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INEQUALITIES AND MONOTONICITY FOR THE RATIO OF GAMMA FUNCTIONS

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Abstract. In this article, using Stirling's formula, the series-expansion of digamma functions and other techniques, some inequalities and monotonicity concerning the ratio of gamma functions are obtained, several inequalities involving the geometric mean of natural numbers are deduced.

1. Introduction

In [1], Dr. H. Alzer proved that the inequalities

$$\frac{n+2\sqrt{2}-1}{n+1} \cdot \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} < \frac{n+2}{n+1} \tag{1}$$

hold for all integers $n \ge 1$. The lower and upper bounds in (1) are the best possible.

He also proved in [2] that the inequality

$$\frac{[\Gamma(x+2)]^{1/(x+1)}}{[\Gamma(x+1)]^{1/x}} < \frac{x+2}{x+1} \tag{2}$$

holds for $x \ge 2$.

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Since $\Gamma(n+1) = n!$, the right hand side in (1) can be deduced from inequality (2) only if we let $x = n \geq 2$. Moreover, the right hand side in (1) refines the inequality

$$\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} < \frac{n+1}{n}, \tag{3}$$

which was obtained in [13] by H. Minc and L. Sathre.

Recently, in [19] and [23], the second author obtained the following

$$\frac{n+k}{n+m+k} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} < \sqrt{\frac{n+k}{n+m+k}} \tag{4}$$

for positive integers n and m and nonnegative integer k.

The inequality (3) was refined by H. Alzer in [3]: Let $n \in \mathbb{N}$, then, for any $r > 0$, we have

$$\frac{n}{n+1} \cdot \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{5}$$

The lower and upper bounds are the best possible.

Many new and simple proofs of the inequalities in (5) and some generalizations were given in [5, 6, 7, 8, 12, 13, 16, 18, 23, 25, 31, 32, 35, 36].

The left hand side of inequality (5) was generalized in [17]: Let n and m be natural numbers, k a nonnegative integer. Then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{6}$$

where r is any given positive real number. The lower bound is the best possible.

The integral analogue of (6) was presented in [9] and [16]: Let $b > a > 0$ and $\delta > 0$ be real numbers, then, for any given positive $r \in \mathbb{R}$, we have

$$\begin{aligned} \frac{b}{b+\delta} &< \left(\frac{b+\delta-a}{b-a} \cdot \frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} \\ &= \left(\frac{1}{b-a}\int_a^b x^r \mathrm{d}x \Big/ \frac{1}{b+\delta-a}\int_a^{b+\delta} x^r \mathrm{d}x\right)^{1/r} \\ &< \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \end{aligned} \tag{7}$$

The lower and upper bounds in (7) are the best possible.

The inequality (7) was generalized to an inequality for linear positive functionals in [8].

Recently, results related to those above were obtained in [20]. These results were generalisations for monotonic sequences involving convex functions as follows:

- For $a > 1$, let $n \in \mathbb{N}$ and $r > 0$, then

$$\left(\frac{1}{n}\sum_{i=1}^{n} a^{ir} \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} a^{ir}\right) > \frac{1}{a}. \tag{8}$$

- For $n, m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ and $r > 0$, we have

$$\frac{1}{a^m} < \left(\frac{1}{a^n}\sum_{i=k+1}^{n+k} a^{ir} \Big/ \frac{1}{a^{n+m}}\sum_{i=k+1}^{n+m+k} a^{ir}\right)^{1/r}, \tag{9}$$

  that is,

$$\frac{1}{a^{m(r+1)}} \cdot \sum_{i=k+1}^{n+k} a^{ir} \Big/ \sum_{i=k+1}^{n+m+k} a^{ir}, \tag{10}$$

  where $a > 1$ is a positive real number.

- If $\{a_i\}_{i\in\mathbb{N}}$ is an increasing, positive sequence such that $\{i(\frac{a_{i+1}}{a_i} - 1)\}_{i\in\mathbb{N}}$ increases, then we have

$$\frac{a_n}{a_{n+1}} \cdot \sqrt[n]{\prod_{i=1}^{n}(a_i + a_n)} \Big/ \sqrt[n+1]{\prod_{i=1}^{n+1}(a_i + a_{n+1})} \cdot \sqrt[n]{\prod_{i=1}^{n} a_i} \Big/ \sqrt[n+1]{\prod_{i=1}^{n+1} a_i}. \tag{11}$$

- If φ is an increasing, convex, positive function defined on $(0, \infty)$ such that $\left\{\varphi(i)\left[\frac{\varphi(i)}{\varphi(i+1)} - 1\right]\right\}_{i\in\mathbb{N}}$ decreases, then

$$\frac{[\varphi(n)]^{n/\varphi(n)}}{[\varphi(n+1)]^{(n+1)/\varphi(n+1)}} \cdot \sqrt[\varphi(n)]{\prod_{i=1}^{n}[\varphi(i) + \varphi(n)]} \Big/ \sqrt[\varphi(n+1)]{\prod_{i=1}^{n+1}[\varphi(i) + \varphi(n+1)]}. \tag{12}$$

These inequalities generalize those obtained in [11], [18], and [23].

In this article, we will prove the following inequalities

Theorem 1. *For $m, n \in \mathbb{N}$ and nonnegative integer k, we have*

$$\frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}} > \frac{n+k+1}{n+m+k+1}. \tag{13}$$

Theorem 2. *The function*

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{14}$$

is decreasing in $x \geq 1$ *for fixed* $y \geq 0$. *For positive real numbers* x *and* y, *we have*

$$\frac{x+y+1}{x+y+2} \cdot \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{15}$$

Remark 1. If we take $x, y \in \mathbb{N}$, then the right hand side of (4) and inequality (13) follow from (15).

2. Proofs of Theorems

Proof of Theorem 1. Inequality (13) can be rearranged so that we have

$$\frac{n+k+1}{\sqrt[n]{(n+k)!/k!}} < \frac{n+m+k+1}{\sqrt[n+m]{(n+m+k)!/k!}},$$

which is equivalent to

$$\frac{n+k+1}{\sqrt[n]{(n+k)!/k!}} < \frac{n+k+2}{\sqrt[n+1]{(n+k+1)!/k!}}. \tag{16}$$

When $k = 0$, inequality (16) follows from the right inequality in (1).

When $k \geq 1$, the inequality (16) can be rewritten as

$$\left[\frac{(n+k)!}{k!}\right]^{1/n} > \frac{(n+k+1)^{n+2}}{(n+k+2)^{n+1}}. \tag{17}$$

In [11] and [14, p. 184], the following inequalities were given for $n \in \mathbb{N}$

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\frac{1}{12n}. \tag{18}$$

Inequality (18) is related to the Stirling's formula.

By substituting the inequalities in (18) into the left term of inequality (17), we see that it is sufficient to prove the following

$$\left[\sqrt{2\pi(n+k)}\left(\frac{n+k}{e}\right)^{n+k}\right]^{1/n} > \frac{(n+k+1)^{n+2}}{(n+k+2)^{n+1}}\left[\sqrt{2\pi k}\left(\frac{k}{e}\right)^k \exp\frac{1}{12k}\right]^{1/n}. \tag{19}$$

Taking logarithm on both sides of inequality (20), simplifying directly and using standard arguments, we obtain

$$\begin{aligned}\frac{2k+1}{2n}\ln\left(1+\frac{n}{k}\right)+(n+1)\ln\left(1+\frac{1}{n+k+1}\right)-\ln\left(1+\frac{1}{n+k}\right)\\-\frac{1}{12kn}-1>0.\end{aligned}\tag{20}$$

In [10, pp. 367-368], [14, pp. 273-274] and [21], we have for $t>0$

$$\ln\left(1+\frac{1}{t}\right)>\frac{2}{2t+1},\tag{21}$$

$$\ln(1+t)<\frac{t(2+t)}{2(1+t)}.\tag{22}$$

Thus, to get inequality (21), it suffices to show that

$$\frac{2(n+1)}{2(n+k+1)+1}+\frac{2k+1}{2k+n}-\frac{2(n+k)+1}{2(n+k)(n+k+1)}-\frac{1}{12kn}-1>0,$$

which can be deduced from the following

$$\begin{aligned}&12kn^4(k-1)+2n^3(n+5k)(k^2-1)+5n^3(k^3-1)+6k^2n^2(kn-1)\\&\quad+3n^2(k^3n-1)+2k(n^2+3k)(k^2n-1)+k^2n(kn^2-1)\\&\quad+9kn(k^2n^2-1)+10k^3(n^3-1)+2k^2n(k+12)(n^2-1)\\&\quad+4k^4(6n^2-1)+6k^3n(3k+10n)+10k^2n^4>0.\end{aligned}\tag{23}$$

Thus we complete the proof. ■

Remark 2. In [26], J. Sándor and L. Debnath proved a new form of the Stirling's formula: For all positive integers $n\geq 2$, we have the double inequality

$$\sqrt{2\pi}\,\mathrm{e}^{-n}n^{n+1/2}<n!<\left(\frac{n}{n-1}\right)^{1/2}\sqrt{2\pi}\,\mathrm{e}^{-n}n^{n+1/2}.\tag{24}$$

Proof of Theorem 2. For a fixed real number $y\geq 0$, define

$$w(x)=\frac{\ln\Gamma(x+y+1)-\ln\Gamma(y+1)}{x}-\ln(x+y+1),\quad x\in[1,\infty).\tag{25}$$

A simple calculation reveals that

$$w'(x)=\frac{\ln\Gamma(y+1)-\ln\Gamma(x+y+1)}{x^2}-\frac{1}{x+y+1}+\frac{\psi(x+y+1)}{x},\tag{26}$$

where $\psi = \Gamma'/\Gamma$ denotes the logarithmic derivatives of the gamma function. It is also called a digamma function in [4, p. 71].

It is well-known that

$$\Gamma(z+1) = z\Gamma(z), \quad \operatorname{Re}(z) > 0; \tag{27}$$

$$\psi(x) < \ln x - \frac{1}{2x}, \quad x > 1; \tag{28}$$

$$\psi'(z) = \sum_{i=0}^{\infty} \frac{1}{(i+z)^2}. \tag{29}$$

The inequality (28) can be found in [10, 13, 14] respectively. For more on formula (29), please refer to formula (8.12) in Theorem 8.3, page 93 in [27].

Using the formulas (27) and (29), inequalities (23) and (28) and from direct computation, we have

$$\begin{aligned}\frac{[x^2 w'(x)]'}{x} &= \psi'(x+y+1) - \frac{x+2y+2}{(x+y+1)^2}\\ &= \sum_{i=1}^{\infty} \frac{1}{(x+y+i)^2} - \frac{x+2y+2}{(x+y+1)^2}\\ &< \frac{1}{(x+y+1)^2} + \int_1^{\infty} \frac{\mathrm{d}t}{(x+y+t)^2} - \frac{x+2y+2}{(x+y+1)^2}\\ &= -\frac{y}{(x+y+1)^2}\\ &< 0,\end{aligned} \tag{30}$$

and

$$\begin{aligned}w'(1) &= \ln\Gamma(1+y) - \ln\Gamma(2+y) + \psi(2+y) - \frac{1}{2+y}\\ &= \psi(2+y) - \ln(1+y) - \frac{1}{2+y}\\ &< \ln(2+y) - \ln(1+y) - \frac{1}{2(2+y)} - \frac{1}{2+y}\\ &= \ln\left(1 + \frac{1}{1+y}\right) - \frac{3}{2(2+y)}\\ &< \frac{2y+3}{2(1+y)(2+y)} - \frac{3}{2(2+y)}\\ &= -\frac{y}{2(1+y)(2+y)}\\ &< 0.\end{aligned} \tag{31}$$

Thus, the function $x^2 w'(x)$ is decreasing, $x^2 w'(x) < w'(1) < 0$, and the function $w(x)$ is decreasing with $x > 1$. That is, the function $[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}/(x+y+1)$ is decreasing with $x > 1$ for fixed $y \geq 0$. This completes the proof. ■

Remark 3. In [22, 28, 30], the second author and others had obtained a lot of inequalities relating to the ratios of gamma and incomplete gamma functions using monotonicity and properties of the generalized weighted mean values with two parameters and other techniques.

3. Open Problem

At last, we pose the following open problem.

Open Problem. *For positive real numbers x and y, we have*

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}} \cdot \sqrt{\frac{x+y}{x+y+1}}, \tag{32}$$

where Γ denotes the gamma function.

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