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SOME PROPERTIES OF VARIABLE SOBOLEV CAPACITY*

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Abstract. This paper deals with relationship between singular measure and corresponding $p(x)$-capacity. It we have proved that the value of capacity of an arbitrary capacitor is completely determined by an absolutely continuous component of the measure, and the contribution of a singular part of the measure appears to be zero.

1. Introduction

In the beginning of the 90's Kovacik and Rakosnik [7] introduced variable exponent Lebesgue and Sobolev spaces. In fact, generalized Lebesgue and Sobolev spaces are special cases of so-called Orlicz-Musielak spaces, and in this form their investigation goes back a bit further, to [6, 8] and, see also [14]. Recently, developments in variable exponent Lebesgue and Sobolev spaces have great rapidly and these spaces have been used different areas such as differential equations, harmoic analysis, norm in inequalities etc. ([11], [12], [13]).

A mathematical application is the study of variational integrals with non-standard growth, see the paper [1]. It seems, however, that there is a good reason to study the particular spaces introduced in [7] and in recent years several papers have appeared along this line of investigation ; it is now known that generalized Lebesgue and Sobolev spaces satisfy several of the properties of their classical equivalents.

Sobolev capacity for fixed exponent has found a great number of uses, see for instance the monographs in ([2], [5], [10]). It was introduced into the study of variable exponent spaces in [3] and has been applied to the investigation of zero boundary values of Sobolev functions in [4]. In [3] we required the assumption $1 < ess \inf p(x) \leq ess \sup p(x) < \infty$ of the variable exponent p to guarantee

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that our set-function is indeed a Choquet capacity.Unfortunately, no information about the singular measure properties of variable Sobolev capacity has been found in literature.

The principal idea of this paper is the reconstruction of the permissible function, when the new function, remaining permissible, appears to be locally constant on the "almost whole" support of the singular measure. Countable semi additivity of the $p(x)$-capacity and its continuity on both increasing and decreasing families of sets allow to study the capacity of compact subsets of the bounded domain $\Omega \subset R^n$.

Let μ be a locally finite non-degenerate Borel measure on Ω and $p : \Omega \longrightarrow [1, \infty)$ be a measurable function (called the variable exponent on). We define $p^- = ess\inf_{x\in\Omega} p(x) p^+ = ess\sup_{x\in\Omega} p(x)$.

We also define that the variable exponent Sobolev space $L^{1,p(x)}(\Omega, \mu)$ to consist of all measurable functions $u : \Omega \to R$ such that

$$\rho_{1,p(x),\mu}(\lambda \nabla u) = \int_{\Omega} |\lambda \nabla u(x)|^{p(x)} \, d\mu$$

for some $\lambda > 0$.

The functions $\rho_{1,p(x),\mu} : L^{1,p(x)}(\Omega, \mu) \to [0, \infty)$ is called the modular of the space $L^{1,p(x)}(\Omega, \mu)$. This space is a Banach space with the Luxemburg norm given by the formula

$$\|u\|_{1,p(x),\mu} = \inf \left\{ \lambda > 0 : \rho_{1,p(x),\mu} \left(\frac{u}{\lambda} \right) \leq 1 \right\}. \tag{1.1}$$

We will call a pair of disjoint compacts G^0, G^1 as a capacitor in the domain Ω. For an arbitrary capacitor $G = (G^0, G^1)$, we will define a family of permissible functions

$$\begin{aligned} T(G) &= \{u \in C_0^1(\Omega) : 0 \leq u(x) \leq 1, \\ u(x) &= \text{ 0 in the neighbourhood } G^0, \\ u(x) &= \text{ 1 in the neighbourhood } G^1\} \end{aligned}$$

and corresponding to $p(x)$-capacity

$$C_{p(x)}(G^0, G^1, \mu) := C_{p(x)}(G, \mu) = \inf_{u \in T(G)} \rho_{1,p(x),\mu}(u).$$

Lemma 1. (see, e.g., [7, 9]).
(i) *For any $p(x)$, $1 \leq p^- \leq p(x) \leq p^+ < \infty$, the following inequalities*

$$\|f\|_{p(x)}^{p^+} \leq \rho_p(f) \leq \|f\|_{p(x)}^{p^-}, \|f\|_{p(x)} \leq 1$$
$$\|f\|_{p(x)}^{p^-} \leq \rho_p(f) \leq \|f\|_{p(x)}^{p^+}, \|f\|_{p(x)} \geq 1$$

hold.

(ii) *The generalization of Hölder's inequality*

$$\left|\int_{\Omega} f(x)\varphi(x)dx\right| \leq c\,\|f\|_{p(x)}\,\|\varphi\|_{p'(x)}$$

holds, where $p'(x) = \frac{p(x)}{p(x)-1}$ *and the constant* $c > 0$ *depends only on* p.

Now, let us consider the relationship between singular measures and corresponding capacity in the case of $n \geq 2$. Assume the following notations. Let m_n - Lebesgue measure in R^n, and $|E|$ - Lebesgue measure of the set $E \subset R^n$. Given any finite regular Borel measure μ in the domain $\Omega \subset R^n$, we will define the Sobolev space $L_0^{1,p(x)}(\Omega, \mu)$ as a complement of the $C_0^1(\Omega)$ space by the norm (1.1). Sobolev spaces $L_0^{1,p(x)}(\Omega, \mu)$ are uniform convex with respect to norm (1.1).

For arbitrary capacitor G, the family of permissible functions is not empty, and its$p(x)$-capacity is finite. In the cases, when the measure μ is the Lebesgue measure m_n, we will omit a measure symbol in notations and write $L_0^{1,p(x)}(\Omega)$, $C_{p(x)}(G)$ and dx instead of dm_n. In the following, we will denote a finite absolute continuous relative to n-dimensional Lebesgue measure in Ω by the symbol ν and we will use the notation $d\nu = \nu dx$, $\nu \geq 0$, $v \in L_1(\Omega)$.

Definition. We will call a finite regular Borel measure σ in Ω ($\sigma(\Omega) > 0$) as $p(x)$ -trivial, if for any capacitor G and any finite absolutely continuous measure ν we have

$$C_{p(x)}(G, \nu + \sigma) = C_{p(x)}(G, \nu).$$

It is easy to show that any $p(x)$-trivial measure is singular.

Remark 1. Assume the contrary, i.e. let a $p(x)$-trivial absolutely continuous measure $\sigma, d\sigma = \omega dx, \omega > 0$ exist on the set $E \subset \Omega$, $|E| > 0$. Since almost any point of the set E is a point of a plane, there will be a spherical ring $R = B(x_0, 2\delta)/B(x_0, \delta)$ such that $|R \cap E| > \frac{1}{2}|R|$. Let $\mu = m_n + \sigma, G^0 = S(x_0, 2\delta), G^1 = S(x_0, \delta)$-surfaces of corresponding spheres. A sequence of functions $\{u_k\}$ allowable for the capacitor $G = (G^0, G^1)$ is such that when $k \to \infty$then $\rho_{1,p(x)}(u_k) \to C_{p(x)}(G)$.

By virtue of uniform convexity of Sobolev spaces $L_0^{1,p(x)}(\Omega, \|.\|_{1,p(x)})$ at $1 \leq p^- \leq p(x) \leq p^+ < \infty$, the sequence of functions $\{u_k\}$ converge in the space $L_0^{1,p(x)}(\Omega)$ to extreme function u_0 which gradient differs from zero everywhere in the ring (see [9] Theor. 1.8 , Theor. 1.10)

$$\int_R |\nabla u_k|^{p(x)}\,\omega dx \to 0$$

and, consequently , $\omega = 0$ almost everywhere in the ring R, which contradicts to our assumption about absolute continuity of the measure σ.

On the other hand, in the case of $n \geq 2$, not every singular measure appears to be $p(x)$-trivial.

Remark 2. Consider arbitrary segment $[\alpha, \beta] \subset R^2$, which length is $d \geq 1$. Let σ be a linear Lebesgue measure on $[\alpha, \beta] \subset R^2$ and $\sigma(E) = 0$ for any set $u(\alpha) = 0$, $u(\beta) = 1$. By using Hölder inequality (see Lemma 1), we obtain

$$1 \leq \int\limits_{[\alpha,\beta]} |\nabla u| \, d\sigma \leq d^{1-\frac{1}{p(x)}} \left\| \nabla u \right\|_{p(x),\sigma}$$

$$\leq d^{1-\frac{1}{p^+}} \max \left\{ \left(\rho_{1,p(x),\sigma}\right)^{\frac{1}{p^-}}, \left(\rho_{1,p(x),\sigma}\right)^{\frac{1}{p^+}} \right\}$$

Thus,

$$\rho_{1,p(x),\sigma} \geq d^{\left(\frac{1}{p^+}-1\right)p^-} = C_1 > 0 if \rho_{1,p(x),\sigma} \geq 1$$

$$\rho_{1,p(x),\sigma} \geq d^{\left(\frac{1}{p^+}-1\right)p^+} = C_2 > 0 if \rho_{1,p(x),\sigma} \leq 1$$

or

$$\int\limits_{[\alpha,\beta]} |\nabla u|^{p(x)} \, d\sigma \geq C > 0.$$

Hence, for any absolutely continuous measure ν and the capacitor G such that $\alpha \in G^0$, $\beta \in G^1$ the inequality $C_{p(x)}(G, \nu + \sigma) = C_{p(x)}(G, \nu) + C$ holds. Thus, at any $n \geq 2$, restriction of a linear measure on the arbitrary segment is not $p(x)$-trivial measure.

Let us show now that any measure support of which is contained in some set of a zero linear Hausdorff measure, is a $p(x)$-trivial. Principal idea lies in such reconstruction of a permissible function during which a new function appears to be locally constant on the "almost whole" support of the singular measure, while remaining permissible.

First, let us prove a lemma. For the fixed function $u : \Omega \to R$ and numbers $a \leq b$, we assume the following notations of sets of the level

$$l_a = \{x \in \Omega : u(x) < a\}, L_a = \{x \in \Omega : u(x) \leq a\}, \Omega_{a,b} = L_b \backslash l_a.$$

Lemma 2. *Let $p : \Omega \to [1, \infty)$ be a measurable bounded function called the variable exponent on Ω. Let $u \in C_0^1(\Omega)$, $M = \max_{x \in \Omega} u(x)$, $0 < a < b < M$, $1 \leq p^- \leq p(x) \leq p^+ < \infty$ and ν be a finite absolutely continuous measure in Ω. Then for any $\varepsilon > 0$ there are numbers $0 < a' < a < b < b' < M$ and a function $u^* \in C_0^1(\Omega)$ such that*

(i) $u^*(x) = u(x)$ *if* $x \in l_{a^/}$
(ii) $u^*(x) = u(x) - (b-a)$ *if* $x \in \Omega / L_{b^/}$
(iii) $u^*(x) = a$ *if* $x \in \Omega_{a,b}$
(iv) $\rho_{1,p(x),\nu}(u^*) \le \rho_{1,p(x),\nu}(u) + \varepsilon.$

Proof. Consider a sequence of functions $\{\lambda_k(t)\}$ which are smooth on the interval $(0,1)$ such that $\lambda_k(t) = 0$ in a neighbourhood of zero, $\lambda_k(t) = t$ in a neighbourhood of one and $\left|\lambda_k^/(t)\right| \le 1 + \frac{1}{k}$.

Let us select numbers$a^/$ and $b^/$so that $0 < a^/ < a < b < b^/ < M$, and construct the followiny sequence of permissible functions

$$u_k^*(x) = \begin{cases} u(x) & \text{if } x \in l_{a^/} \\ a - (a - a^/)\lambda_k \left(\dfrac{a - u(x)}{a - a^/}\right) & \text{if } x \in \Omega_{a^/,a} \\ a & \text{if } x \in \Omega_{a,b} \\ a - (b^/ - b)\lambda_k \left(\dfrac{u(x) - b}{b^/ - b}\right) & \text{if } x \in \Omega_{b,b^/} \\ u(x) - (b-a) & \text{if } x \in \Omega / L_{b^/} \end{cases}$$

Let us construct, $\nabla u_k^* = 0$ in a neighbourhood of the set $\Omega_{a,b}$, and $\nabla u_k^* = \nabla u$ in a neighbourhood of the level sets, on which $u = a^/$ and $u = b^/$. Consequently, the functions $u_k^* \in C_0^1(\Omega)$ and conditions (i)-(iii) hold true for those. Because the inequality $|\nabla u_k^*| \le \left(1 + \frac{1}{k}\right) |\nabla u|$ takes place everywhere in Ω, then for sufficiently large values of k

$$\rho_{1,p(x),\nu}(u_k^*) \le \rho_{1,p(x),\nu}(u) + \varepsilon.$$

It is seen from proof of the lemma that numbers $a^/$ and $b^/$can be chosen as arbitrarily close to a and b, correspondingly. If $a = 0$ or $b = 0$, then it is sufficient to reconstruct the function from one side of the corresponding set of the level. Finity of function u is required for finiteness of Sobolev norm for any absolutely continuous measure.

2. Main Results

Theorem 3. *Let $p : \Omega \to [1, \infty)$ be a measurable bounded function called the variable exponent on Ω , $1 \le p^- \le p(x) \le p^+ < \infty$ and $E \subset \Omega$ be a set of the zero linear Hausdorff measure. Then, any finite regular Borel measure, support of which is in the set E, is $p(x)$-trivial.*

Proof. Let ν be an absolutely continuous measure in Ω, and σ be a singular measure with a support in the set E. Consider a capacitor $G \subset E$ and an arbitrary permissible function $u \in T(G)$. We will show that for any $\varepsilon > 0$ a permissible function $u^* \in T(G)$ exists such that

$$\rho_{1,p(x),\nu}(u^*) \leq \rho_{1,p(x),\nu}(u) + \varepsilon.$$

Since σ is a finite regular Borel measure which support is in the set E, for arbitrary $\varepsilon_1 > 0$, such compact $G \subset E$ exists that $\sigma(E/G) < \varepsilon_1$. Set E has the linear Hausdorff measure , and it can be covered with such a sequence of open spheres $\{B_i(r_i)\}$ that $\sum_{i=1}^{\infty} r_i < \varepsilon_1$, at the same time compact G will be covered with a finite collection of spheres. Consequently, a finite system of spheres $B_1(r_1)$,...,$B_k(r_k)$ can be found such that $\sum_{i=1}^{k} r_i < \varepsilon_1$ and for the set $F = E/\left(\bigcup_{i=1}^{k} B_i(r_i)\right)$, the estimate $\sigma(F) < \varepsilon$ holds.

Denote $c = \max_{x\in\Omega} |\nabla u(x)|$, $A_i = \inf_{x\in B_i} u(x)$, $B_i = \sup_{x\in B_i} u(x)$. We have

$$\sum_{i=1}^{k} (B_i - A_i) < 2c\sum_{i=1}^{k} r_i < 2c\varepsilon_1.$$

Let I^* be an interior of the set $\bigcup_{i=1}^{k} (A_i, B_i)$. Then $I^* = \bigcup_{j=1}^{m} (a_i, b_i), m \leq k$ and here the intervals (a_i, b_i) are of positive distance one from another.

By reconstructing function u, as in the lemma, in the neighbourhood of a set of the level Ω_{a_1,b_1}, we will construct function a $u_1 \in C_0^1(\Omega)$ such that $u_1 = const$ on the set Ω_{a_1,b_1} and

$$\rho_{1,p(x),\nu}(u_1) \leq \rho_{1,p(x),\nu}(u) + \frac{\varepsilon_1}{2}.$$

By continuing the reconstruction to the function u_{j-1} we will construct a function $u_j = const$ on the set Ω_{a_j,b_j}such that

$$\rho_{1,p(x),\nu}(u_j) \leq \rho_{1,p(x),\nu}(u_{j-1}) + \frac{\varepsilon_1}{2^j} \ \ (j = 1, ..., m).$$

In consequence we will obtain a function $u_m \in C_0^1(\Omega)$ which is constant on every set Ω_{a_j,b_j}, and, hence , on any sphere $B_1(r_1)$,...,$B_k(r_k)$, and here

$$\rho_{1,p(x),\nu}(u_m) \leq \rho_{1,p(x),\nu}(u) + \varepsilon_1$$

$|\nabla u_m| \leq 2|\nabla u| \leq 2c$ holds everywhere in Ω and $|\nabla u_m| = 0$ on the set $\bigcup_{i=1}^{k} B_i(r_i)$, therefore

$$\int_{\Omega} |\nabla u_m|^{p(x)} d\sigma = \int_{F} |\nabla u_m|^{p(x)} d\sigma \leq (2c)^{p^+} \varepsilon_1.$$

We have $u_m = 0$ in a neighbourhood of the compact G^0 and $u = 1 - \delta$ in the neighbourhood of a compact G^1, where

$$\delta \leq \sum_{j=1}^{m}(a_j - b_j) < (2c)^{p^+}\,\varepsilon_1.$$

When $\varepsilon_1 < \frac{1}{(2c)^{p^+}}$, the function $u^* = \frac{u_m}{1-\delta}$ will be permissible for the capacitor G and

$$\int_\Omega |\nabla u^*|^{p(x)}\,d(\nu+\sigma) \leq (1-\delta)^{-p^+}\left(\int_\Omega |\nabla u|^{p(x)}\,d\nu + (2c)^{p^+}\,\varepsilon_1\right).$$

For any $\varepsilon > 0$ we can choose$\varepsilon_1 > 0$ such that the inequality

$$C_{p(x)}(G,\nu+\sigma) \leq \rho_{1,p(x),\nu+\sigma}(u^*) \leq \rho_{1,p(x),\nu}(u) + \varepsilon$$

holds.

By virtue of arbitrariness of the number $\varepsilon > 0$ and the permissible function u we obtain

$$C_{p(x)}(G,\nu+\sigma) = C_{p(x)}(G,\nu)$$

which means the $p(x)$ -triviality of the measure σ. The theorem is proved.

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