TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 3, pp. 703-717, June 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

INEQUALITIES OF HERMITE-HADAMARD-FEJÉR TYPE FOR CONVEX FUNCTIONS AND CONVEX FUNCTIONS ON THE CO-ORDINATES IN A RECTANGLE FROM THE PLANE

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Abstract. In this paper, we establish some inequalities of Hermite-Hadamard-Fejér type for convex functions and convex functions on the co-ordinates defined in a rectangle from the plane.

1. Introduction

If $f : [a, b] \to R$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

is known as Hermite-Hadamard inequality $[6]$.

In $[5]$, Fejér established the following weighted generalization of the inequalities (1):

Theorem A. *If $f : [a, b] \to R$ is a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx \le \int_a^b f(x)\,w(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx \tag{2}$$

holds, where $w : [a, b] \to R$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$.

For some results which generalize, improve and extend the inequalities (1) and (2) see $[1-17]$.

Received March 26, 2006, accepted October 14, 2006.
Communicated by H. M. Srivastava.
2000 *Mathematics Subject Classification*: 26D15.
Key words and phrases: Hermite-Hadamard inequality, Fejér inequality, Convex function, Convex function on the co-ordinates.

In $[11,$ Remark 6] and $[14,$ Theorem 1], Yang and Tseng proved the following two theorems which refine the inequality (2).

Theorem B. *Let f and w be defined as in Theorem 1. If $P:[a,b]\to R$ are defined by*

$$P(t):=\int_a^b f\left[tx+(1-t)\frac{a+b}{2}\right]w(x)\,dx \tag{3}$$

then P is convex, increasing on $[0,1]$ and, for all $t\in[0,1]$,

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx=P(0)\le P(t)\le P(1)=\int_a^b f(x)\,w(x)\,dx$$

If we choose $w(x)\equiv\frac{1}{b-a}$ in Theorem B, then

$$P(t)=\frac{1}{b-a}\int_a^b f\left[tx+(1-t)\frac{a+b}{2}\right]dx \tag{4}$$

is reduced to a result established by Dragomir [2].

Theorem C. *Let $f:[a,b]\to R$ be convex and let $w:[a,b]\to R$ be positive, integrable and symmetric about $\frac{a+b}{2}$. If $G:[0,1]\to R$ is defined by*

$$G(t):=\frac{1}{\int_a^b w(x)\,dx}\int_a^b\int_a^b f[tx+(1-t)y]\,w(x)\,w(y)\,dxdy, \tag{5}$$

then (a) G is convex on $[0,1]$, symmetric about $\frac{1}{2}$, decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$,

$$\sup_{t\in[0,1]}G(t)=G(0)=G(1)=\int_a^b f(x)\,w(x)\,dx$$

and

$$\inf_{t\in[0,1]}G(t)=G\left(\frac{1}{2}\right)=\frac{1}{\int_a^b w(x)\,dx}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)w(x)\,w(y)\,dxdy;$$

(b) we have:

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx\le G\left(\frac{1}{2}\right)$$

and

$$P(t)\le G(t)\quad(t\in(0,1))$$

where P is defined as in (3).

If we choose $w(x) \equiv \frac{1}{b-a}$ in Theorem 3, then

$$G(t) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f[tx + (1-t)y]\,dxdy \tag{6}$$

is reduced to a result established by Dragomir [2].

Recently Dragomir [4] has proved some results for convex functions and convex functions on the co-ordinates defined in rectangle from the plane related to (1), (4) and (6). In this paper, we shall establish some inequalities for convex functions and convex functions on the co-ordinates defined in rectangle from the plane related to Theorems A-C.

2. Main Results

Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in R^2 with $a < b$ and $c < d$. A function $F : \Delta \to R$ will be called *convex on the co-ordinates on* Δ if the partial mapping $F_y : [a,b] \to R$, $F_y(u) := F(u,y)$ is convex on $[a,b]$ for each $y \in [c,d]$, and the partial mapping $F_x : [c,d] \to R$, $F_x(v) := F(x,v)$ is convex on $[c,d]$ for each $x \in [a,b]$. A function $H : \Delta \to R$ will be called *increasing on the co-ordinates on* Δ if the partial mapping $H_y : [a,b] \to R$, $H_y(u) := H(u,y)$ is *increasing* on $[a,b]$ for each $y \in [c,d]$, and the partial mapping $H_x : [c,d] \to R$, $H_x(v) := H(x,v)$ is *increasing* on $[c,d]$ for each $x \in [a,b]$. A function $g : \Delta \to R$ will be called *symmetric on the co-ordinates on* Δ if the partial mapping $g_y : [a,b] \to R$, $g_y(u) := g(u,y)$ is symmetric about $\frac{a+b}{2}$ for each $y \in [c,d]$, and the partial mapping $g_x : [c,d] \to R$, $g_x(v) := g(x,v)$ is symmetric about $\frac{c+d}{2}$ for each $x \in [a,b]$.

The following theorems hold:

Theorem 1. *Let* $0 \le \gamma,\ \rho \le 1$. *If* $F : \Delta \to R$ *is convex on the co-ordinates and* $g : \Delta \to R$ *is nonnegative, integrable and symmetric on the co-ordinates, then the inequality*

$$\begin{aligned}
& F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d g(x,y)\,dydx \\
\le & \int_a^b \int_c^d \left[\gamma F\left(x, \frac{c+d}{2}\right) + (1-\gamma) F\left(\frac{a+b}{2}, y\right)\right] g(x,y)\,dydx \\
\le & \int_a^b \int_c^d F(x,y)\, g(x,y)\,dydx \\
\le & \int_a^b \int_c^d \left[\frac{\rho}{2}\left(F(x,c) + F(x,d)\right)\right.
\end{aligned} \tag{7}$$

$$+\frac{1-\rho}{2}\left(F\left(a,y\right)+F\left(b,y\right)\right)\bigg]\,g\left(x,y\right)dydx$$
$$\leq\frac{1}{4}\left[F\left(a,c\right)+F\left(a,d\right)+F\left(b,c\right)+F\left(b,d\right)\right]\int_a^b\int_c^d g\left(x,y\right)dydx$$

holds. The inequality (8) *is sharp.*

Proof. Since F is convex on the co-ordinates on Δ and g is nonnegative, integrable and symmetric on the co-ordinates on Δ, we have the identities

(8) $$g\left(x,v\right)=g\left(a+b-x,v\right)\quad\left(\left(x,v\right)\in\Delta\right)$$

(9) $$g\left(u,y\right)=g\left(u,c+d-y\right)\quad\left(\left(u,y\right)\in\Delta\right)$$

and

(10)
$$\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d g\left(x,y\right)dydx\\
&=\int_a^b\int_c^d\bigg[\gamma F\left(\frac{x}{2}+\frac{a+b-x}{2},\frac{c+d}{2}\right)\\
&\quad+\left(1-\gamma\right)F\left(\tfrac{a+b}{2},\tfrac{y}{2}+\tfrac{c+d-y}{2}\right)\bigg]g\left(x,y\right)dydx\\
&\leq\int_a^b\int_c^d\bigg[\frac{\gamma}{2}\left(F\left(x,\frac{c+d}{2}\right)+F\left(a+b-x,\frac{c+d}{2}\right)\right)\\
&\quad+\frac{1-\gamma}{2}\left(F\left(\frac{a+b}{2},y\right)+F\left(\frac{a+b}{2},c+d-y\right)\right)\bigg]g\left(x,y\right)dydx\\
&=\int_c^d\int_a^b\frac{\gamma}{2}\bigg[F\left(x,\frac{c+d}{2}\right)g\left(x,y\right)\\
&\quad+F\left(a+b-x,\frac{c+d}{2}\right)g\left(a+b-x,y\right)\bigg]dxdy\\
&\quad+\int_a^b\int_c^d\frac{1-\gamma}{2}\bigg[F\left(\frac{a+b}{2},y\right)g\left(x,y\right)\\
&\quad+F\left(\frac{a+b}{2},c+d-y\right)g\left(x,c+d-y\right)\bigg]dydx\\
&=\int_a^b\int_c^d\left[\gamma F\left(x,\frac{c+d}{2}\right)+\left(1-\gamma\right)F\left(\frac{a+b}{2},y\right)\right]g\left(x,y\right)dydx\\
&=\int_a^b\int_c^d\bigg[\gamma F\left(x,\frac{y}{2}+\frac{c+d-y}{2}\right)\\
&\quad+\left(1-\gamma\right)F\left(\frac{x}{2}+\frac{a+b-x}{2},y\right)\bigg]g\left(x,y\right)dydx
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_a^b \int_c^d \left[\frac{\gamma}{2}\left(F(x,y)+F(x,c+d-y)\right)\right. \\
&\quad \left. +\frac{1-\gamma}{2}\left(F(x,y)+F(a+b-x,y)\right)\right] g(x,y)\,dydx \\
&= \int_a^b \int_c^d \frac{\gamma}{2}\left[F(x,y)\,g(x,y)+F(x,c+d-y)\,g(x,c+d-y)\right] dydx \\
&\quad +\int_c^d \int_a^b \frac{1-\gamma}{2}\left[F(x,y)\,g(x,y)+F(a+b-x,y)\,g(a+b-x,y)\right] dxdy \\
&= \int_a^b \int_c^d F(x,y)\,g(x,y)\,dydx.
\end{aligned}
$$

If $x,y\in[a,b]$ then $0\leq \frac{b-x}{b-a},\frac{x-a}{b-a},\frac{d-y}{d-c},\frac{y-c}{d-c}\leq 1$, $\frac{b-x}{b-a}+\frac{x-a}{b-a}=1$, $\frac{d-y}{d-c}+\frac{y-c}{d-c}=1$, $x=\frac{b-x}{b-a}a+\frac{x-a}{b-a}b$, $a+b-x=\frac{x-a}{b-a}a+\frac{b-x}{b-a}b$, $y=\frac{d-y}{d-c}c+\frac{y-c}{d-c}d$ and $c+d-y=\frac{y-c}{d-c}c+\frac{d-y}{d-c}d$. It follows from the above conclusions, the convexity of F on the co-ordinates on Δ and the identities (9) and (10), that we have

$$
\begin{aligned}
&\int_a^b \int_c^d F(x,y)\,g(x,y)\,dydx \\
&= \int_a^b \int_c^d \frac{\rho}{2}\left[F(x,y)+F(x,c+d-y)\right] g(x,y)\,dydx \\
&\quad +\int_c^d \int_a^b \frac{1-\rho}{2}\left[F(x,y)+F(a+b-x,y)\right] g(x,y)\,dxdy \\
&= \int_a^b \int_c^d \frac{\rho}{2}\left[F\left(x,\frac{d-y}{d-c}c+\frac{y-c}{d-c}d\right)\right. \\
&\quad \left. +F\left(x,\frac{y-c}{d-c}c+\frac{d-y}{d-c}d\right)\right] g(x,y)\,dydx \\
&\quad +\int_c^d \int_a^b \frac{1-\rho}{2}\left[F\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b,y\right)\right. \\
&\quad \left. +F\left(\frac{x-a}{b-a}a+\frac{b-x}{b-a}b,y\right)\right] g(x,y)\,dxdy \\
&\leq \int_a^b \int_c^d \frac{\rho}{2}\left[\frac{d-y}{d-c}F(x,c)\right. \\
&\quad \left. +\frac{y-c}{d-c}F(x,d)+\frac{y-c}{d-c}F(x,c)+\frac{d-y}{d-c}F(x,d)\right] g(x,y)\,dydx \\
&\quad +\int_c^d \int_a^b \frac{1-\rho}{2}\left[\frac{b-x}{b-a}F(a,y)\right.
\end{aligned} \tag{11}
$$

$$+\frac{x-a}{b-a}F(b,y)+\frac{x-a}{b-a}F(a,y)+\frac{b-x}{b-a}F(b,y)\bigg]\,g(x,y)\,dxdy$$
$$=\int_a^b\int_c^d\left[\frac{\rho}{2}(F(x,c)+F(x,d))\right.$$
$$\left.+\frac{1-\rho}{2}(F(a,y)+F(b,y))\right]g(x,y)\,dydx.$$

Similarly, we have

$$\int_a^b\int_c^d\left[\frac{\rho}{2}(F(x,c)+F(x,d))\right.$$
$$\left.+\frac{1-\rho}{2}(F(a,y)+F(b,y))\right]g(x,y)\,dydx$$
$$\leq\int_a^b\int_c^d\left[\frac{\rho}{4}(F(a,c)+F(b,c)+F(a,d)+F(b,d))\right. \tag{12}$$
$$\left.+\frac{1-\rho}{4}(F(a,c)+F(a,d)+F(b,c)+F(b,d))\right]g(x,y)\,dydx$$
$$=\frac{1}{4}[F(a,c)+F(a,d)+F(b,c)+F(b,d)]\int_a^b\int_c^d g(x,y)\,dydx.$$

Combining (10)-(12), we get (7).

If in (7) we choose $F(x,y)=xy$ and $g(x,y)\equiv 1$ $((x,y)\in\Delta)$, then the inequality (7) becomes an equality, which shows that the inequality (7) is sharp. This completes the proof.

Remark 1. Let f and w be defined as in Theorem 1. If we choose $\gamma=\rho=1$, $F(x,y)=\frac{f(x)}{d-c}$ and $g(x,y)=w(x)$ $((x,y)\in\Delta)$, then Theorem 1 reduces to Theorem A.

Remark 2. In Theorem 1, if we choose $\gamma=\rho=\frac{1}{2}$ and $g(x,y)\equiv\frac{1}{(b-a)(d-c)}$ $((x,y)\in\Delta)$, then Theorem 1 reduces to a result established by Dragomir [4, Theorem 1].

Theorem 2. *Let F and g be defined as in Theorem 1 and let $H:[0,1]^2\to R$ be defined by*

$$H(t,s):=\int_a^b\int_c^d F\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)g(x,y)\,dydx. \tag{13}$$

Then:

(a) *The function* H *is convex on the co-ordinates on* $[0,1]^2$.

(b) *The function* H *is increasing on the co-ordinates on* $[0,1]^2$,

$$\sup_{(t,s)\in[0,1]^2} H(t,s) = H(1,1) = \int_a^b \int_c^d F(x,y)\, g(x,y)\, dydx$$

and

$$\inf_{(t,s)\in[0,1]^2} H(t,s) = H(0,0) = F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d g(x,y)\, dydx.$$

Proof. (a) Fix $s \in [0,1]$. Since F is convex on the co-ordinates on Δ and g is nonnegative on Δ, we have for t_1, $t_2 \in [0,1]$ and α, $\beta \geq 0$ with $\alpha + \beta = 1$ that

$$\begin{aligned}
&H(\alpha t_1 + \beta t_2, s) \\
&= \int_a^b \int_c^d F\bigg((\alpha t_1 + \beta t_2)\, x + (1 - \alpha t_1 - \beta t_2) \frac{a+b}{2}, \\
&\qquad sy + (1-s) \frac{c+d}{2} \bigg)\, g(x,y)\, dydx \\
&= \int_a^b \int_c^d F\bigg(\alpha \left(t_1 x + (1-t_1) \frac{a+b}{2} \right) \\
&\qquad + \beta \left(t_2 x + (1-t_2) \frac{a+b}{2} \right), sy + (1-s) \frac{c+d}{2} \bigg) \times g(x,y)\, dydx \\
&\leq \int_a^b \int_c^d \bigg[\alpha F\left(t_1 x + (1-t_1) \frac{a+b}{2}, sy + (1-s) \frac{c+d}{2} \right) \\
&\qquad + \beta F\left(t_2 x + (1-t_2) \frac{a+b}{2}, sy + (1-s) \frac{c+d}{2} \right) \bigg]\, g(x,y)\, dydx \\
&= \alpha H(t_1, s) + \beta H(t_2, s).
\end{aligned}$$

Similarly, if t is fixed in $[0,1]$, then for s_1, $s_2 \in [0,1]$ and α, $\beta \geq 0$ with $\alpha + \beta = 1$, we have

$$H(t, \alpha s_1 + \beta s_2) \leq \alpha H(t, s_1) + \beta H(t, s_2)$$

and the statement is proved.

(b) Since F is convex on the co-ordinates on Δ and g is nonnegative, integrable and symmetric on the co-ordinates on Δ, using the identities (9) and (10), we have, for all $(t,s) \in [0,1]^2$,

$$\begin{aligned}
&H(t,s)\\
&=\int_a^b\int_c^d \frac{1}{2}\left[F\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)+\right.\\
&\quad\left.F\left(tx+(1-t)\frac{a+b}{2},s(c+d-y)+(1-s)\frac{c+d}{2}\right)\right]g(x,y)\,dydx \qquad (14)\\
&\geq \int_a^b\int_c^d F\left(tx+(1-t)\frac{a+b}{2},\frac{c+d}{2}\right)g(x,y)\,dydx\\
&=H(t,0)
\end{aligned}$$

and

$$\begin{aligned}
&H(t,s)\\
&=\int_c^d\int_a^b \frac{1}{2}\left[F\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right.\\
&\quad\left.+F\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]g(x,y)\,dxdy \qquad (15)\\
&\geq \int_c^d\int_a^b F\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)g(x,y)\,dxdy\\
&=H(0,s).
\end{aligned}$$

If $0 \leq t_1 < t_2 \leq 1$ and $0 \leq s_1 < s_2 \leq 1$, then, for all $(t,s) \in [0,1]^2$, it follows from the convexity of H on the co-ordinates on $[0,1]^2$, (14) and (15) that

$$\frac{H(t_2,s)-H(t_1,s)}{t_2-t_1} \geq \frac{H(t_1,s)-H(0,s)}{t_1-0} \geq 0$$

and

$$\frac{H(t,s_2)-H(t,s_1)}{s_2-s_1} \geq \frac{H(t,s_1)-H(t,0)}{s_1-0} \geq 0$$

which show that H is increasing on the co-ordinates on $[0,1]^2$. Hence

$$\sup_{(t,s)\in[0,1]^2} H(t,s) = H(1,1) = \int_a^b\int_c^d F(x,y)\,g(x,y)\,dydx$$

and

$$\inf_{(t,s)\in[0,1]^2} H(t,s) = H(0,0) = F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d g(x,y)\,dydx.$$

This completes the proof.

Remark 3. In Theorem 2, if we choose $g(x,y) \equiv \frac{1}{(b-a)(d-c)}$, then Theorem 2 reduces to a result established by Dragomir [4, Theorem 2].

Remark 4. Let f and w be defined as in Theorem B. In Theorem 2, if we choose $F(x,y) = \frac{f(x)}{d-c}$ and $g(x,y) = w(x)$ $((x,y) \in \Delta)$, then $H(t,s)$ reduces to (3).

Theorem 3. *Let g be defined as in Theorem 1 and let $F : \Delta \to R$ be convex. Then:*

(a) *H is convex on $[0,1]^2$ where H is defined as in (13).*

(b) *Define $h : [0,1] \to R$ by $h(t) := H(t,t)$. Then h is convex, increasing on $[0,1]$,*

$$\sup_{t\in[0,1]} h(t) = h(1) = \int_a^b \int_c^d F(x,y)\, g(x,y)\, dydx \tag{16}$$

and

$$\inf_{t\in[0,1]} h(t) = h(0) = F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d g(x,y)\, dydx. \tag{17}$$

Proof. (a) Since F is convex and g is nonnegative, we have for (t_1, s_1), $(t_2, s_2) \in [0,1]^2$ and α, $\beta \geq 0$ with $\alpha + \beta = 1$ that

$$\begin{aligned}
&H(\alpha(t_1,s_1) + \beta(t_2,s_2)) \\
&= H(\alpha t_1 + \beta t_2, \alpha s_1 + \beta s_2) \\
&= \int_a^b \int_c^d F\Bigg((\alpha t_1 + \beta t_2)x + (1 - (\alpha t_1 + \beta t_2))\frac{a+b}{2}, \\
&\quad (\alpha s_1 + \beta s_2)y + (1 - (\alpha s_1 + \beta s_2))\frac{c+d}{2}\Bigg) g(x,y)\, dydx \\
&= \int_a^b \int_c^d F\Bigg(\alpha\left(t_1 x + (1-t_1)\frac{a+b}{2}, s_1 y + (1-s_1)\frac{c+d}{2}\right) \\
&\quad + \beta\left(t_2 x + (1-t_2)\frac{a+b}{2}, s_2 y + (1-s_2)\frac{c+d}{2}\right)\Bigg) g(x,y)\, dydx \\
&\leq \int_a^b \int_c^d \Bigg[\alpha F\left(t_1 x + (1-t_1)\frac{a+b}{2}, s_1 y + (1-s_1)\frac{c+d}{2}\right) \\
&\quad + \beta F\left(t_2 x + (1-t_2)\frac{a+b}{2}, s_2 y + (1-s_2)\frac{c+d}{2}\right)\Bigg] g(x,y)\, dydx \\
&= \alpha H(t_1,s_1) + \beta H(t_2,s_2),
\end{aligned}$$

which shows that H is convex on $[0,1]^2$.

(b) Let t_1, $t_2 \in [0,1]$ and α, $\beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned} & h(\alpha t_1 + \beta t_2) \\ = & H(\alpha(t_1, t_1) + \beta(t_2, t_2)) \\ \leq & \alpha H(t_1, t_1) + \beta H(t_2, t_2) \\ = & \alpha h(t_1) + \beta h(t_2) \end{aligned}$$

which shows that h is convex on $[0,1]$. By Theorem 2, we have that, for $0 \leq t_1 < t_2 \leq 1$,

$$h(t_1) = H(t_1, t_1) \leq H(t_2, t_1) \leq H(t_2, t_2) = h(t_2)$$

which show that h is increasing on $[0,1]$. Since h is increasing on $[0,1]$, (16) and (17) hold. This completes the proof.

Remark 5. In Theorem 3, if we choose $g(x,y) \equiv \frac{1}{(b-a)(d-c)}$, then Theorem 10 reduces to a result established by Dragomir [4, Theorem 3].

Theorem 4. *Let F and g be defined as in Theorem 1 and let $K : [0,1]^2 \to R$ with*

$$K(t,s) := \int_a^b \int_a^b \int_c^d \int_c^d F(tx + (1-t)y, sz + (1-s)u)\, g(x,z)\, g(y,u)\, dzdudxdy. \tag{18}$$

Then:

(a) *K is symmetric on the co-ordinates on $[0,1]^2$ and convex on the co-ordinates on $[0,1]^2$.*

(b) *$K(\cdot, s)$ is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$ for all $s \in [0,1]$, $K(t, \cdot)$ is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$ for all $t \in [0,1]$,*

$$\begin{aligned} & \inf_{(t,s)\in[0,1]^2} K(t,s) = K\left(\frac{1}{2}, \frac{1}{2}\right) \\ = & \int_a^b \int_a^b \int_c^d \int_c^d F\left(\frac{x+y}{2}, \frac{z+u}{2}\right) g(x,z)\, g(y,u)\, dzdudxdy \end{aligned} \tag{19}$$

and

$$\begin{aligned} & \sup_{(t,s)\in[0,1]^2} K(t,s) = K(0,0) = K(1,1) \\ = & \int_a^b \int_c^d F(x,z)\, g(x,z)\, dzdx \cdot \int_a^b \int_c^d g(x,z)\, dzdx. \end{aligned} \tag{20}$$

(c) *For all* $(t,s)\in[0,1]^2$,

$$\begin{aligned}
&K(t,s)\\
&\geq \max\{H(t,s),H(1-t,s),H(t,1-s),H(1-t,1-s)\}\\
&\quad\times\int_a^b\int_c^d g(x,z)\,dzdx \qquad (21)\\
&\geq F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\cdot\left(\int_a^b\int_c^d g(x,y)\,dydx\right)^2
\end{aligned}$$

where H *is defined as in* (13).

Proof. (a) From the definition of K, it is obvious that K is symmetric on the co-ordinates on $[0,1]^2$. Using a similar argument as the proof of Theorem 2, we have that K is convex on the co-ordinates on $[0,1]^2$.

(b) Since F is convex on the co-ordinates on Δ and g is nonnegative on Δ, we have, for $(t,s)\in[0,1]^2$,

$$\begin{aligned}
&K(t,s)\\
&=\int_a^b\int_a^b\int_c^d\int_c^d \frac{1}{2}\left[F(tx+(1-t)y,sz+(1-s)u)+\right.\\
&\qquad\left.F(tx+(1-t)y,(1-s)z+su)\right]g(x,z)\,g(y,u)\,dzdudxdy \qquad (22)\\
&\geq\int_a^b\int_a^b\int_c^d\int_c^d F\left(tx+(1-t)y,\frac{z+u}{2}\right)g(x,z)\,g(y,u)\,dzdudxdy\\
&=K\left(t,\frac{1}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
&K(t,s)\\
&=\int_a^b\int_a^b\int_c^d\int_c^d \frac{1}{2}\left[F(tx+(1-t)y,sz+(1-s)u)+\right.\\
&\qquad\left.F((1-t)x+ty,sz+(1-s)u)\right]g(x,z)\,g(y,u)\,dzdudxdy \qquad (23)\\
&\geq\int_a^b\int_a^b\int_c^d\int_c^d F\left(\frac{x+y}{2},sz+(1-s)u\right)g(x,z)\,g(y,u)\,dzdudxdy\\
&=K\left(\frac{1}{2},s\right).
\end{aligned}$$

If $0 \le s_1 < s_2 \le \frac{1}{2} \le s_3 < s_4 \le 1$ and $0 \le t_1 < t_2 \le \frac{1}{2} \le t_3 < t_4 \le 1$, then it follows from (23), (24) and the convexity of K on the co-ordinates on $[0,1]^2$ that

$$\frac{K(t,s_2)-K(t,s_1)}{s_2-s_1} \le \frac{K\left(t,\frac{1}{2}\right)-K(t,s_1)}{\frac{1}{2}-s_1} \le 0,\ t\in[0,1],$$

$$\frac{K(t,s_4)-K(t,s_3)}{s_4-s_3} \ge \frac{K(t,s_4)-K\left(t,\frac{1}{2}\right)}{s_4-\frac{1}{2}} \ge 0,\ t\in[0,1],$$

$$\frac{K(t_2,s)-K(t_1,s)}{t_2-t_1} \le \frac{K\left(\frac{1}{2},s\right)-K(t_1,s)}{\frac{1}{2}-t_1} \le 0,\ s\in[0,1],$$

and

$$\frac{K(t_4,s)-K(t_3,s)}{t_4-t_3} \ge \frac{K(t_4,s)-K\left(\frac{1}{2},s\right)}{t_4-\frac{1}{2}} \ge 0,\ s\in[0,1]$$

which show that $K(t,\cdot)$, $K(\cdot,s)$ are decreasing on $\left[0,\frac{1}{2}\right]$ and $K(t,\cdot)$, $K(\cdot,s)$ are increasing on $\left[\frac{1}{2},1\right]$ for $(t,s)\in[0,1]^2$. Hence (19) and (20) hold.

(c) By the definition of g, we have the identities (8) and (9) again, hence we get, for $(t,s)\in[0,1]^2$

$$\begin{aligned}
K(t,s) &= \int_a^b\int_a^b\int_c^d\int_c^d \frac{1}{2}\left[F(tx+(1-t)y, sz+(1-s)u)+\right.\\
&\quad \left. F(tx+(1-t)y, sz+(1-s)(c+d-u))\right] g(x,z)\\
&\quad g(y,u)\,dudzdxdy\\
&\ge \int_a^b\int_a^b\int_c^d\int_c^d F\left(tx+(1-t)y, sz+(1-s)\frac{c+d}{2}\right)\\
&\quad g(x,z)\,g(y,u)\,dudzdxdy\\
&= \int_a^b\int_a^b\int_c^d\int_c^d \frac{1}{2}\left[F\left(tx+(1-t)y, sz+(1-s)\frac{c+d}{2}\right)\right.\\
&\quad \left.+F\left(tx+(1-t)(a+b-y), sz+(1-s)\frac{c+d}{2}\right)\right]\\
&\quad g(x,z)\,g(y,u)\,dydudzdx\\
&\ge \int_a^b\int_c^d F\left(tx+(1-t)\frac{a+b}{2}, sz+(1-s)\frac{c+d}{2}\right)\\
&\quad g(x,z)\,dzdx\cdot\int_a^b\int_c^d g(x,z)\,dzdx\\
&= H(t,s)\cdot\int_a^b\int_c^d g(x,z)\,dzdx.
\end{aligned} \tag{24}$$

From the conclusion in (a), it follows that

$$K(t,s) = K(1-t,s) = K(t,1-s) = K(1-t,1-s) \tag{25}$$

for all $(t,s) \in [0,1]^2$. Therefore, by the inequality (24), the identity (25) and the conclusion of Theorem 2, we deduce the inequality (21). This comletes the proof.

Remark 6. In Theorem 4, if we choose $g(x,y) \equiv \frac{1}{(b-a)(d-c)}$, then Theorem 4 reduces to a result established by Dragomir [4, Theorem 4].

Remark 7. Let f and w be defined as in Theorem C. In Theorem 4, if we choose $F(x,y) = \frac{f(x)}{(d-c)^2 \int_a^b w(x)dx}$ and $g(x,y) = w(x)$ $((x,y) \in \Delta)$, then Theorem 4 reduces to Theorem C.

Theorem 5. *Let F and g be defined as in Theorem 3 and let $k : [0,1] \to R$ with $k(t) := K(t,t)$ where K is defined as in (18). Then:*

(a) *K is convex on $[0,1]^2$ and k is convex on $[0,1]$.*

(b) *k is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$,*

$$\begin{aligned}\sup_{t\in[0,1]} k(t) &= k(0) = k(1) \\ &= \int_a^b \int_c^d F(x,y)\, g(x,y)\, dydx \cdot \int_a^b \int_c^d g(x,y)\, dydx\end{aligned}$$

and

$$\begin{aligned}\inf_{t\in[0,1]} k(t) &= k\left(\frac{1}{2}\right) \\ &= \int_a^b \int_a^b \int_c^d \int_c^d F\left(\frac{x+y}{2}, \frac{z+u}{2}\right) g(x,z)\, g(y,u)\, dzdudxdy.\end{aligned}$$

(c) *For all $t \in [0,1]$,*

$$k(t) \geq \max\{h(t), h(1-t)\} \cdot \int_a^b \int_c^d g(x,y)\, dydx$$

where h is defined as in Theorem 3.

Proof. The proof is similar to that of Theorem 3.

Remark 8. In Theorem 5, if we choose $g(x,y) \equiv \frac{1}{(b-a)(d-c)}$, then Theorem 5 reduces to a result established by Dragomir [4, Theorem 5].

References

1. J. L. Brenner and H. Alzer, Integral inequalities for concave functions with applications to special functions, *Proc. Roy. Soc. Edinburgh A*, **118** (1991), 173-192.

2. S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.*, **167** (1992), 49-56.

3. S. S. Dragomir, Y. J. Cho and S. S. Kim, Inequalities of Hadamard's type for Lipschitzian mappings and their applications, *J. Math. Anal. Appl.*, **245** (2000), 489-501.

4. S. S. Dragomir, On the Hadamard's inequality for convex functions of the co-ordinates in a rectangle from the plane, *Taiwanese J. Math.*, **5(4)** (2001), 775-788.

5. L. Fejér, Über die Fourierreihen, II, *Math. Naturwiss. Anz Ungar. Akad. Wiss.*, **24** (1906), 369-390. (In Hungarian).

6. J. Hadamard, Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann, *J. Math. Pures Appl.*, **58** (1893), 171-215.

7. K. C. Lee and K. L. Tseng, On a weighted generalization of Hadamard's inequality for G-convex functions, *Tamsui Oxford Journal of Math. Sci.*, **16(1)** (2000), 91-104.

8. M. Matić and J. Pecarić, Note on inequalities of Hadamard's type for Lipschitzian mappings, *Tamkang J. Math.*, **32(2)** (2001), 127-130.

9. C. E. M. Pecarce and J. Pecarić, On some inequalities of Brenner and Alzer for concave functions, *J. Math. Anal. Appl.*, **198** (1996), 282-288

10. K. L. Tseng, S. R. Hwang and S. S. Dragomir, On some new inequalities of Hermite-Hadamard-Fejér type involving convex functions, *RGMIA Research Report Collection*, **8(4)** (2005) Article 9. htpp://rgmia.vu.edu.au/

11. K. L. Tseng, G. S. Yang and S. Dragomir, On quasi convex functions and Hadamard's inequality, *RGMIA Research Report Collection*, **6(3)** (2003), Article 1. htpp://rgmia.vu.edu.au/

12. K. L. Tseng, G. S. Yang and S. Dragomir, Hadamard inequalities for Wright-Convex functions, *Demonstratio Mathematica*, **XXXVII(3)** (2004), 525-532.

13. G. S. Yang and M. C. Hong, A note on Hadamard's inequality, *Tamkang J. Math.*, **28(1)** (1997), 33-37.

14. G. S. Yang and K. L. Tseng, On certain integral inequalities related to Hermite-Hadamard inequalities, *J. Math. Anal. Appl.*, **239** (1999), 180-187.

15. G. S. Yang and K. L. Tseng, Inequalities of Hadamard's type for Lipschitzian mappings, *J. Math. Anal. Appl.*, **260** (2001), 230-238.

16. G. S. Yang and K. L. Tseng, On certain multiple integral inequalities related to Hermite-Hadamard inequalities, *Utilitas Math.* , **62** (2002), 131-142.

17. G. S. Yang and K. L. Tseng, Inequalities of Hermite-Hadamard-Fejér type for convex functions and Lipschitzian functions, *Taiwanese J. Math.*, **7(3)** (2003), 433-440.

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