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MULTIPLICATIVE LINEAR FUNCTIONALS OF CONTINUOUS FUNCTIONS ARE COUNTABLY EVALUATED

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Abstract. We prove that each nonzero algebra homomorphism $\pi : C(X) \longrightarrow \mathbb{R}$ is countably evaluated. This is applied to give a simple and direct proof (from the algebraic view) of the fact that each Lindelöf space is realcompact.

0. Introduction

We refer to standard books [1, 3, 7], for the notations and terminology for this paper. Let X be a topological space. The algebra (under the pointwise operations) of real valued continuous functions on X is denoted by $C(X)$. An algebra A on X means a subalgebra of $C(X)$ containing the constant functions. Recall that X is called a Lindelöf space if each open cover of X has a countable subcover. X is called *realcompact* space if it is homeomorphic to a closed subspace of the product space of real lines. In [4] by using very elementary arguments, it is proved that a Tychonoff space X is a Lindelöf space if and only if each countably evaluated (algebra) homomorphism π from any algebra A on X into $\mathbb{R}$ is point evaluated (see also [2]). Without using of Axiom of Choice, a direct and easy proof of the fact that a Tychonoff space X is realcompact if and only if each nonzero algebra homomorphism $\pi : C(X) \longrightarrow \mathbb{R}$ is point evaluated, is given in [5]. Let A be an algebra on X and $\pi : A \longrightarrow \mathbb{R}$ be an algebra homomorphism. Recall that π is called *point evaluated* if there exists $x \in X$ such that $\pi(f) = f(x)$ for each $f \in A$. Let α be a cardinal number. If for each subset $B \subset A$ with $card(B) \leq \alpha$ there exists x in X such that $\pi(f) = f(x)$ for each $f \in B$, then π is called *α-evaluated*. In the case $\alpha = card(\mathbb{N})$ we call that π is *countably evaluated*.

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1. Some Remarks on the Algebra Homomorphisms and Its Consequences

Theorem 1. *Let X be a topological space and $\pi : C(X) \longrightarrow \mathbb{R}$ be a nonzero algebra homomorphism. Then π is countably evaluated.*

Proof. It is clear that π is also a Riesz homomorphism, that is $\pi(|f|) = |\pi(f)|$ for each $f \in C(X)$. Let us call a sequence (f_n) point evaluated, if there exits $x \in X$ such that $\pi(f_n) = f_n(x)$ for each n. Suppose that π is not countably evaluated. Then there exists a sequence (f_n) in $C(X)$ which is not point evaluated. For each n, let

$$g_n := ((\pi(f_n)\mathbf{1} - f_n))^2 \wedge n^{-2}\mathbf{1}.$$

That is, $g_n(x) = min\{(\pi(f_n) - f_n(x))^2, n^{-2}\}$. Then it is clear that the sequence (g_n) is not point evaluated. Let $g : X \longrightarrow \mathbb{R}$ be defined by $g(x) := \Sigma_n g_n(x)$. Then $g \in C(X)$ and g is the uniform limit of the sequence $(\Sigma_{i=1}^n g_n)$ in the subalgebra $C_b(X)$ on X of bounded functions in $C(X)$. Let π_0 be restriction of π into $C_b(X)$. Then as π_0 is continuous (it is positive, that is, $\pi(f) \geq 0$ whenever $f(x) \geq 0$ for each $x \in X$) and $\pi_0(g_n) = 0$ for each n, then $\pi(g) = \pi_0(g) = 0$. Then there exists $x \in X$ such that $g(x) = 0$. Indeed, if $g(x) \neq 0$ for each $x \in X$, then the inverse g^{-1} exists. Then we have the following contradiction.

$$1 = \pi(\mathbf{1}) = \pi(gg^{-1}) = \pi(g)\pi(g^{-1}) = 0.$$

Let $x \in X$ with $g(x) = 0$. Then for each n, $\pi(f_n) = f_n(x)$. This contradicts to our assumption and completes the proof. ∎

Let A be an Archemedean f-algebra with unit e and let B be an Archimedean semiprime f-algebra. Then a Riesz homomorphism π from A into B is an algebra homomorphism if and only if $\pi(e)$ is idempotent (see [10], p. 98). This implies that a map π between Archimedean f-algebras A and B with units e_A and e_B, respectively, with $\pi(e_A) = e_B$ is a Riesz homomorphism if and only if it is an algebra homomorphisms, this is due to Putten [12]. Although the proof of this is not very elementary, in the case $A = C(K)$ and $B = C(M)$, where K and M are compact Hausdorff spaces, the proof is very elementary. By using this, to make the paper is self contained we give the following lemma with a proof.

Lemma 2. *Let K be an arbitrary topological space and $\pi : C(K) \longrightarrow \mathbb{R}$ be a map with $\pi(\mathbf{1}) = 1$. Then π is a Riesz homomorphism if and only if it is an algebra homomorphism.*

Proof. It is clear that π is Riesz homomorphism whenever it is an algebra homomorphism. Suppose that π is a Riesz homomorphism. Let $0 \leq f \in C(X)$ be given. Let $n \in \mathbb{N}$ be given so that $(\pi(f))^2 < n$. Then as the restriction π_0 of π into $C_b(K)$ is a homomorphism we have

$$(\pi(f))^2 = (\pi(f) \wedge \sqrt{n}\mathbf{1})^2 = (\pi(f \wedge \sqrt{n}\mathbf{1}))^2 = \pi((f \wedge \sqrt{n}\mathbf{1})^2) = \pi(f^2 \wedge n\mathbf{1})$$

On the other hand $\pi(f^2 \wedge n\mathbf{1}) = \pi(f^2) \wedge n$. As $(\pi(f))^2 < n$ we have that $\pi(f^2) = (\pi(f))^2$. Now from the fact that $4fg = ((f+g)^2 - (f-g)^2)$ that π is an algebra homomorphism.

Corollary 3. *Let X be a topological space and $\pi : C(X) \longrightarrow \mathbb{R}$ be a nonzero Riesz homomorphism. Then π is countably evaluated.*

Remarks 1.

1. A subalgebra A of $C(X)$ is called *inverse-closed* if $f \in A$ and $f(x) \neq 0$ for each $x \in X$, then $f^{-1} \in A$. A is called *uniformly closed* if $f \in A$ whenever there exists a sequence (f_n) in A with $sup_x|f_n(x) - f(x)| \longrightarrow 0$. Theorem 1 can be generalized as follows: Every real valued nonzero algebra homomorphism π on a uniformly closed, inverse closed algebra A is countably evaluated. Indeed, suppose that it is not. Then there exits a sequence (f_n) which is not countably evaluated. Let

$$g_n = 2^{-n}(\pi(f_n)\mathbf{1} - f_n))^2(1 + ((\pi(f_n)\mathbf{1} - f_n))^2)^{-1}.$$

Then (g_n) is not countably evaluated. As A is uniformly closed $g = \Sigma_n g_n$ in A. From the fact $g_n \leq (\pi(f_n)\mathbf{1} - f_n)^2$ we have $\pi(g_n) = 0$. Then it is clear that $\pi(g) = 0$. This implies that $g(x) = 0$ for some $x \in X$, because other wise g^{-1} exists and in A and this implies that $1 = \pi(gg^{-1})) = 0$. This contradiction shows that there exists $x \in X$ such that $\pi(f_n) = f_n(x)$ for each n.

2. It is well known that if A is inverse closed subalgebra of $C(X)$ with unit then for each nonzero algebra homomorphism $\pi : A \longrightarrow \mathbb{R}$ and finite subset $F \subset A$ there exits $a_F \in A$ such that $\pi(f) = f(a_F)$ for each $f \in F$ (see [7]).

Let X be a topological space. Then it is well known that X is compact if and only if each nonzero algebra homomorphism on $C_b(X)$ is point evaluated. By using this and the above results we have the following corollary.

Corollary 4. *Let X be a Lindelöf space. Then X is compact if and only if each nonzero algebra homomorphism $\pi : C_b(X) \longrightarrow \mathbb{R}$ is countably evaluated.*

Corollary 4 shows that in remark 1 the condition "inverse-closed" can not be dropped. By combining the above arguments we have a re-proof of the following well known important theorem which is more direct and easier than most of the the well known proofs. (see [3], p. 216).

Theorem 2.2. (Hewitt, [6]) *Every Lindelöf space is realcompact.*

Proof. Let X be a Lindelöf space. Then for each algebra A on X, each countable nonzero algebra homomorphism $\pi : A \longrightarrow \mathbb{R}$ is point evaluated (see [2,4]). Then as from Theorem 1, any nonzero algebra homomorphism $\pi : C(X) \longrightarrow \mathbb{R}$ is countably evaluated π is point evaluated. So, X is realcompact space. ∎

An alternative proof of the above Theorem is also given in [12]. In [11] it is observed that any ring homomorphism $\pi : C(X) \longrightarrow \mathbb{R}$ is an algebra homomorphism. Now we have the following main result of the paper.

Theorem 6. *Let X be a topological space and $\pi : C(X) \longrightarrow \mathbb{R}$ be a nonzero map. Then the followings are equivalent.*

(i) *π is an algebra homomorphism*

(ii) *π is a Riesz homomorphism with $\pi(\mathbf{1}) = 1$.*

(iii) *π is a ring homomorphism with $\pi(\mathbf{1}) = 1$.*

(iv) *There exists a net (x_α) in X such that $\pi(f) = lim f(x_\alpha)$ for each $f \in C(X)$.*

(v) *π is countably evaluated.*

(vi) *π is n-evaluated for each $n \in \mathbb{N}$.*

(vii) *π is 3-evaluated.*

Proof. $(vii) \Longrightarrow (i)$: Let $f, g, h \in C(X)$ be given. As π is 3-evaluated there exits $a, b \in X$ such that

$$\pi(f+g) = (f+g)(a), \quad \pi(f) = f(a) \quad \text{and} \quad \pi(g) = g(a)$$

and

$$\pi(fg) = (fg)(b), \quad \pi(f) = f(b) \quad \text{and} \quad \pi(g) = g(b).$$

This shows that

$$\pi(f+g) = \pi(f) + \pi(g) \quad \text{and} \quad \pi(fg) = \pi(f)\pi(g).$$

It is also clear that $\pi(\lambda f) = \lambda \pi(f)$ for each $\lambda \in \mathbb{R}$. That is, π is an algebra homomorphism. Suppose that *(i)* holds. As there exists a realcompact space Y with $C(X)$ and $C(Y)$ are algebraic isomorphic (we can choose Y is the closure of $\prod_{f \in C(X)} f(X)$ in the product space $\prod_{f \in C(X)} \mathbb{R}$) under the map $\alpha : C(Y) \longrightarrow C(X)$, $\alpha(h)(x) = h(i(x))$, where $i(x) = (f(x))_{f \in C(X)}$. (see Theorem 3.9 of [7] and p. 218 of [3]), since each nonzero algebra homomorphism from $C(Y)$ into $\mathbb{R}$ is point evaluated. This implies *(iv)*. Rest of the proof is more or less clear. ∎

Corollary 7. [14]) *(Let X be a realcompact space. Then each nonzero Riesz homomorphism π from $C(X)$ into $\mathbb{R}$ is point evaluated.*

Remarks

1. Let X be a topological space and $\pi : C(X) \to \mathbb{R}$ be a nonzero and 2-evaluated map. Then it is clear that for each $f \in C(X), \lambda \in \mathbb{R}$

$$\pi(f^2) = \pi(f)^2, \quad \pi(\lambda + f) = \lambda + \pi(f) \quad \text{and} \quad \pi(\lambda f) = \lambda \pi(f).$$

Moreover, the referee suspects that π is an algebra homomorphism, by applying a result of [9].

2. For each topological space X there exists a completely regular Hausdorff space Y such that $C(X)$ and $C(Y)$ are algebraic isomorphic (see [7]). So, when we study the algebraic properties of $C(X)$ without loss of the generality we can suppose that X is a completely regular Hausdorff space. In this way, some arguments of the paper may be simplified.

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