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CHARACTERIZATIONS OF BOUNDED APPROXIMATION PROPERTIES

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Abstract. New necessary and sufficient conditions for Banach spaces to have bounded approximation properties are established, which are easier to check than known ones. Also using these it is shown that for a Banach space X, the dual X^* has the bounded approximation property if and only if X has the bounded approximation property approximated by the weak adjoint operator topology, and if X^* has the bounded weak approximation property, then X has the bounded weak approximation property approximated by the weak adjoint operator topology.

1. Introduction and Main Results

There are known characterizations of bounded approximation properties. The purpose of this paper is to develop other characterizations and to obtain applications of these characterizations.

Let X, Y be Banach spaces and $\lambda > 0$. Throughout this paper, we use the following notations:

T^* The adjoint of an operator T

$\mathcal{B}(X,Y)$ The space of bounded linear operators from X into Y.

$\mathcal{F}(X,Y)$ The space of bounded and finite rank linear operators from X into Y.

$\mathcal{K}(X,Y)$ The space of compact operators from X into Y.

$\mathcal{K}(X,Y,\lambda)$ The collection of compact operators T from X into Y satisfying $\|T\| \leq \lambda$.

$\mathcal{K}^*(X,Y) = \{T^* : T \in \mathcal{K}(X,Y)\}$.

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$\mathcal{K}^*(X, Y, \lambda) = \{T^* : T \in \mathcal{K}(X, Y, \lambda)\}$.

We similarly define $\mathcal{F}(X, Y, \lambda)$, $\mathcal{F}^*(X, Y)$, $\mathcal{F}^*(X, Y, \lambda)$, $\mathcal{B}(X, Y, \lambda)$, $\mathcal{B}^*(X, Y)$, and $\mathcal{B}^*(X, Y, \lambda)$. For convenience we denote $\mathcal{B}(X, X)$, $\cdots$ by $\mathcal{B}(X)$, $\cdots$.

Notice that $\mathcal{B}^*(X, Y)$ is the space of w^*-to-w^* continuous and bounded linear operators from Y^* into X^*, where w^* is the $weak^*$ topology on X^* and Y^*.

A Banach space X is said to have the *λ-bounded approximation property* (in short, λ-BAP) if for every compact $K \subset X$ and $\epsilon > 0$, there is a $T \in \mathcal{F}(X, \lambda)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. If X has the λ-bounded approximation property for some $\lambda > 0$, then we say that X has the *bounded approximation property* (in short, BAP). For some results of the BAP one may see Casazza [1, Section 3] and Lindenstrauss and Tzafriri [6, Section 1,e].

We now have the following characterizations of a Banach space to have the BAP, which are easier to check than those in Lindenstrauss and Tzafriri [6, Proposition 1.e.14].

Theorem 1.1. *Let X be a Banach space. Then the following are equivalent.*

(a) *X has the λ-BAP.*

(b) *There is a net (T_α) in $\mathcal{F}(X, \lambda)$ such that $x^*T_\alpha x \longrightarrow x^*x$ for each $x \in X$ and $x^* \in X^*$.*

(c) *$\mathcal{F}(X, \lambda)$ is wo-dense in $\mathcal{B}(X, 1)$.*

(d) *For every Banach space Y, $\mathcal{F}(Y, X, \lambda)$ is wo-dense in $\mathcal{B}(Y, X, 1)$.*

(e) *For every Banach space Y, $\mathcal{F}(X, Y, \lambda)$ is wo-dense in $\mathcal{B}(X, Y, 1)$.*

(f) *For every $(x_n)_{n=1}^m \subset X$ and $(x_n^*)_{n=1}^m \subset X^*$, if $|\sum_{n=1}^m x_n^*(Sx_n)| \leq 1$ for all S in $\mathcal{F}(X, \lambda)$, then $|\sum_{n=1}^m x_n^* x_n| \leq 1$.*

(g) *For every $(x_n)_{n=1}^m \subset X$ and $(x_n^*)_{n=1}^m \subset X^*$, if $|\sum_{n=1}^m \sum_{k=1}^l x_n^*(z_k) z_k^*(x_n)| \leq 1$ for all $(z_k)_{k=1}^l \subset X$ and $(z_k^*)_{k=1}^l \subset X^*$ satisfying $\sup_{\|x\|=1} \|\sum_{k=1}^l z_k^*(x) z_k\| \leq \lambda$, then $|\sum_{n=1}^m x_n^* x_n| \leq 1$.*

In Section 2, we introduce the *weak operator topology* (in short, *wo*) on $\mathcal{B}(X, Y)$ and in Section 3, we prove the main theorems.

For the dual X^* of a Banach space X we have some different characterizations from Theorem 1.1. The following characterizations say that if X^* has the BAP, then X^* has some stronger properties than those in Theorem 1.1.

Theorem 1.2. *Let X be a Banach space. Then the following are equivalent.*

(a) *X^* has the λ-BAP.*

(b) *There is a net (T_α^*) in $\mathcal{F}^*(X, \lambda)$ such that $x^{**}T_\alpha^* x^* \longrightarrow x^{**}x^*$ for each $x^* \in X^*$ and $x^{**} \in X^{**}$.*

(c) *$\mathcal{F}^*(X,\lambda)$ is wo-dense in $\mathcal{B}^*(X,1)$.*

(d) *For every Banach space Y, $\mathcal{F}^*(Y,X,\lambda)$ is wo-dense in $\mathcal{B}^*(Y,X,1)$.*

(e) *For every Banach space Y, $\mathcal{F}^*(X,Y,\lambda)$ is wo-dense in $\mathcal{B}^*(X,Y,1)$.*

(f) *For every $(x_n^*)_{n=1}^m \subset X^*$ and $(x_n^{**})_{n=1}^m \subset X^{**}$, if $|\sum_{n=1}^m x_n^{**}(S^*x_n^*)| \leq 1$ for all S^* in $\mathcal{F}^*(X,\lambda)$, then $|\sum_{n=1}^m x_n^{**}x_n^*| \leq 1$.*

(g) *For every $(x_n^*)_{n=1}^m \subset X^*$ and $(x_n^{**})_{n=1}^m \subset X^{**}$, if $|\sum_{n=1}^m \sum_{k=1}^l x_n^{**}(z_k^*)x_n^*(z_k)| \leq 1$ for all $(z_k)_{k=1}^l \subset X$ and $(z_k^*)_{k=1}^l \subset X^*$ satisfying $\sup_{\|x^*\|=1} \| \sum_{k=1}^l x^*(z_k)z_k^*\| \leq \lambda$, then $|\sum_{n=1}^m x_n^{**}x_n^*| \leq 1$.*

A Banach space X is said to have the *λ-commuting bounded approximation property* (in short, λ-CBAP) if there is a net (T_α) in $\mathcal{F}(X,\lambda)$ satisfying for all α, β $T_\alpha T_\beta$=$T_\beta T_\alpha$ such that $T_\alpha x \longrightarrow x$ for each $x \in X$. If X has the λ-commuting bounded approximation property for some $\lambda > 0$, then we say that X has the *commuting bounded approximation property* (in short, CBAP). For some sesults of the CBAP one may see [1, Section 4].

For the CBAP we have the following characterizations. The following characterization (b) means that the pointwise convergence in the above definition can be replaced by some weaker convergence than the convergence.

Theorem 1.3. *Let X be a Banach space. Then the following are equivalent.*

(a) *X has the λ-CBAP.*

(b) *There is a net (T_α) in $\mathcal{F}(X,\lambda)$ satisfying for all α, β $T_\alpha T_\beta$=$T_\beta T_\alpha$ such that $x^*T_\alpha x \longrightarrow x^*x$ for each $x \in X$ and $x^* \in X^*$.*

(c) *There is a balanced convex set $\mathcal{C}$ in $\mathcal{F}(X,\lambda)$ satisfying for all $S, T \in \mathcal{C}$ TS=ST such that for every $(x_n)_{n=1}^m \subset X$ and $(x_n^*)_{n=1}^m \subset X^*$, if $|\sum_{n=1}^m x_n^*(Sx_n)| \leq 1$ for all S in $\mathcal{C}$, then $|\sum_{n=1}^m x_n^*x_n| \leq 1$.*

A Banach space X is said to have the *λ-bounded compact approximation property* (in short, λ-BCAP) if for every compact $K \subset X$ and $\epsilon > 0$, there is a $T \in \mathcal{K}(X,\lambda)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. If X has the λ-bounded compact approximation property for some $\lambda > 0$, then we say that X has the *bounded compact approximation property* (in short, BCAP). For some results of the BCAP one may see [1, Section 8].

For the BCAP we have the following characterizations.

Theorem 1.4. *Let X be a Banach space. Then the following are equivalent.*

(a) *X has the λ-BCAP.*

(b) *There is a net (T_α) in $\mathcal{K}(X,\lambda)$ such that $x^*T_\alpha x \longrightarrow x^*x$ for each $x \in X$ and $x^* \in X^*$.*

(c) *$\mathcal{K}(X,\lambda)$ is wo-dense in $\mathcal{B}(X,1)$.*

(d) *For every Banach space Y, $\mathcal{K}(Y,X,\lambda)$ is wo-dense in $\mathcal{B}(Y,X,1)$.*

(e) *For every Banach space Y, $\mathcal{K}(X,Y,\lambda)$ is wo-dense in $\mathcal{B}(X,Y,1)$.*

(f) *For every $(x_n)_{n=1}^m \subset X$ and $(x_n^*)_{n=1}^m \subset X^*$ if $|\sum_{n=1}^m x_n^*(Sx_n)| \le 1$ for all S in $\mathcal{K}(X,\lambda)$, then $|\sum_{n=1}^m x_n^* x_n| \le 1$.*

A Banach space X is said to have the *bounded weak approximation property* (in short, BWAP) if for every $T \in \mathcal{K}(X)$, there exists a $\lambda_T > 0$ such that for every compact $K \subset X$ and $\epsilon > 0$, there is a $T_0 \in \mathcal{F}(X,\lambda_T)$ such that $\|T_0x - Tx\| < \epsilon$ for all $x \in K$. Recently Choi and Kim [2] introduced the BWAP and the *weak approximation property* (in short, WAP), which are weak versions of the *approximation property*.

We now have the following characterizations of a Banach space to have the BWAP, which are easier to check than those in [2, Theorem 3.9].

Theorem 1.5. *Let X be a Banach space. Then the following are equivalent.*

(a) *X has the BWAP.*

(b) *For every $T \in \mathcal{K}(X)$, there exists a $\lambda_T > 0$ such that there is a net (T_α) in $\mathcal{F}(X,\lambda_T)$ such that $x^*T_\alpha x \longrightarrow x^*Tx$ for each $x \in X$ and $x^* \in X^*$.*

(c) *For every $T \in \mathcal{K}(X)$, there exists a $\lambda_T > 0$ such that for every $(x_n)_{n=1}^m \subset X$ and $(x_n^*)_{n=1}^m \subset X^*$, if $|\sum_{n=1}^m x_n^*(Sx_n)| \le 1$ for all S in $\mathcal{F}(X,\lambda_T)$, then $|\sum_{n=1}^m x_n^*(Tx_n)| \le 1$.*

(d) *For every $T \in \mathcal{K}(X)$, there exists a $\lambda_T > 0$ such that for every $(x_n)_{n=1}^m \subset X$ and $(x_n^*)_{n=1}^m \subset X^*$, if $|\sum_{n=1}^m \sum_{k=1}^l x_n^*(z_k)z_k^*(x_n)| \le 1$ for all $(z_k)_{k=1}^l \subset X$ and $(z_k^*)_{k=1}^l \subset X^*$ satisfying $\sup_{\|x\|=1} \|\sum_{k=1}^l z_k^*(x)z_k\| \le \lambda_T$, then $|\sum_{n=1}^m x_n^*(Tx_n)| \le 1$.*

For X^* we have the following.

Theorem 1.6. *Let X be a Banach space. Then the following are equivalent.*

(a) *X^* has the BWAP.*

(b) *For every $T \in \mathcal{K}(X^*)$, there exists a $\lambda_T > 0$ such that there is a net (T_α^*) in $\mathcal{F}^*(X,\lambda_T)$ such that $x^{**}T_\alpha^*x^* \longrightarrow x^{**}Tx^*$ for each $x^* \in X^*$ and $x^{**} \in X^{**}$.*

(c) *For every $T \in \mathcal{K}(X^*)$, there exists a $\lambda_T > 0$ such that for every $(x_n^*)_{n=1}^m \subset X^*$ and $(x_n^{**})_{n=1}^m \subset X^{**}$, if $|\sum_{n=1}^m x_n^{**}(S^*x_n^*)| \le 1$ for all S^* in $\mathcal{F}^*(X,\lambda_T)$, then $|\sum_{n=1}^m x_n^{**}(Tx_n^*)| \le 1$.*

(d) *For every $T \in \mathcal{K}(X^*)$, there exists a $\lambda_T > 0$ such that for every $(x_n^*)_{n=1}^m \subset X^*$ and $(x_n^{**})_{n=1}^m \subset X^{**}$, if $|\sum_{n=1}^m \sum_{k=1}^l x_n^{**}(z_k^*) x_n^*(z_k)| \le 1$ for all $(z_k)_{k=1}^l \subset X$ and $(z_k^*)_{k=1}^l \subset X^*$ satisfying $\sup_{\|x^*\|=1} \|\sum_{k=1}^l x^*\ (z_k) z_k^*\| \le \lambda_T$, then $|\sum_{n=1}^m x_n^{**}(Tx_n^*)| \le 1$.*

A Banach space X is said to have the *commuting bounded weak approximation property* (in short, CBWAP) if for every $T \in \mathcal{K}(X)$, there exists a $\lambda_T > 0$ such that there is a net (T_α) in $\mathcal{F}(X, \lambda_T)$ satisfying for all α, β $T_\alpha T_\beta$=$T_\beta T_\alpha$ such that $T_\alpha x \longrightarrow Tx$ for each $x \in X$.

We now have the following characterizations of a Banach space to have the CBWAP.

Theorem 1.7. *Let X be a Banach space. Then the following are equivalent.*

(a) *X has the CBWAP.*

(b) *For every $T \in \mathcal{K}(X)$, there exists a $\lambda_T > 0$ such that there is a net (T_α) in $\mathcal{F}(X, \lambda_T)$ satisfying for all α, β $T_\alpha T_\beta$=$T_\beta T_\alpha$ such that $x^* T_\alpha x \longrightarrow x^* T x$ for each $x \in X$ and $x^* \in X^*$.*

(c) *For every $T \in \mathcal{K}(X)$, there exists a $\lambda_T > 0$ such that there is a balanced convex set $\mathcal{C}$ in $\mathcal{F}(X, \lambda_T)$ satisfying for all $S, T \in \mathcal{C}$ TS=ST such that for every $(x_n)_{n=1}^m \subset X$ and $(x_n^*)_{n=1}^m \subset X^*$, if $|\sum_{n=1}^m x_n^*(Sx_n)| \le 1$ for all S in $\mathcal{C}$, then $|\sum_{n=1}^m x_n^*(Tx_n)| \le 1$.*

2. Three Important Topologies on $\mathcal{B}(X, Y)$ and Their Relations

At first, we introduce two topologies on $\mathcal{B}(X, Y)$ generated by some subbases.

Grothendieck [4] initiated the study of the approximation properties and the relations between them. One important tool he used was the following topology which is also called the *topology of compact convergence*.

Definition 2.1. Let X and Y be Banach spaces. For compact $K \subset X$, $\epsilon > 0$, and $T \in \mathcal{B}(X, Y)$ we put

$$N(T, K, \epsilon) = \{R \in \mathcal{B}(X, Y) : \sup_{x \in K} \|Rx - Tx\| < \epsilon\}.$$

Let $\mathcal{S}$ be the collection of all such $N(T, K, \epsilon)$'s. Then the τ-*topology* (in short, τ) on $\mathcal{B}(X, Y)$ is the topology generated by $\mathcal{S}$.

We can check that τ is a locally convex topology and for a net $(T_\alpha) \subset \mathcal{B}(X, Y)$ and $T \in \mathcal{B}(X, Y)$

$$\begin{aligned} T_\alpha \longrightarrow T \text{ in } (\mathcal{B}(X, Y), \tau) &\Longleftrightarrow \text{for every compact} \\ K \subset X\ \ &\sup\nolimits_{x \in K} \|T_\alpha x - Tx\| \longrightarrow 0. \end{aligned} \tag{2.1}$$

By the definitions of BAP, BCAP, and BWAP, we see the following :

$$X \text{ has the } \lambda - BAP \Longleftrightarrow I \in \overline{\mathcal{F}(X,\lambda)}^{\tau}. \tag{2.2}$$

$$X \text{ has the } \lambda - BCAP \Longleftrightarrow I \in \overline{\mathcal{K}(X,\lambda)}^{\tau}. \tag{2.3}$$

$$\begin{aligned} X \text{ has the } BWAP \iff & \text{ for every } T \in \mathcal{K}(X), \\ & \text{there exists a } \lambda_T > 0 \text{ such that } T \in \overline{\mathcal{F}(X,\lambda_T)}^{\tau}. \end{aligned} \tag{2.4}$$

Here I is the identity in $\mathcal{B}(X)$.

The following topology is also called the *topology of pointwise convergence*.

Definition 2.2. Let X and Y be Banach spaces. For $x \in X$, $\epsilon > 0$, and $T \in \mathcal{B}(X,Y)$ we put

$$N(T,x,\epsilon) = \{R \in \mathcal{B}(X,Y) : \|Rx - Tx\| < \epsilon\}.$$

Let $\mathcal{S}$ be the collection of all such $N(T,x,\epsilon)$'s. Then the *strong operator topology* (in short, *sto*) on $\mathcal{B}(X,Y)$ is the topology generated by $\mathcal{S}$.

We can check that *sto* is a locally convex topology and for a net $(T_\alpha) \subset \mathcal{B}(X,Y)$ and $T \in \mathcal{B}(X,Y)$

$$T_\alpha \longrightarrow T \text{ in } (\mathcal{B}(X,Y), sto) \Longleftrightarrow \text{ for every } x \in X \ \|T_\alpha x - Tx\| \longrightarrow 0. \tag{2.5}$$

Now we introduce another topology on $\mathcal{B}(X,Y)$ generated by a subspace of the vector space of all linear functionals on $\mathcal{B}(X,Y)$.

Definition 2.3. Let X and Y be Banach spaces. Let $\mathcal{Z}$ be the linear span of all linear functionals f on $\mathcal{B}(X,Y)$ of the form $f(T) = y^*Tx$ for $x \in X$ and $y^* \in Y^*$. Then the *weak operator topology* (in short, *wo*) on $\mathcal{B}(X,Y)$ is the topology generated by $\mathcal{Z}$.

We see that the *wo* is a locally convex topology (See Megginson [7, Proposition 2.4.4 and Theorem 2.4.11]) and for a net $(T_\alpha) \subset \mathcal{B}(X,Y)$ and $T \in \mathcal{B}(X,Y)$

$$\begin{aligned} T_\alpha \longrightarrow T \text{ in } (\mathcal{B}(X,Y), wo) \iff & \text{ for} \\ & \text{each } x \in X \text{ and } y^* \in Y^* \ \ y^*T_\alpha x \longrightarrow y^*Tx. \end{aligned} \tag{2.6}$$

From (2.1), (2.5), and (2.6), τ is stronger than *sto* and *sto* is stronger than *wo*.

Now we obtain the following theorem which is an essential tool to prove characterizations of bounded approximation properties.

Theorem 2.4. *Let X be a Banach space. Then the following statements hold.*

(a) *X has the λ-BAP if and only if $I \in \overline{\mathcal{F}(X,\lambda)}^{wo}$.*

(b) *X has the λ-BCAP if and only if $I \in \overline{\mathcal{K}(X,\lambda)}^{wo}$.*

(c) *X has the BWAP if and only if for every $T \in \mathcal{K}(X)$, there exists a $\lambda_T > 0$ such that $T \in \overline{\mathcal{F}(X,\lambda_T)}^{wo}$.*

To prove Theorem 2.4, we need the following relations between τ, *sto*, and *wo*.

Proposition 2.5. *Let X and Y be Banach spaces. If $\mathcal{A}$ is a bounded set in $\mathcal{B}(X,Y)$, then τ=sto on $\mathcal{A}$.*

Proof. It is enough to show $\tau \leq$ *sto* on $\mathcal{A}$. Since $\mathcal{A}$ is bounded, $\mathcal{A} \subset \mathcal{B}(X,Y,\lambda)$ for some $\lambda > 0$. Let $(T_\alpha) \subset \mathcal{A}$ be a net and $T \in \mathcal{A}$ with $T_\alpha \longrightarrow T$ in $(\mathcal{B}(X,Y), sto)$. Let compact $K \subset X$ and $\epsilon > 0$. Then there is a finite $F \subset K$ such that whenever $x \in K$ we have

$$\|x-y\| < \frac{\epsilon}{3\lambda}$$

for some $y \in F$. Since $T_\alpha \longrightarrow T$ in $(\mathcal{B}(X,Y), sto)$, there is a β such that $\alpha \succeq \beta$ implies $\|T_\alpha y - Ty\| < \epsilon/3$ for every $y \in F$. Now let $x \in K$. Then by the triangle inequality $\alpha \succeq \beta$ implies

$$\|T_\alpha x - Tx\| < \epsilon.$$

Hence $T_\alpha \longrightarrow T$ *in* $(\mathcal{B}(X,Y),\tau)$. This completes the proof. ■

Lemma 2.6. [3, p. 447, Theorem 4] *Let X and Y be Banach spaces.*
Then $(\mathcal{B}(X,Y), sto)^*$=$(\mathcal{B}(X,Y), wo)^*$ and the form of the bounded linear functionals f on $\mathcal{B}(X,Y)$ is $f(T)$= $\sum_{n=1}^m y_n^*(Tx_n)$, $(x_n)_{n=1}^m \subset X$ and $(y_n^*)_{n=1}^m \subset Y^*$.

Lemma 2.7. [7, Corollary 2.2.29] *Suppose that a vector space X has two locally convex topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ such that the dual spaces of X under the two topologies are the same. Let C be a convex subset of X. Then the $\mathcal{T}_1$-closure of C is the same as its $\mathcal{T}_2$-closure.*

From Lemma 2.6 and Lemma 2.7, we have the following proposition.

Proposition 2.8. *Let X and Y be Banach spaces. If $\mathcal{C}$ is a convex set in $\mathcal{B}(X,Y)$, then $\overline{\mathcal{C}}^{sto}$=$\overline{\mathcal{C}}^{wo}$.*

From Proposition 2.5 and Proposition 2.8, we have the following conclusion.

Corollary 2.9. *Let X and Y be Banach spaces. If $\mathcal{C}$ is a bounded convex set in $\mathcal{B}(X,Y)$, then $\overline{\mathcal{C}}^{\tau}$=$\overline{\mathcal{C}}^{wo}$.*

Now we can prove Theorem 2.4.

Proof of Theorem 2.4. Since $\mathcal{F}(X,\lambda)$ and $\mathcal{K}(X,\lambda)$ are bounded and convex for $\lambda > 0$, Theorem 2.4 follows from Corollary 2.9, (2.2), (2.3), and (2.4). ■

3. Proofs of the Main Theorems

To prove characterizations of bounded approximation properties, we need the following two lemmas.

Lemma 3.1. *Let X be a Banach space. Suppose that $\mathcal{C}$ is a balanced convex subset of $\mathcal{B}(X)$. Let $T \in \mathcal{B}(X)$. Then the following are equivalent.*

(a) *T belongs to the wo-closure of $\mathcal{C}$.*

(b) *For every $f \in (\mathcal{B}(X), wo)^*$ such that $|f(S)| \leq 1$ for all $S \in \mathcal{C}$, we have $|f(T)| \leq 1$.*

Proof. (a)$\Longrightarrow$(b). By continuity.

(b)$\Longrightarrow$(a). Suppose that T dose not belongs to the *wo*-closure of $\mathcal{C}$.

By an application of the separation theorem (See [7, Theorem 2.2.28]), there is a $f \in (\mathcal{B}(X), wo)^*$ such that for all S in the *wo*-closure of $\mathcal{C}$ we have

$$\mathrm{Re} f(S) \leq 1 < \mathrm{Re} f(T).$$

Observe that $|f(S)| \leq 1$ for all S in $\mathcal{C}$ because $\mathcal{C}$ is balanced. This contradicts (b). ■

The following lemma is due to Johnson [5, Lemma 1]. A concrete proof is in [2, Lemma 3.11].

Lemma 3.2. *Let X be a Banach space and $\lambda > 0$. Then $\overline{\mathcal{F}(X^*,\lambda)}^{\tau} = \overline{\mathcal{F}^*(X,\lambda)}^{\tau}$.*

Since for $\lambda > 0$ $\mathcal{F}^*(X,\lambda)$ is bounded and convex, by Corollary 2.9 and Lemma 3.2 we have the following conclusion.

Corollary 3.3. *Let X be a Banach space and $\lambda > 0$. Then $\overline{\mathcal{F}(X^*,\lambda)}^{\tau} = \overline{\mathcal{F}^*(X,\lambda)}^{wo}$.*

Now we are ready for the proof of characterizations of bounded approximation properties

Proof of Theorem 1.1. Recall Theorem 2.4(a). Then (a)$\Longleftrightarrow$(b) follows.

(a)$\Longrightarrow$(d). Let $T \in \mathcal{B}(Y, X, 1)$. Then there is a net (T_α) in $\mathcal{F}(X, \lambda)$ such that $x^*T_\alpha Ty \longrightarrow x^*Ty$ for each $y \in Y$ and $x^* \in X^*$. Since each $T_\alpha T \in \mathcal{F}(Y, X, \lambda)$ and $T_\alpha T \longrightarrow T$ in $(\mathcal{B}(Y, X), wo)$, we have the conclusion (d).

(a)$\Longrightarrow$(e). Let $T \in \mathcal{B}(X, Y, 1)$. Then there is a net (T_α) in $\mathcal{F}(X, \lambda)$ such that $y^*TT_\alpha x \longrightarrow y^*Tx$ for each $x \in X$ and $y^* \in Y^*$ because $y^*T \in X^*$. Since each $TT_\alpha \in \mathcal{F}(X, Y, \lambda)$ and $TT_\alpha \longrightarrow T$ in $(\mathcal{B}(X, Y), wo)$, we have the conclusion (e). (d)$\Longrightarrow$(a) and (e)$\Longrightarrow$(a) are clear.

(a)$\Longleftrightarrow$(c). By (d)$\Longrightarrow$(c)$\Longrightarrow$(a)$\Longrightarrow$(d).

(a)$\Longleftrightarrow$(f). Since $\mathcal{F}(X, \lambda)$ is balanced and convex for $\lambda > 0$, Lemma 2.6 and Lemma 3.1 prove the equivalence.

(f)$\Longleftrightarrow$(g). For each $S \in \mathcal{F}(X)$, the form of S is $Sx = \sum_{k=1}^{l} z_k^*(x) z_k$ for some $(z_k)_{k=1}^{l} \subset X$ and $(z_k^*)_{k=1}^{l} \subset X^*$. This proves the equivalence. ■

Proof of Theorem 1.2. Notice that X^* has the λ-BAP if and only if $I \in \overline{\mathcal{F}^*(X, \lambda)}^{wo}$ by (2.2) and Corollary 3.3, where I is the identity in $\mathcal{B}(X^*)$.

From this (a)$\Longleftrightarrow$(b) follows.

(a)$\Longrightarrow$(d). Let $T^* \in \mathcal{B}^*(Y, X, 1)$. Then there is a net (T_α^*) in $\mathcal{F}^*(X, \lambda)$ such that

$$y^{**}(T_\alpha T)^* x^* = y^{**}T^*T_\alpha^* x^* \longrightarrow y^{**}T^*x^*$$

for each $x^* \in X^*$ and $y^{**} \in Y^{**}$ because $y^{**}T^* \in X^{**}$. Since each $(T_\alpha T)^* \in \mathcal{F}^*(Y, X, \lambda)$ and $(T_\alpha T)^* \longrightarrow T^*$ in $(\mathcal{B}(X^*, Y^*), wo)$, we have the conclusion (d).

(a)$\Longrightarrow$(e). Let $T^* \in \mathcal{B}^*(X, Y, 1)$. Then there is a net (T_α^*) in $\mathcal{F}^*(X, \lambda)$ such that

$$x^{**}(TT_\alpha)^* y^* = x^{**}T_\alpha^* T^* y^* \longrightarrow x^{**}T^*y^*$$

for each $y^* \in Y^*$ and $x^{**} \in X^*$. Since each $(TT_\alpha)^* \in \mathcal{F}^*(X, Y, \lambda)$ and $(TT_\alpha)^* \longrightarrow T^*$ in $(\mathcal{B}(Y^*, X^*), wo)$, we have the conclusion (e). (d)$\Longrightarrow$(a) and (e)$\Longrightarrow$(a) are clear.

(a)$\Longleftrightarrow$(c). By (d)$\Longrightarrow$(c)$\Longrightarrow$(a)$\Longrightarrow$(d).

(a)$\Longleftrightarrow$(f). Since $\mathcal{F}^*(X, \lambda)$ is balanced and convex for $\lambda > 0$, Lemma 2.6 and Lemma 3.1 prove the equivalence.

(f)$\Longleftrightarrow$(g). For each $S^* \in \mathcal{F}^*(X)$, the form of S^* is $S^*x^* = \sum_{k=1}^{l} x^*(z_k) z_k^*$ for some $(z_k)_{k=1}^{l} \subset X$ and $(z_k^*)_{k=1}^{l} \subset X^*$. This proves the equivalence. ■

Proof of Theorem 1.3. (a)$\Longrightarrow$(b). By $sto \geq wo$.

(b)$\Longrightarrow$(a). Let (T_α) be a net satisfying (b). Consider $co(\{T_\alpha\})$. Then by Proposition 2.8 $\overline{co}^{sto}(\{T_\alpha\})$=$\overline{co}^{wo}(\{T_\alpha\})$. Since $I \in \overline{co}^{wo}(\{T_\alpha\})$, there is a net (S_α) in $co(\{T_\alpha\})$ such that

$$S_\alpha x \longrightarrow x$$

for each $x \in X$. We can check that $S_\alpha S_\beta$=$S_\beta S_\alpha$ for every $S_\alpha, S_\beta \in \{S_\alpha\}$ and $(S_\alpha) \subset \mathcal{F}(X, \lambda)$. Hence (S_α) is a desired net which proves the conclusion.

(b)$\Longrightarrow$(c). Let (T_α) be a net satisfying (b). Consider $co(\bigcup_{|\delta|\le 1}\{\delta T_\alpha\})$. Then we can check that $co(\bigcup_{|\delta|\le 1}\{\delta T_\alpha\})$ is a balanced convex set in $\mathcal{F}(X,\lambda)$ and TS=ST for every $T,S \in co(\bigcup_{|\delta|\le 1}\{\delta T_\alpha\})$. Since $I \in \overline{co}^{wo}(\bigcup_{|\delta|\le 1}\{\delta T_\alpha\})$, by Lemma 2.6 and Lemma 3.1 we have the conclusion.

(c)$\Longrightarrow$(b). Let $\mathcal{C}$ be a set satisfying (c). By Lemma 2.6 and Lemma 3.1 $I \in \overline{\mathcal{C}}^{wo}$. Then there is a net (T_α) in $\mathcal{C}$ such that

$$T_\alpha \longrightarrow I$$

in $(\mathcal{B}(X), wo)$. By the property of $\mathcal{C}$ $(T_\alpha) \subset \mathcal{F}(X,\lambda)$ and $T_\alpha T_\beta$=$T_\beta T_\alpha$ for every $T_\alpha, T_\beta \in \{T_\alpha\}$. Hence (T_α) is a desired net which proves the conclusion. ■

Proof of Theorem 1.4. See the proof of Theorem 1.1. ■

Proof of Theorem 1.5. Recall Theorem 2.4(c). Then (a)$\Longleftrightarrow$(b) follows.

(a)$\Longleftrightarrow$(c). Since $\mathcal{F}(X,\lambda)$ is balanced and convex for $\lambda > 0$, Lemma 2.6 and Lemma 3.1 prove the equivalence.

(c)$\Longleftrightarrow$(d). See the proof of Theorem 1.1 (f)$\Longleftrightarrow$(g). ■

Proof of Theorem 1.6. Notice that X^* has the BWAP if and only if for every $T \in \mathcal{K}(X^*)$, there exists a $\lambda_T > 0$ such that $T \in \overline{\mathcal{F}^*(X,\lambda_T)}^{wo}$ by (2.4) and Corollary 3.3.

From this (a)$\Longleftrightarrow$(b) follows.

(a)$\Longleftrightarrow$(c). Since $\mathcal{F}^*(X,\lambda)$ is balanced and convex for $\lambda > 0$, Lemma 2.6 and Lemma 3.1 prove the equivalence.

(c)$\Longleftrightarrow$(d). See the proof of Theorem 1.2 (f)$\Longleftrightarrow$(g). ■

Proof of Theorem 1.7. See the proof of Theorem 1.3. ■

4. Applications

At first, we introduce one more topology on $\mathcal{B}(X,Y)$ generated by a subspace of the vector space of all linear functionals on $\mathcal{B}(X,Y)$.

Definition 4.1. Let X and Y be Banach spaces. Let $\mathcal{Z}$ be the linear span of all linear functionals f on $\mathcal{B}(X,Y)$ of the form $f(T) = x^{**}T^*y^*$ for $x^{**} \in X^{**}$ and $y^* \in Y^*$. Then the *weak adjoint operator topology* (in short, *wao*) on $\mathcal{B}(X,Y)$ is the topology generated by $\mathcal{Z}$.

We see that the wao is a locally convex topology (See [7, Proposition 2.4.4 and Theorem 2.4.11]) and for a net $(T_\alpha) \subset \mathcal{B}(X,Y)$ and $T \in \mathcal{B}(X,Y)$

$$T_\alpha \longrightarrow T \ \text{ in } \ (\mathcal{B}(X,Y), wao) \Longleftrightarrow \quad \text{for each} \ \ x^{**} \in X^{**}$$

$$\text{and} \;\; y^* \in Y^* \;\; x^{**}T_\alpha^* y^* \longrightarrow x^{**}T^*y^*.$$

From (2.6) we see that *wao* is stronger than *wo*.

It is well-known that for a Banach space X, if X^* has the λ-BAP, then X has the λ-BAP but the converse is not true in general (See [1]). Roughly speaking, the following theorem means that if we give $\mathcal{B}(X)$ the weak adjoint operator topology, then λ-BAP of X and X^* are equivalent.

Theorem 4.2. *Let X be a Banach space. Then X^* has the λ-BAP if and only if X has the λ-BAP approximated by wao.*

Proof. Suppose that X^* has the λ-BAP. Then by Theorem 1.2 there is a net (T_α^*) in $\mathcal{F}^*(X, \lambda)$ such that $x^{**}T_\alpha^* x^* \longrightarrow x^{**}x^*$ for each $x^* \in X^*$ and $x^{**} \in X^{**}$. It follows that $S_\alpha \longrightarrow I$ in $(\mathcal{B}(X), wao)$, where I is the identity in $\mathcal{B}(X)$. Hence X has the λ-BAP approximated by wao.

Suppose the converse. Then there is a net (T_α) in $\mathcal{F}(X, \lambda)$ such that $T_\alpha \longrightarrow I$ in $(\mathcal{B}(X), wao)$, that is, $x^{**}T_\alpha^* x^* \longrightarrow x^{**}I^*x^*$ for each $x^* \in X^*$ and $x^{**} \in X^{**}$. Since I^* is the identity in $\mathcal{B}(X^*)$ and $(T_\alpha^*) \subset \mathcal{F}(X^*, \lambda)$, by Theorem 1.1 X^* has the λ-BAP. ∎

Since $wao \geq wo$, from Theorem 4.2 and Theorem 1.1 we have the following well-known conclusion as a corollary of Theorem 4.2.

Corollary 4.3. *Let X be a Banach space. If X^* has the λ-BAP, then X has the λ-BAP.*

For BWAP we have the following.

Theorem 4.4. *Let X be a Banach space. If X^* has the BWAP, then X has the BWAP approximated by wao.*

Proof. Suppose that X^* has the BWAP. Let $T \in \mathcal{K}(X)$. Then $T^* \in \mathcal{K}(X^*)$. Thus by Theorem 1.6 there exists a $\lambda_T > 0$ such that there is a net (T_α^*) in $\mathcal{F}^*(X, \lambda_T)$ such that $x^{**}T_\alpha^* x^* \longrightarrow x^{**}T^*x^*$ for each $x^* \in X^*$ and $x^{**} \in X^{**}$. It follows that $T_\alpha \longrightarrow T$ in $(\mathcal{B}(X), wao)$. Hence X has the BWAP approximated by wao. ∎

In [2] it was shown that for a Banach space X, if X^* has the BWAP, then X has the BWAP. In view of Theorem 1.5 and $wao \geq wo$, this is a corollary of Theorem 4.4.

Corollary 4.5. *Let X be a Banach space. If X^* has the BWAP, then X has the BWAP.*

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