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BLOW-UP SOLUTIONS TO THE NONLINEAR SECOND ORDER DIFFERENTIAL EQUATION $u''(t) = u(t)^p(c_1 + c_2u'(t)^q)$ (I)

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Abstract. In this paper we study the following initial value problem for the nonlinear equation,

$$\begin{cases} u''(t) = u(t)^p(c_1 + c_2u'(t)^q),\ p,\ q \geq 1,\ c_1 \geq 0, c_2 \geq 0, \\ u(0) = u_0,\ u'(0) = u_1. \end{cases}$$

We are interested in the properties of solutions of the above problem. We have found blow-up phenomena and obtained some results on blow-up rates, blow-up constants and life-spans.

0. Introduction

Consider the nonlinear equation

$$\begin{cases} u'' = u^p(c_1 + c_2\,(u'\,(t))^q), \\ u(0) = u_0,\ u'(0) = u_1, \end{cases}$$

where u^p and $(u')^q$ are well-definded functions. We are interested in the properties of solutions of the problem, particularly in phenomena on blow-up, blow-up rates, blow-up constants and life-spans for $p \geq 1,\ q \geq 1,\ c_1 + c_2 > 0,\ c_1 \geq 0, c_2 \geq 0$.

To gain a rough estimate of the life-span of the solution for the initial value problem (0.1) below, we reconsider the existence of the solutions of the nonlinear equation:

$$\text{(0.1)} \qquad \begin{cases} u''(t) = u(t)^p(c_1 + c_2u'(t)^q),\ p \geq 1,\ q \geq 1,\ c_1^2 + c_2^2 \neq 0, \\ u(0) = u_0, u'(0) = u_1. \end{cases}$$

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There are more discussion which concern nonlinear differential equation in [3] and [4].

For $p \in \mathbb{Q}$, we say that p is odd (even, respectively) if $p = r/s$, $r \in \mathbb{N}$, $s \in 2\mathbb{N}+1, (r,s) = 1$ (common factor) and r is odd (even, respectively). Define

$$T_1^* = \min\left\{\frac{N-|u_1|}{K}, \frac{-|u_1| + \sqrt{u_1^2 - 4K(|u_0| - M)}}{2K}\right\},$$
$$T_2^* = \min\left\{T_1^*, \sqrt{\frac{1}{k_1+k_2}}\right\},$$

where $N = |u_1| + 1,\ M = |u_0| + 1,\ K = M^p(|c_1| + |c_2| N^q),\ k_2 = qN^{q-1}M^p$, $k_1 = pM^{p-1}\left(|c_1|\ M^2 + |c_2|\ N^q\right)$ and $X_T = \{u \in H2 : \| u \|_\infty \leq M\ ,\ \| u' \|_\infty \leq N\},\ H2 := C^2[0,T]$.

By the standard arguments of existence of solutions to ordinary differential equations, one can easily prove the following result:

For any initial values u_0 and u_1, there exists a constant T given as above such that the problem (0.1) *possesses exactly one solution u in X_T.*

In particular $c_2 = 0 < c_1$ we have $u'' = c_1 u^p$ and $\left(c_1^{\frac{1}{p-1}} u\right)'' = \left(c_1^{\frac{1}{p-1}} u\right)^p$. We make some notations

$$E_{1,\ p} = c_1^{\frac{2}{p-1}}\left(u_1^2 - \frac{2c_1}{p+1} {u_0}^{p+1}\right),\ \bar{a}(t) = c_1^{\frac{2}{p-1}} u(t)^2,\ v_0 = c_1^{\frac{1}{p-1}} u_0,\ v_1 = c_1^{\frac{1}{p-1}} u_1.$$

To estimate the life-span of the solution to the equation (0.1), we separate this section into three parts, $E_{1,\ p} < 0,\ E_{1,\ p} = 0$ and $E_{1,\ p} > 0$. Here the life-span T^* of u means that u is the solution of problem (0.1) and u exists only in $[0, T^*)$ so that the problem (0.1) possesses the solution $u \in H2$ for $T < T^*$. We have considered the cases :

(i) $E_{1,\ p} < 0,\ \bar{a}'(0) \geq 0$
(ii) $E_{1,\ p} < 0,\ \bar{a}'(0) < 0$
(iii) $E_{1,\ p} = 0,\ \bar{a}'(0) > 0$
(iv) $E_{1,\ p} > 0,\ \bar{a}'(0)^2 > 4\bar{a}(0)E_{1,\ p}$ (v) $E_{1,\ p} > 0,\ \bar{a}'(0)^2 = 4\bar{a}(0)E_{1,\ p}$ and $u_1 > 0$ (vi) $E_{1,\ p} > 0,\ \bar{a}'(0)^2 = 4\bar{a}(0)E_{1,\ p},\ u_1 < 0$ and p is odd

and obtained some results on the blow-up time, blow-up rate and blow-up constant $[1, 5]$. Here we discuss the problem (0.1) in two parts: "$c_1 = 0 < c_2$" and "$c_1 > 0, 0 < c_2$".

Part I. $c_1 = 0 < c_2$

In this part we study the following initial value problem for the nonlinear equation,

$$(0.2)\qquad \begin{cases} u''(t) = c_2 u'(t)^q u(t)^p,\ p,\ q \geq 1,\ c_2 > 0, \\ u(0) = u_0,\ u'(0) = u_1. \end{cases}$$

We are interested in properties of solutions of the problem, particularly in phenomena on blow-up, blow-up rates, blow-up constants and life-spans. In next section, we separate q into three parts, $1 \leq q < 2$, $q = 2$ and $q > 2$. And we find the blow-up time, blow-up rate and blow-up constant of u. Define

$$T = \min\left\{\frac{1}{|u_1|},\ \frac{1}{|c_2|\,M^qN^p},\frac{-|u_1| + \sqrt{u_1^2 + 2\,|c_2|\,M^qN^p}}{|c_2|\,M^qN^p}, -1 + \sqrt{1 + \frac{1}{\alpha_3}}\right\},$$

where $N = |u_0| + 1,\ M = |u_1| + 1,\ \alpha_3 = |c_2|\,qN^pM^{q-1}$ and

$$X_T = \left\{u \in H2 : \| u \|_\infty \leq N \text{ and } \| u' \|_\infty \leq M\right\}.$$

By the standard arguments of existence of solutions to ordinary differential equations, one can easily prove the following result:

Theorem 0.1. *For any initial values u_0 and u_1, there exists a constant T given as above such that the problem* (1) *possesses exactly one solution u in X_T.*

1. Fundamental Lemmas

For $u_1 = 0$, the solution u of problem (1) must be constant. For $u_1 \neq 0$ and $t \in [0, T^*)$, where $T^* = \inf\{t > 0 : u'(t) = 0\}$, we have the relations between $u(t)$ and $u'(t)$.

$$(1.1)\qquad \begin{cases} u'(t)^{2-q} = (2-q)(\dfrac{c_2}{p+1}u(t)^{p+1} + E(0)) \quad \text{for } q \neq 2, \\ E(0) = \frac{u_1^{2-q}}{2-q} - \dfrac{c_2}{p+1}u_0^{p+1} \end{cases}$$

and

$$(1.2)\qquad \begin{cases} \ln|u'(t)| = (\dfrac{c_2}{p+1}u(t)^{p+1} + E_1(0)) \text{ for } q = 2, \\ E_1(0) = \ln|u_1| - \dfrac{c_2}{p+1}u_0^{p+1}. \end{cases}$$

Lemma 1.1. *Suppose that $f \in C^1[t_0, \infty) \cap C^2(t_0, \infty)$, $f(t_0) > 0$, $f'(t_0) < 0$ and $f''(t) \leq 0$ for $t > t_0$. Then there exists a finite positive number $T > t_0$ such that $f(T) = 0$.*

Proof. Since $f \in C^1[t_0, \infty)$ and $f''(t) \leq 0$ for $t > t_0$, we obtain that $f'(t) \leq f'(t_0) < 0$ and $f(t) \leq f(t_0) + f'(t_0)(t - t_0)$. Hence there exists $t_1 > t_0$ such that $f(t_1) < 0$. By the continuity of f in $[t_0, \infty)$, there exists $T \in (t_0, t_1)$ such that $f(T) = 0$. ■

Lemma 1.2. *Suppose that* u *is the solution of* (1). *If* $u_0 \geq 0$, $c_2 > 0, u_1 > 0$, *then* $u(t) > 0, u'(t) > 0, u''(t) > 0$ *for* $t \in [0, T)$, *where* T *is the life-span of* u.

Proof. Suppose that there exists a positive number t_0 such that $u'(t_0) \leq 0$. Since $u \in C^2$ and $u_1 > 0$, there exists a positive number t_1, defined by

$$t_1 = \inf\{t \in (0, t_0] : u'(t) = 0\},$$

such that $u'(t_1) = 0$ and $u'(t) \geq 0$ for $t \in [0, t_1]$. For $t \in [0, t_1]$, $u'(t) \geq 0$, we have $u(t)^p \geq 0$, $u''(t) \geq 0$. Therefore, $u'(t_1) \geq u_1 > 0$. This result contradicts with $u'(t_1) = 0$; thus we conclude that $u'(t) > 0$ for $t \in [0, T)$. Together the equation (1) and the continuities of u, u' and u'', the lemma follows. ■

By Theorem 0.1, there exists the unique solution to the (1) on $[0, T)$, where T depends on the initial values as follows

$$T(u_0, u_1) = \min\left\{\frac{1}{|u_1|}, \frac{1}{|c_2|\, M^q N^p}, \frac{-|u_1| + \sqrt{u_1^2 + 2\,|c_2|\, M^q N^p}}{|c_2|\, M^q N^p}, -1 + \sqrt{1 + \frac{1}{\alpha_3}}\right\}$$

and $N = |u_0| + 1$, $M = |u_1| + 1, \alpha_3 = |c_2|\, q N^p M^{q-1}$. The function $T(u_0, u_1)$ has the following monotonicity property.

Lemma 1.3. *If* $u_0 \leq u_0^*$ *and* $u_1 \leq u_1^*$, *then* $T(u_0, u_1) \geq T(u_0^*, u_1^*)$.

Proof. Let $N^* = |u_0^*| + 1$, $M^* = |u_1^*| + 1, \alpha_3^* = |c_2|\, q N^{*p} M^{*q-1}$.

(1) If $T(u_0, u_1) = \dfrac{1}{|u_1|}$, then by $u_1 \leq u_1^*$, $T(u_0, u_1) \geq \dfrac{1}{|u_1^*|} \geq T(u_0^*, u_1^*)$.

(2) If $T(u_0, u_1) = -1 + \sqrt{1 + \frac{1}{\alpha_3}}$, using the fact that $u_1 \leq u_1^*$, $p, q \geq 1$, we have $\alpha_3^* \geq \alpha_3 \geq 0$,

$$T(u_0, u_1) \geq -1 + \sqrt{1 + \frac{1}{\alpha_3^*}} \geq T(u_0^*, u_1^*).$$

(3) If $T(u_0, u_1) = \dfrac{1}{|\, c_2 \,|\, M^q N^p}$, then by the conditions $u_0 \leq u_0^*$, $u_1 \leq u_1^*$ and $p \geq 1$, $q \geq 1$, we obtain that $M^{*q} \geq M^q$ and $N^{*p} \geq N^p$. Thus

$$T(u_0, u_1) \geq \frac{1}{|c_2|\, M^{*q} N^{*p}} \geq T(u_0^*, u_1^*).$$

(4) If $T(u_0, u_1) = \dfrac{- \mid u_1 \mid + \sqrt{u_1^2 + 2 \mid c_2 \mid M^q N^p}}{\mid c_2 \mid M^q N^p}$, then from $u_0 \leq u_0^*$ and $u_1 \leq u_1^*$, it follows that $M^{*q} \geq M^q$, $N^{*p} \geq N^p$ and

$$\begin{aligned} T(u_0, u_1) &= \frac{2}{|u_1| + \sqrt{{u_1}^2 + 2\,|c_2|\, M^q N^p}} \\ &\geq \frac{2}{|u_1^*| + \sqrt{{u_1^*}^2 + 2\,|c_2|\, M^{*q} N^{*p}}} \geq T(u_0^*, u_1^*). \end{aligned}$$

Lemma 1.4. *Suppose that* u *is the solution of* (1) *for* $q \in [1, 2]$. *If* u *exists locally and* t_1^* *is the life-span of* u*, then* u *blows up at* $t = t_1{}^*$.

Proof. Assume that $\lim_{t \to t_1^{*-}} u(t) = M < \infty$. By (1.1), (1.2) and $q \in [1, 2]$, we have

$$\lim_{t \to t_1^{*-}} u'(t) = \begin{cases} \left((2-q) \left(\dfrac{c_2}{p+1} M^{p+1} + E(0) \right) \right)^{\frac{1}{2-q}} & \text{if } 1 \leq q < 2, \\ \exp\left(\dfrac{c_2}{p+1} M^{p+1} + E_1(0) \right) & \text{if } q = 2. \end{cases}$$

Now we consider the following differential equation

$$\begin{cases} v''(t) = c_2 v'(t)^q v(t)^p), \\ v(0) = u(t_1^{*-}), v'(0) = u'(t_1^{*-}). \end{cases}$$

Let $v(t)$ be the existing unique solution to the above equation on $[0, T_v)$. Since $u(t_1^{*-})$ and $u'(t_1^{*-})$ are finite, so $T_v > 0$. Let

$$U(t) = \begin{cases} u(t) \text{ if } t \in [0, t_1^{*-}), \\ v(t - t_1^{*-}) \text{ if } t \in [t_1^{*-}, t_1^{*-} + T_v), \end{cases}$$

the problem(1) can be solved beyond the time $t_1{}^*$, this contradicts with the assumption of $t_1{}^*$. Therefore, u blows up at $t = t_1{}^*$. ■

We would use the following two lemmas can be proved in a similar way as Lemma 5.1 and Lemma 6.1, for the fluquency for the writting, we postpone the proofs to Lemma 5.1 and Lemma 6.1.

Lemma 1.5. *Suppose that* u *is a positive solution of problem* (1) *and that* $c_2 > 0,\ u_0 \geq 0,\ u_1 > 0$. *For* $1 \leq q \leq 2$, $u(t)$ *and* $u'(t)$ *blow up simultaneously; and so does* u''. *For* $q > 2$, $u'(t)$ *and* u'' *blow up at the same time.*

Lemma 1.6. *Suppose that* u *is the solution of* (0.1). *If* $u_0 \geq 0, u_1 > 0$ *and* $c_2 > 0$*, then* $u(t) > 0, u'(t) > 0, u''(t) > 0$ *for* $t \in [0, T)$*, where* T *is the life-span of* u.

According to the similarity, the proof of the lemma 1.7 below will be postponed to Theorem 8.1.

Lemma 1.7. *For* $q > 2$*, if* u *is the solution of* (0.1) *and* $c_2 > 0,\ u_0 \geq 0,\ u_1 > 0$, *then* u^{p+1} *is bounded by* $u_0^{p+1} + (p+1)\, u_1^{2-q} / (q-2)\, c_2$.

2. Blow-up Phenomena of u

To discuss blow-up phenomena of u with $u_1 \neq 0$, we separate this subsection into three parts $1 \leq q < 2$, $q > 2$ and $q = 2$. We have some blow-up results.

Theorem 2. *Suppose that* u *is the positive solution of* (1) *and* $c_2 > 0,\ u_0 \geq 0,\ u_1 > 0$. *Then*

(I) *for* $q \in [1, 2),$ u *blows up at finite time* $t = T_{11}$ *for some finite real number* $T_{11} > 0$*; further, we have*

$$\lim_{t \to T_{11}^-} (T - t)^{\frac{2-q}{p+q-1}} u(t) = \left(\frac{p+q-1}{2-q} \right)^{-\frac{2-q}{p+q-1}} \left((2-q) \frac{c_2}{p+1} \right)^{\frac{-1}{p+q-1}}.$$

(II) *for* $q = 2$*, then* u *blows up logarithmically at finite time* $t = T_{12}$ *and*

$$\lim_{t \to T_{12}^-} \left(\frac{1}{-\ln (T_{12} - t)} \right)^{\frac{1}{p+1}} u(t) = \left(\frac{c_2}{p+1} \right)^{-\frac{1}{p+1}}.$$

(III) *for* $q > 2$*, if* u *is the positive solution of* (1) *and* $c_2 > 0,\ u_0 \geq 0,\ u_1 > 0$*, then* u *is bounded in* $[0, T)$*, where* T *is the life span of* u.

Remark 2. *If we don't restrict ourself to the positiveness of the solution* u *to the equation* (1)*, then we also have the following blow-up results:*
If u *is the solution of equation* (1), $q \in [1, 2]$ and one of the followings is valid:

(1) p is even, q is odd, $c_2 > 0,\ u_0 \leq 0,\ u_1 < 0,\ u_0^p \geq 0$,
(2) p is odd, q is even, $c_2 > 0,\ u_0 \leq 0,\ u_1 < 0,\ u_0^p \leq 0$,
(3) p is even, q is even, $c_2 < 0,\ u_0 \leq 0,\ u_1 < 0,\ u_0^p \geq 0$,
(4) p is odd, q is odd, $c_2 < 0,\ u_0 \leq 0,\ u_1 < 0,\ u_0^p \leq 0$,

then u blows up in finite time.
For a given function u in this work we use the following abbreviations

$$a(t) = u(t)^2,\ J(t) = a(t)^{-m},\ m = \frac{1}{2} \left(\frac{1}{2-q} - 1 \right).$$

Proof of Theorem 2. Suppose that u is a global solution of equation (1).

(I-1) For $q = 1$, $u''(t) = c_2u'(t)u(t)^p$, by (1.1) and lemma 1.6, we obtain that

$$\int_{u_0}^{u(t)} \frac{1}{\frac{c_2}{p+1}r^{p+1} + E(0)}dr = t \text{ for all } t > 0$$

and

$$u(t) > u_0 \text{ for } t > 0.$$

Using the fact that $\frac{c_2}{p+1}r^{p+1} + E(0) > 0$ for $r \geq u_0$, we get

$$\int_{u_0}^{u(t)} \frac{1}{\frac{c_2}{p+1}r^{p+1} + E(0)}dr \leq \int_{u_0}^{\infty} \frac{1}{\frac{c_2}{p+1}r^{p+1} + E(0)}dr \text{ for all } t > 0$$

and then

$$\int_{u_0}^{\infty} \frac{1}{\frac{c_2}{p+1}r^{p+1} + E(0)}dr \geq \lim_{t\to\infty} \int_{u_0}^{u(t)} \frac{1}{\frac{c_2}{p+1}r^{p+1} + E(0)}dr = \lim_{t\to\infty} t.$$

Since the integral $\int_{u_0}^{\infty} \dfrac{1}{\frac{c_2}{p+1}r^{p+1} + E(0)}dr$ is finite, this leads to a contradictory conclusion with the above last estimate. Hence we can conclude that u only exists on $[0, T_{11})$, where T_{11} is the life-span of u. By Lemma 1.4, we obtain that u blows up at $t = T_{11}$.

(I-2) For $1 < q < 2$, $m = \frac{1}{2}(\frac{1}{2-q} - 1) > 0$, and we claim that there exists a finite time $T_{11} > 0$ such that $J(T_{11}) = 0$. According to Lemma 1.5, we find that u' and u blow up simultaneously. Thus $u \in C^2[0, T)$, where T is a blow-up time of u. By (1) and Lemma 1.6,

$$u'(t)^{2-q} = (2 - q)\left(\frac{c_2}{p+1}u(t)^{p+1} + E(0)\right) \text{ for all } t > 0.$$

By direct computation, we obtain that

$$\begin{aligned}
J'(t) &= -ma(t)^{-(m+1)}a'(t) = -ma(t)^{-(m+1)}2u(t)u'(t),\\
a''(t) &= 2u'(t)^2 + 2c_2u'(t)^q u(t)^{p+1}\\
&= 2\left(1 + \frac{1}{2-q}\right)a'(t)^2 + 2u'(t)^q\left(\frac{pc_2}{p+1}u(t)^{p+1} - E(0)\right)
\end{aligned}$$

and

$$a(t)a''(t) = \frac{1}{2}\left(1 + \frac{1}{2-q}\right)a'(t)^2 + 2a(t)u'(t)^q\left(\frac{pc_2}{p+1}u(t)^{p+1} - E(0)\right).$$

Hence we have

$$\begin{aligned} J''(t) &= -ma(t)^{-(m+2)}\left(a(t)a''(t)-(m+1)a'(t)^2\right) \\ &= -ma(t)^{-(m+2)}2a(t)u'(t)^q\left(\frac{pc_2}{p+1}u(t)^{p+1}-E(0)\right). \end{aligned}$$

With the help of Lemma 1.6, $u(t), u'(t), u''(t) > 0$ for all $t > 0$, and there exists a finite time $t_1 > 0$ such that

$$\frac{pc_2}{p+1}u(t_1)^{p+1} - E(0) \geq 0.$$

Herewith, $J(t_1) > 0$, $J'(t_1) < 0$ and $J''(t) \leq 0$ for $t \geq t_1$. These and Lemma 1.1 imply that there exists a finite positive number $T_{11} > t_1$ such that $J(T_{11}) = 0$. Thus u blows up in finite time. This leads to contradiction and we have shown that u exists locally and by Lemma 1.4, u blows up in finite time.

(I-3) We estimate the blow-up rate and blow-up constant. Set $i = \frac{p+q-1}{2-q}$. By some calculations on (1) using L. Hôpital's rule we obtain

$$\begin{aligned} \lim_{t\to T_{11}^-}\frac{u^{-i}}{T_{11}-t} &= \lim_{t\to T_{11}^-} i\frac{\left((2-q)(\frac{c_2}{p+1}u(t)^{p+1}+E(0))\right)^{\frac{1}{2-q}}}{u(t)^{i+1}} \\ &= \frac{p+q-1}{2-q}\left((2-q)\frac{c_2}{p+1}\right)^{\frac{1}{2-q}}. \end{aligned}$$

Thus

$$\lim_{t\to T_{11}^-}(T-t)^{\frac{2-q}{p+q-1}}u(t) = \left(\frac{p+q-1}{2-q}\right)^{-\frac{2-q}{p+q-1}}\left((2-q)\frac{c_2}{p+1}\right)^{\frac{-1}{p+q-1}}.$$

(II) For $q = 2$, assume that u is a global solution of (1). By (1.2) and Lemma 1.6,

$$\ln\left|u'(t)\right| = \frac{c_2}{p+1}u(t)^{p+1} + E_1(0) \text{ for all } t > 0.$$

Since $u(t)$, $u'(t)$ blow up simultaneously (by Lemma 1.5), $u \in C^2[0, T_{12})$, where T_{12} is blow-up time of u.

Let $K(t) = a(t)^{-1}$, then

$$K'(t) = -a(t)^{-2}a'(t) = -2a(t)^{-2}u(t)u'(t)$$

and

$$K''(t) = -a(t)^{-3}\left(a(t)a''(t) - 2a'(t)^2\right) = -a(t)^{-3}a'(t)^2\left(\frac{1}{2}\left(1 + c_2u(t)u(t)^p\right) - 2\right).$$

By Lemma 1.6, $u(t), u'(t), u''(t) > 0$ for $t > 0$. Hence there exists $t_0 > 0$ such that $u(t) \geq \left(\frac{3}{c_2}\right)^{\frac{1}{p}} + 1$ for $t \geq t_0$ and $\frac{1}{2}\left(1 + c_2u(t)u(t)^p\right) - 2 \geq 0$ for $t \geq t_0$. We conclude that

$$K(t_0) > 0, \quad K'(t) < 0 \text{ and } K''(t) < 0 \text{ for } t \geq t_0,$$

thus by Lemma 1.1 there exists positive number T_{12} such that $K(T_{12}) = 0$ and u blows up at time $t = T_{12}$. This result contradicts with our assumption that u is a global solution of problem (1). Therefore u can exist only locally. By Lemma 1.4, u blows up in finite time. After some computations we get

$$\begin{aligned}\lim_{t\to T_{12}^-} -\ln\left(T_{12} - t\right) u(t)^{-(p+1)} &= \lim_{t\to T_{12}^-} \frac{u(t)^{-p}u'(t)^{-1}}{(p+1)(T_{12} - t)} \\ &= \lim_{t\to T_{12}^-} \frac{pu(t)^{-(p+1)} + u(t)^{-p}u'(t)^{-2}u''(t)}{p+1}.\end{aligned}$$

Using (1), we obtain $u''(t) = c_2u'(t)^2u(t)^p$ and

$$\lim_{t\to T_{12}^-} -\ln\left(T_{12} - t\right)u(t)^{-(p+1)} = \lim_{t\to T_{12}^-} \frac{pu(t)^{-(p+1)} + c_2}{p+1} = \frac{c_2}{p+1}.$$

(III) For $q > 2$, integrating the equation (1) from 0 to t,

$$\frac{u'(t)^{2-q}}{2-q} - \frac{u_1^{2-q}}{2-q} = \frac{c_2}{p+1}u(t)^{p+1} - \frac{c_2}{p+1}u_0^{p+1}.$$

For $t \in [0, T)$, by Lemma 1.6, $u(t), u'(t) > 0$ and

$$\frac{u_1^{2-q}}{q-2} > \frac{c_2}{p+1}u(t)^{p+1} - \frac{c_2}{p+1}u_0^{p+1}.$$

Since that $c_2 > 0$ and $u(t) > 0$ for $t \in [0, T)$, u is bounded in $[0, T)$. ■

Proof of Remark 2. The arguments are similar to the proof of Theorem 2, we only mention the case (1).
Let $v(t) = -u(t)$. By the fact that p is even and q is odd, we have $v(t)^p = u(t)^p$ and $v'(t)^q = -u'(t)^q$. We get

$$\begin{cases} v''(t) = -u''(t) = -c_2u'(t)^qu(t)^p = c_2v'(t)^qv(t)^p, \\ v(0) = v_0 = -u_0, v'(0) = v_1 = -u_1. \end{cases}$$

Since $u_0 \leq 0, u_0^p \geq 0, u_1 < 0$ and p is even, we have $v_0 \geq 0, v_1 > 0$ and $v_0^p = u_0^p \geq 0$. By Theorem 2 and Theorem 3 below, v blows up, so does u. ■

3. Blow-up Phenomena of u'

In this subsection we come back to the consideration of blow-up phenomena of u'.

Theorem 3. *For $q \geq 1$, if u is a positive solution of (1) and $c_2 > 0$, $u_0 \geq 0$, $u_1 > 0$, then u' blows up at time $t = T_{21}$. Further, we have*

$$\lim_{t\to T_{21}^-} (T_{21}-t)^{\frac{p+1}{p+q-1}} u'(t)$$
$$= \left(\frac{c_2(p+q-1)}{p+1} \left(\frac{c_2(2-q)}{p+1} \right)^{\frac{-p}{p+1}} \right)^{\frac{-(p+1)}{p+q-1}} \quad \text{for } 1 \leq q < 2,$$

$$\lim_{t\to T_{22}^-} [-\ln(T_{22}-t)]^{\frac{p}{p+1}} (T_{22}-t) u'(t) = c_2^{\frac{-1}{p+1}} \left(\frac{1}{p+1} \right)^{\frac{p}{p+1}} \quad \text{for } q = 2,$$

$$\lim_{t\to T_{23}^-} (T_{23}-t)^{\frac{1}{q-1}} u'(t) = (c_2(q-1)u(T_{23})^p)^{\frac{1}{1-q}} \quad \text{for } q > 2.$$

Proof. We separate this proof into three parts: $1 \leq q < 2$, $q = 2$ and $q > 2$.

(I) For $1 \leq q < 2$, by Theorem 2 and Lemma 1.5, u and u' blow up in finite time simultaneously. According to (1), L. Hôpital's rule and Theorem 2 we have

$$\lim_{t\to T_{21}^-} \frac{u'(t)^{\frac{1-p-q}{p+1}}}{(T_{21}-t)} = \lim_{t\to T_{21}^-} \frac{c_2\,(p+q-1)}{p+1} \left((2-q) \left(\frac{c_2}{p+1} u(t)^{p+1} + E(0) \right) \right)^{\frac{-p}{p+1}} u(t)^p$$
$$= \frac{c_2(p+q-1)}{p+1} \left(\frac{c_2(2-q)}{p+1} \right)^{\frac{-p}{p+1}}.$$

Thus

$$\lim_{t\to T_{21}^-} (T_{21}-t)^{\frac{p+1}{p+q-1}} u'(t) = \left(\frac{c_2(p+q-1)}{p+1} \left(\frac{c_2(2-q)}{p+1} \right)^{\frac{-p}{p+1}} \right)^{\frac{-(p+1)}{p+q-1}}.$$

(II) For $q = 2$, using Theorem 2 and Lemma 1.5, then u and u' blow up in finite time simultaneously. By (1), L. Hôpital's rule and Theorem 2 we have

$$\lim_{t\to T_{22}^-} [-\ln(T_{22}-t)]^{\frac{p}{p+1}} (T_{22}-t) u'(t)$$
$$= \lim_{t\to T_{22}^-} \frac{\frac{p}{p+1}[-\ln(T_{22}-t)]^{\frac{-1}{p+1}}(T_{22}-t) - [-\ln(T_{22}-t)]^{\frac{p}{p+1}}}{-c_2 u(t)^p}$$
$$= c_2^{\frac{-1}{p+1}} (\frac{1}{p+1})^{\frac{p}{p+1}}.$$

(III) In the case $q > 2$, let

$$b(t) = u'(t)^2,\ L(t) = b(t)^{-\alpha}, \alpha = \frac{1}{2}(q-1),$$

we have $L'(t) = -\alpha b(t)^{-(\alpha+1)}b'(t) = -2\alpha b(t)^{-(\alpha+1)}u'(t)u''(t)$ and

$$\begin{aligned} L''(t) &= -\alpha b(t)^{-(\alpha+2)}\left(\left(\frac{1}{2}(1+q)-(\alpha+1)\right)b'(t)^2 + 2c_2pb(t)u(t)^{p-1}u'(t)^{q+2}\right) \\ &= -2pc_2\alpha b(t)^{-(\alpha+1)}u(t)^{p-1}u'(t)^{q+2}. \end{aligned}$$

From Lemma 1.6, $u(t) > 0$, $u'(t) > 0$ and $u''(t) > 0$ for $t > 0$, we obtain that $L'(t), L''(t) < 0$ for $t > 0$. Now we need to check that u doesn't blow up earlier than u'. By Lemma 1.7, u is bounded. Using Lemma 1.1, there exists a finite number T_{21} such that $L(T_{21}) = 0$. Since $q > 2$, we $\alpha > 0$, we obtain that u' blows up at finite time $t = T_{21}$.

For $q > 2$, by (1) and L. Hôpital's rule we have

$$\lim_{t\to T_{23}^-}\frac{u'(t)^{1-q}}{(T_{23}-t)} = \lim_{t\to T_{23}^-}(1-q)u'(t)^{-q}u''(t)(-1) = c_2(q-1)u(T_{23})^p.$$

Thus

$$\lim_{t\to T_{23}^-}(T_{23}-t)^{\frac{1}{q-1}}u'(t) = (c_2(q-1)u(T_{23})^p)^{\frac{1}{1-q}}\ .$$ ■

3. Blow-up Phenomena of u''

We want to calculate blow-up rate and blow-up constant of u'' in the this subsection.

Theorem 4. *Under the conditions in Theorem 3 suppose that u is a positive solution of* (1). *For $q \geq 1$, then u'' blows up at time $t = T_{31}$ for some $T_{31} > 0$. Furthermore, for*

(I) $q \in [1, 2)$, *the blow-up rate of u'' is* $\frac{q(p+1)}{p+q-1} + \frac{p(2-q)}{p+q-1}$ *and the blow-up constant is*

$$c_2^{\frac{-1}{p+q-1}}(2-q)^{\frac{p}{p+q-1}}(p+1)^{\frac{p+q}{p+q-1}}(p+q-1)^{\frac{-(2p+q)}{p+q-1}}.$$

(II) $q = 2$, *then u'' blows up logarithmically at time $t = T_{32}$ for some $T_{32} > 0$ and*

$$\begin{aligned} &\lim_{t\to T_{32}^-}\left\{(-\ln(T_{32}-t))^{\frac{p}{p+1}}(T_{32}-t)\right\}^q\left\{(-\ln(T_{32}-t))^{\frac{-1}{p+1}}\right\}^p u''(t) \\ &= c_2^{\frac{1-q}{p+1}}(p+1)^{\frac{p(1-q)}{p+1}}. \end{aligned}$$

(III) $q > 2$, *then* u'' *blows up at time* $t = T_{33}$ *for some* $T_{33} > 0$, *the blow-up rate of* u'' *is* $\frac{q}{q-1}$ *and the blow-up constant is*

$$(q-1)^{\frac{q}{1-q}}(c_2 u(T_{33})^p)^{\frac{1}{1-q}}.$$

Proof. According to Theorem 3 and Lemma 1.5, u' and u'' blow up at the same time $t = T_{31}$.

(I) For $1 \leq q < 2$, by Lemma 1.5, u, u' and u'' possess the same blow-up time . Using (1.1), Theorem 2 and Theorem 3, we conclude that

$$\begin{aligned}
&\lim_{t\to T_{31}^-}(T_{31}-t)^{\frac{q(p+1)}{p+q-1}+\frac{p(2-q)}{p+q-1}}u''(t)\\
&= \lim_{t\to T_{31}^-} c_2(T_{31}-t)^{\frac{q(p+1)}{p+q-1}}u'(t)^q(T_{31}-t)^{\frac{p(2-q)}{p+q-1}}u(t)^p\\
&= c_2^{\frac{-1}{p+q-1}}(2-q)^{\frac{p}{p+q-1}}(p+1)^{\frac{p+q}{p+q-1}}(p+q-1)^{\frac{-(2p+q)}{p+q-1}}.
\end{aligned}$$

(II) For $q = 2$, using Lemma 1.5, u, u' and u'' have the same blow-up time. Thus T_3 is also blow-up time of u and u'. By (1.1), Theorem 2 and Theorem 3, we conclude that

$$\begin{aligned}
&\lim_{t\to T_{32}^-}\{[-\ln(T_{32}-t)]^{\frac{p}{p+1}}(T_{32}-t)\}^q\{[-\ln(T_{32}-t)]^{\frac{-1}{p+1}}\}^p u''(t)\\
&= \lim_{t\to T_{32}^-} c_2\{[-\ln(T_{32}-t)]^{\frac{p}{p+1}}(T_{32}-t)\}^q u'(t)^q\{[-\ln(T_{32}-t)]^{\frac{-1}{p+1}}\}^p u(t)^p\\
&= c_2^{\frac{1-q}{p+1}}(p+1)^{\frac{p(1-q)}{p+1}}.
\end{aligned}$$

(III) For $q > 2$, by Lemma 1.5, u'' and u' blow up contemporaneously in finite time. Thanks to Lemma 1.6 we have $u(t) > 0$ and $u(t)^p \geq 0$. Since $c_2 > 0$, $c_2 u(t)^p > 0$. By (1) and Theorem 3, we conclude that

$$\begin{aligned}
\lim_{t\to T_{33}^-}(T_{33}-t)^{\frac{q}{q-1}}u''(t) &= \lim_{t\to T_{33}^-} c_2(T_{33}-t)^{\frac{q}{q-1}}u'(t)^q u(t)^p\\
&= (q-1)^{\frac{q}{1-q}}(c_2 u(T_{33})^p)^{\frac{1}{1-q}}. \qquad \blacksquare
\end{aligned}$$

5. Estimations for the Life-spans

To estimate the life-span of the solution of the equation (1), we separate this section into two parts, $1 \leq q < 2$ and $q = 2$. Here the life-span T of u means that u is the solution of problem (1) and the existence interval of u is contained only in $[0, T)$ so that the problem (1) has the solution $u \in C^2[0, T)$. We have the following results.

Lemma 5.1. *Suppose that u is a positive solution of problem (1) and that $c_2 > 0,\ u_0 \geq 0,\ u_1 > 0$. For $1 \leq q \leq 2$, $u(t)$ and $u'(t)$ blow up simultaneously; and so does u''. For $q > 2$, $u'(t)$ and u'' blow up at the same time.*

Proof. (I) For $1 \leq q < 2$, by (1) we have

$$u'(t)^{2-q} = (2-q)(\frac{c_2}{p+1}u(t)^{p+1} + E(0)).$$

(1) First, we claim that if u blows up in finite time, then so does u'. According to Theorem 2.1, u blows up at time $t = T_{11}$. Since $\lim_{t\to T_{11}^-} \frac{1}{u(t)} = 0$, we have

$$\lim_{t\to T_{11}^-} \frac{1}{u'(t)^{2-q}} = \lim_{t\to T_{11}^-} \frac{1}{(2-q)(\frac{c_2}{p+1}u(t)^{p+1} + E(0))}$$
$$= \lim_{t\to T_{11}^-} \frac{\frac{1}{u(t)^{p+1}}}{(2-q)(\frac{c_2}{p+1} + \frac{E(0)}{u(t)^{p+1}})} = 0.$$

Therefore, $\lim_{t\to T_{11}^-} \dfrac{1}{u'(t)} = 0$. Thus, u' blows up at the same finite time.

(2) We claim that if u' blows up in finite time, then so does u. With the help of Theorem 8.3 below, u' blows up at time $t = T_{21}$. Assume that u doesn't blow up at time $t = T_{21}$. Let $\lim_{t\to T_{21}^-} u(t) = M < \infty$. Then

$$\lim_{t\to T_{21}^-} u'(t)^{2-q} = \lim_{t\to T_{21}^-} (2-q)(\frac{c_2}{p+1}u(t)^{p+1} + E(0))$$
$$= (2-q)(\frac{c_2}{p+1}M^{p+1} + E(0)) < \infty.$$

This result contradicts with the fact that $u'(t)$ blows up at time $t = T_{21}$. It deduces that u blows up at time $t = T_{21}$. Combining 1) with 2), we conclude that u and u' blow up simultaneously.

(II) For the case $q = 2$, by (1.2), we have

$$\ln\left|u'(t)\right| = \frac{c_2}{p+1}u(t)^p + E_1(0).$$

(3) We claim that if u blows up in finite time, then so does u'. By Theorem 2.2 and lemma 6.1 below, u blows up at time $t = T_{12}$ and $u(t), u'(t) > 0$ for $0 \leq t < T_{12}$. Since that $c_2 > 0$ and u blows up toward positive direction, $\ln|u'|$ also blows up toward positive direction. Thus u' blows up at time $t = T_{12}$.

(4) We now prove that u' blows up then so does u. Using Theorem 3.1 and Lemma 1.6, u' blows up at time $t = T_{21}$ and $u(t), u'(t) > 0$ for $0 \leq t < T_{12}$. Assume that u doesn't blow up at time $t = T_{21}$. Set

$$\lim_{t \to T_{21}^-} u(t) = M < \infty.$$

Then

$$\begin{aligned}\lim_{t \to T_{21}^-} \ln \left|u'(t)\right| &= \lim_{t \to T_{21}^-} \left(\frac{c_2}{p+1} u(t)^{p+1} + E_1(0) \right) \\ &= (2-q) \left(\frac{c_2}{p+1} M^{p+1} + E_1(0) \right) < \infty.\end{aligned}$$

This result is contradictory to the fact that u' blows up in finite time. It deduces that u blows up at time $t = T_{21}$. Together 3) and 4), we conclude that u and u' blow up simultaneously. Since that u and u' blow up toward positive direction at the same time and $c_2 > 0$, u'' blows up toward positive direction.

(III) Under $q > 2$, according to Theorem 8.3 below, u' blows up at time $t = T_{21}$. By Lemma 1.7, we obtain that u is bounded in $[0, T_{21})$, and, by Lemma 1.6, we have $u'(t) > 0$ for $t \in [0, T_{21})$. Thus the limit exists, $\lim_{t \to T_{21}^-} c_2 u(t)^p$. Since $u_0 \geq 0$ and $u'(t) > 0$ for $t \in [0, T_{21})$, we have $\lim_{t \to T_{21}^-} c_2 u(t)^p > 0$. From $u''(t) = c_2 u'(t)^q u(t)^p$, it deduces that u' and u'' blow up simultaneously. ■

We have the following estimates for the life-span of solution to the equation (1).

Theorem 5.2. *Suppose that u is the positive solution of (1) and T is life-span of u and that T_{11}^* is blow-up time of u. Under the same conditions as in Theorem 2.1, T is bounded. For $1 \leq q < 2$, we have the estimation*

$$T \leq T_{11}^* = (2-q)^{\frac{1}{q-2}} \int_{u_0}^{\infty} (\frac{c_2}{p+1} r^{p+1} + E(0))^{\frac{1}{q-2}} dr.$$

For $q = 2$, we have

$$T \leq T_{12}^* := \int_{u_0}^{\infty} \frac{1}{\exp\left(\frac{c_2}{p+1} r^{p+1} + E_1(0)\right)} dr,$$

where $E_1(0) = \ln|u_1| - \frac{c_2}{p+1} u_0^{p+1}$.

Proof. (I) For $1 \leq q < 2$, using the fact

$$u'(t) = \left((2-q) \left(\frac{c_2}{p+1} u(t)^{p+1} + E(0) \right) \right)^{\frac{1}{2-q}} > 0 \text{ for } t \in [0, T_{11}^*),$$

we have

$$\int_{u_0}^{u(t)} \frac{1}{\left(\frac{c_2}{p+1}r^{p+1} + E(0)\right)^{\frac{1}{2-q}}}\, dr = (2-q)^{\frac{1}{2-q}} t. \tag{5.1}$$

We claim that $T_{11}^* < \infty$. By $u_0 \geq 0$ and

$$\frac{c_2}{p+1}r^{p+1} + E(0) = \frac{u_1^{2-q}}{2-q} + \int_{u_0}^{r} (c_1 + c_2 s^p)\, ds,$$

we obtain that $\frac{c_2}{p+1}r^{p+1} + E(0) > 0$ for $r \geq u_0$. And it is continuous on $[u_0, a]$ for $a \geq u_0$. Therefore the function $\left(\frac{c_2}{p+1}r^{p+1} + E(0)\right)^{\frac{-1}{2-q}}$ is integrable and positive on $[u_0, a]$ for $a \geq u_0$. Thus T_{11}^* is bounded and $T \leq T_{11}^*$.

(II) For $q = 2$, by (1.2), $\ln|u'(t)| = \dfrac{c_2}{p+1}u(t)^{p+1} + E_1(0)$. Seeing that $u'(t) > 0$, we have

$$\int_{u_0}^{u(t)} \frac{1}{\exp\left(\frac{c_2}{p+1}r^{p+1} + E_1(0)\right)} dr = t.$$

We next claim that $T_{12}^* < \infty$. Set $f(r) = \dfrac{c_2}{p+1}r^{p+1} + E_1(0)$. Then $f'(r) \geq 0$ for $r^p \geq 0$ and $f''(r) \geq 0$ for $r \geq 0$. So there exists $r_0 > 0$, $r_0^p \geq 0$, such that $f(r) > 0$ for $r \geq r_0$. We calculate

$$\begin{aligned}
&\int_{u_0}^{\infty} \frac{1}{\exp\left(\frac{c_2}{p+1}r^{p+1} + E_1(0)\right)} dr \\
= &\int_{u_0}^{r_0} \frac{1}{\exp\left(\frac{c_2}{p+1}r^{p+1} + E_1(0)\right)} dr + \int_{r_0}^{\infty} \frac{1}{\exp\left(\frac{c_2}{p+1}r^{p+1} + E_1(0)\right)} dr
\end{aligned}$$

Since $\frac{1}{\exp\left(\frac{c_2}{p+1}r^{p+1}+E_1(0)\right)}$ is a continuous function on $[u_0, r_0]$, the first integrand is bounded. From $\exp\left(\dfrac{c_2}{p+1}r^{p+1} + E_1(0)\right) > \dfrac{c_2}{p+1}r^{p+1} + E_1(0) > 0$ for $r \geq r_0$, we obtain that $\dfrac{1}{\exp\left(\frac{c_2}{p+1}r^{p+1} + E_1(0)\right)} < \dfrac{1}{\frac{c_2}{p+1}r^{p+1} + E_1(0)}$ for $r \geq r_0$. By $\int_{r_0}^{\infty} \frac{1}{\frac{c_2}{p+1}r^{p+1}+E_1(0)} dr < \infty$ and the comparison test, the second integrand is bounded. Therefore, T_{12}^* is bounded and $T \leq T_{12}^*$. ∎

Part II. $c_{1>0}, c_{2>0}$

6. Blow-up Phenomena for $1 \leq q < 2$

In this section we study the blow-up phenomena of the solution to the initial value problem (0.1).

Lemma 6.1. *Suppose that* u *is the solution of* (0.1)*. If* $u_0 \geq 0, u_1 > 0, c_1 > 0$ *and* $c_2 > 0$*, then* $u(t) > 0, u'(t) > 0, u''(t) > 0$ *for* $t \in [0, T)$*, where* T *is the life-span of* u*.*

Proof. We only prove lemma 6.1. Suppose that there exists a positive number t_0 such that $u'(t_0) \leq 0$. Since $u \in C^2$ and $u_1 > 0$, there exists a positive number t_1, defined by

$$t_1 = \inf\{t \in (0, t_0] : u'(t) \leq 0\},$$

then $u'(t_1) = 0, u'(t) > 0, u(t) > 0$ and $u''(t) > 0$ for $t \in [0, t_1)$. Therefore, $u'(t_1) \geq u_1 > 0$. This result contradicts with $u'(t_1) = 0$; thus we conclude that

$$u'(t) > 0 \text{ for } t \in [0, T),$$

where T is the life-span of u. Together the equation (0.1) and the continuities of u, u' and u'', the lemma follows. ■

For a given function u in this work we use the following abbreviations

$$a(t) = u(t)^2, \;\; \bar{J}(t) = a(t)^{-k}, \;\; k = \frac{p-1}{4}.$$

Using lemma 6.1 one can easily obtain the following lemmas after some computations:

Lemma 6.2. *Suppose that* u *is the solution of* (0.1) *and that* T^*_{31} *is the life-span of* u*, then for every* $c \in R$*,*

$$\lim_{t \to T_{31}{}^{*-}} \frac{u'(t)^{2-q}}{u(t)^{p+1}} = \lim_{t \to T_{31}{}^{*-}} \frac{\left((u(t)+c)'\right)^{2-q}}{(u(t)+c)^{p+1}};$$

further, for $u_0 > 0, u_1 > 0$ *and* $c_1 > 0, c_2 > 0$*, then* $\lim_{t \to T_{31}{}^{*-}} \dfrac{\int_0^t u(r)^p \, u'(r)^{1-q} \, dr}{u(t)^{p+1}}$ $= 0 \; for \; q > 1$ *and*

$$\lim_{t \to T_{31}{}^{*-}} \frac{u'(r)^{2-q}}{u(t)^{p+1}} = \frac{2-q}{p+1} c_2 \;\; for \; q \in (1, 2). \tag{6.1}$$

Lemma 6.3. *Suppose that* u *is the solution of* (0.1) *and that* T_{31}^* *is the life-span of* u, *then for* $t \in [0, T_{31}{}^*)$ *we have:*

$$E_2(t) = u'(t)^2 - \frac{2c_1}{p+1}u(t)^{p+1} - 2c_2\int_0^t u(r)^p u'(r)^{1+q}\,dr = E_2(0), \tag{6.2}$$

$$a'(t)^2 = 4E_2(0)\,a(t) + \frac{8c_1}{p+1}u(t)^{p+3} + 8c_2 a\int_0^t u(r)^p u'(r)^{1+q}\,dr, \tag{6.3}$$

$$\begin{aligned} a''(t) &= 2E_2(0) + 2u(t)^{p+1}\left(\frac{p+3}{p+1}c_1 + 2c_2u'(t)^q\right) \\ &+4c_2\textstyle\int_0^t u(r)^p u'(r)^{1+q}\,dr \end{aligned} \tag{6.4}$$

$$\begin{aligned} \bar{J}''(t) = &\frac{p^2-1}{4}E_2(0)\,a(t)^{-\frac{p+3}{4}} - \frac{p-1}{2}c_2 a(t)^{-\frac{p+3}{4}} \\ &\left(u(t)^{p+1}u'(t)^q + (p+5)\int_0^t u(r)^p u'(r)^{1+q}\,dr\right) \end{aligned} \tag{6.5}$$

To discuss blow-up phenomena of u with $u_1 \neq 0$, we separate this subsection into three parts $1 \leq q < 2$, $q = 2$ and $q > 2$.
For $1 \leq q < 2$, we have blow-up results.

Theorem 6.4. *Suppose that* $q \in [1, 2)$ *and* u *is the solution of* (0.1) *with* $E_2(0) \leq 0, u_0 \geq 0,\ u_1 > 0,\ c_2 > 0$, *then* u *blows up at finite time* $T_{31}^* \leq \frac{2u_0}{(p-1)u_1}$ *and the blow-up rate* α_1 *and blow-up constant* β_1 *for* u *are* $\frac{2-q}{p+q-1}$ *and* $\left(\frac{p+q-1}{2-q}\left(\frac{2-q}{p+1}c_2\right)^{\frac{1}{2-q}}\right)^{\frac{q-2}{p+q-1}}$ *respectively.*

Proof.

Step 1. We prove there exists a bounded positive real number T such that $J(T) = 0$.

By lemma 6.1, 6.3 and $E_2(0) \leq 0$, we get that $u(t), u'(t), u''(t)$ are all positive for $t \in [0, T_1^*)$, and

$$\bar{J}'(t) < 0, \bar{J}''(t) < 0 \quad for\ t \in [0, T_{31}^*).$$

Using lemma 1.1, there exists T such that $\bar{J}(T) = 0$ and $\bar{J}(T) \leq \bar{J}(0) + \bar{J}'(0)\,T$,

$$T_{31}^* \leq \frac{2u_0}{(p-1)\,u_1}.$$

Step 2. We compute the blow-up rate and blow-up constant for u.

For $q = 1$, $u''(t) = u(t)^p(c_1 + c_2u(t))$, $\alpha_1 = \frac{1}{p}$; using lemmas 6.2, 6.3, we obtain that

$$\begin{aligned}\lim_{t\to T3_1^{*-}} (T_1^* - t)^{-1} u(t)^{-\frac{1}{\alpha_1}} &= \lim_{t\to T3_1^{*-}} \frac{u'(t)}{\alpha_1 u(t)^{1+\frac{1}{\alpha_1}}}\\ &= \lim_{t\to T_{31}^{*-}} \frac{u(t)^p (c_1 + c_2 u'(t))}{(1+\alpha_1) u(t)^{\frac{1}{\alpha_1}} u'(t)} = \frac{p}{p+1}c_2.\end{aligned}$$

For $q \neq 1, \alpha_1 = \frac{2-q}{p+q-1}$, inducing lemma 6.2, we conclude that

$$\lim_{t\to T_{31}^{*-}} (T_{31}^* - t)^{-1} u(t)^{-\frac{1}{\alpha_1}} = \lim_{t\to T_{31}^{*-}} \frac{u'(t)}{\alpha_1 u(t)^{1+\frac{1}{\alpha_1}}} = \frac{1}{\alpha_1}\left(\frac{2-q}{p+1}c_2\right)^{\frac{1}{2-q}}.$$

This means,

$$\lim_{t\to T_{31}^{*-}} (T_{31}^* - t)^{\alpha_1} u(t) = \left(\frac{p+q-1}{2-q}\left(\frac{2-q}{p+1}c_2\right)^{\frac{1}{2-q}}\right)^{\frac{q-2}{p+q-1}}. \blacksquare$$

To estimate the blow-up rate of u' and u'', we need the following lemma:

Lemma 6.5. *Under the condition of Theorem 6.4 then* u' *and* u'' *blow up at the same finite* T_{31}^{*-}.

Proof. According to (6.3) and Theorem 6.4 we obtain

$$\begin{aligned}&0 < \frac{1}{u'(t)^2} \leq \frac{1}{E_2(0) + \frac{2}{p+1}c_1 u(t)^{p+1}} \quad \forall t \in [0, T_{31}^*),\\ &0 \leq \lim_{t\to T_{31}^{*-}} \frac{1}{u'(t)^2} \leq \lim_{t\to T_{31}^{*-}} \frac{1}{E_2(0) + \frac{2}{p+1}c_1 u(t)^{p+1}} = 0\end{aligned}$$

and

$$\begin{aligned}0 \leq \lim_{t\to T_{31}^{*-}} \frac{1}{u''(t)} &= \lim_{t\to T_{31}^{*-}} \frac{1}{u(t)^p (c_1 + c_2 u'(t)^q)}\\ &\leq \lim_{t\to T_{31}^{*-}} \frac{1}{\frac{2}{p+1}c_1 u(t)^{p+1}} = 0. \qquad \blacksquare\end{aligned}$$

Due to this lemma, we have results concerning blow-up rate and blow-up constant for u' and u''.

Theorem 6.6. *Under the condition of Theorem 6.4, then the blow-up rate* α_2 *and blow-up constant* β_2 *of* u' *are* $\frac{p+1}{p+q-1}$ *and* $\left(\frac{p+q-1}{p+1}\left(\frac{p+1}{2-q}c_2\right)^{\frac{p}{p+1}}\right)^{\frac{-p-1}{p+q-1}}$ *; and the blow-up rate* α_3 *and blow-up constant* β_3 *of* u'' *are* $\frac{2p+q}{p+q-1}$ *and* $c_2^{\frac{pq-p}{p+q-1}}\ (2-q)^{\frac{p}{p+q-1}}$ $(p+1)^{\frac{p+q}{p+q-1}}\ (p+q-1)^{\frac{-2p-q}{p+q-1}}$ *respectively.*

Proof. By lemmas 6.2 and 6.5, u' blows up at T_{31}^* and

$$\lim_{t\to T_{31}^{*-}} \left(T_1^* - t\right)^{-1} u'(t)^{-\frac{1}{\alpha_2}} = \lim_{t\to T_{31}^{*-}} \frac{u''(t)}{\alpha_2 u(t)^{1+\frac{1}{\alpha_2}}} = \frac{1}{\alpha_2}\left(\frac{2-q}{p+1}c_2\right)^{\frac{-p}{p+1}},$$

$$\lim_{t\to T_{31}^{*-}} \left(T_{31}^* - t\right)^{\alpha_2} u'(t) = \left(\frac{p+q-1}{p+1}\left(\frac{p+1}{2-q}c_2\right)^{\frac{p}{p+1}}\right)^{\frac{-p-1}{p+q-1}}.$$

Using lemmas 6.3, 6.5 and Theorem 6.4, we obtain

$$\begin{aligned}
&\lim_{t\to T_{31}^{*-}} \left(T_{31}^* - t\right)^{\alpha_3} u''(t) \\
&= \lim_{t\to T_{31}^{*-}} u(t)^p \left(c_1 + c_2 u'(t)^q\right) = \lim_{t\to T_{31}^{*-}} c_2 u(t)^p u'(t)^q \\
&= \left(c_2^{pq-p}(2-q)^p (p+1)^{p+q}(p+q-1)^{-2p-q}\right)^{\frac{1}{p+q-1}}.
\end{aligned}$$

■

Remark 6.7. The life-span $T_{31}^* := T_{31}^*(p,q,c_1,c_2)$of the solution to equation (0.1) is still unknown under the condition of Theorem 6.4 and it would has the properties:

(i) $T_{31}^*(p,q,c_1,c_2) \le \min\left\{T_{11}^*(q,c_2), T_{31}^*(p,0,0,c_1)\right\}$

(ii) $T_{31}^*(p,q,c_1,c_2) \to T_{11}^*(q,c_2)$ as $c_1 \to 0$.

(iii) $T_{31}^*(p,q,c_1,c_2) \to T_{31}^*(p,0,0,c_1)$ as $c_2 \to 0$.

7. Blow-up Phenomena for $q = 2$

In the particular case of $q = 2$, we obtain an interesting blow-up result and special blow-up constant.

Lemma 7.1. *Suppose* $q = 2$ *and* u *is the solution of* (0.1) *with* $E_2(0) \le 0, u_0 \ge 0,\ u_1 > 0,\ c_2 > 0,$ *then*

$$u'(t)^2 = \frac{c_1 + c_2 u_1^2}{c_2} e^{\frac{2c_2}{p+1}\left(u^{p+1} - u_0^{p+1}\right)} - \frac{c_1}{c_2}, \tag{7.1}$$

u, u' and u'' blow up at the same T_{32}^*; further,

$$T_{32}^* = \int_{u_0}^{\infty} \frac{dr}{\sqrt{\frac{c_1+c_2u_1^2}{c_2} e^{\frac{2c_2}{p+1}\left(r^{p+1}-u_0^{p+1}\right)} - \frac{c_1}{c_2}}}. \tag{7.2}$$

Proof. By (0.5), then

$$\ln\left|\frac{c_1+c_2u'(t)^2}{c_1+c_2u_1^2}\right| = \frac{2c_2}{p+1}\left(u(t)^{p+1}-u_0^{p+1}\right)$$

and (7.1) follows. Inducing lemma 6.1, u, u' and u'' are all large than 0; by (6.5) and (7.1), u blows up in finite time and also

$$\frac{u'(t)}{\sqrt{\frac{c_1+c_2u_1^2}{c_2} e^{\frac{2c_2}{p+1}\left(u^{p+1}-u_0^{p+1}\right)} - \frac{c_1}{c_2}}} = 1$$

and then (7.2) is obtained. Using (7.1) and (0.5) again, u' and u'' blow up at the same T_{32}^*. ■

Theorem 7.2. *Under the assumption of Lemma 10.1, we have*

$$\lim_{t\to T_{32}^{*-}} \left(\frac{-1}{\ln(T_{32}^*-t)}\right)^{\frac{1}{p+1}} u(t) = \left(\frac{c_2}{p+1}\right)^{-\frac{1}{p+1}}, \tag{7.3}$$

$$\lim_{t\to T_{32}^{*-}} \left(-\ln(T_{32}^*-t)\right)^{\frac{p}{p+1}} (T_{32}^*-t)\, u'(t) = \frac{1}{c_1+c_2u_1^2}\left(\frac{c_2}{p+1}\right)^{\frac{p}{p+1}} \tag{7.4}$$

and

$$\lim_{t\to T_{32}^{*-}} \left(-\ln(T_{32}^*-t)\right)^{\frac{p}{p+1}} (T_{32}^*-t)^2\, u''(t) = \left(\frac{1}{c_1+c_2u_1^2}\right)^2\left(\frac{c_2}{p+1}\right)^{\frac{p}{p+1}}. \tag{7.5}$$

Proof. By lemma 6.1 and (7.1), u, u' and u'' are all large than 0; after some computations we obtain

$$\begin{aligned}
&\lim_{t\to T_{32}^{*-}} -\ln(T_{32}^{*-}-t)u(t)^{-(p+1)} = \lim_{t\to T_{32}^{*-}} \frac{u(t)^{-p}u'(t)^{-1}}{(p+1)(T_{32}^{*-}-t)} \\
&= \lim_{t\to T_{32}^{*-}} \frac{pu(t)^{-(p+1)}+u(t)^{-p}u'(t)^{-2}u''(t)}{p+1} = \frac{c_2}{p+1}.
\end{aligned}$$

Using (7.3) and lemma 1.1 , we have

$$\begin{aligned}
&\lim_{t\to T_2^{*-}} \left(-\ln\left(T_2^*-t\right)\right)^{\frac{p}{p+1}} \left(T_2^*-t\right) u'(t) \\
&= \lim_{t\to T_2^{*-}} \frac{2\left(-\ln\left(T_2^*-t\right)\right)^{\frac{p}{p+1}}}{(p+1)\,u(t)^p} \frac{p+1}{2\left(c_1+c_2u_1^2\right)} = \frac{1}{c_1+c_2u_1^2}\left(\frac{c_2}{p+1}\right)^{\frac{p}{p+1}}.
\end{aligned}$$

Inducing (7.3) and (7.4), we conclude

$$\begin{aligned}
&\lim_{t\to T_{32}^{*-}} \left(-\ln\left(T_{32}^*-t\right)\right)^{\frac{p}{p+1}} \left(T_{32}^*-t\right)^2 u''(t) \\
&= \lim_{t\to T_{32}^{*-}} \left(-\ln\left(T_{32}^*-t\right)\right)^{\frac{p}{p+1}} \left(T_{32}^*-t\right)^2 \left(c_1u\left(t\right)^p + c_2u\left(t\right)^p u'\left(t\right)^q\right) \\
&= \left(\frac{1}{c_1+c_2u_1^2}\right)^2 \left(\frac{c_2}{p+1}\right)^{\frac{p}{p+1}}.
\end{aligned}$$

■

8. Blow-up Phenomena for $q > 2$

Under $q > 2$ we have the boundedness. for the solution u and estimate for the blow-up rate and blow-up constant for u' and u''.

Theorem 8.1. *For $q > 2$, if u is the solution of* (0.1) *and $c_1 > 0,\ c_2 > 0,\ u_0 \geq 0,\ u_1 > 0$, then u^{p+1} is bounded by $u_0^{p+1} + (p+1)\,u_1^{2-q}/\left(q-2\right)c_2$.*

Proof. By lemma 6.1 and $c_1 > 0,\ c_2 > 0,\ u' > 0, u'' > 0$, we have

$$\frac{u_1^{2-q} - u'\left(t\right)^{2-q}}{\left(q-2\right)c_2} = \int_0^t \frac{u'\left(r\right)u''\left(r\right)}{c_2u'\left(r\right)}dr \geq \int_0^t \frac{u'\left(r\right)u''\left(r\right)}{c_1+c_2u'\left(r\right)}dr = \frac{u\left(t\right)^{p+1} - u_0^{p+1}}{p+1}$$

and then

$$\frac{u_1^{2-q}}{\left(q-2\right)c_2} \geq \frac{u(t)^{p+1}}{p+1} - \frac{u_0^{p+1}}{p+1}.$$

After some computations one can easily obtain the following lemma used below to estimate the blow-up rate and blow-up constant for u' and u''. ■

Lemma 8.2. *Suppose u is the solution for the problem* (1.1) *and $b\left(t\right) = u'\left(t\right)^2,\ I\left(t\right) = b\left(t\right)^{-\frac{q-2}{4}}$. Then we have*

$$b'\left(t\right)^2 = 4b\left(t\right)u\left(t\right)^{2p}\left(c_1 + c_2u'\left(t\right)^q\right)^2, \tag{8.1}$$

$$b''(t) = 2u(t)^{2p}\left(c_1^2 + (2+q)c_1c_2u'(t)^q + c_2^2(1+q)u'(t)^{2q}\right)$$
$$+pu(t)^{p-1}u'(t)^2\left(c_1 + c_2u'(t)^q\right),$$

(8.3)
$$I''(t) = -\frac{q-2}{4}b^{-\frac{q+2}{4}}$$
$$\left(q\left(c_2^2u'(t)^2 - c_1^2\right) + 2pu(t)^{p-1}u'(t)^2\left(c_1 + c_2u'(t)^q\right)\right).$$

Theorem 8.3. *Under the assumption of Theorem 8.1 and $c_2u_1 \geq c_1$, then u' and u'' blow up at some finite time $t = T_{33}^*$. Further, the blow-up rate and blow-up constant for u' are $\frac{1}{q-1}$ and $((q-1)c_2u(T_{33}^*)^p)^{\frac{1}{1-q}}$ respectively; and the blow-up rate and blow-up constant for u'' are $\frac{q}{q-1}$ and $\left((q-1)^q c_2^{q+1}u(T_{33}^*)^{p(1+q)}\right)^{\frac{q}{1-q}}$ respectively, where $u(T_{33}^*)$ is given by*

$$u(T_{33}^*)^{p+1} = u_0^{p+1} + (p+1)\int_{u_1}^{\infty}\frac{rdr}{c_1 + c_2r^q}.$$

Proof. From lemma 6.1, $u(t) > 0$, $u'(t) > 0$ and $u''(t) > 0$ for $t > 0$. For $c_2u_1 \geq c_1$, using (8.3) we obtain that $I'(t)$, $I''(t) < 0$ for $t > 0$; thus there exists a finite number T_{33}^* so that $I(T_{33}^*) = 0$ and

$$\begin{aligned}
0 &\leq \lim_{t\to T_{33}^{*-}}\frac{1}{u''(t)} = \lim_{t\to T_{33}^{*-}}\frac{1}{u(t)^p\left(c_1 + c_2u'(t)^q\right)} \\
&\leq \lim_{t\to T_{33}^{*-}}\frac{1}{c_2u(t)^pu'(t)^q} = 0, \lim_{t\to T_{33}^{*-}}\left(T_{33}^* - t\right)u'(t)^{1-q} \\
&= \lim_{t\to T_{33}^{*-}}(q-1)u''(t)u'(t)^{-q} \\
&= \lim_{t\to T_{33}^{*-}}(q-1)u(t)^pu'(t)^{-q}\left(c_1 + c_2u'(t)^q\right) = (q-1)c_2u(T_{33}^*)^p
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to T_{33}^{*-}}\left(T_{33}^* - t\right)^{\frac{q}{q-1}}u''(t) &= \lim_{t\to T_{33}^{*-}}\left(T_{33}^* - t\right)^{\frac{q}{q-1}}u(t)^p\left(c_1 + c_2u'(t)^q\right) \\
&= \lim_{t\to T_{33}^{*-}}c_2\left(T_{33}^* - t\right)^{\frac{q}{q-1}}u(t)^pu'(t)^q \\
&= \left((q-1)^q c_2^{q+1}u(T_{33}^*)^{p(1+q)}\right)^{\frac{q}{1-q}}.
\end{aligned}$$

∎

Remark 8.4. The life-span $T_{33}^* := T_{33}^*(p, q, c_1, c_2)$of u' is still unknown under the condition of Theorem 8.1 and it would has the properties:

(i) $T_{33}^*(p, q, c_1, c_2) \leq \min\{T_{21}^*(q, c_2), T_{33}^*(p, 0, 0, c_1)\}$

(ii) $T_{33}^*(p, q, c_1, c_2) \to T_{21}^*(q, c_2)$ as $c_1 \to 0$.

(iii) $T_{33}^*(p, q, c_1, c_2) \to T_{33}^*(p, 0, 0, c_1)$ as $c_2 \to 0$.

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