F_2^I([0,t])F_{01}^I([0,t])$ to write

$$L_T^{I-}(dt) = \frac{H_0(dt)}{F_2^I([0,t)) - H([0,t))}.$$

In Turnbull's model [relations (8) and (9)] we have

$$L_T^-(dt) = \frac{H_0(dt)}{F_L([0,t)) - H([0,t))}.$$

Next, the definition of M_2^-, the last equation in (8) and equation (9) imply

$$M_2^-(dt) = \frac{F_L([0,t])F_T([0,t])}{F_L([0,t])F_T([0,t]) + F_R([0,t])F_T((t,\infty])} M_L^-(dt).$$

Deduce that the measure M_2^- is smaller than the measure M_L^-. Therefore, $F_2^I([0,t)) \ge F_L([0,t))$, $\forall t \ge 0$. Hence, the measure L_T^{I-} is smaller than the measure L_T^-, which implies $F_T^I([0,t]) \le F_T([0,t])$, $\forall t \ge 0$.

On the other hand, for Model II, use the general relationship between M^+ and M^-, the definition (7) and $H((t,\infty]) = F_1^{II}((t,\infty])F_{02}^{II}((t,\infty])$ and write

$$M_T^{II+}(dt) = \frac{H_0(dt)}{F_1^{II}([t,\infty]) - H([t,\infty])}.$$

Meanwhile, in Turnbull's model,

$$M_T^+(dt) = \frac{H_0(dt)}{F_R([t,\infty]) - H([t,\infty])}.$$

Next, use the definition of L_1^-, the general relationship between L^+ and L^-, the second equation in (3) and the equality $H((t,\infty]) = F_L((t,\infty])F_T([0,t]) + F_R((t,\infty])F_T((t,\infty])$ [this is a consequence of (9)] to deduce

$$L_1^+(dt) = \frac{F_R((t,\infty])F_T((t,\infty])}{F_L((t,\infty])F_T([0,t]) + F_R((t,\infty])F_T((t,\infty])} L_R^+(dt).$$

Clearly, the measure L_1^+ is smaller than the measure L_R^+ and, therefore, $F_1^{II}([t,\infty]) \geq F_R([t,\infty])$, $\forall t \geq 0$. Hence, the measure M_T^{II+} is smaller than the measure M_T^+ and this implies $F_T^{II}([0,t]) \geq F_T([0,t])$, $\forall t \geq 0$.

5. Product-limit estimators. If we replace in the expressions of F_T^I and F_T^{II} the subdistributions H_0, H_1 and H_2 by their empirical counterparts, we obtain the product-limit estimators F_{nT}^I and F_{nT}^{II}, respectively. For this, denote by $\{Z_j : 1 \leq j \leq M\}$ the distinct values in increasing order of Y_i in a set of independent identically distributed (i.i.d.) observations $\{(Y_i, A_i) : 1 \leq i \leq n\}$. Define

$$D_{kj} = \sum_{1\leq i\leq n} \mathbb{1}_{\{Y_i = Z_j, A_i = k\}}, \qquad N_j = \sum_{1\leq i\leq n} \mathbb{1}_{\{Y_i \leq Z_j\}}, \qquad \overline{N}_j = \sum_{1\leq i\leq n} \mathbb{1}_{\{Y_i \geq Z_j\}},$$

$k = 0, 1, 2$. With these definitions, the product-limit estimator of F_T in Model I is

$$F_{nT}^I((Z_j,\infty]) = \prod_{1\leq k\leq j} \left\{1 - \frac{D_{0k}}{U_{k-1} - N_{k-1}}\right\},$$

where

$$U_{j-1} = n \prod_{j\leq k\leq M} \left\{1 - \frac{D_{2k}}{N_k}\right\}.$$

The product-limit estimator of F_T in Model II is given by

$$F_{nT}^{II}([0,Z_j]) = \prod_{j<k\leq M} \left\{1 - \frac{D_{0k}}{V_k - \overline{N}_k + D_{0k}}\right\},$$

where

$$V_j = n \prod_{1\leq k\leq j} \left\{1 - \frac{D_{1k}}{\overline{N}_{k+1} + D_{1k}}\right\}.$$

When the doubly censored data model is considered, our product-limit estimators represent lower and upper bounds for Turnbull's estimator. These bounds may serve for the numerical algorithms used to compute Turnbull's estimator.

6. Strong convergence. We study the strong (almost sure or a.s.) uniform convergence of F_{nT}^I and F_{nT}^{II}. Since, in fact, the estimators F_{nT}^I and F_{nT}^{II} are built as explicit functionals of the empirical distribution, we deduce their asymptotic behavior, in particular, the strong convergence, whatever the properties of the underlying censoring mechanism are. Hereafter, we use the following rule: the subscript n indicates the empirical version of the quantities we consider. Moreover, if μ is a nonnegative measure on $(\overline{\mathbb{R}}^+, \overline{\mathscr{B}}^+)$ and f is a measurable function, $\mu(f) = \int f(t)\mu(dt)$.

For the strong convergence, we recall a result of Rolin [12], an extension of the strong law under right-censorship proved by Stute and Wang [13]. Let $H =$

$\sum_{1\le r\le g} H_r$ be a probability distribution decomposed into g subdistributions. If $\mathcal{I}\subset\mathcal{K}=\{r:1\le r\le g\}$, let $H_{\mathcal{I}}=\sum_{r\in\mathcal{I}} H_r$. For $\mathcal{J}_k\subset\mathcal{I}_k\subset\mathcal{K}$, $k=1,2$, define

$$L_k^-(dt)=\frac{H_{\mathcal{J}_k}(dt)}{H((t,\infty])+H_{\mathcal{I}_k}(\{t\})}$$

and consider the measure $G(dt)=\exp\{-L_2([0,t))\}L_1^-(dt)$.

THEOREM 6.1. *If $G(f)<\infty$, then $G_n(f)\to G(f)$ a.s. and in the mean.*

The same result holds if we define the predictable reverse hazard measures

$$M_k^-(dt)=\frac{H_{\mathcal{J}_k}(dt)}{H([0,t))+H_{\mathcal{I}_k}(\{t\})},\qquad k=1,2,$$

and consider the measure $G(dt)=\exp\{-M_2((t,\infty])\}M_1^-(dt)$.

Let us extend the number of hazard measures associated with Model I by defining

$$M_0^-(dt)=\frac{H_0(dt)}{H([0,t))+H_{01}(\{t\})},\qquad M_1^-(dt)=\frac{H_1(dt)}{H([0,t))+H_1(\{t\})}.$$

Consider F_0^I, F_1^I, the corresponding distributions. Deduce that $M_{01}=M_0+M_1$, where M_0, M_1 and M_{01} are the reverse hazard measures associated with M_0^-, M_1^- and M_{01}^- [see (5)], respectively. In view of equations (2), deduce $H([0,t))=F_2^I([0,t))F_{01}^I([0,t))$ and $H_{01}(\{t\})=F_2^I([0,t))F_{01}^I(\{t\})$. Therefore,

$$H_0(dt)=F_2^I([0,t))F_{01}^I([0,t])M_0^-(dt).$$

Consequently, in the expression of the predictable hazard measure defining F_T^I [see (6)], we get rid of F_2^I and obtain

$$L_T^{I-}(dt)=\frac{F_{01}^I([0,t])}{F_{01}^I([t,\infty])}M_0^-(dt).$$

THEOREM 6.2. *If f is a nonnegative Borel measurable function defined on $(\overline{\mathbb{R}}^+,\overline{\mathcal{B}}^+)$ such that $L_T^{I-}(f)<\infty$, then, almost surely as $n\to\infty$, $L_{nT}^{I-}(f)\to L_T^{I-}(f)$.*

Theorem 6.2 is a direct consequence of the following lemma.

LEMMA 6.3. (i) *If $L_T^{I-}(f\mathbb{1}_{[0,t]})<\infty$ and $F_{01}^I([t,\infty])>0$, then a.s.*

$$L_{nT}^{I-}(f\mathbb{1}_{[0,t]})\to L_T^{I-}(f\mathbb{1}_{[0,t]}),\qquad n\to\infty.$$

(ii) *If $L_T^{I-}(f\mathbb{1}_{[t,\infty]})<\infty$ and $F_2^I([0,t))>0$, then almost surely as $n\to\infty$,*

$$L_{nT}^{I-}(f\mathbb{1}_{[t,\infty]})\to L_T^{I-}(f\mathbb{1}_{[t,\infty]}).$$

PROOF. (i) First, Theorem 6.1 implies that any empirical distribution function defined by the empirical reverse hazard measures of Model I converges uniformly on $[0, \infty]$. Now,

$$\left| L_{nT}^{I-}(f \mathbb{1}_{[0,t]}) - \int_{(0,t]} \frac{f(s)}{F_{01}^I([s,\infty])} F_{n01}^I([0,s]) M_{n0}^-(ds) \right|$$
$$\leq \frac{\|F_{n01}^I - F_{01}^I\|}{F_{n01}^I([t,\infty])} \int_{(0,t]} \frac{f(s)}{F_{01}^I([s,\infty])} F_{n01}^I([0,s]) M_{n0}^-(ds).$$

The second member of the inequality tends to zero almost surely because $F_{n01}^I([t,\infty]) \to F_{01}^I([t,\infty]) > 0$ a.s. and, by Theorem 6.1 applied for $G(ds) = \exp\{-M_{01}((s,\infty])\} M_0^-(ds)$,

$$\int_{(0,t]} \frac{f(s)}{F_{01}^I([s,\infty])} F_{n01}^I([0,s]) M_{n0}^-(ds) \to L_T^{I-}(f \mathbb{1}_{[0,t]}), \qquad \text{a.s.}$$

(ii) First, looking at the definition of M_{01}^-, by a simple computation,

$$H_{01}([s,\infty]) \leq F_{01}^I([s,\infty]) \leq \frac{H_{01}([s,\infty])}{H_{01}([0,\infty])}.$$

Using definition (6) for the predictable hazard measure defining F_T^I, we have

$$\left| L_{nT}^{I-}(f \mathbb{1}_{[t,\infty]}) - \int_{[t,\infty]} \frac{f(s)}{F_2^I([0,s))} \frac{H_{n0}(ds)}{F_{n01}^I([s,\infty])} \right|$$
$$\leq \frac{\|F_{n2}^I - F_2^I\|}{F_{n2}^I([0,t))} \int_{[t,\infty]} \frac{f(s)}{F_2^I([0,s))} \frac{H_{n0}(ds)}{F_{n01}^I([s,\infty])}$$
$$\leq \frac{\|F_{n2}^I - F_2^I\|}{F_{n2}^I([0,t))} \int_{[t,\infty]} \frac{f(s)}{F_2^I([0,s))} \frac{H_{n0}(ds)}{H_{n01}([s,\infty])}.$$

Now, almost surely $F_{n2}^I([0,t)) \to F_2^I([0,t))$, which is strictly positive. Since

$$\int_{[t,\infty]} \frac{f(s)}{F_2^I([0,s))} \frac{H_0(ds)}{H_{01}([s,\infty])} \leq H_{01}([0,\infty])^{-1} L_T^{I-}(f \mathbb{1}_{[t,\infty]}) < \infty,$$

a new application of Theorem 6.1 provides the result. □

Denote by t_{0k} the left endpoint and by t_{1k} the right endpoint of the support of H_k, $k = 0, 1, 2$. We have the following corollary of Theorem 6.2. Note that the strong uniform convergence of F_{nT}^I is obtained without any additional assumption, apart from that of i.i.d. observations and the condition $H_0(\{0\}) = H_1(\{0\}) = 0$.

COROLLARY 6.4. (a) *If* $L_T^{I-}([0, t_{10})) < \infty$, *then, almost surely,*

$$\sup_{0 \leq t < t_{10}} |L_{nT}^I([0,t]) - L_T^I([0,t])| \to 0$$

and $L^I_{nT}(\{t_{10}\}) \to L^I_T(\{t_{10}\})$. *If* $L^{I-}_T([0, t_{10})) = \infty$, *then, almost surely,*

$$\sup_{0 \le s \le t} |L^I_{nT}([0, s]) - L^I_T([0, s])| \to 0$$

for all $t < t_{10}$ *and* $L^I_{nT}([0, t_{10})) \to \infty$.

(b) *Almost surely,* $\|F^I_{nT} - F^I_T\| = \sup_{0 \le t \le \infty} |F^I_{nT}([0, t]) - F^I_T([0, t])| \to 0$.

PROOF. The Glivenko–Cantelli theorem provides the result in (a) with L^I_T and L^I_{nT} replaced by L^{I-}_T and L^{I-}_{nT}, respectively. The similar result for the hazard measure L^I_{nT} is obtained by taking care of the fact that $L^I_T(\{t_{10}\}) = \infty$ if $L^{I-}_T(\{t_{10}\}) = 1$. This happens if $t_{10} \ge t_{11}$, $H_0(\{t_{10}\}) > 0$ and $H_1(\{t_{10}\}) = 0$. The convergence of F^I_{nT} is implied by the convergence of the associated hazard measure L^I_{nT}. □

The strong uniform convergence of F^{II}_{nT} can be obtained in a similar way. Define

$$L^-_0(dt) = \frac{H_0(dt)}{H([t, \infty])}, \qquad L^-_2(dt) = \frac{H_2(dt)}{H((t, \infty]) + H_1(\{t\}) + H_2(\{t\})}$$

and consider F^{II}_0, F^{II}_2, the corresponding distributions. After some manipulations we can get rid of F^{II}_1 in the definition (7):

$$M^{II-}_T(dt) = \frac{F^{II}_2([t, \infty]F^{II}_0([t, \infty])}{1 - F^{II}_2([t, \infty])F^{II}_0((t, \infty])} L^-_0(dt).$$

Next, apply Theorem 6.1 (see [10] for the details).

7. Asymptotic normality. Let $(D[a, b], \|\cdot\|)$ be the space of càdlàg functions defined on $[a, b] \subset [0, \infty]$, endowed with the supremum norm. $BV_C[a, b] \subset D[a, b]$ is the set of càdlàg functions with total variation bounded by C. The integrals with respect to functions which are not of bounded variation have to be understood via partial integration. Finally, weak convergence is denoted by $\rightsquigarrow$ and is in the sense considered by Pollard [11], that is, $D[a, b]$ is endowed with the ball σ-field.

Given the explicit form of F^I_{nT} and F^{II}_{nT}, a convenient approach for proving weak convergence is the delta method (e.g., [5] and [15], Section 3.9). For proving Hadamard differentiability, the denominators appearing in the maps used to define F^I_T and F^{II}_T should stay away from zero. Therefore, we have to complete the delta method with a tool for treating the endpoints of the intervals on which weak convergence is finally proved.

LEMMA 7.1 ([11], page 70). *Let* $X, X_1, X_2, \ldots$ *be random elements of* $(D[a, b], \|\cdot\|)$ *with the distribution of* X *concentrated on a separable set. Suppose,* *for each* $\varepsilon, \delta > 0$ *there exist approximating random elements* $AX, AX_1, AX_2, \dots$ *such that* $AX_n \rightsquigarrow AX$, $P(\|X - AX\| > \delta) < \varepsilon$ *and*

$$\limsup_{n\to\infty} P(\|X_n - AX_n\| > \delta) < \varepsilon. \tag{10}$$

Then $X_n \rightsquigarrow X$.

For brevity, we consider only the asymptotic normality of F^I_{nT}; similar arguments apply for F^{II}_{nT}. The empirical central limit theorem yields

$$\sqrt{n}(H_n - H, H_{0n} - H_0, H_{2n} - H_2) \rightsquigarrow (G, G_0, G_2) \qquad \text{in } D^3([0,\infty]).$$

Now, we prove that $\sqrt{n}(M^-_{n2} - M^-_2)$ and $\sqrt{n}(F^I_{n2} - F^I_2)$ converge weakly to Gaussian limits. The computation of the covariance structures for the limit processes in this section is elementary, albeit tedious (see [10] for some formulae).

LEMMA 7.2. *Let* $M^-_{2t} = M^-_2((t,\infty])$ *and* M^-_{n2t} *be the corresponding estimator. Assume that*

$$\int_{(t_{00},\infty]} \frac{M^-_2(du)}{H([0,u])} = \int_{(t_{00},\infty]} \frac{H_2(du)}{H([0,u])^2} < \infty, \tag{11}$$

where $t_{00} = \inf\{t : H_0([0,t]) > 0\}$. *Then*

$$\sqrt{n}(H_n - H, H_{n0} - H_0, M^-_{n2} - M^-_2) \rightsquigarrow (G, G_0, G_M) \qquad \text{in } D^3[t_{00},\infty], \tag{12}$$

where (G, G_0, G_M) *is a zero-mean Gaussian process with*

$$G_{Mt} = \int_{(t,\infty]} \frac{dG_{2u}}{H([0,u])} - \int_{(t,\infty]} \frac{G_u}{H([0,u])^2} H_2(du). \tag{13}$$

Moreover, if $F^I_{2t} = F^I_2([0,t])$, *then*

$$\sqrt{n}(H_n - H, H_{n0} - H_0, F^I_{n2} - F^I_2) \rightsquigarrow (G, G_0, G_3) \qquad \text{in } D^3[t_{00},\infty],$$

where (G, G_0, G_3) *is a zero-mean Gaussian process with*

$$G_{3t} = F^I_2([0,t]) \int_{(t,\infty]} \frac{dG_{Mu}}{1 - M^-_2(\{u\})}. \tag{14}$$

PROOF. The map $(A, B) \to \int_{(\cdot,\infty]} (1/A)\,dB$ is Hadamard-differentiable on a domain of the type $\{(A,B) : A \in D[a,b], B \in BV_C[a,b], A \geq \epsilon\}$, $C, \epsilon > 0$, at every point such that $1/A$ is of bounded variation. The derivative map is given by $(\alpha, \beta) \to \int_{(\cdot,\infty]} (1/A)\,d\beta - \int_{(\cdot,\infty]} (\alpha/A^2)\,dB$. Therefore, the delta method for the map $(H, H_0, H_2) \to (H, H_0, M^-_2)$ yields the weak convergence of $\sqrt{n}(H_n - H, H_{n0} - H_0, M^-_{n2} - M^-_2)$ in $D^3[\sigma,\infty]$, provided that $H([0,\sigma]) > 0$.

For the weak convergence in $D^3[t_{00}, \infty]$, consider the pathwise limit of $G_{M\sigma}$ as $\sigma \downarrow t_{00}$, which exists in view of (11). It remains to verify (10) when $H([0, t_{00}]) = 0$. It suffices to prove the following: (a) for any $\varepsilon, \delta > 0$, there exists $\sigma = \sigma(\varepsilon, \delta) > t_{00}$ such that

$$\limsup_{n\to\infty} P\left(\sup_{U \le t \le \sigma} \sqrt{n}|M_{n2}^-([t, \sigma)) - M_2^-([t, \sigma))| > \delta \right) < \varepsilon; \tag{15}$$

and (b) $\sqrt{n}M_2^-((t_{00}, U)) \to 0$, in probability, where $U = \min_i Y_i$. To ensure (a), reverse the time and apply the arguments usually used to check the "tightness at $\tau_H = \sup\{t : H([0, t]) < 1\}$" when proving weak convergence for Nelson–Aalen and Kaplan–Meier estimators (see [3], Theorem 6.2.1, [4]). For (b), first note that (11) ensures $M_2^-((t_{00}, \infty])$ is finite. This implies $F_2^I([0, t_{00}]) > 0$ (use, e.g., arguments as in Lemma 6 of [6]). Since in general M^- is smaller than M, deduce

$$M_2^-((t_{00}, U)) \le M_2((t_{00}, U)) = \ln \frac{F_2^I([0, U))}{F_2^I([0, t_{00}])} \le \frac{F_2^I((t_{00}, U))}{F_2^I([0, t_{00}])}.$$

Let $u_n^\lambda = \sup\{s : \sqrt{n}F_2^I((t_{00}, s)) \le \lambda\}$ (see also [16]). We have

$$\begin{aligned} P\big(\sqrt{n}F_2^I((t_{00}, U)) > \lambda\big) &\le P(U > u_n^\lambda) = H((u_n^\lambda, \infty])^n \\ &\le \{1 - F_2^I((t_{00}, u_n^\lambda])F_{01}^I([0, u_n^\lambda])\}^n \\ &\le \left(1 - \frac{\lambda^2}{n}\frac{F_{01}^I([0, u_n^\lambda])}{F_2^I((t_{00}, u_n^\lambda])}\right)^n \to 0. \end{aligned}$$

The convergence to zero is true because, in view of (11),

$$\frac{F_2^I((t_{00}, u_n^\lambda])}{F_{01}^I([0, u_n^\lambda])} \le \int_{(t_{00}, u_n^\lambda]} \frac{F_2^I(ds)}{F_{01}^I([0, s])} = \int_{(t_{00}, u_n^\lambda]} \frac{M_2^-(ds)}{H([0, s])} \to 0, \qquad n \to \infty.$$

Now, (b) is clear. For the last part of the lemma, apply the delta method for the map $A \to \pi_{(\cdot,\infty]}(1 - A(ds))$ defined on $BV_C[t_{00}, \infty]$, for some $C > 0$. $\square$

REMARK. In view of the variance of the process G_3, it seems possible to relax condition (11) when $F_2^I([0, t_{00}]) = 0$ (see also [4]). However, in the following, due to the lack of an obvious martingale structure for $L_{nT}^{I-} - L_T^{I-}$, it is convenient to keep the denominator appearing in the definition of L_T^{I-} away from zero when $t \downarrow t_{00}$. For this, we have to impose $F_2^I([0, t_{00}]) > 0$ and, in this case, (11) is needed to bound the variance of G_{3t} when $t \downarrow t_{00}$.

Now we state the asymptotic normality for L_{nT}^{I-} and F_{nT}^I. The notation A_- means that we consider the left-limits of the process A.

THEOREM 7.3. *Suppose condition* (11) *holds*. *Let* $t_{00} < \tau$ *such that* $H_{01}([0, \tau)) < 1$. *If* $L^{I-}_{Tt} = L^{I-}_T([0,t])$, *then* $\sqrt{n}(L^{I-}_{nT} - L^{I-}_T) \rightsquigarrow V$ *in* $D[0,\tau]$, *where*

$$V_t = \int_{(0,t]} \frac{dG_{0u}}{(F^I_2 - H)([0,u))} - \int_{(0,t]} \frac{G_{3u-} - G_{u-}}{(F^I_2 - H)^2([0,u))} H_0(du), \qquad t \in [0,\tau],$$

is a zero-mean Gaussian process. *Moreover, if* $F^I_{Tt} = F^I_T([0,t])$, *then we have* $\sqrt{n}(F^I_{nT} - F^I_T) \rightsquigarrow W$ *in* $D[0,\tau]$, *with* W *the zero-mean Gaussian process*

$$W_t = F^I_T((t,\infty]) \int_{(0,t]} \frac{dV_u}{1 - L^{I-}_T(\{u\})}.$$

PROOF. Since $F^I_{01}([\tau,\infty]) > 0$ and, by (11), $F^I_2([0,t_{00}]) > 0$, we have $\inf_{(t_{00},\tau]}(F^I_2 - H)([0,s)) > \epsilon$, for some $\epsilon > 0$. Thus, if $H_0(\{t_{00}\}) = 0$, the weak convergence of $\sqrt{n}(L^{I-}_{nT} - L^{I-}_T)$ is obtained by the delta method for the map $(A,B) \to \int_{(t_{00},\cdot]}(1/A_-)\,dB$ (see [15], pages 382–384).

When $H_0(\{t_{00}\}) > 0$ (hence, necessarily $t_{00} > 0$), in the definition of L^{I-}_T, we also have to take into account $F^I_2([0,t_{00}))$. For this, extend the weak convergence in (12) on $D^3[0,\infty]$ by considering a modified predictable reverse hazard function

$$M^-_{2t} = M^-_2((t,\infty]) = \int_{(t,\infty]} \frac{H_2(du)}{H([0, u \vee t_{00}])}, \qquad t \in [0,\infty].$$

Let M^-_{n2} be the empirical counterpart. Since the denominator in the last display stays away from zero, the weak convergence of $\sqrt{n}(H_n - H, H_{n0} - H_0, F^I_{n2} - F^I_2)$ in $D^3[0,\infty]$ is easily obtained by the delta method, where now F^I_2, F^I_{n2} correspond to the modified M^-_2, M^-_{n2}, respectively. Note that now $F^I_2([0,t_{00})) > 0$. The processes G_M and G_3 are still defined according to (13) and (14), respectively. Since the modification of M^-_{2t} and M^-_{n2} does not change the definitions of L^{I-}_T and L^{I-}_{nT}, the delta method yields the weak convergence of $\sqrt{n}(L^{I-}_{nT} - L^{I-}_T)$. The last part of the theorem is obtained by the delta method for the product-integration map. □

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