TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 3, pp. 635-648, June 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

SOME RESULTS ON EXACT CONTROLLABILITY OF PARABOLIC SYSTEMS

Zhongcheng Zhou, Ping Lin* and Hang Gao

Abstract. In this paper, we first discuss the exact controllability of a linear parabolic system governed by bilinear control as a coefficient like uy and then give a result of the locally exact null-controllability of a semilinear parabolic system. Our main method is based on the Implicit Function Theorem.

1. Introduction

Let $\Omega \subset R^n$, $n \in N$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. We will first consider the exact controllability (reachability) of the following bilinear control system

$$(1.1) \qquad \begin{cases} y_t - \Delta y = uy, & \text{in } \ Q_T, \\ y(x,t) = g(x), & \text{on } \ \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \ \Omega, \end{cases}$$

where $Q_T = \Omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$.

In the context of heat-transfer, the term uy describes the heat-exchange at point x at time t of the given substance according to Newton's Law (see e.g. [19] pp. 155-156). In this case, u is proportional to the heat-transfer coefficient, which depends on the substance at hand, its surface area and the environment. More generally, the

Received March 10, 2006, accepted June 1, 2006.
Communicated by Yingfei Yi.
2000 *Mathematics Subject Classification*: 35K05, 35K55, 93B05.
Key words and phrases: Parabolic system, Exact controllability, Bilinear control, Implicit function theorem.
This work was partially supported by the NSF of Chongqing City under grant CSTC 2005EB2048, the Youth Foundation of Southwest Normal University under grant SWNUQ2005031, and Natural Science Youth Foundation of Northeast Normal University under grant 20080102. and the NSF of China under grant 10471021.
*Corresponding author.

bilinear control u can be linked to the use of various means ("catalysts") that can accelerate or decelerate the reaction at hand.

Let us recall that, in its general form, it is said that the system at hand is approximately controllable in the given (linear phase-) space H at time $T > 0$ if, by selecting a suitable available ("traditionally" linear additive) control, it can be steered in H from any initial state into any neighborhood of any desirable target state at time T. In turn, it is exact controllable if it can be steered in H from any initial state within the given time-interval [0, T] to the given state exactly.

We refer to the early paper [12] by J.M. Ball, J.E. Mardsen and M. Slemrod on controllability of an abstract infinite dimensional bilinear system, which appears to be the first work on this subject in the framework of pde's. In [12], the global approximate controllability of the rod equation $u_{tt} + u_{xxxx} + k(t)u_{xx} = 0$ with hinged ends and of the wave equation $u_{tt} - u_{xx} + k(t)u = 0$ with Dirichlet boundary conditions, where k is control (the axial load), was shown making use of the nonharmonic Fourier series approach under the additional (nontraditional) assumption that all the modes in the initial data are active. We refer to just one additional paper, [13], on bilinear controllability for pde's, dealing with the simultaneous control of the rod equation and a simple Schr*ö*dinger equation. In [10], A.Y. Khapalov discussed the non-negative approximate controllability of a parabolic system with superlinear term governed by a bilinear control, and in [11], he also discussed the bilinear null-controllability of a parabolic system with the reaction term satisfying Newton's Law.

Our first main result is on the exact controllability (reachability) of (1.1).

Theorem 1.1. *Let $n = 1$, $\Omega = (0, 1)$, $Q_T = (0, 1) \times (0, T)$. Assume that $\theta \in W^{2,\infty}(\Omega)$, $\theta > 0$ in $\overline{\Omega}$, and $\Delta\theta \geq 0$ a.e. in Ω, $g(\cdot) \in C(\overline{\Omega})$, $g(x) > 0$ in $\overline{\Omega}$ and $\theta(x) = g(x)$, $\forall x \in \{0, 1\}$, then there exists a $T(\theta) > 0$ such that for any $y_0(x) \in L^2(\Omega)$, there exists a control $u \in L^2(Q_T)$ such that the corresponding solution to (1.1) in $C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ satisfies*

$$y(x, T) = \theta(x) \ a.e. \ in \ \Omega.$$

Next, we will consider the locally exact null-controllability of the following semilinear parabolic system governed in a nonempty subdomain $\omega \subset \Omega$ by a locally additive control u,

$$\begin{cases} y_t - \Delta y + f(y) = \chi_\omega u, & \text{in } Q_T, \\ y(x,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases} \tag{1.2}$$

where χ_ω is the characteristic function of ω, $u \in L^2(Q_T)$ is the control, $y_0 \in L^2(\Omega)$, $f(\cdot) \in C^1(R)$, $f(0) = 0$ and satisfies the following conditions

$$
(1.3) \qquad \begin{aligned} & |f(s_1) - f(s_2) - f'(0)(s_1 - s_2)| \\ & \le C(|s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|, \forall s_1, s_2 \in R, \end{aligned}
$$

where $C > 0$, $p > 1$ such that $p \le (n+4)/4$. It is known (e.g.[4]) that there exists $C > 0$ and $\eta > 0$ such that when

$$
(1.4) \qquad \|y_0\|_{L^2(\Omega)} + \|u\|_{L^2(Q_T)} \le \eta,
$$

the solution to (1.2) exists and is unique in $C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$. Moreover,

$$
(1.5) \qquad \begin{aligned} & \|y_1 - y_2\|_{L^\infty([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))} \\ & \le C(\|y_0^1 - y_0^2\|_{L^2(\Omega)} + \|u_1 - u_2\|_{L^2(Q_T)}), \end{aligned}
$$

for any pair of (y_0^i, u_i) as in (1.4), $y^i,\ i = 1, 2$ being the solution to (1.2) with those data.

The controllability of linear and semilinear parabolic systems with traditionally additive control has been analysed in several recent papers. Among them, let us mention [2,3,6,7,8,20] in what concerns null controllability and for approximate controllability, we refer to [2,4,5,6]. We note also that it is shown in [6] that for any $\beta > 2$, there exists functions $f = f(s)$ with $f(0) = 0$ and $f(s) \sim |s| log^\beta(1+|s|)$ as $|s| \to \infty$ such that (1.2) is not null controllable for all $T > 0$. For general systems of the form (1.2), the best one one can expect is the local null controllability i.e., the exact null controllability for initial data in a neigborhood of the origin.

Our second main result is as follows.

Theorem 1.2. *Let $f(\cdot) \in C^1(R)$ satisfies (1.3) and (1.4), then there exists $\rho > 0$ such that for all $y_0 \in L^2(\Omega)$, $\|y_0\|_{L^2(\Omega)} \le \rho$, there are $y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ and $u \in L^2(Q_T)$ which satisfies (1.2) and such that $y(\cdot, T) = 0$.*

The rest of this paper is organized as follows. Section 2 is devoted to giving some technical lemmas we will use below. In section 3 and section 4, we will prove theorems 1.1 and 1.2 by implicit function theorem.

2. Preliminary Lemmas

In order to prove the above theorems, we need the following results ([21]). Denote by $\mathcal{L}(X, Y)$ the space of continuous linear mappings from X into Y, where X and Y are Banach spaces. Let D be a dense subset of X.

Definition 2.1. The function $G : X \to Y$ is strongly $Fréchet$ differentiable at $a \in X$ if $\forall\ \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$
\|G(v) - G(x) - M(v - x)\| < \varepsilon \|v - x\|,
$$

whenever $x,\ v$ satisfies $\|x-a\|,\ \|v-a\| < \delta(\varepsilon)$. Here the linear mapping $M = G'(a) : D \to Y$ is the $Fréchet$ derivative of G at a.

Definition 2.2. The function $G : X \to Y$ is Hadamard differentiable at $a \in X$ if there exists $M \in \mathcal{L}(X, Y)$ such that, for any continuous function $\omega : [0,1] \to X$ for which $\omega'(0^+)$ exists and $\omega(0) = a$, the function $F = G \circ \omega$ is differentiable at 0^+, with $F'(0^+) = M\omega'(0^+)$, thus

$$G(\omega(t)) - G(\omega(0)) - M\omega'(0^+)t = o(t)\ as\ t \downarrow 0,$$

where M is the Hadamard derivative.

Definition 2.3. The function $G : X \to Y$ is strongly Hadamard differentiable at $a \in X$ if $F = G \circ \omega$ is strongly differentiable at 0^+, that is

$$\lim_{(t,\ u)\downarrow(0,\ 0)} (u-t)^{-1}[F(u) - F(t)] = F'(0^+).$$

Lemma 2.1. *Let $G : X \to Y$ have a $Gâteaux$ variation $\delta G(x; h)$ at all points in a convex neigborhood Ω of $x_0 \in X$ and all $h \in X$. If $\delta G(\cdot\ ;\ \cdot)$ is continous at $(x_0,\ 0)$, then G is strongly Hadamard differentiable at x_0.*

Definition 2.4. The function $G : X \to Y$ is restricted strongly Hadamard differentiable at a if the strongly Hadamard differentiable property holds, with ω restricted to be strongly differentiable.

Definition 2.5. The linear mapping $M : D \to Y$ is called approximately outer invertible if, for each $\mu \in (0, 1)$, there exists a bounded linear mapping $B_\mu : Y \to X$, and a bound Γ, depending on μ, for which

$$\|(B_\mu M B_\mu - B_\mu)y\| \le \mu\|B_\mu y\|\ and\ \|B_\mu y\| \le \tau_\mu\|y\|,\ \ \forall\ y \in Y,$$

then each B_μ is called an approximately outer invertible of M, with bound function $\Gamma(\cdot)$.

Lemma 2.2. *Let H_1 and H_2 be two real Hilbert spaces; Let $M : H_1 \to H_2$ be a compact linear operator, then M is approximately outer invertible.*

The following lemma will be the main tool in our proof of the above theorems.

Lemma 2.3. (Implicit Function Theorem) *Let X and Y be real Banach spaces, with $a \in X$. Let S be a closed convex cone in Y. Let the function $G : X \to Y$ be restricted strongly Hadamard defferentiable at a. Let b:=G(a) and assume $b \in S$. Let the Hadamard derivative $M = G'(a) : X \to Y$ be bounded linear* *with approximate outer inverse B_μ and bound function $\Gamma(\mu) = k_0\mu^{-\gamma}$, with $\gamma < 1$. Then for sufficiently small μ, whenever c satisfies $-[G(a) + G'(a)c] \in S$, and $\|c\| = 1$, there exists a solution $x = a + tc + \eta(t) \in X$ to $-G(x) \in S$, valid for all sufficiently small $t < 0$, with $x \neq a$. With an appropriate choice of $\mu = \mu(t) \downarrow 0$ as $t \downarrow 0$, $\|\eta(t)\|_{\mu(t)=o(t)}$ as $t \downarrow 0$.*

Lemma 2.4. [6] *For the system*

$$\begin{cases} q_t - \Delta q + a(x,t)q = \chi_\omega v(x,t), & in \ \ Q_T, \\ q = 0, & on \ \Sigma_T, \\ q(x,0) = q_0(x), & in \ \ \Omega, \end{cases} \tag{2.1}$$

we have, for any $q_0 \in L^2(\Omega)$, $a \in L^\infty(Q_T)$, there exists a control $v \in L^\infty(Q_T)$ such that the corresponding solution to (2.1) satisfies

$$q(\cdot, T) = 0.$$

Moreover,

$$\|v\|_{L^2(Q_T)} \leq C(T, \|a\|_{L^\infty(Q_T)})\|q_0\|_{L^2(\Omega)}.$$

3. Proof of Theorem 1.1

Proof.

Step 1. Let $z = y - \theta(x)$, $z_0(x) = y_0(x) - \theta(x)$, from (1.1), we have z satisfies

$$\begin{cases} z_t - z_{xx} = u(z + \theta(x)) + \theta_{xx}, & \text{in } Q_T, \\ z|_{x=0,1} = 0, & \\ z(x,0) = z_0(x), & \text{in } \ \Omega. \end{cases} \tag{3.1}$$

It is known (e.g.[9]) that for any $T > 0$, system (3.1) admits a unique solution in $C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$.

In order to prove theorem 1.1, it is sufficient to prove that there exists a $T(\theta) > 0$ such that (3.1) is exact null controllable.

Notice that $\theta \in W^{2,\,\infty}(\Omega)$, we have $\theta \in C(\overline{\Omega})$ by Sobolev Embedding Theorem. Hence, $\frac{\theta_{xx}}{\theta} \in L^\infty(\Omega)$. Define the mapping $G : L^2(\Omega) \times L^2(Q_T) \to L^2(\Omega)$, $G(z_0,\ u) = z(x,T)$, where z is the solution to (3.1). We will first prove that G is strongly Hadamard differentiable at $(0, -\frac{\theta_{xx}}{\theta})$.

By Lemma 2.1, we are sufficient to prove G has a $G\hat{a}teaux$ variation $\delta G((z_0,\ u); (h_0,\ v))$ at all points $(z_0,\ u) \in L^2(\Omega) \times L^2(Q_T)$ and all $(h_0,\ v) \in L^2(\Omega) \times L^2(Q_T)$ and $\delta G(\cdot\ ;\ \cdot)$ is continous at $((0, -\frac{\theta_{xx}}{\theta}); (0,0))$.

In fact, for any $(z_0,\ u)\in L^2(\Omega)\times L^2(Q_T)$, any $(h_0,\ v)\in L^2(\Omega)\times L^2(Q_T)$ and $\varepsilon>0$, let Z^ε and Z satisfy the following systems

$$\begin{cases} Z^\varepsilon_t - Z^\varepsilon_{xx} = (u+\varepsilon v)(Z^\varepsilon+\theta(x))+\theta_{xx}, & \text{in } \ Q_T,\\ Z^\varepsilon|_{x=0,1}=0, & \\ Z^\varepsilon(x,0)=z_0+\varepsilon h_0(x), & \text{in } \ \Omega, \end{cases} \tag{3.2}$$

$$\begin{cases} Z_t - Z_{xx} = u(Z+\theta(x))+\theta_{xx}, & \text{in } \ Q_T,\\ Z|_{x=0,1}=0, & \\ Z(x,0)=z_0, & \text{in } \ \Omega, \end{cases} \tag{3.3}$$

respectively.
Let $w^\varepsilon(x,t)=(Z^\varepsilon-Z)/\varepsilon$, from (3.2) and (3.3), we have

$$\begin{cases} w^\varepsilon_t - w^\varepsilon_{xx} = uw^\varepsilon+v(Z^\varepsilon+\theta(x)), & \text{in } \ Q_T,\\ w^\varepsilon|_{x=0,1}=0, & \\ w^\varepsilon(x,0)=h_0(x), & \text{in } \ \Omega. \end{cases} \tag{3.4}$$

Let $V_2(Q_T)=C([0,T];L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega))$ with the norm

$$\|y\|_{V_2(Q_T)}=\|y\|_{L^\infty([0,T];L^2(\Omega))}+\|y\|_{L^2(0,T;H^1_0(\Omega))},$$

from (3.3) and (3.4), it is easily seen that (see e.g. [9])

$$\|w^\varepsilon\|_{V_2(Q_T)}\le C, \tag{3.5}$$

where C is independent of ε.
From (3.5), we have

$$w_\varepsilon \to w \quad \begin{array}{l}\text{strongly in } L^2(Q_T),\\ \text{weakly in } L^2(0,T;H^1(\Omega)) \text{ and}\\ \text{weakly star in } L^\infty(0,T;L^2(\Omega)),\end{array} \tag{3.6}$$

where w satisfies

$$\begin{cases} w_t - w_{xx} = uw+v(Z+\theta(x)), & \text{in } \ Q_T,\\ w|_{x=0,1}=0, & \\ w(x,0)=h_0(x), & \text{in } \ \Omega, \end{cases} \tag{3.7}$$

where Z is the solution to (3.3).

It is easily seen that the solution w to (3.7) satisfies $\delta G((z_0,\ u);(h_0,\ v)) = w(x,\ T)$. Let $\{(z_0^k, u^k)\}$ and $\{(h_0^k, v^k)\} \in L^2(\Omega)\times L^2(Q_T)$ satisfy

$$(z_0^k, u^k) \to (0, -\frac{\theta_{xx}}{\theta}) \quad \text{in } L^2(\Omega)\times L^2(Q_T), \tag{3.8}$$

$$(h_0^k, v^k) \to (0,0) \quad \text{in } L^2(\Omega)\times L^2(Q_T), \tag{3.8}$$

then we have $\delta G((z_0^k,\ u^k);(h_0^k,\ v^k)) = w^k(x,\ T)$, where w^k, Z^k satisfy the following systems

$$\begin{cases} w_t^k - w_{xx}^k = u^k w^k + v^k(Z^k+\theta(x)), & \text{in } \ Q_T, \\ w^k|_{x=0,1} = 0, & \\ w^k(x,0) = h_0^k(x), & \text{in } \ \Omega, \end{cases} \tag{3.10}$$

$$\begin{cases} Z_t^k - Z_{xx}^k = u^k(Z^k+\theta(x)) + \theta_{xx}, & \text{in } \ Q_T, \\ Z^k|_{x=0,1} = 0, & \\ Z^k(x,0) = z_0^k, & \text{in } \ \Omega, \end{cases} \tag{3.11}$$

respectively, and $\delta G((0, -\frac{\theta_{xx}}{\theta}); (0\ 0)) = 0$.

In view of $n = 1$, we have $V_2(Q_T) \hookrightarrow L^6(Q_T)$. By (3.8), (3.9) and (3.11), we have

$$\|Z^k\|_{L^6(Q_T)} \le C_1\|Z^k\|_{V_2(Q_T)} \le C_2 \tag{3.12}$$

(where C_2 is independent of k). Furthermore, by (3.8), (3.9), (3.10) and Hölder inequality, we have (see e.g. [9])

$$\begin{aligned} &\|w^k\|_{V_2(Q_T)} \\ &\le C_3(\|h_0^k\|_{L^2(\Omega)} + \|v^k Z^k\|_{L^{\frac{6}{5}}(Q_T)} + \|v^k\theta\|_{L^2(Q_T)}) \\ &\le C_4(\|h_0^k\|_{L^2(\Omega)} + \|Z^k\|_{L^6(Q_T)}\|v^k\|_{L^2(Q_T)} + \|\theta\|_{L^\infty(\Omega)}\|v^k\|_{L^2(Q_T)}) \\ &\le C_4(\|h_0^k\|_{L^2(\Omega)} + C_2\|v^k\|_{L^2(Q_T)} + \|\theta\|_{L^\infty(\Omega)}\|v^k\|_{L^2(Q_T)}) \le C_5. \end{aligned} \tag{3.13}$$

where C_5 is independent of k.
By (3.8), (3.9) and (3.13), we have

$$\|w^k(x,\ T)\|_{L^2(Q_T)} \to 0,$$

this implies $\delta G(\cdot\,;\,\cdot)$ is continous at $((0,\ -\frac{\theta_{xx}}{\theta});(0\ 0))$.

Step 2. In this step, we will prove that the linear operator $G'(0,\ -\frac{\theta_{xx}}{\theta}) : L^2(\Omega)\times L^2(Q_T)\to L^2(\Omega)$ is compact. In fact, $G(0,-\frac{\theta_{xx}}{\theta})=0$. Let U be a bounded domain of $L^2(\Omega)\times L^2(Q_T)$, then for any $(h_0,\ v)\in U$, $G'(0,-\frac{\theta_{xx}}{\theta})(h_0,v)=w(x,\ T)$, where w is the solution to the following system,

$$\begin{cases} w_t - w_{xx} = -\dfrac{\theta_{xx}}{\theta}w + v\theta(x), & \text{in } \ Q_T,\\ w|_{x=0,1}, & \\ w(x,0)=h_0(x), & \text{in } \ \Omega. \end{cases} \tag{3.14}$$

Notice that $\frac{\theta_{xx}}{\theta}\in L^\infty(\Omega)$, (3.14) enjoys the following properties: for any $(h_0,v)\in U$, the corresponding solution $w_{h_0}^v$ to (3.14) satisfies

$$\begin{aligned} &\{w_{h_0}^v\} && \text{is bounded in } L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega)),\\ &\{\sqrt{t}\ (w_{h_0}^v)_t\} && \text{is bounded in } L^2(Q_T),\\ &\{\sqrt{t}w_{h_0}^v\} && \text{is bounded in } L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H^1(\Omega)). \end{aligned}$$

Hence, we can select a sequence $\{w_{h_0^k}^{v^k}\}\subset U$ such that

$$\begin{aligned} &w_{h_0^k}^{v^k}\to\widehat{w} && \text{weakly in } L^2(0,T;H^1(\Omega)) \text{ and}\\ & && \text{weak-star in } L^2(Q_T),\\ &\sqrt{t}\ (w_{h_0^k}^{v^k})_t\to\sqrt{t}\ \widehat{w}_t && \text{weakly in } L^2(Q_T),\\ &w_{h_0^k}^{v^k}(t)\to\widehat{w}(t) && \text{strongly in } L^2(\Omega) \text{ and}\\ & && \text{uniformly in } L^2(\Omega) \text{ on every compact interval } [\delta,T]. \end{aligned}$$

This means that $G'(0,-\frac{\theta_{xx}}{\theta}) : L^2(\Omega)\times L^2(Q_T)\to L^2(\Omega)$ is compact.

Step 3. In this step, we will prove that the system (3.14) is exact null controllable, that is, for any $h_0\in L^2(\Omega)$, there exists a control v_{h_0} such that the corresponding solution to (3.14) satisfies $w(x,T)=0$.

In fact, set

$$J(\varphi^0)=\frac{1}{2}\int_{Q_T}\theta\varphi^2dxdt+\int_\Omega h_0(x)\varphi(x,0)dx,\ \forall\ \varphi^0\in L^2(\Omega),$$

where φ is the corresponding solution to the dual system of (3.14),

$$\begin{cases} \varphi_t+\varphi_{xx}-\dfrac{\theta_{xx}}{\theta}\varphi=0, & \text{in } Q_T,\\ \varphi|_{x=0,1}=0, & \\ \varphi(x,T)=\varphi^0(x), & \text{in } \ \Omega. \end{cases} \tag{3.15}$$

Similarly to the argument of [2], noticing that $\theta(x) \geq C > 0\ in\ \Omega$, we have, for any $h_0 \in L^2(\Omega)$, the functional J achieves its minimum at a unique point $\widehat{\varphi}^0$. If we set $v_{h_0} = \widehat{\varphi}$, where $\widehat{\varphi}$ is the solution to (3.15) with $\varphi^0 = \widehat{\varphi}^0$, we have the correponding solution to (3.14) with $v_{h_0} = \widehat{\varphi}$ satisfies $w(x, T) = 0$.

In fact, $\forall\ \psi^0 \in L^2(\Omega)$ and $\forall\ \rho \in R$, we have

$$J(\widehat{\varphi}^0 + \rho\psi^0) \geq J(\widehat{\varphi}^0),$$

hence,

$$\int_{Q_T} \theta\hat{\varphi}\psi dxdt + \int_\Omega h_0(x)\psi(x, 0)dx = 0, \tag{3.16}$$

where ψ is the solution to (3.15) with $\varphi^0 = \psi^0$.
Multiplying (3.14) by ψ and (3.15) (with $\varphi = \psi,\ \varphi^0 = \psi^0$) by w and integrating by parts, we have

$$\int_{Q_T} \theta\hat{\varphi}\psi dxdt = \int_\Omega w(x, T)\psi^0(x)dx - \int_\Omega h_0(x)\psi(x, 0)dx. \tag{3.17}$$

By (3.16) and (3.17), we have

$$\int_\Omega w(x, T)\psi^0(x)dx = 0,\ \forall\ \psi^0 \in L^2(\Omega).$$

Hence,

$$w(x, T) = 0.$$

Furthermore, similar to [6], we have

$$\|v_{h_0}\|_{L^2(Q_T)} \leq C\|h_0\|_{L^2(\Omega)}.$$

Step 4. In this step, we will prove that the system (3.1) is locally exact null controllable by lemma 2.3.

In fact, $G(0, -\frac{\theta_{xx}}{\theta}) = 0$. From step 3, we have, for any $h_0 \in L^2(\Omega)$, there exists a control $v_{h_0} \in L^2(\Omega)$ such that $G'(0, -\frac{\theta_{xx}}{\theta})(h_0, v_{h_0}) = 0$. Let $c = (h_0, v_{h_0})$ (we may assume that $\|c\|_{L^2(\Omega)\times L^2(Q_T)} = 1$) and $S = \{0\} \subset L^2(\Omega)$, we have $-(G(0, -\frac{\theta_{xx}}{\theta}) + G'(0, -\frac{\theta_{xx}}{\theta})c) = 0 \in S$. From step 1 to step 3, G satisfies all the conditions of lemma 2.3, hence, by lemma 2.3, we have the equation $G(z_0,\ u) = 0$ has local solution,

$$z_0 = 0 + th_0 + \eta_1(t),$$

$$u = -\frac{\theta_{xx}}{\theta} + tv_{h_0} + \eta_2(t),$$

for $t > 0$ sufficiently small, $(z_0, u) \neq (0, -\frac{\theta_{xx}}{\theta})$, $\|\eta_1(t)\|_{L^2(\Omega)} = o(t)$ $(t \to 0)$, $\|\eta_2(t)\|_{L^2(Q_T)} = o(t)$ $(t \to 0)$, this implies the system (3.1) is locally null controllable, that is, there exists $\varepsilon_0 > 0$ sufficiently small such that for any $z_0 \in L^2(\Omega)$ satisfies $\|z_0\|_{L^2(\Omega)} \leq \varepsilon_0$, there exists a control $u \in L^2(Q_T)$ such that the corresponding solution to (3.1) satisfies

$$z(x, T) = 0.$$

Step 5. Multiplifying (3.1) by $z(x,t)$ and integrating over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\int\limits_\Omega z^2(x,t)dx + 2\int\limits_\Omega |z_x(x,t)|^2 dx \\
=\ & 2\int\limits_\Omega uz^2(x,t)dx + 2\int\limits_\Omega u\theta(x)z(x,t)dx + 2\int\limits_\Omega \theta_{xx}(x)z(x,t)dx \\
\leq\ & 2\int\limits_\Omega uz^2(x,t)dx + \|u\|_{L^\infty(Q_T)}\int\limits_\Omega z^2(x,t)dx \\
& +\|u\|_{L^\infty(Q_T)}\int\limits_\Omega \theta^2(x)dx + \int\limits_\Omega \theta_{xx}^2(x)dx + \int\limits_\Omega z^2(x,t)dx \\
=\ & \int\limits_\Omega (2u + \|u\|_{L^\infty(Q_T)} + 1)z^2(x,t)dx \\
& +\|u\|_{L^\infty(Q_T)}\int\limits_\Omega \theta^2(x)dx + \int\limits_\Omega \theta_{xx}^2(x)dx \\
\leq\ & esssup_{Q_t}(2u + \|u\|_{L^\infty(Q_T)} + 1)\int\limits_\Omega z^2(x,t)dx \\
& +\|u\|_{L^\infty(Q_T)}\int\limits_\Omega \theta^2(x)dx + \int\limits_\Omega \theta_{xx}^2(x)dx.
\end{aligned}$$

Let $r(t) = \int\limits_\Omega z^2(x,t)dx$, we have

$$\frac{dr(t)}{dt} \leq esssup_{Q_t}(2u+\|u\|_{L^\infty(Q_T)}+1)r(t)+\|u\|_{L^\infty(Q_T)}\int\limits_\Omega \theta^2(x)dx+\int\limits_\Omega \theta_{xx}^2(x)dx.$$

Let $r(0) = \|z(\cdot,0)\|_{L^2(\Omega)}^2$, $u < 0$ be the constant , we have

$$\frac{dr(t)}{dt} \leq (u+1)r(t) + |u|\int\limits_\Omega \theta^2(x)dx + \int\limits_\Omega \theta_{xx}^2(x)dx.$$

Hence,

$$r(t) \le e^{(u+1)t}\|z(\cdot,0)\|^2_{L^2(\Omega)} + \int_0^t e^{(u+1)(t-\tau)}\left(|u|\int_\Omega \theta^2(x)dx + \int_\Omega \theta^2_{xx}(x)dx\right)d\tau$$
$$= e^{(u+1)t}\|z(\cdot,0)\|^2_{L^2(\Omega)} + \left(|u|\int_\Omega \theta^2(x)dx + \int_\Omega \theta^2_{xx}(x)dx\right)\int_0^t e^{(u+1)(t-\tau)}d\tau,$$

thus

$$\begin{aligned}\|z(\cdot,t)\|^2_{L^2(\Omega)} \le\ & e^{(u+1)t}\|z(\cdot,0)\|^2_{L^2(\Omega)} \\ & + \frac{e^{(u+1)t}-1}{u+1}\left(|u|\int_\Omega \theta^2(x)dx + \int_\Omega \theta^2_{xx}(x)dx\right) \\ =\ & e^{(u+1)t}\|z(\cdot,0)\|^2_{L^2(\Omega)} + \frac{|u|(e^{(u+1)t}-1)}{u+1}\int_\Omega \theta^2(x)dx \\ & + \frac{e^{(u+1)t}-1}{u+1}\int_\Omega \theta^2_{xx}(x)dx.\end{aligned}$$

Hence, given $T_1 > 0$, we can select the constant $u_1 < 0$ (u_1 depends on z_0) such that $|u_1|$ is sufficiently large, then there exists $M_1 > 0$ (M_1 depends on θ, but independent of z_0), such that $\|z(\cdot,\ T_1)\|_{L^2(\Omega)} \le M_1$. Furthermore, noticing that $\theta > 0$ and $\theta_{xx} \ge 0$, if we select $u = \frac{-\theta_{xx}}{\theta}$ in $(T_1,\ T_2)$, then for

$$\text{(3.18)} \qquad \begin{cases} z_t - z_{xx} = \dfrac{-\theta_{xx}}{\theta} z, & \text{in } \Omega \times (T_1,\ T_2), \\ z = 0, & \text{on } \partial\Omega \times (T_1,\ T_2), \\ z(x,T_1) = z(x,T_1), & \text{in } \ \Omega, \end{cases}$$

we have

$$\text{(3.19)} \qquad \|z(T_2)\|_{L^2(\Omega)} \le e^{-\lambda(T_2-T_1)}\|z(T_1)\|_{L^2(\Omega)} \le M_1 e^{-\lambda(T_2-T_1)},$$

where $\lambda > 0$ the first eigenfunction of $-\Delta$ in $H^1_0(\Omega)$.

Hence for any $\varepsilon_0 > 0$, there exists $T_2(\theta) > 0$ sufficiently large such that $\|z(T_2)\|_{L^2(\Omega)} \le \varepsilon_0$.

Then by step 4, for the following system,

$$\text{(3.20)} \qquad \begin{cases} z_t - z_{xx} = u(z+\theta(x)) + \theta_{xx}(x), & \text{in } \Omega \times (T_2, T_2+1), \\ z = 0, & \text{on } \partial\Omega \times (T_2, T_2+1), \\ z(x,T_2) = z(T_2), & \text{in } \ \Omega, \end{cases}$$

there exists a control $u_2 \in L^2(Q_T)$ such that the corresponding solution to (3.20) satisfies

$$z(x, T_2 + 1) = 0.$$

Thus let $T = T_2 + 1$, if we select u as follows

$$u = \begin{cases} 0, & \text{in } \Omega \times (0, T_1), \\ \dfrac{-\theta_{xx}}{\theta}, & \text{in } \Omega \times (T_1, T_2), \\ u_2, & (T_2, T), \end{cases}$$

then the corresponding solution to (3.1) satifies

$$z(x, T) = 0.$$

This completes the proof of theorem 1.1.

4. Proof of Theorem 1.2

The proof of theorem 1.2 is similar to that of theorem 1.1, we just give a sketch of it.

Proof. Define the mapping $G : L^2(\Omega) \times L^2(Q_T) \to L^2(\Omega)$, $G(y_0,\ u) = y(x, T)$, where y is the solution to (1.2).

With the similar argument in theorem 1.1, we have $\delta G((y_0,\ u);\ (h_0,\ v)) = p(x, T)$, where p is the solution to the following system.

$$\begin{cases} p_t - \Delta p + f'(Y(x, t, y_0, u))p = \chi_\omega v, & \text{in } Q_T, \\ p = 0, & \text{on } \Sigma_T, \\ p(x, 0) = h_0(x), & \text{in } \Omega, \end{cases} \tag{4.1}$$

where Y satisfies

$$\begin{cases} Y_t - \Delta Y + f(Y) = \chi_\omega u, & \text{in } Q_T, \\ Y = 0, & \text{on } \Sigma_T, \\ Y(x, 0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{4.2}$$

Similar to theorem 1.1, we can easily prove that $\delta G(\cdot\,;\,\cdot)$ is continous at $((0, 0); (0, 0))$ which implies G is strongly Hadamard differentiable at $(0, 0)$ and $G'(0, 0) : L^2(\Omega) \times L^2(Q_T) \to L^2(\Omega)$ is compact by lemma 2.1 and lemma 2.2. Hence lemma 2.3 and lemma 2.4 implies theorem 1.2.

References

1. V. Barbu, Controllability of parabolic and Navier-Stokes equations, *Sci. Math. Japonica*, **6** (2002), 143-211.
2. C. Fabre, J. P. Puel and E. Zuazua, Approximate controllability of the semilinear heat equation, *Proceeding of the Royal Society of Edinburgh*, **125A** (1995), 31-61.
3. E. Zuazua, Finite dimensional null controllability for the semilinear heat equation, *Math. Pure. Appl.*, **76** (1997), 237-264.
4. Lue de Teresa, Approximate controllability of a semilinear heat equation in R^N, *Siam. J. Control Optim.*, **36** (1998), 2128-2147.
5. E. Zuazua, Approximate controllability for semilinear heat equations with globally Lipschitz nonlinearities, *Control Cybernet.*, **28** (1999), 665-683.
6. E. Fernánez-Cara and E. Zuazua, Null and approximate controllability for weakly blowing-up semilinear heat equations, *Non Linéaire.*, **17** (2000), 583-616.
7. O. Yu. Imanuvilov, Controllability of Evolution Equations, *Mat. Sb.*, **186** (1995), 102-132.
8. E. Fernánez-Cara, Null controllability of the semilinear heat equation, *Esiam: Control Optim. Calc. Var.*, **2** (1997), 87-107.
9. O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Mono., 23, AMS, Providence RI, 1968.
10. A. Y. Khapalov, Controllability of the semilinear parabolic equation governed by a multiplicative control in the reaction term: a qualitative approach, *Siam J. Control Optim.*, **41** (2003), 1886-1990.
11. A. Y. Khapalov, On bilinear controllability of the parabolic equation with the reaction-diffusion term satisfying Newton's Law. *J. Comput. Appl. Math.*, **21** (2002), 1-23.
12. J. M. Ball, J. E. Mardsen and M. Slemrod, Controllability for distributed bilinear systems, *Siam J. Control Optim.*, **20** (1982), 575-597.
13. K. Kime, Simultaneous control of a rod equation and a simple Schrödinger equation, *Systems Control Lett.*, **24** (1995), 301-306.
14. A. Baciotti, *Local Stabilizability of Nonlinear Control Systems*, Series on Advances in Mathematics and Applied Sciences, vol 8, World Scientific, Singapore, 1992.
15. J. M. Ball and M. Slemrod, Feedback stabilization of semilinear control systems, *Appl. Math. Opt.*, **5** (1979), 169-179.
16. J. M. Ball and M. Slemrod, Nonharmonic Fourier series and the stabilization of distributed semi-linear control systems, *Comm. Pure. Appl. Math.*, **32** (1979), 555-587.
17. S. Müller, Strong convergence and arbitrarily slow decay of energy for a class of bilinear control problems, *J. Differential Equations*, **81** (1989), 50-67.

18. V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional System*, Academic Press, Boston, 1993.

19. A. N. Tichonov and A. A. Samarski, *Partial Differential Equations of Mathematical Physics*, Vol. 1, Holden-Day, 1964.

20. A. Doubova, E. Fernandez-cara and E. Zuazua, On the controllability of parabolic systems with a nonlinear term involving the state and the gradient, *Siam. J. Control Optim.*, **41** (2003), 798-819.

21. B. D. Craven and M. Z. Nashed, Generalized implicit function theorems when the derivative has no bounded inverse, *Nonlinear Analysis, Theory, Methods and Applications*, **6** (1982), 375-387.

Zhongcheng Zhou
School of Mathematics and Statistics,
Southwest University,
Chongqing 400715,
P. R. China
E-mail: zhouzc@swu.edu.cn

Ping Lin and Hang Gao
School of Mathematics and Statistics,
Northeast Normal University,
Changchun 130024,
P. R. China
E-mail: linp258@nenu.edu.cn