A COMPLEMENTARY DESIGN THEORY FOR DOUBLING

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Chen and Cheng [*Ann. Statist.* **34** (2006) 546–558] discussed the method of doubling for constructing two-level fractional factorial designs. They showed that for $9N/32 \leq n \leq 5N/16$, all minimum aberration designs with N runs and n factors are projections of the maximal design with $5N/16$ factors which is constructed by repeatedly doubling the 2^{5-1} design defined by $I = ABCDE$. This paper develops a general complementary design theory for doubling. For any design obtained by repeated doubling, general identities are established to link the wordlength patterns of each pair of complementary projection designs. A rule is developed for choosing minimum aberration projection designs from the maximal design with $5N/16$ factors. It is further shown that for $17N/64 \leq n \leq 5N/16$, all minimum aberration designs with N runs and n factors are projections of the maximal design with N runs and $5N/16$ factors.

1. Introduction. Fractional factorial designs are widely used in various experiments and are typically chosen according to the *minimum aberration* criterion. There is much recent work on the theory and construction of minimum aberration designs; see Wu and Hamada [11] for a good summary. Chen and Cheng [4] discussed the method of doubling for constructing two-level fractional factorial designs, in particular, those of resolution IV. Suppose that X is an $N \times n$ matrix with two distinct entries, 1 and -1. Then the *double* of X is the $2N \times 2n$ matrix $\begin{pmatrix} X & X \\ X & -X \end{pmatrix}$. Doubling plays an important role in the construction of maximal designs. A regular design of resolution IV or higher is called *maximal* if its resolution reduces to three whenever an extra factor is added. It is obvious that every regular design of resolution IV or higher is a projection of some maximal regular design of resolution IV or higher. Some recent results in the literature of finite projective geometry essentially characterize all maximal regular designs of resolution IV with $n \geq N/4 + 1$. These results imply that for $n \geq N/4 + 1$, a maximal regular design of resolution IV or higher must have

$$n \in \{N/2, 5N/16, 9N/32, 17N/64, 33N/128, \ldots\},$$

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where N is a power of 2 and n is the number of factors; see Chen and Cheng [4]. All these maximal regular designs can be constructed by repeatedly doubling some small designs. For example, the maximal regular design with $N/2$ factors can be constructed by repeatedly doubling the 2^1 design. The maximal regular design with $5N/16$ factors is constructed by repeatedly doubling the 2^{5-1} design defined by $I = ABCDE$. An immediate result is that all regular designs of resolution IV with $5N/16 < n \le N/2$ must be projections of the maximal regular design with $N/2$ factors and that all regular designs of resolution IV with $9N/32 < n \le 5N/16$ must be projections of the maximal regular design with either $N/2$ or $5N/16$ factors. Chen and Cheng [4] showed that for $9N/32 \le n \le 5N/16$, all minimum aberration designs with N runs and n factors are projections of the maximal design with $5N/16$ factors. The motivation of this paper is to develop a complementary design theory that guides the construction of minimum aberration designs from the maximal design with $5N/16$ factors. Previously, Chen and Hedayat [6] and Tang and Wu [10] developed a complementary design theory for saturated regular designs of resolution III. Butler [3] and Chen and Cheng [5] developed a complementary design theory for the maximal regular designs with $N/2$ factors. Note that the maximal regular designs with $N/2$ factors are even designs (i.e., all defining words have even lengths) and are the only even designs that are maximal. We shall call them maximal even designs.

In Section 2, we review some basic concepts and background. In Section 3, we develop a general complementary design theory for doubling. For each design constructed by repeated doubling, we express the wordlength pattern of a projection design in terms of a linear combination of the wordlength pattern of its complement and some terms that depend on the frequencies of the columns being doubled. We illustrate how previous complementary design theories for saturated regular designs of resolution III and maximal even designs can be easily derived as special cases of our result. In Section 4, we apply the general theory to the maximal design with $5N/16$ factors and obtain a general rule on constructing its minimum aberration projection designs. As conjectured by Chen and Cheng [4], we show that the minimum aberration projection designs have equal or nearly equal frequencies of the columns being doubled. In Section 5, we consider the maximal design with $9N/32$ factors and derive a lower bound for the wordlength patterns of its projections. This lower bound is then used in Section 6 to show that for $17N/64 \le n \le 5N/16$, all minimum aberration designs with N runs and n factors are projections of the maximal design with $5N/16$ factors.

2. Basic concepts and background.   An N-run design with n factors is represented by an $N \times n$ matrix, where each row corresponds to a run and each column corresponds to a factor. A two-level design takes on only two symbols which, for convenience, are denoted by 0 and 1 in the rest of the paper. Regular designs are those which can be constructed by using defining relations and are discussed in many textbooks on experimental design; see, for example, [9, 11]. A regular

2^{n-p} design is defined by p independent defining words and has $N = 2^{n-p}$ runs and n factors. The p independent defining words together generate $2^p - 1$ defining words. The *resolution* is the length of the shortest defining word. Let A_i be the number of defining words of length i. Then the vector $(A_1, \ldots, A_n)$ is called the *wordlength pattern*. The *minimum aberration* criterion introduced by Fries and Hunter [7] chooses a design by sequentially minimizing $A_1, A_2, \ldots, A_n$. Throughout the paper, we will consider only two-level regular designs.

For a two-level regular $N \times n$ design D (with symbols 0 and 1), let $w_i(D)$ be the *Hamming weight* (i.e., the number of nonzero components) of the ith row of D. For integers $k > 0$, define the moments $M_k(D) = \sum_{i=1}^{N} w_i(D)^k$. Using the Pless power moment identities [8], a fundamental result in coding theory, Xu [12–14] expressed M_k as a linear combination of $A_1, \ldots, A_k$ as follows.

LEMMA 1. *For a two-level regular $N \times n$ design D and integers $k > 0$,*

$$M_k(D) = N \sum_{i=0}^{\min(n,k)} Q_k(i; n) A_i(D),$$

where $Q_k(i; n) = (-1)^i \sum_{j=0}^{k} j! S(k, j) 2^{-j} \binom{n-i}{j-i}$, $S(k, j) = (1/j!) \times \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^k$ *is a Stirling number of the second kind and $A_0(D) = 1$.*

It is easy to verify that $S(k, k) = 1$ and $Q_k(k; n) = (-1)^k k!/2^k$.

3. Doubling and complementary designs. For a two-level regular $N_0 \times m_0$ design $X_0 = (x_{ij})$ with $x_{ij} = 0$ or 1. Define its double as

$$\begin{pmatrix} X_0 & X_0 \\ X_0 & X_0 + 1 \end{pmatrix} \qquad (\text{mod } 2).$$

Let X be the regular $N \times m$ design which is obtained by repeatedly doubling X_0 t times, where $N = 2^t N_0$ and $m = 2^t m_0$.

Suppose that D and $\overline{D}$ are a pair of complementary projection designs of X, that is, $D \cup \overline{D} = X$, where D has n columns and $\overline{D}$ has u columns, with $n + u = m$. Note that each column of X_0 generates 2^t columns of X. For $i = 1, \ldots, m_0$, let f_i be the number of columns of $\overline{D}$ that are generated from the ith column of X_0. Clearly $\sum_{i=1}^{m_0} f_i = u$.

It is easy to verify the relationship of the Hamming weights of D and $\overline{D}$ as follows:

$$w_i(\overline{D}) = \sum_{j=1}^{m_0} f_j x_{ij} \qquad \text{for } i = 1, \ldots, N_0,$$

$$w_i(D) = \sum_{j=1}^{m_0} (2^t - f_j) x_{ij} = 2^t w_i(X_0) - w_i(\overline{D}) \qquad \text{for } i = 1, \ldots, N_0,$$

$$w_i(D) + w_i(\overline{D}) = m/2 \qquad \text{for } i = N_0 + 1, \ldots, N.$$

Then

$$
M_k(D) = \sum_{i=1}^{N} w_i(D)^k = \sum_{i=1}^{N} [m/2 - w_i(\overline{D})]^k + \Delta_k(D, \overline{D})
$$

$$
= \sum_{i=1}^{N} \sum_{j=0}^{k} \binom{k}{j} (m/2)^{k-j} (-1)^j w_i(\overline{D})^j + \Delta_k(D, \overline{D})
$$

$$
= \sum_{j=0}^{k} \binom{k}{j} (m/2)^{k-j} (-1)^j M_j(\overline{D}) + \Delta_k(D, \overline{D}),
$$

where

$$
(1) \qquad \Delta_k(D, \overline{D}) = \sum_{i=1}^{N_0} [w_i(D)^k - (m/2 - w_i(\overline{D}))^k]
$$

depends on the frequencies $f_1, \ldots, f_{m_0}$ and X_0 only. Using Lemma 1, we obtain the following relationship between the wordlength patterns of D and $\overline{D}$.

THEOREM 1. *Let D and $\overline{D}$ be a pair of complementary projection designs of the regular $N \times m$ design X constructed via doubling. For integers $k > 0$,*

$$
\sum_{i=0}^{k} Q_k(i; n) A_i(D) = \sum_{i=0}^{k} \left[\sum_{j=i}^{k} \binom{k}{j} \left(\frac{m}{2} \right)^{k-j} (-1)^j Q_j(i; u) \right] A_i(\overline{D}) + \frac{\Delta_k(D, \overline{D})}{N},
$$

where $Q_k(i; n)$ is defined in Lemma 1, $\Delta_k(D, \overline{D})$ is defined in (1), $A_0(D) = A_0(\overline{D}) = 1$, $A_i(D) = 0$ for $i > n$ and $A_i(\overline{D}) = 0$ for $i > u$.

Because $Q_k(k; n) = (-1)^k k!/2^k$, we can express $A_k(D)$ in terms of a linear combination of $A_k(\overline{D}), \ldots, A_1(\overline{D})$, plus some terms that depend on the frequencies $f_1, \ldots, f_{m_0}$ and X_0 as follows:

$$
(2) \qquad
\begin{aligned}
A_k(D) = (-1)^k A_k(\overline{D}) + c_{k-1} A_{k-1}(\overline{D}) + \cdots + c_1 A_1(\overline{D}) + c_0 \\
+ d_k \Delta_k(D, \overline{D}) + \cdots + d_1 \Delta_1(D, \overline{D}),
\end{aligned}
$$

where c_i and d_i are constants that do not depend on D or $\overline{D}$. If X has resolution r, then $A_i(D) = A_i(\overline{D}) = 0$ for $1 \leq i < r$ and Theorem 1 implies that $\Delta_i(D, \overline{D})$ is a constant for $1 \leq i < r$. In this paper, we are interested in resolution IV designs. The next result follows from Theorem 1.

COROLLARY 1. *If X has resolution IV or higher, then $\Delta_1(D, \overline{D}) = 0$, $\Delta_2(D, \overline{D}) = N(2n - m)/4$, $\Delta_3(D, \overline{D}) = 3Nn(2n - m)/8$ and*

$$
A_4(D) = A_4(\overline{D}) - (2n - m)(6n^2 + 3m - 2)/24 + 2\Delta_4(D, \overline{D})/(3N).
$$

The following two examples illustrate how complementary design theories for saturated regular designs of resolution III and maximal even designs can be derived from Theorem 1.

EXAMPLE 1. Let

$$X_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Repeatedly doubling X_0 t times yields an $N \times m$ design X, where $N = m = 2^{t+1}$. Note that the first column of X is a vector of 0's. Deleting the first column, we obtain a saturated regular design of resolution III, denoted by X'. Following the previous notation, we have $w_1(\overline{D}) = w_1(D) = 0$, $w_2(\overline{D}) = f_2$ and $w_2(D) = 2^t - f_2$. Then

$$\Delta_k(D, \overline{D}) = \sum_{i=1}^{2} [w_i(D)^k - (m/2 - w_i(\overline{D}))^k]$$

$$= (2^t - f_2)^k - (2^t)^k - (2^t - f_2)^k = -2^{tk},$$

which does not depend on the frequencies f_1 and f_2 at all. By (2), sequentially minimizing $A_1(D), A_2(D), \ldots$ is equivalent to sequentially minimizing $-A_1(\overline{D}), A_2(\overline{D}), -A_3(\overline{D}), A_4(\overline{D}), \ldots$. Note that $A_1(\overline{D})$ is maximized if and only if $\overline{D}$ contains the first column of X, a vector of 0's. Let $\overline{D}'$ be the complement of D with respect to X', the saturated design of resolution III. It is easy to see that $A_1(\overline{D}') = A_2(\overline{D}') = 0$ and $A_i(\overline{D}) = A_i(\overline{D}') + A_{i-1}(\overline{D}')$ for $1 \le i \le u$. Therefore, sequentially minimizing $A_1(D), A_2(D), \ldots$ is equivalent to sequentially minimizing $-A_3(\overline{D}'), A_4(\overline{D}'), -A_4(\overline{D}'), A_5(\overline{D}'), \ldots$. This result is equivalent to the rules developed by Chen and Hedayat [6] and Tang and Wu [10].

EXAMPLE 2. Let $X_0 = \binom{0}{1}$. Repeatedly doubling X_0 t times yields an $N \times m$ design X, where $N = 2^{t+1}$ and $m = 2^t$. According to Chen and Cheng [4], X is a maximal even design. Following the previous notation, we have $w_1(\overline{D}) = w_1(D) = 0$, $w_2(\overline{D}) = u$ and $w_2(D) = 2^t - u$. Then

$$\Delta_k(D, \overline{D}) = \sum_{i=1}^{2} [w_i(D)^k - (m/2 - w_i(\overline{D}))^k] = (2^t - u)^k - (2^{t-1})^k - (2^{t-1} - u)^k,$$

which is a constant. Since X is an even design, $A_i(D) = A_i(\overline{D}) = 0$ for all odd i. Then by (2), sequentially minimizing $A_4(D), A_6(D), \ldots$ is equivalent to sequentially minimizing $A_4(\overline{D}), A_6(\overline{D}), \ldots$. Combining this with the fact that all regular designs of resolution IV with $5N/16 < n \le N/2$ must be projections of the maximal even design X, that is, they must be even designs, we reach the conclusion that for $5N/16 < n \le N/2$, D has minimum aberration among all regular designs

with N runs and n factors if and only if $\overline{D}$ has minimum aberration among all even designs. This is indeed Theorem 3 of Butler [3]. Chen and Cheng [5] derived explicit relationships between the wordlength patterns of complementary projection designs of a maximal even design.

4. Complementary projection designs of the maximal regular design of resolution IV with $5N/16$ factors. Let X_0 be the unique 16×5 design of resolution V defined by $I = ABCDE$. Repeatedly doubling X_0 t times, we obtain the maximal regular design of resolution IV with $5N/16$ factors, denoted by X, where $N = 16 \cdot 2^t$. Following the previous notation, the Hamming weights of $\overline{D}$ [i.e., $w_i(\overline{D})$] are 0, $f_i + f_j$ $(i < j)$ and $u - f_i$ $(i = 1, \ldots, 5)$. The Hamming weights of D [i.e., $w_i(D)$] are 0, $2^{t+1} - f_i - f_j$ $(i < j)$ and $2^{t+2} - u + f_i$ $(i = 1, \ldots, 5)$. Then

$$\Delta_k(D, \overline{D}) = \sum_{i=1}^{5}[(2^{t+2} - u + f_i)^k - (m/2 - u + f_i)^k]$$

$$+ \sum_{1 \leq i < j \leq 5}[(2^{t+1} - f_i - f_j)^k - (m/2 - f_i - f_j)^k] - (m/2)^k,$$

where $m = 5 \cdot 2^t$. According to Corollary 1, $\Delta_1(D, \overline{D}) = 0$, $\Delta_2(D, \overline{D}) = 2^{t+2}(2n - 5 \cdot 2^t)$ and $\Delta_3(D, \overline{D}) = 3 \cdot 2^{t+1}n(2n - 5 \cdot 2^t)$, which can be verified easily. For $k \geq 4$, $\Delta_k(D, \overline{D})$ depends on the frequencies $f_1, \ldots, f_5$. The next lemma states that when $u \leq 15 \cdot 2^{t-3}$, $\Delta_4(D, \overline{D})$ achieves the minimum value if and only if all f_i are equal or nearly equal.

LEMMA 2. *When $u \leq 15 \cdot 2^{t-3}$, $\Delta_4(D, \overline{D})$ achieves the minimum value if and only if pairwise differences of the frequencies $f_1, \ldots, f_5$ do not exceed 1, that is, $|f_i - f_j| \leq 1$ for all $1 \leq i < j \leq 5$.*

PROOF. We show that $\Delta_4(D, \overline{D})$ does not attain the minimum value if $|f_i - f_j| > 1$ for some $i < j$. Without loss of generality, assume that $f_1 \geq \cdots \geq f_5 \geq 0$ and $f_1 - f_5 > 1$. Consider another pair of complementary projection designs D' and $\overline{D}'$ with frequencies $f_1 - 1, f_2, f_3, f_4, f_5 + 1$. It can be verified that

$$\Delta_4(D, \overline{D}) - \Delta_4(D', \overline{D}') = 3 \cdot 2^{t+2}(f_1 - f_5 - 1)(3 \cdot 2^t + 2f_1 + 2f_5 - 2u).$$

Because $f_1 \geq \cdots \geq f_5 \geq 0$ and $u \leq 15 \cdot 2^{t-3}$, we have $f_1 + f_5 > u/5$, $3 \cdot 2^t + 2f_1 + 2f_5 - 2u > 3 \cdot 2^t - 8u/5 \geq 0$ and $\Delta_4(D, \overline{D}) > \Delta_4(D', \overline{D}')$. This completes the proof. $\square$

When $u > 15 \cdot 2^{t-3}$, Lemma 2 does not hold. For example, when $t = 2$ and $u = 8$, $\Delta_k(D, \overline{D})$ has the same value for (a) $f_1 = f_2 = f_3 = 2$ and $f_4 = f_5 = 1$ and (b) $f_1 = f_2 = f_3 = f_4 = 2$ and $f_5 = 0$.

Note that $\Delta_k(D, \overline{D})$ is uniquely determined when the frequencies $f_1, \ldots, f_5$ differ from one another by at most 1. Theorem 1 and Lemma 2 together lead to the following result.

THEOREM 2. *Let D and $\overline{D}$ be a pair of complementary projection designs of the maximal regular design X of resolution IV with $N = 16 \cdot 2^t$ runs and $5N/16$ factors. When $25N/128 \le n \le 5N/16$, D has minimum aberration among all possible n-factor projections of X if the following two conditions hold*:

(i) $A_4(\overline{D}), -A_5(\overline{D}), A_6(\overline{D}), -A_7(\overline{D}), \ldots$ *are sequentially minimized.*
(ii) $|f_i - f_j| \le 1$ *for all $1 \le i < j \le 5$.*

The construction of $\overline{D}$ satisfying both conditions (i) and (ii) is challenging. It is possible that there may not exist designs satisfying both conditions. In such cases, Theorem 1 has to be used instead. Here we present complementary designs $\overline{D}$ satisfying both conditions for $u = 1$–11. For $i = 1, \ldots, 5$, the ith column of X_0 generates columns $5k + i$, $0 \le k \le 2^t - 1$, of X. For $u = 1, 2, 3, 4$, we can choose $\overline{D}$ as the first u columns of X, which satisfies both conditions. For $u = 5$, $\overline{D} = \{1, 2, 3, 4, 5\}$ (i.e., the first five columns of X) satisfies both conditions because $A_4(\overline{D}) = 0$ is minimized and $A_5(\overline{D}) = 1$ is maximized. For $u = 6$–11 and $N \ge 128$, let $S = \{1, 2, 3, 4, 5, 6, 12, 18, 24, 30, 31\}$ and $\overline{D}$ be the projection design of X whose column indexes are the first u elements of S. Clearly $\overline{D}$ satisfies the second condition. It can be verified (via complete computer search) that $A_4(\overline{D}) = 0$ is minimized and $A_5(\overline{D})$ is maximized among all possible regular designs. Furthermore, except for $u = 9$, such designs are unique up to isomorphism and therefore satisfy both conditions. For $u = 9$ and $N \ge 128$, there are two non-isomorphic designs with $A_4(\overline{D}) = 0$ and maximum $A_5(\overline{D}) = 2$ as follows:

$$\overline{D}_1 = \{1, 2, 3, 4, 5, 6, 12, 18, 24\}, \qquad (A_4, \ldots, A_9) = (0, 2, 1, 0, 0, 0);$$

$$\overline{D}_2 = \{1, 2, 3, 4, 5, 6, 12, 23, 39\}, \qquad (A_4, \ldots, A_9) = (0, 2, 0, 0, 1, 0).$$

The second design has a smaller A_6; therefore, we shall choose $\overline{D} = \overline{D}_2$.

5. Complementary projection designs of the maximal regular design of resolution IV with $9N/32$ factors.

There is a unique maximal regular design of resolution IV with $9N/32$ factors [4], which can be constructed by repeatedly doubling a regular 32×9 design. For convenience, we choose X_0 as the 2^{9-5} design with defining contrast subgroup

$$I = 1235 = 2346 = 3457 = 4561 = 5672 = 6713 = 7124 = 123456789$$

$$= 13489 = 24589 = 35689 = 46789 = 57189 = 61289 = 72389.$$

Note that factors 1–7 are cyclic (i.e., replacing factors 1–7 with 2–7, 1 yields the same design).

The explicit expression of $\Delta_k(D, \overline{D})$ is quite complicated. Nevertheless, it is cyclic in $f_1, \ldots, f_7$ and symmetric in f_8 and f_9. In particular, with some tedious algebra, we have

$$
\begin{aligned}
\Delta_4(D, \overline{D}) = {} & 9534T^4 - 8T^3(535F_1 + 511G_1) \\
& + 12T^2(51F_1^2 + 3F_2 + 94F_1G_1 + 47G_1^2 + 7G_2) \\
& - 8T(4F_1^3 + 9F_1^2G_1 + 3F_2G_1 \\
& \qquad + 12G_1G_2 + 9F_1G_1^2 + 3F_1G_2 - 8G_3) \\
& + 48T(f_1f_3f_4 + f_2f_4f_5 + f_3f_5f_6 \\
& \qquad + f_4f_6f_7 + f_5f_7f_1 + f_6f_1f_2 + f_7f_2f_3),
\end{aligned}
$$

where $T = 2^t$, $F_1 = \sum_{i=1}^{7} f_i$, $F_2 = \sum_{i=1}^{7} f_i^2$, $G_1 = f_8 + f_9$, $G_2 = f_8^2 + f_9^2$ and $G_3 = f_8^3 + f_9^3$.

The next lemma gives necessary conditions for $\Delta_4(D, \overline{D})$ to achieve the minimum value and a lower bound.

LEMMA 3. *Let D and $\overline{D}$ be a pair of complementary projection designs of the maximal regular design with $32 \cdot 2^t$ runs and $9 \cdot 2^t$ factors defined in this section. When $u \le 3 \cdot 2^{t-1}$,*

(a) *necessary conditions for $\Delta_4(D, \overline{D})$ to achieve the minimum value are* (i) $f_8 = f_9 = 0$ *and* (ii) $|f_i - f_j| \le 1$ *for all $1 \le i < j \le 7$;*

(b) $\Delta_4(D, \overline{D}) \ge 2^{t+1}(760n^3 - 5400n^2 2^t + 17380n2^{2t} - 39477 \cdot 2^{3t})/49$.

PROOF.   (a) Let $T = 2^t$ and $F(f_1, \ldots, f_9) = \Delta_4(D, \overline{D})$. Without loss of generality, assume that f_1 is the smallest among all f_i, $1 \le i \le 7$. It can be verified that

$$
\begin{aligned}
& F(f_1, \ldots, f_9) - F(f_1 + f_8, f_2, f_3, f_4, f_5, f_6, f_7, 0, f_9) \\
& \quad = 192T^3 f_8 - 24T^2(7f_1 + 4f_2 + 4f_3 + 4f_4 + 4f_5 + 4f_6 + 4f_7)f_8 \\
& \qquad + 48Tf_8(f_1f_2 + f_1f_3 + f_2f_3 + f_1f_4 + f_2f_4 + f_1f_5 \\
& \qquad\qquad + f_2f_5 + f_3f_5 + f_4f_5 + f_1f_6 + f_3f_6 + f_4f_6 \\
& \qquad\qquad + f_5f_6 + f_1f_7 + f_2f_7 + f_3f_7 + f_4f_7 + f_6f_7 + f_1f_9) \\
& \quad > 192T^3 f_8 - 24T^2(5u)f_8 \ge 0
\end{aligned}
$$

when $u \le 3 \cdot 2^{t-1}$. Therefore, a necessary condition for $\Delta_4(D, \overline{D})$ to achieve the minimum value is $f_8 = 0$. By the symmetry of f_8 and f_9, another necessary condition is $f_9 = 0$. Next, we show that $F(f_1, \ldots, f_9)$ cannot be the minimum if $|f_i - f_j| \ge 2$ for some $1 \le i < j \le 7$. Without loss of generality, assume that $f_2 - f_1$ is the largest among all possible $f_i - f_j$ for $1 \le i, j \le 7$; for the other

cases, the proof is the same due to the special structure of X_0. It is sufficient to show that if $f_2 - f_1 \geq 2x > 0$, then

$$(3) \qquad F(f_1, \ldots, f_9) > F(f_1 + x, f_2 - x, f_3, f_4, f_5, f_6, f_7, f_8, f_9).$$

It is straightforward to verify that

$$
\begin{aligned}
& F(f_1, \ldots, f_9) - F(f_1 + x, f_2 - x, f_3, f_4, f_5, f_6, f_7, f_8, f_9) \\
(4) \quad & = 24Tx\big[(f_2 - f_1 - x)(3T - 2(f_6 + f_8 + f_9)) - 2(f_3 - f_5)(f_4 - f_7)\big].
\end{aligned}
$$

By the assumption that $f_2 - f_1$ is the largest among all $f_i - f_j$ for $1 \leq i, j \leq 7$, it is obvious that

$$-2(f_3 - f_5)(f_4 - f_7) \geq -(f_2 - f_1)(f_3 + f_4 + f_5 + f_7).$$

If $f_2 - f_1 \geq 2x > 0$, then $f_2 - f_1 - x \geq (f_2 - f_1)/2$ and

$$
\begin{aligned}
(f_2 & - f_1 - x)(3T - 2(f_6 + f_8 + f_9)) - 2(f_3 - f_5)(f_4 - f_7) \\
& \geq ((f_2 - f_1)/2)(3T - 2(f_6 + f_8 + f_9)) \\
(5) \qquad & \qquad - (f_2 - f_1)(f_3 + f_4 + f_5 + f_7) \\
& = ((f_2 - f_1)/2)(3T - 2(f_6 + f_8 + f_9 + f_3 + f_4 + f_5 + f_7)) \\
& > (f_2 - f_1)(3T - 2u)/2 \geq 0,
\end{aligned}
$$

because $u \leq 3T/2 = 3 \cdot 2^{t-1}$. Then (3) follows from (4) and (5). This completes the proof.

(b) Following the proof of (a), when $f_i = u/7 = (9 \cdot 2^t - n)/7$ for $i = 1, \ldots, 7$ and $f_8 = f_9 = 0$, $F(f_1, \ldots, f_9)$ achieves the lower bound. The lower bound is achievable only if u is a multiple of 7. $\square$

A result similar to Theorem 2 can be developed. However, such a result is not of interest because, as we will show in the next section, projection designs of the maximal regular design with $9N/32$ factors often do not have minimum aberration. The following lower bound of A_4, a direct result of Corollary 1 and Lemma 3(b), is important in the development.

LEMMA 4. *Let D be an n-factor projection design of the maximal regular design of resolution IV with $N = 32 \cdot 2^t$ runs and $9N/32$ factors. For $15N/64 \leq n \leq 9N/32$,*

$$A_4(D) \geq L(n) = [a(n) - b(n)2^t + c(n)2^{2t} - 39477 \cdot 2^{3t}]/1176,$$

where $a(n) = 196n + 172n^3$, $b(n) = 882 + 2646n + 2754n^2$ and $c(n) = 11907 + 17380n$.

6. Some results on minimum aberration designs. Chen and Cheng [4] showed that for $9N/32 \le n \le 5N/16$, minimum aberration designs are projection designs of the maximal regular design of resolution IV with $5N/16$ factors. Here we strengthen their result and show that this is also true for $17N/64 \le n < 9N/32$.

We first establish a general upper bound of wordlengths for projection designs.

LEMMA 5. *Let X be a regular design of resolution r with m factors and $A_r(X)$ words of length r. For $r \le n \le m$, there exists an n-factor projection design having at most $A_r(X)\binom{n}{r}/\binom{m}{r}$ words of length r.*

PROOF. We show that such a design can be constructed by deleting one factor at a time. First, there must be a factor that appears in at least $r \cdot A_r(X)/m$ words of length r. Deleting this factor yields a design with $m - 1$ factors and at most $A_r(X) - r \cdot A_r(X)/m = A_r(X)(m-r)/m$ words of length r. Repeat this procedure until we get an n-factor projection design which has at most

$$A_r(X)\frac{m-r}{m}\frac{m-1-r}{m-1}\cdots\frac{n+1-r}{n+1} = A_r(X)\binom{n}{r}\bigg/\binom{m}{r}$$

words of length r. □

For the maximal regular design X of resolution IV with $5N/16 = 5 \cdot 2^t$ factors, Chen and Cheng [4] showed that $A_4(X) = (65 \cdot 2^{3t-2} - 75 \cdot 2^{2t-2} + 5 \cdot 2^{t-1})/6$. Applying Lemma 5, we obtain the following upper bound of A_4 for minimum aberration designs.

COROLLARY 2. *For $N = 16 \cdot 2^t$, $t \ge 0$, and $n \le 5N/16$, there exists a regular $N \times n$ design D of resolution IV with*

$$A_4(D) \le ((65 \cdot 2^{3t-2} - 75 \cdot 2^{2t-2} + 5 \cdot 2^{t-1})/6)\binom{n}{4}\bigg/\binom{5 \cdot 2^t}{4}.$$

Corollary 2 is used next to show that for $N/4 + 1 \le n \le 5N/16$, projection designs of the maximal regular designs of resolution IV with $N/2$ or $9N/32$ factors do not have minimum aberration.

LEMMA 6. *For $N = 16 \cdot 2^t$, $t \ge 0$, and $N/4 + 1 \le n \le 5N/16$, projection designs of the maximal regular design of resolution IV with $N/2$ factors do not have minimum aberration.*

PROOF. Let D be an n-factor projection design of the maximal regular design of resolution IV with $N/2 = 2^{t+3}$ factors. It follows from (2.2) and (4.3) of [4] that

$$A_4(D) \ge \left[\binom{n}{2}^2\bigg/(2^{t+3} - 1) - \binom{n}{2}\right]\bigg/6.$$

It is sufficient to show that the upper bound given in Corollary 2 is less than the above lower bound. It is equivalent to show that

$$[(65 \cdot 2^{3t-2} - 75 \cdot 2^{2t-2} + 5 \cdot 2^{t-1})](n-2)(n-3)$$

(6)
$$< \left[\frac{6n(n-1)}{2^{t+3}-1} - 12\right]\binom{5 \cdot 2^t}{4}.$$

Given t, both the left-hand side and the right-hand side are polynomials in n of degree 2. It can easily be verified that (6) holds for $t \geq 0$ and $4 \cdot 2^t + 1 \leq n \leq 5 \cdot 2^t$. □

LEMMA 7. *For $N = 32 \cdot 2^t$, $t \geq 0$, and $N/4 + 1 \leq n \leq 9N/32$, projection designs of the maximal regular design of resolution IV with $9N/32$ factors do not have minimum aberration.*

PROOF. It is sufficient to show that the upper bound (with $N = 16 \cdot 2^{t+1}$) given in Corollary 2 is less than the lower bound given in Lemma 4, that is,

$$U(n) = [(65 \cdot 2^{3t+1} - 75 \cdot 2^{2t} + 5 \cdot 2^t)/6]\binom{n}{4} \bigg/ \binom{5 \cdot 2^{t+1}}{4} < L(n),$$

where $L(n)$ is defined in Lemma 4. Let $T = 2^t$. Given $t \geq 0$, $F(n) = L(n) - U(n)$ is a polynomial of degree 4. It is sufficient to show that $F(n)$ increases strictly when $8T \leq n \leq 9T$ and that $F(8T + 1) > 0$. Let $F^{(i)}(n)$ be the ith derivative of $F(n)$. It is straightforward to verify that $F^{(3)}(9T) > 0$ and $F^{(4)}(n) < 0$ when $8T \leq n \leq 9T$; therefore, $F^{(3)}(n) > 0$ for $8T \leq n \leq 9T$. It is easy to verify that $F^{(2)}(8T) > 0$, which implies $F^{(2)}(n) > 0$ for $8T \leq n \leq 9T$. It is also easy to verify that $F^{(1)}(8T) > 0$, leading to $F^{(1)}(n) > 0$ for $8T \leq n \leq 9T$. Therefore, $F(n)$ increases strictly when $8T \leq n \leq 9T$. Finally, it can be verified that $F(8T + 1) > 0$. This completes the proof. □

THEOREM 3. *For $N = 32 \cdot 2^t$, $t \geq 0$, and $17N/64 \leq n \leq 5N/16$, a minimum aberration design with N runs and n factors must be a projection of the maximal regular design of resolution IV with $5N/16$ factors.*

PROOF. As explained in the Introduction, for $17N/64 \leq n \leq 5N/16$, every regular design must be a projection of maximal regular designs of resolution IV with $N/2$, $5N/16$, $9N/32$ or $17N/64$ factors. Lemmas 6 and 7 state that projection designs of maximal regular designs of resolution IV with $N/2$ or $9N/32$ factors do not have minimum aberration. We need only show that maximal designs of resolution IV with $17N/64$ factors do not have minimum aberration. According to Chen and Cheng [4], there are five maximal designs of resolution IV with $17N/64$ factors that are constructed by repeatedly doubling five maximal regular 64×17 designs of resolution IV. According to Block and Mee [2], all five maximal regular

64×17 designs do not have minimum aberration. Then it follows from Corollary 2.4 of [4] that all five maximal regular designs of resolution IV with $17N/64$ factors do not have minimum aberration. This completes the proof. $\square$

Theorems 2 and 3 together lead to the following result on minimum aberration designs.

THEOREM 4. *Let D and $\overline{D}$ be a pair of complementary projection designs of the maximal regular design of resolution IV with $5N/16$ factors. For $N = 32 \cdot 2^t$, $t \geq 0$, and $17N/64 \leq n \leq 5N/16$, D has minimum aberration among all regular designs with N runs and n factors if $\overline{D}$ satisfies the following two conditions*:

(i) $A_4(\overline{D}), -A_5(\overline{D}), A_6(\overline{D}), -A_7(\overline{D}), \ldots$ *are sequentially minimized among all u-factor projection designs of the maximal regular design with $5N/16$ factors, where $u = 5N/16 - n$.*
(ii) $|f_i - f_j| \leq 1$ *for all $1 \leq i < j \leq 5$.*

As a numerical illustration, consider the maximal regular design of resolution IV with 256 runs and 80 factors. In Section 4 we constructed complementary designs $\overline{D}$ satisfying both conditions (i) and (ii) for $u = 1$–11. By Theorem 4, deleting them yields minimum aberration designs among all regular designs with $n = 69$–79 factors. Block [1] previously considered the construction of minimum aberration designs from this maximal regular design via a naive projection (i.e., deleting one column at a time). Our theoretical result confirms that the designs given by Block [1] with $n = 69$–79 factors have minimum aberration except for $n = 71$.

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