

SELECTION PRINCIPLES RELATED TO α_i -PROPERTIES

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Abstract. We investigate selection principles which are motivated by Arhangel'skiĭ's α_i -properties, $i = 1, 2, 3, 4$, and their relations with classical selection principles. It will be shown that they are closely related to the selection principle S_1 and often are equivalent to it.

1. INTRODUCTION

In this paper we use the usual topological notation and terminology [8] and consider infinite Hausdorff spaces.

Let us fix some more notation and terminology regarding selection principles and families of open covers of a topological space which are necessary for this exposition. For more information in connection with selection principles we refer the interested reader to the survey papers [10, 19, 21].

Let \mathcal{A} and \mathcal{B} be collections of sets of an infinite set X .

The symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

For each sequence $(A_n : n \in \mathbb{N})$ of elements of \mathcal{A} there is a sequence $(b_n : n \in \mathbb{N})$ such that $b_n \in A_n$ for each $n \in \mathbb{N}$ and $\{b_n : n \in \mathbb{N}\}$ is an element of \mathcal{B} .

When both \mathcal{A} and \mathcal{B} are the collection \mathcal{O} of open covers of a space X , then $S_1(\mathcal{O}, \mathcal{O})$ defines the classical *Rothberger covering property* (see [16]).

There is an infinite game, denoted $G_1(\mathcal{A}, \mathcal{B})$, corresponding to $S_1(\mathcal{A}, \mathcal{B})$. Two players, ONE and TWO, play a round for each natural number n . In the n -th round ONE chooses a set $A_n \in \mathcal{A}$ and TWO responds by an element b_n from A_n . A

Received January 25, 2006, accepted August 9, 2006.

Communicated by Sen-Yen Shaw.

2000 *Mathematics Subject Classification*: 54D20, 54B20, 54D55, 54H11.

Key words and phrases: Selection principles, ω -cover, k -Cover, γ -Cover, γ_k -Cover, Boundedness of topological groups, Upper Fell topology, Upper Vietoris topology.

Supported, in part, by MNZŽS RS.

play $A_1, b_1; \dots; A_n, b_n; \dots$ is won by TWO if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

It is easy to see that if ONE does not have a winning strategy in the game $G_1(\mathcal{A}, \mathcal{B})$, then the corresponding selection hypothesis $S_1(\mathcal{A}, \mathcal{B})$ is true. However, the converse implication is not always true.

We introduce now new selection principles. The motivation for these definitions is the Arhangel'skii definition of α_i -properties, $i = 1, 2, 3, 4$, introduced in [1]. \mathcal{A} and \mathcal{B} are as above.

Definition 1. The symbol $\alpha_i(\mathcal{A}, \mathcal{B})$, $i = 1, 2, 3, 4$, denotes the following selection hypothesis:

For each sequence $(A_n : n \in \mathbb{N})$ of infinite elements of \mathcal{A} there is an element $B \in \mathcal{B}$ such that:

$\alpha_1(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \setminus B$ is finite;

$\alpha_2(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;

$\alpha_3(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;

$\alpha_4(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is nonempty.

Evidently, if all members of \mathcal{A} are infinite, then

$$\alpha_1(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_2(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_3(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_4(\mathcal{A}, \mathcal{B})$$

and

$$S_1(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_4(\mathcal{A}, \mathcal{B}).$$

However, if \mathcal{A} contains finite members, then $\alpha_1(\mathcal{A}, \mathcal{B})$ does not imply $\alpha_2(\mathcal{A}, \mathcal{B})$, while $\alpha_3(\mathcal{A}, \mathcal{B})$ fails (see [23]).

If for a space X and a point $x \in X$, Σ_x denotes the family of nontrivial sequences in X that converge to x , then X has the Arhangel'skii α_i -property, $i = 1, 2, 3, 4$, if for each $x \in X$ the property $\alpha_i(\Sigma_x, \Sigma_x)$, $i = 1, 2, 3, 4$, holds.

It is known that the four properties $\alpha_i(\Sigma_x, \Sigma_x)$ are different from each other [1, 14] and that the same holds in topological groups [20, 15]. However, it was shown in [18] that in function spaces $C_p(X)$ and in some hyperspaces [7] the properties α_2 , α_3 and α_4 are equivalent to each other and to the corresponding S_1 property. We shall see here that for some classes \mathcal{A} and \mathcal{B} the properties $\alpha_2(\mathcal{A}, \mathcal{B})$, $\alpha_3(\mathcal{A}, \mathcal{B})$ and $\alpha_4(\mathcal{A}, \mathcal{B})$ are closely related (and often equivalent) to $S_1(\mathcal{A}, \mathcal{B})$.

Let X be a topological space, $x \in X$, $A \subset X$. Then we use the following notation.

- \mathcal{O} : the collection of open covers of X ;
- Ω : the collection of ω -covers of X ;
- \mathcal{K} : the collection of k -covers of X ;

- Γ : the collection of γ -covers;
- Γ_k : the collection of γ_k -covers;
- Ω_x : the set $\{A \subset X \setminus \{x\} : x \in \overline{A}\}$;
- Σ_x : the set of all nontrivial sequences in X that converge to x .

An open cover \mathcal{U} of a space X is called an ω -cover (a k -cover) if every finite (compact) subset of X is contained in a member of \mathcal{U} and X is not a member of \mathcal{U} (i.e. we consider non-trivial covers).

An open cover \mathcal{U} of X is said to be a γ -cover (γ_k -cover) if it is infinite, and for each finite (compact) subset A of X the set $\{U \in \mathcal{U} : A \not\subseteq U\}$ is finite.

Observe that each infinite subset of a γ -cover (γ_k -cover) is still a γ -cover (γ_k -cover). So, we may suppose that such covers are countable. Each finite (compact) subset of an infinite (non-compact) space belongs to infinitely many elements of an ω -cover (k -cover) of the space.

Recall that a space X is said to be ω -Lindelöf (k -Lindelöf) if every ω -cover (k -cover) of X contains a countable ω -cover (k -cover).

2. GENERAL RESULTS

In this section we discuss covering and closure-type properties $\alpha_i(\mathcal{A}, \mathcal{B})$, $i = 2, 3, 4$, in topological spaces and identify some classes \mathcal{A} and \mathcal{B} for which these properties are equivalent to $S_1(\mathcal{A}, \mathcal{B})$.

We have already mentioned that every space X satisfying the Rothberger covering property $S_1(\mathcal{O}, \mathcal{O})$ satisfies also $\alpha_4(\mathcal{O}, \mathcal{O})$. The real line \mathbb{R} satisfies all the properties $\alpha_i(\mathcal{O}, \mathcal{O})$, $i = 2, 3, 4$, but \mathbb{R} does not have the Rothberger property.

However, we have the following result.

Theorem 2. *For an ω -Lindelöf space X the following are equivalent:*

- (1) X satisfies $\alpha_2(\Omega, \Gamma)$;
- (2) X satisfies $\alpha_3(\Omega, \Gamma)$;
- (3) X satisfies $\alpha_4(\Omega, \Gamma)$;
- (4) X satisfies $S_1(\Omega, \Gamma)$.

Proof. (3) \Rightarrow (4): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω -covers of X . Assume that for each $n \in \mathbb{N}$ we have $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ define

$$\mathcal{V}_n = \{U_{1,m_1} \cap \cdots \cap U_{n,m_n} : n < m_1 < m_2 < \cdots < m_n, U_{i,m_i} \in \mathcal{U}_i, i \leq n\} \setminus \{\emptyset\}.$$

Then each \mathcal{V}_n is an ω -cover of X . By (3) and the fact that each infinite subset of a γ -cover is also a γ -cover, there is an increasing sequence $n_1 < n_2 < \cdots$ in \mathbb{N}

and a γ -cover $\mathcal{V} = \{V_{n_i} : i \in \mathbb{N}\}$ such that for each $i \in \mathbb{N}$, $V_{n_i} \in \mathcal{V}_{n_i}$. Let for each $i \in \mathbb{N}$,

$$V_{n_i} = U_{1,m_1} \cap \cdots \cap U_{n_i,m_{n_i}}, \quad U_{j,m_j} \in \mathcal{U}_j, \quad j \leq n_i.$$

Put $n_0 = 0$. For each $i \geq 0$ and each n with $n_i < n \leq n_{i+1}$ let H_n be the n -th coordinate in the chosen representation of $V_{n_{i+1}}$:

$$H_n = U_{n,m_{n_{i+1}}}.$$

For each $n \in \mathbb{N}$, $H_n \in \mathcal{U}_n$ and the set $\mathcal{H} := \{H_n : n \in \mathbb{N}\}$ is a γ -cover of X because \mathcal{V} is a refinement of \mathcal{H} , and $X \notin \mathcal{H}$. Therefore, X satisfies $S_1(\Omega, \Gamma)$.

(4) \Rightarrow (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω -covers of X and let for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. We shall use the fact that $S_1(\Omega, \Gamma)$ is equivalent to ONE has no winning strategy in the game $G_1(\Omega, \Gamma)$ on X [17]. Define the following strategy σ for ONE. ONE's first move is $\sigma(\emptyset) = \mathcal{U}_1$. Assuming that the set $U_{1,m_{i_1}} \in \mathcal{U}_1$ is TWO's response, ONE plays $\sigma(U_{1,m_{i_1}})$ to be $\mathcal{V}(1, m_{i_1}) = \{U_{1,m} : m > m_{i_1}\}$, still an ω -cover of X . If TWO now chooses a set $U_{1,m_{i_2}} \in \mathcal{V}(1, m_{i_1})$, ONE plays $\sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}}) = \mathcal{V}(1, m_{i_2}) = \{U_{1,m} : m > m_{i_2}\}$ which is still an ω -cover of X . Then TWO chooses a set $U_{1,m_{i_3}} \in \sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}})$. And so on.

[Note: For each $n \in \mathbb{N}$ and each \mathcal{U}_n moves of ONE form a new sequence of ω -covers and ensure that from each \mathcal{U}_n TWO chooses infinitely many elements.]

Since σ is not a winning strategy for ONE, consider a σ -play

$$\sigma(\emptyset), U_{1,m_{i_1}}; \sigma(U_{1,m_{i_1}}), U_{1,m_{i_2}}; \sigma(U_{1,m_{i_1}}, U_{1,m_{i_2}}), U_{1,m_{i_3}}; \cdots$$

lost by ONE. That means that the sequence \mathcal{W} consisting of TWO's moves is a γ -cover of X . As it contains infinitely many elements from each \mathcal{U}_n , $n \in \mathbb{N}$, \mathcal{W} witnesses for the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ that X has property $\alpha_2(\Omega, \Gamma)$. ■

Similarly to the proof of Theorem 2 we can prove the following two theorems. For that we use:

- (i) X satisfies $S_1(\mathcal{K}, \Gamma)$ iff ONE has no winning strategy in the game $G_1(\mathcal{K}, \Gamma)$ on X (see [6]).
- (ii) X satisfies $S_1(\mathcal{K}, \Gamma_k)$ iff ONE has no winning strategy in the game $G_1(\mathcal{K}, \Gamma_k)$ on X (see [12]).

Theorem 3. For a k -Lindelöf non-compact space X , the properties $\alpha_2(\mathcal{K}, \Gamma)$, $\alpha_3(\mathcal{K}, \Gamma)$, $\alpha_4(\mathcal{K}, \Gamma)$ and $S_1(\mathcal{K}, \Gamma)$ are equivalent.

Theorem 4. For a k -Lindelöf non-compact space X , the properties $\alpha_2(\mathcal{K}, \Gamma_k)$, $\alpha_3(\mathcal{K}, \Gamma_k)$, $\alpha_4(\mathcal{K}, \Gamma_k)$ and $S_1(\mathcal{K}, \Gamma_k)$ are equivalent.

We also have the following results.

Theorem 5. For a space X and $\mathcal{B} \in \{\Gamma, \Gamma_k\}$ the following statements are equivalent:

- (1) X satisfies $\alpha_2(\Gamma_k, \mathcal{B})$;
- (2) X satisfies $\alpha_3(\Gamma_k, \mathcal{B})$;
- (3) X satisfies $\alpha_4(\Gamma_k, \mathcal{B})$;
- (4) X satisfies $S_1(\Gamma_k, \mathcal{B})$.

Proof. We have to prove only (3) \Rightarrow (4) and (4) \Rightarrow (1).

(3) \Rightarrow (4): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of γ_k -covers of X . Enumerate every \mathcal{U}_n bijectively as $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. For all $n, m \in \mathbb{N}$ define

$$V_{n,m} = U_{1,m} \cap U_{2,m} \cap \dots \cap U_{n,m}.$$

Then for each n the set $\mathcal{V}_n = \{V_{n,m} : m \in \mathbb{N}\}$ is a γ_k -cover of X , because \mathcal{U}_n 's are γ_k -covers. By (4) applied to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ there is an increasing sequence $n_1 < n_2 < \dots$ in \mathbb{N} and a cover $\mathcal{V} = (V_{n_i, m_i} : i \in \mathbb{N}) \in \mathcal{B}$ such that for each $i \in \mathbb{N}$, $V_{n_i, m_i} \in \mathcal{V}_{n_i}$. Put $n_0 = 0$. For each $i \geq 0$, each j with $n_i < j \leq n_{i+1}$ and each $V_{n_{i+1}, m_{i+1}} = U_{1, m_{i+1}} \cap \dots \cap U_{n_{i+1}, m_{i+1}}$ put

$$H_j = U_{j, m_{i+1}}.$$

For each $j \in \mathbb{N}$, $H_j \in \mathcal{U}_j$ and the set $\{H_j : j \in \mathbb{N}\}$ is in \mathcal{B} because this set is refined by \mathcal{V} which is in \mathcal{B} . So, X satisfies $S_1(\Gamma_k, \mathcal{B})$.

(4) \Rightarrow (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of γ_k -covers of X . Suppose that for each $n \in \mathbb{N}$, we have $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Choose an increasing sequence $k_1 < k_2 < \dots < k_p < \dots$ of positive integers and for each n and each k_i consider $\mathcal{V}(n, k_i) := \{U_{n,m} : m \geq k_i\}$. Then each $\mathcal{V}(n, k_i)$, $n, i \in \mathbb{N}$, is a γ_k -cover of X . Apply now (1) to the sequence $(\mathcal{V}(n, k_i) : i \in \mathbb{N}, n \in \mathbb{N})$ from Γ_k and find a sequence $(V_{n, k_i} : i, n \in \mathbb{N})$ such that for each $(n, i) \in \mathbb{N} \times \mathbb{N}$, $V_{n, k_i} \in \mathcal{V}(n, k_i)$ and the set $\mathcal{W} := \{V_{n, k_i} : n, i \in \mathbb{N}\} \in \mathcal{B}$. It is easy to see that \mathcal{W} can be chosen in such a way that for each $n \in \mathbb{N}$ the set $\mathcal{U}_n \cap \mathcal{W}$ is infinite. Therefore, \mathcal{W} witnesses for the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ that X has property $\alpha_2(\Gamma_k, \mathcal{B})$. ■

Notice that in a similar way one can prove that for a space X the properties $\alpha_2(\Gamma, \Gamma)$, $\alpha_3(\Gamma, \Gamma)$, $\alpha_4(\Gamma, \Gamma)$ and $S_1(\Gamma, \Gamma)$ are equivalent.

As B. Tsaban observed [23], the property $\alpha_1(\Gamma, \Gamma)$ is strictly stronger than $S_1(\Gamma, \Gamma)$.

3. APPLICATIONS TO TOPOLOGICAL GROUPS

Let (G, \cdot, τ) be a topological group with the neutral element e and let \mathcal{B}_e be a local base at e . For each $U \in \mathcal{B}_e$ with $U \neq G$ define

$$\begin{aligned} o(U) &= \{x \cdot U : x \in G\}, \\ \mathcal{O}(e) &= \{o(U) : U \in \mathcal{B}_e\}; \\ \omega(U) &= \{F \cdot U : F \in \mathbb{F}(G)\}, \\ \Omega(e) &= \{\omega(U) : U \in \mathcal{B}_e \text{ and there is no } F \in \mathbb{F}(G) \text{ with } F \cdot U = G\}; \\ k(U) &= \{K \cdot U : K \in \mathbb{K}(G)\}, \\ \mathcal{K}(e) &= \{k(U) : U \in \mathcal{B}_e \text{ and there is no } K \in \mathbb{K}(G) \text{ with } K \cdot U = G\}. \end{aligned}$$

Then clearly, $\mathcal{O}(e) \subset \mathcal{O}$, $\Omega(e) \subset \Omega$; $\mathcal{K}(e) \subset \mathcal{K}$.

In [2] (see also [9, 13, 22]) Menger-bounded, Rothberger-bounded and Hurewicz-bounded topological groups have been studied. A topological group G is *Menger-bounded* (*Rothberger-bounded*, *Hurewicz-bounded*) if it satisfies the selection principle $\mathcal{S}_1(\Omega(e), \mathcal{O})$ ($\mathcal{S}_1(\mathcal{O}(e), \mathcal{O})$, $\mathcal{S}_1(\Omega(e), \Gamma)$).

We have the following results. Their proofs are similar, so we prove only the first of them.

Theorem 6. *For a topological group G the following are equivalent:*

- (1) G satisfies $\alpha_4(\Omega(e), \Gamma)$;
- (2) G satisfies $\mathcal{S}_1(\Omega(e), \Gamma)$;
- (3) G satisfies $\mathcal{S}_1(\mathcal{K}(e), \Gamma)$.

Proof. The implications (2) \Rightarrow (1) and (2) \Rightarrow (3) are obvious.

(1) \Rightarrow (2): Let $(U_n : n \in \mathbb{N})$ be a sequence of elements of \mathcal{B}_e . For each $n \in \mathbb{N}$ let $V_n = U_1 \cap U_2 \cap \dots \cap U_n$ be a member of \mathcal{B}_e . If we now apply (1) to the sequence $(V_n : n \in \mathbb{N})$ we find an increasing sequence $n_1 < n_2 < \dots$ in \mathbb{N} and finite sets $F_{n_i} \subset G$, $i \in \mathbb{N}$, so that $\{F_{n_i} \cdot V_{n_i} : n \in \mathbb{N}\}$ is a γ -cover of G . If $n_0 = 0$, then for each positive integer n with $n_{i-1} < n \leq n_i$, $i \in \mathbb{N}$, put $F_n = F_{n_i}$ and U_n to be the n -th component in the representation $U_1 \cap \dots \cap U_{n_i}$ of V_{n_i} . Evidently, $\{F_n \cdot U_n : n \in \mathbb{N}\}$ is a γ -cover of G , i.e. the sequence $(F_n : n \in \mathbb{N})$ guaranties for $(U_n : n \in \mathbb{N})$ that G satisfies $\mathcal{S}_1(\Omega(e), \Gamma)$.

(3) \Rightarrow (2): Let $(U_n : n \in \mathbb{N})$ be a sequence of elements of \mathcal{B}_e . For each n pick a $V_n \in \mathcal{B}_e$ so that $V_n^2 \subset U_n$. By (3) choose a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of G such that $\{K_n \cdot V_n : n \in \mathbb{N}\}$ is a γ -cover of G . Next, for

each n pick a finite set F_n in G such that $K_n \subset F_n \cdot V_n$. Then for each n we have $K_n \cdot V_n \subset (F_n \cdot V_n) \cdot V_n \subset F_n \cdot U_n$ and one concludes that $\{F_n \cdot U_n : n \in \mathbb{N}\}$ is a γ -cover of G . ■

Notice that $\alpha_2(\Omega(e), \Gamma)$ and $\alpha_3(\Omega(e), \Gamma)$ are also equivalent to the properties listed in the theorem above.

Theorem 7. *For a topological group G the following are equivalent:*

- (1) G satisfies $\alpha_4(\Omega(e), \Gamma_k)$;
- (2) G satisfies $S_1(\Omega(e), \Gamma_k)$;
- (3) G satisfies $S_1(\mathcal{K}(e), \Gamma_k)$.

4. $\alpha_i(\mathcal{A}, \mathcal{B})$ PROPERTIES IN HYPERSPACES

In this section we consider $\alpha_i(\mathcal{A}, \mathcal{B})$ properties, $i = 2, 3, 4$, in hyperspaces. We begin with some definitions that we need.

For a (Hausdorff) space X by 2^X we denote the family of all closed subsets of X . $\mathbb{K}(X)$ is the collection of all non-empty compact subsets of X , and $\mathbb{F}(X)$ denotes the family of all non-empty finite subsets of X . If A is a subset of X and \mathcal{A} a family of subsets of X , then we write

$$A^+ = \{F \in 2^X : F \subset A\}, \quad \mathcal{A}^+ = \{A^+ : A \in \mathcal{A}\}.$$

Notice that we use the same symbol F to denote a closed subset of X and the point F in 2^X ; from the context it will be clear what F is.

The *upper Fell topology* F^+ on 2^X is the topology whose base is the collection

$$\{(K^c)^+ : K \in \mathbb{K}(X)\} \cup \{2^X\},$$

while the *upper Vietoris topology* V^+ has basic sets of the form U^+ , U open in X . It is clear that $(\mathbb{K}(X), V^+)$ and $(\mathbb{F}(X), V^+)$ are considered as subspaces of $(2^X, V^+)$.

In [7] it was shown that in $(2^X, F^+)$ each of Arhangel'skii's α_2 , α_3 and α_4 properties is equivalent to $S_1(\Sigma_E, \Sigma_E)$, $E \in 2^X$. We discuss here some other properties. For similar consideration see [5, 11].

Theorem 8. *If X is a space whose all open subspaces are k -Lindelöf and $E \in 2^X$, then the following statements are equivalent:*

- (1) $(2^X, F^+)$ satisfies $\alpha_4(\Omega_E, \Sigma_E)$;
- (2) $(2^X, F^+)$ satisfies $S_1(\Omega_E, \Sigma_E)$.

Proof. We have to prove only (1) implies (2). Let $(\mathcal{A}_n : n \in \mathbb{N})$ be a sequence of elements of Ω_E . Since each open subspace of X is k -Lindelöf, $(2^X, \mathbb{F}^+)$ has countable tightness (see [4, 3]) and we may assume that for each $n \in \mathbb{N}$, \mathcal{A}_n is countable, say $\mathcal{A}_n = \{A_{n,m} : m \in \mathbb{N}\}$. For each n let \mathcal{B}_n be the collection of all sets of the form

$$A_{1,m_1} \cup A_{2,m_2} \cup \cdots \cup A_{n,m_n}, \quad A_{i,m_i} \in \mathcal{A}_i, \quad i \leq n.$$

Then each \mathcal{B}_n belongs to Ω_E . Apply (1) to the sequence $(\mathcal{B}_n : n \in \mathbb{N})$ of elements of Ω_E . There exist an increasing sequence $n_1 < n_2 < \cdots$ in \mathbb{N} and a sequence $\mathcal{B} := (B_{n_i} : i \in \mathbb{N}) \in \Sigma_E$ such that for each $i \in \mathbb{N}$, $B_{n_i} \in \mathcal{B}_{n_i}$. Put $n_0 = 0$ and define the sequence $(S_n : n \in \mathbb{N})$ in the following manner:

If $i \geq 0$, then for each n with $n_i < n \leq n_{i+1}$ define S_n to be A_{n,m_n} in the chosen representation of $B_{n_{i+1}}$.

Note that for each $n \in \mathbb{N}$, $S_n \in \mathcal{A}_n$ and evidently the sequence $\mathcal{S} := (S_n : n \in \mathbb{N})$ is an element of Σ_E . So, \mathcal{S} is a selector for the original sequence $(\mathcal{A}_n : n \in \mathbb{N})$ showing that $(2^X, \mathbb{F}^+)$ satisfies (2). ■

In what follows we shall need the following two simple lemmas. Because their proofs are similar we prove only the first of them.

Lemma 9. *For a space X and an open cover \mathcal{W} of $(\mathbb{K}(X), \mathbb{V}^+)$ the following holds: \mathcal{W} is an ω -cover of $(\mathbb{K}(X), \mathbb{V}^+)$ if and only if $\mathcal{U}(\mathcal{W}) := \{U \subset X : U \text{ is open in } X \text{ and } U^+ \subset W \text{ for some } W \in \mathcal{W}_n\}$ is a k -cover of X .*

Proof. Let \mathcal{W} be an ω -cover of $(\mathbb{K}(X), \mathbb{V}^+)$ and let K be a compact subset of X . Then there exists $W \in \mathcal{W}$ such that $K \in W$ and consequently there is an open set $U \subset X$ with $K \in U^+ \subset W$. It is understood, $U \in \mathcal{U}(\mathcal{W})$. On the other hand, $K \subset U$, i.e. $\mathcal{U}(\mathcal{W})$ is a k -cover of X .

Conversely, let $\mathcal{U}(\mathcal{W})$ be a k -cover of X and let $\{K_1, \dots, K_m\}$ be a finite subset of $(\mathbb{K}(X), \mathbb{V}^+)$. Then $K = \bigcup_{i=1}^m K_i$ is a compact subset of X and thus K is contained in some $U \in \mathcal{U}(\mathcal{W})$; pick $W \in \mathcal{W}$ such that $U^+ \subset W$. From $K_i \subset U$ for each $i \leq m$, it follows $\{K_1, \dots, K_m\} \subset U^+ \subset W$ which just means that \mathcal{W} is an ω -cover of $(\mathbb{K}(X), \mathbb{V}^+)$. ■

Lemma 10. *For a space X and an open cover \mathcal{W} of $(\mathbb{F}(X), \mathbb{V}^+)$ the following holds: \mathcal{W} is an ω -cover of $(\mathbb{F}(X), \mathbb{V}^+)$ if and only if $\mathcal{U}(\mathcal{W}) := \{U \subset X : U \text{ is open in } X \text{ and } U^+ \subset W \text{ for some } W \in \mathcal{W}_n\}$ is an ω -cover of X .*

We use now the last two lemmas to prove the next two propositions.

Proposition 11. *A space X is k -Lindelöf if and only if $(\mathbb{K}(X), \mathbb{V}^+)$ is ω -Lindelöf.*

Proof. Let X be a k -Lindelöf space and let \mathcal{W} be an ω -cover of $(\mathbb{K}(X), \mathbb{V}^+)$. By Lemma 9 (and notation from that lemma), $\mathcal{U}(\mathcal{W})$ is a k -cover of X . Choose a countable family $\{U_i : i \in \mathbb{N}\} \subset \mathcal{U}(\mathcal{W})$ which is a k -cover of X . For each $i \in \mathbb{N}$ pick $W_i \in \mathcal{W}$ such that $U_i^+ \subset W_i$. Again by Lemma 9 $\{W_i : i \in \mathbb{N}\} \subset \mathcal{W}$ is an ω -cover of $(\mathbb{K}(X), \mathbb{V}^+)$.

Let us show the converse. Let \mathcal{U} be a k -cover of X . It is easy to check that \mathcal{U}^+ is an ω -cover of $(\mathbb{K}(X), \mathbb{V}^+)$. Choose a countable collection $\{U_i^+ : i \in \mathbb{N}\} \subset \mathcal{U}^+$ which is an ω -cover of $(\mathbb{K}(X), \mathbb{V}^+)$. Then $\{U_i : i \in \mathbb{N}\} \subset \mathcal{U}$ is a k -cover of X , i.e. X is a k -Lindelöf space. ■

Similarly, by using Lemma 10, one obtains

Proposition 12. *A space X is ω -Lindelöf if and only if $(\mathbb{F}(X), \mathbb{V}^+)$ is ω -Lindelöf.*

Theorem 13. *For a k -Lindelöf space X the following are equivalent:*

- (1) $(\mathbb{K}(X), \mathbb{V}^+)$ satisfies $\alpha_2(\Omega, \Gamma)$;
- (2) $(\mathbb{K}(X), \mathbb{V}^+)$ satisfies $\alpha_3(\Omega, \Gamma)$;
- (3) $(\mathbb{K}(X), \mathbb{V}^+)$ satisfies $\alpha_4(\Omega, \Gamma)$;
- (4) $(\mathbb{K}(X), \mathbb{V}^+)$ satisfies $S_1(\Omega, \Gamma)$;
- (5) X satisfies $S_1(\mathcal{K}, \Gamma_k)$.

Proof. (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) hold for any space.

(4) \Rightarrow (1): By Proposition 11 the space $(\mathbb{K}(X), \mathbb{V}^+)$ is ω -Lindelöf. It remains to apply Theorem 12.

(4) \Rightarrow (5): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of k -covers of X . Then $(\mathcal{U}_n^+ : n \in \mathbb{N})$ is a sequence of ω -covers of $(\mathbb{K}(X), \mathbb{V}^+)$. Indeed, fix n and let $\{K_1, \dots, K_m\}$ be a finite subset of $\mathbb{K}(X)$. Then $K = K_1 \cup \dots \cup K_m$ is a compact subset of X and thus there is $U \in \mathcal{U}$ with $K \subset U$. This means that for each $i \leq m$, $K_i \subset U$, i.e. $K_i \in U^+$. Therefore $\{K_1, \dots, K_m\} \subset U^+$ and \mathcal{U}_n is an ω -cover of $\mathcal{K}(X)$. By (4) for each n , choose an element U_n^+ in \mathcal{U}_n^+ such that the set $\mathcal{U}^+ = \{U_n^+ : n \in \mathbb{N}\}$ is a γ -cover of $(\mathbb{K}(X), \mathbb{V}^+)$. We prove that $\{U_n : n \in \mathbb{N}\}$ is a γ_k -cover of X . Let K be a compact subset of X . Then there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ we have $K \in U_n^+$, hence $K \subset U_n$. It shows that $\{U_n : n \in \mathbb{N}\}$ is really a γ_k -cover of X , i.e. that (5) holds.

(5) \Rightarrow (4): Let $(\mathcal{W}_n : n \in \mathbb{N})$ be a sequence of ω -covers of $(\mathbb{K}(X), \mathbb{V}^+)$. For

each n let

$$\mathcal{U}_n = \{U \subset X : U \text{ is open in } X \text{ and } U^+ \subset W \text{ for some } W \in \mathcal{W}_n\}.$$

By Lemma 9 each \mathcal{U}_n is a k -cover of X . By (5) applied to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ one can find a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and the set $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a γ_k -cover of X . For each $U_n \in \mathcal{U}$ pick an element $W_n \in \mathcal{W}_n$ so that $U_n^+ \subset W_n$. We claim that $\{W_n : n \in \mathbb{N}\}$ is a γ -cover of $(\mathbb{K}(X), \mathcal{V}^+)$ and so it witnesses for $(\mathcal{W}_n : n \in \mathbb{N})$ that (4) is satisfied. Let $K \in \mathbb{K}(X)$. Then there is n_0 such that for each $n \geq n_0$, $K \subset U_n$, i.e. $K \in U_n^+ \subset W_n$. ■

It is not difficult to verify that in a similar way, using Proposition 12 and Theorem 2, one obtains the following theorem.

Theorem 14. *For an ω -Lindelöf space X the following are equivalent:*

- (1) $(\mathbb{F}(X), \mathcal{V}^+)$ satisfies $\alpha_2(\Omega, \Gamma)$;
- (2) $(\mathbb{F}(X), \mathcal{V}^+)$ satisfies $\alpha_3(\Omega, \Gamma)$;
- (3) $(\mathbb{F}(X), \mathcal{V}^+)$ satisfies $\alpha_4(\Omega, \Gamma)$;
- (4) $(\mathbb{F}(X), \mathcal{V}^+)$ satisfies $S_1(\Omega, \Gamma)$;
- (5) X satisfies $S_1(\Omega, \Gamma)$.

REFERENCES

1. A. V. Arhangel'skiĭ, The frequency spectrum of a topological space and the classification of spaces, *Soviet Math. Doklady*, **13** (1972), 1185-1189.
2. L. Babinkostova, Lj. D. R. Kočinac, M. Scheepers, Combinatorics of open covers (XI): Menger- and Rothberger-bounded groups, *Topology Appl.*, **154** (2007), 1269-1280.
3. A. Caserta, G. Di Maio, Lj. D. R. Kočinac, E. Meccariello Applications of k -covers II, *Topology Appl.*, **153** (2006), 3277-3293.
4. C. Costantini, Ľ. Holá and P. Vitolo, Tightness, character and related properties of hyperspace topologies, *Topology Appl.*, **142** (2004), 245-292.
5. G. Di Maio, Lj. D. R. Kočinac and E. Meccariello, Selection principles and hyperspace topologies, *Topology Appl.* **153** (2005), 912-923.
6. G. Di Maio, Lj. D. R. Kočinac, E. Meccariello, Applications of k -covers, *Acta Math. Sinica, English Series*, **22** (2006), 1191-1160.
7. G. Di Maio, Lj. D. R. Kočinac and T. Nogura, Convergence properties of hyperspaces, *J. Korean Math. Soc.*, **44** (2007), 845-854.
8. R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

9. Lj. D. R. Kočinac, Selection principles in uniform spaces, *Note di Mat.*, **22(2)** (2003/2004), 127-139.
10. Lj. D. R. Kočinac, Selected results on selection principles, In: *Sh. Rezapour (ed.), Proc. 3rd Seminar on Geometry and Topology*, (July 15-17, 2004, Tabriz, Iran), 71-104.
11. Lj. D. R. Kočinac, The Reznichenko property and the Pytkeev property in hyperspaces, *Acta Math. Hungar.*, **107** (2005), 225-233.
12. Lj. D. R. Kočinac, γ -sets, γ_k -sets and hyperspaces, *Math. Balkanica*, **19** (2005), 109-118.
13. M. Machura and B. Tsaban, The combinatorics of the Baer-Specker group, *Israel J. Math.*, to appear.
14. T. Nogura, The product of $\langle \alpha_i \rangle$ -spaces, *Topology Appl.*, **21** (1985), 251-259.
15. T. Nogura, D. Shakhmatov and Y. Tanaka, α_4 -property versus A -property in topological spaces and groups, *Stud. Sci. Math. Hung.*, **33** (1997), 351-362.
16. F. Rothberger, Eine Verschärfung der Eigenschaft C , *Fund. Math.*, **30** (1938), 50-55.
17. M. Scheepers, Combinatorics of open covers I: Ramsey theory, *Topology Appl.*, **69** (1996), 31-62.
18. M. Scheepers, $C_p(X)$ and Arhangel'skii's α_i -spaces, *Topology Appl.*, **89** (1998), 265-275.
19. M. Scheepers, Selection principles and covering properties in Topology, *Note di Mat.*, **22(2)** (2003/2004), 3-41.
20. D.B. Shakhmatov, α_i -properties in Fréchet-Urysohn topological groups, *Topology Proc.*, **15** (1990), 143-183.
21. B. Tsaban, *Some new directions in infinite-combinatorial topology*, In: J. Bagaria, S. Todorčević (eds.), *Topics in Set Theory and its Applications*, Birkhäuser, 2006, 225-255.
22. B. Tsaban, o -bounded groups and other topological groups with strong combinatorial properties, *Proc. Amer. Math. Soc.*, **134** (2006), 881-891.
23. B. Tsaban, On the Kocinac α_i properties, *Topology Appl.*, **155** (2007), 141-145.

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