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RECOGNIZING HINGE-FREE LINE GRAPHS AND TOTAL GRAPHS

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Abstract. In this paper, we characterize line graphs and total graphs that are hinge-free, i.e., there is no triple of vertices x, y, z such that the distance between y and z increases after x is removed. Based on our characterizations, we show that given a graph G with n vertices and m edges, determining its line graph and total graph to be hinge-free can be solved in O($n + m$) time. Moreover, characterizations of hinge-free iterated line graphs and total graphs are also discussed.

1. Introduction

All graphs considered in this paper are undirected without self-loops and multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. We call $V(G) \cup E(G)$ the set of *elements* of G, and write $n = |V(G)|$ to be the *order* of G and $m = |E(G)|$ to be the *size* of G. Two elements are said to be *associated* if they are either adjacent or incident. The *distance* $d_G(x, y)$ of two elements $x, y \in V(G) \cup E(G)$ is the length (i.e., the number of edges) of a shortest path joining x and y in G, but not including x and y (if $x \in E(G)$ or $y \in E(G)$). A shortest path joining x and y is called an x-y *geodesic*.

A vertex u in a graph G is called a *hinge vertex* if there exist two other vertices x and y such that $d_{G-u}(x, y) > d_G(x, y)$, where $G - u$ denotes the subgraph of G induced by the vertex set $V(G) \backslash \{u\}$. That is, u is a hinge vertex if and only if every x-y geodesic in G must pass through u. Graphs without hinge vertices are called *hinge-free graphs*. The study of hinge-free graphs arises naturally from network design [5, 6, 9]. Because many interconnection networks can be constructed using line (di)graph iterations, such as Kautz networks [10], de Bruijn networks

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[4] and Imase-Itoh networks [13], this provides with a motivation for us to study characterizations of hinge-free (iterated) line graphs.

The *line graph* of G, denoted by $L(G)$, is the intersection graph whose vertices correspond to the edges of G, and two vertices of $L(G)$ are joined by an edge if and only if the corresponding edges in G are adjacent. A natural extension of line graphs is the *total graph*. The total graph $T(G)$ is the graph whose vertex set is the set of all elements of G, and two vertices are adjacent if and only if the corresponding elements are associated in G. For example, Figure 1 shows a graph G and its line graph and total graph. More generally, the *iterated* line graphs and total graphs are defined as follows: $L^1(G) = L(G)$ (resp. $T^1(G) = T(G)$) and $L^i(G) = L(L^{i-1}(G))$ (resp. $T^i(G) = T(T^{i-1}(G))$) for $i \geq 2$.

In this paper, we characterize hinge-free line graphs and total graphs. Moreover, we extend these results to iterated line graphs and total graphs. Two interesting results acquired from our study are these:

Theorem 1. *The line graph $L(G)$ is hinge-free if and only if G is P_4-free.*

Theorem 2. *The total graph $T(G)$ is hinge-free if and only if G is both hinge-free and P_4-free.*

A graph is *P_k-free* if it contains no induced path of length $k-1$. For the case $k = 4$, many structural and algorithmic properties of P_4-free graphs have been discovered (see e.g. [7, 8, 14]). A familiar synonym of P_4-free graph is *complement reducible graph* (abbreviated to *cograph*). Corneil *et al.* [8] showed that cographs can be recognized in O($n+m$) time by constructing a unique tree representation. Therefore, our first result indicates that if the line graph model (i.e., the root graph) G is given, determining whether its line graph is hinge-free or not can be solved in the same time complexity.

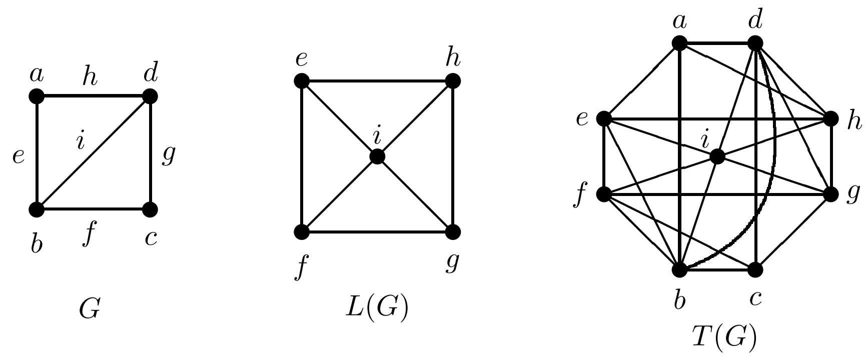


FIG. 1.

Due to the fact that $T(G)$ always contains both G and $L(G)$ as induced subgraphs, if $T(G)$ is P_k-free then G and $L(G)$ are also P_k-free, but the converse is not true. The second result seems to be not intuitive since, unlike the P_k-free property, the hinge-free property is not hereditary, i.e., not every induced subgraph of a hinge-free graph is hinge-free. It is well-known that cographs are properly contained in a class of graphs called *distance-hereditary graphs*, i.e., graphs in which every pair of vertices has the same distance in every connected induced subgraph containing them. Distance-hereditary graphs were first introduced by Howorka [12] and further characterized by Bandelt and Mulder [1]. It is obvious from the definition that every hinge vertex in a distance-hereditary graph must be a cut vertex. Thus, the depth-first search algorithm on graphs (see e.g. [3]) can be used for finding all hinge vertices of a distance-hereditary graph (cograph). An immediate consequence obtained from Theorem 2 is that the hinge-free total graph recognition problem can be solved in linear time once its root graph is given.

2. Preliminaries

Throughout the rest of this paper, we assume that a graph G is connected and nontrivial. For a vertex $u \in V(G)$, the *neighborhood* $N_G(u)$ is the set of all vertices of G adjacent to u. When no ambiguity arises, the subscript G can be omitted. Note that the term "path" always refers to a simple path, i.e., no vertex appears more than once. In particular, a path is called *trivial* if it has a single vertex. Two nontrivial paths joining x and y are *vertex-disjoint* (resp. *edge-disjoint*) if they have no vertices (resp. edges), excluding x and y, in common. Notations and terminologies not given here may be found in any standard textbook on graphs.

A point of view proposed in [6] showed that to identify a hinge vertex of an arbitrary graph, we only need to inspect the neighborhood of this vertex instead of examining the distances among all the vertex-pairs. Based on this property, linear-time algorithms for finding all hinge vertices for some special graphs were found [5, 11].

Lemma 1. (Chang *et al.* [6]) *A vertex v in a graph G is a hinge vertex if and only if there exist two nonadjacent vertices $x, y \in N(v)$ such that $N(x) \cap N(y) = \{v\}$.*

An undirected graph G is *k-connected* if the removal of at least k vertices is necessary to disconnect G or reduce it to a single vertex. In [9], Entringer et al. defined that a graph G is *k-geodetically connected* (k-GC for short) if G is k-connected and the removal of at least k vertices is required to increase the distance of at least two vertices. That is, the structure of k-GC graphs can tolerate any $k-1$ vertices failures without increasing the distance among all the remaining vertices.

In fact, the class of hinge-free graphs is identical to the class of 2-GC graphs. A necessary and sufficient condition for a graph to be hinge-free (see Lemma 2) was proved in [5]. This result suggests that a hinge-free graph recognition algorithm can easily be implemented in O(nm) time. Indeed, a characterization which generalizes the result of Lemma 2 for k-GC graphs, $k \geq 2$, was also provided in [5].

Lemma 2. (Chang and Ho [5]) *A graph G is hinge-free if and only if every pair of nonadjacent vertices in G are joined by at least two vertex-disjoint geodesics.*

3. Hinge-Free Line Graphs

It is well-known that a line graph does not contain $K_{1,3}$ (claw) as an induced subgraph. A complete list of the forbidden induced subgraphs for the family of line graphs was characterized by Beineke [2]. In this section, we characterize hinge-free (iterated) line graphs.

We first give some observations which can easily be derived from the definition of a line graph. For an arbitrary graph G, there is a one-to-one correspondence between the nontrivial paths of G and the induced paths of $L(G)$; i.e., if G contains a path $P = v_0v_1 \ldots v_k$ of length $k \geq 1$ which consists of edges $e_i = v_{i-1}v_i$, then the corresponding vertices of e_i in $L(G)$ form an induced path $P' = e_1e_2 \ldots e_k$ of length $k-1$, and vice versa. Further, P is a v_0-v_k geodesic in G if and only if P' is an e_1-e_k geodesic in $L(G)$, and two geodesics in G are edge-disjoint if and only if the corresponding induced geodesics in $L(G)$ are vertex-disjoint. Thus, we have the following properties.

Proposition 1. *The line graph $L(G)$ is P_k-free if and only if G contains no path of length k.*

Proposition 2. *Let G be a graph and $l \geq 2$ be an integer. Two vertices of $L(G)$ are joined by k vertex-disjoint geodesics of length l if and only if the corresponding edges in G are joined by k edge-disjoint geodesics of length $l-1$.*

Proof of Theorem 1. By Lemma 2, if $L(G)$ is hinge-free then every pair of nonadjacent vertices in $L(G)$ are joined by at least two vertex-disjoint geodesics. It follows from Proposition 2 that every pair of nonadjacent edges in G is joined by at least two edge-disjoint geodesics. Let $wxyz$ be any path of G. Since edges wx and yz are joined by at least two edge-disjoint geodesics, at least one of edges wy, wz, and xz must exist. Thus G contains no induced P_4.

Conversely, suppose that w is a hinge vertex of $L(G)$. By Lemma 1, there exist two nonadjacent vertices $x, y \in N_{L(G)}(w)$ such that $N_{L(G)}(x) \cap N_{L(G)}(y) = \{w\}$.

That is, xwy is the unique x-y geodesic in $L(G)$. Let $x = ab$ and $y = cd$ be two such corresponding edges of G. Since ab and cd are nonadjacent, by Proposition 2, they are joined by only one edge w. Hence, in G, w must be one of the following: ac, bd, ad, or bc. Therefore, a, b, c, d induce a P_4 in G. ■

Given a connected graph G, we write kG for the graph with k components each isomorphic with G. For two vertex-disjoint graphs G_1 and G_2, the *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph having $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2, denoted by $G_1 + G_2$, is the graph consisting of the union $G_1 \cup G_2$ together with $\{uv : u \in V(G_1)$ and $v \in V(G_2)\}$. Define H_k as the star $K_{1,k+2}$ with one additional edge added, i.e., $H_k \cong K_1 + (kK_1 \cup K_2)$. Note that $H_0 \cong K_3$ and each H_k for $k \geq 2$ contains a claw as an induced subgraph.

In what follows, the hinge-free iterated line graphs will be characterized. Obviously, for every graph G of order $n < 3$, $L^i(G)$, $i \geq 2$, does not exist. By Proposition 1 and Theorem 1, $L^2(G)$ is hinge-free if and only if every path of G has length at most 3. Thus, if G has order 4 or less, $L^2(G)$ is trivially hinge-free.

Theorem 3. *Let G be a graph of order at least 5. Then $L^2(G)$ is hinge-free if and only if G is a tree of diameter at most 3 or one of the graphs H_k for $k \geq 2$.*

Proof. Clearly, if G is a tree of diameter no more than 3 then G contains no path of length 4. We now consider the graphs with order at least 5 and containing a cycle. Let G be a graph of order $n \geq 5$ and let C be a longest cycle of G (without induced). Since G is connected and $n \geq 5$, if $|V(C)| \geq 4$ then G contains a path of length 4. For $|V(C)| = 3$, it is easy to verify that if $G \notin \{H_k : k = 2, 3, \ldots\}$, then G contains an induced subgraph isomorphic to W_1, W_2 or W_3 (see Figure 2), and in each case G always contains a path of length 4. On the contrary, if $G \in \{H_k : k = 2, 3 \ldots\}$ then G has no path of length 4. Thus, the graphs $L^2(H_k)$ for $k \geq 2$ are hinge-free. ■

From above, the family of graphs with order $n \geq 3$ containing no path of length 4 is precisely $\{T : T$ is a tree of diameter 2 or $3\} \cup \{H_k : k = 0, 1, 2, \ldots\} \cup \{C_4, K_4, K_4 - e\}$, where $K_4 - e$ is a 4-vertex complete graph by deleting any edge.

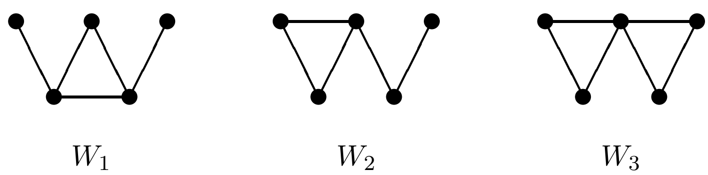
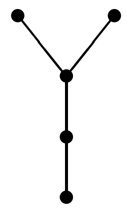


FIG. 2.



Y

FIG. 3.

Furthermore, by Proposition 1 and Theorem 1, $L^3(G)$ is hinge-free if and only if $L(G)$ contains no path of length 4. Thus, we can characterize $L^3(G)$ to be hinge-free by considering those graphs whose corresponding line graphs appear in the above family.

Obviously, not every graph containing no path of length 4 is the line graph of some graph. Note that, except for $K_{1,3}$, every forbidden induced subgraph of a line graph (provided by Beineke [2]) contains a path of length 4. Thus, we only need to restrict our attention to the $K_{1,3}$ inspection in the above family when we consider line graphs without a path of length 4. Since a line graph that is a tree must be a path if it has no claw, and since each H_k for $k \geq 2$ contains an induced claw, we characterize $L^3(G)$ to be hinge-free as follows: $L(P_4) \cong P_3$, $L(P_5) \cong P_4$, $L(K_3) \cong L(K_{1,3}) \cong K_3 \cong H_0$, $L(Y) \cong H_1$ (see Figure 3 for the graph Y), $L(C_4) \cong C_4$, $L(K_{1,4}) \cong K_4$ and $L(H_1) \cong K_4 - e$. Therefore, we have the following theorem.

Theorem 4. *$L^3(G)$ is hinge-free if and only if $G \in \{P_4, P_5, K_3, K_{1,3}, Y, C_4, K_{1,4}, H_1\}$.*

4. Hinge-Free Total Graphs

In this section, the hinge-free and P_k-free properties for total graphs are considered. Let P be an induced path of $T(G)$. We say that P is *vertex-unified* (resp. *edge-unified*) if all the corresponding elements of the vertices in P are vertices (resp. edges) of G. For convenience, we say that P is *unified* if it is either vertex-unified or edge-unified. Clearly, every trivial path is unified. If P is not unified, then it can be divided into maximal unified subpaths such that vertex-unified subpaths and edge-unified subpaths alternate along P. The following properties are directly obtained from the fact that $T(G)$ contains G (resp. $L(G)$) as an induced subgraph.

Proposition 3. *Let P be a vertex-unified induced path in $T(G)$ of length l. Then $V(P)$ induces a path of the same length in G.*

Proposition 4. *Let P be an edge-unified induced path in $T(G)$ of length l. Then the corresponding edges of $V(P)$ constitute a path of length $l+1$ in G.*

Let P be an induced path of $T(G)$ having $L_1, \ldots, L_j$ as its maximal unified subpaths. By Propositions 3 and 4, for each L_i, there is a corresponding path L'_i in G. Since P is an induced path, the collection of paths L'_i in G still forms a path. For instance, we consider an induced path $P = abfg$ of $T(G)$ in Figure 1. Then P can be divided into ab and fg maximal unified subpaths. The vertex set $\{a, b\}$ in G induces a path of length 1 and the edge set $\{f = bc, g = cd\}$ in G yields a path of length 2. Consequently, the elements a, b, f, g in G produce a path $abcd$ of length 3. We now prove three geodetic properties related to the graphs G and $T(G)$, which are helpful to establish the main result for hinge-free total graphs.

Lemma 3. *Two nonadjacent vertices of a graph G are joined by k vertex-disjoint geodesics of length l if and only if their corresponding vertices in $T(G)$ are joined by k vertex-disjoint geodesics with the same length.*

Proof. The "only if" part follows immediately from the fact that G is an induced subgraph of $T(G)$. Conversely, we show that for any two nonadjacent vertices x and y in G, every x-y geodesic in $T(G)$ must be vertex-unified. Thus the result follows from Proposition 3.

Let P be an x-y geodesic of length l in $T(G)$. Suppose that P is not vertex-unified. Then P contains at least one maximal edge-unified subpath. We may assume that $P = x \cdots v_0 e_1 e_2 \cdots e_j v_j \cdots y$, where $v_0, v_j \in V(G)$, $e_i = v_{i-1}v_i \in E(G)$ and $j \geq 1$. This means that $e_1 \cdots e_j$ is a maximal edge-unified subpath of P. It is easy to see that $x \cdots v_0 v_1 \cdots v_{j-1} v_j \cdots y$ forms another path in $T(G)$ of length $l - 1$. This contradicts the fact that P is an x-y geodesic in $T(G)$. ■

Lemma 4. *Two nonadjacent edges of a graph G are joined by k edge-disjoint geodesics of length l if and only if their corresponding vertices in $T(G)$ are joined by k vertex-disjoint geodesics of length $l + 1$.*

Proof. The "only if" part follows immediately from the fact that $L(G)$ is an induced subgraph of $T(G)$. Conversely, a similar proof of Lemma 3 can show that every x-y geodesic in $T(G)$ is edge-unified, where x and y are any two nonadjacent edges in G. Thus the result follows from Proposition 4. ■

Lemma 5. *For any two nonassociated elements $x \in V(G)$ and $y = uv \in E(G)$, the corresponding vertices of x and y in $T(G)$ are joined by at least two vertex-disjoint geodesics.*

Proof. Since G is connected, without loss of generality, we may assume that $P = w_1w_2\cdots w_k$ is an x-y geodesic of G where $w_1 = x$, $w_k = u$ and $k \geq 2$. Let $e_i = w_iw_{i+1}$. Then we can find two vertex-disjoint paths of length k joining x and y in $T(G)$, namely, $P' = w_1w_2w_3\cdots w_ky$ and $P'' = w_1e_1e_2\cdots e_{k-1}y$. Also, if $T(G)$ contains another x-y path of length less than k, then P cannot be an x-y geodesic in G. Thus P' and P'' are vertex-disjoint geodesics in $T(G)$. ■

Now, we complete the proof of Theorem 2.

Proof of Theorem 2. Suppose that $T(G)$ is hinge-free. By Lemma 2, every pair of nonadjacent vertices in $T(G)$ are joined by at least two vertex-disjoint geodesics. Since two vertices $x, y \in V(G)$ are nonadjacent if and only if the corresponding vertices of x and y in $T(G)$ are also nonadjacent, it follows from Lemma 3 that every two nonadjacent vertices of G are joined by at least two vertex-disjoint geodesics. Thus, by Lemma 2, G is hinge-free. To show that G is P_4-free, by Theorem 1 it suffices to show that $L(G)$ is hinge-free. Let x and y be nonadjacent vertices in $L(G)$. Since $T(G)$ contains $L(G)$ as an induced subgraph, x and y are also nonadjacent in $T(G)$. Since $T(G)$ is hinge-free, there exist at least two vertex-disjoint geodesics joining x and y in $T(G)$. By Lemma 4, the corresponding edges of x and y in G are joined by at least two edge-disjoint geodesics. Thus, by Proposition 2 and Lemma 2, we conclude that $L(G)$ is hinge-free.

Conversely, let G be a P_4-free and hinge-free graph and assume that $T(G)$ contains a hinge vertex w. By Lemma 1, there exist two nonadjacent vertices $x, y \in N_{T(G)}(w)$ such that $N_{T(G)}(x) \cap N_{T(G)}(y) = \{w\}$. That is, the corresponding elements of x and y in G are nonassociated, and the induced path xwy in $T(G)$ is the unique x-y geodesic. We now consider all possible cases about the elements x and y to be either vertices or edges of G as follows.

Case 1: x and y are nonadjacent vertices of G. Since xwy is the unique x-y geodesic in $T(G)$, by Lemma 3, there is only one geodesic with length 2 between x and y in G. Thus, by Lemma 2, G is not hinge-free, a contradiction.

Case 2: $x = ab$ and $y = cd$ are nonadjacent edges of G. Since xwy is the unique x-y geodesic in $T(G)$, by Lemma 4, ab and cd in G must be joined by only one edge. Thus, G contains an induced P_4, a contradiction.

Case 3: $x \in V(G)$ and $y \in E(G)$ or $x \in E(G)$ and $y \in V(G)$ are nonassociated elements. By Lemma 5, x and y in $T(G)$ are joined by at least two vertex-disjoint geodesics. This violates the fact that xwy is the unique x-y geodesic in $T(G)$. ■

As immediate consequences, we obtain the following corollaries.

Corollary 1. *The total graph $T(G)$ is hinge-free if and only if both G and $L(G)$ are hinge-free.*

Corollary 2. *The following statements are equivalent for a graph G:*

(1) *$T(L(G))$ is hinge-free.*

(2) *$L(G)$ is both hinge-free and P_4-free.*

(3) *Both G and $L(G)$ are P_4-free.*

(4) *G is P_4-free and every path of G has length at most* 3.

(5) $G \in \{C_4, K_4, K_4 - e\} \cup \{H_k : k = 0, 1, 2, \dots\} \cup \{K_{1,n} : n = 1, 2, 3, \dots\}$.

Proof. The equivalences of statements (1), (2), (3) and (4) are established by Theorems 2, 1 and Proposition 1. (4)⇔(5) can be proved similarly to Theorem 3 by restricting G without an induced P_4. ■

In what follows, we present some properties of total graphs without induced P_k and then use these properties to characterize the hinge-free iterated total graphs. Let P be an induced path of a total graph $T(G)$. We first show that the number of maximal unified subpaths with respect to P has a bound.

Lemma 6. *Every induced path of length $k-1$ in $T(G)$ can be divided into at most $\lfloor \frac{k}{2} \rfloor + 1$ maximal unified subpaths.*

Proof. Let $P = x_1 x_2 \cdots x_k$ be an induced path of $T(G)$ which consists of j maximal unified subpaths $L_1, \dots, L_j$. Consider three consecutive vertices x_i, x_{i+1} and x_{i+2} in P, where $i = 1, \dots, k-2$. Clearly, if the corresponding elements of these three vertices in G satisfy $x_i, x_{i+2} \in V(G)$ and $x_{i+1} \in E(G)$ or $x_i, x_{i+2} \in E(G)$ and $x_{i+1} \in V(G)$, then x_i and x_{i+2} are two associated elements of G (corresponding to two adjacent vertices of $T(G)$). This implies that P is not an induced path of $T(G)$. Thus, each subpath L_i, excluding L_1 and L_j, contains at least two vertices. So P has $k \geq 2(j-2)+2$ vertices. Since j must be an integer, we have $j \leq \lfloor \frac{k}{2} \rfloor + 1$. ■

Theorem 5. *Let G be a graph and let $k \geq 2$. Then $T(G)$ is P_k-free if G contains no path of length $\lceil \frac{3k}{4} \rceil - 1$.*

Proof. We will show that if $T(G)$ is not P_k-free, then G contains a path of length at least $\lceil \frac{3k}{4} \rceil - 1$. Assume that there is an induced path $P = x_1 x_2 \cdots x_k$ of length $k-1$ in $T(G)$ which is divided into $L_1, \dots, L_j$ maximal unified subpaths such that $x_1 \in V(L_1)$ and $x_k \in V(L_j)$. For $i = 1, \dots, j$, let L_i' be the corresponding path of L_i in G. By Propositions 3 and 4, the length of L_i' can be determined by the length of L_i. Let P' be the path in G that is constituted from the set of subpaths $L'_1, \ldots, L'_j$. Then $|P'| = \sum_{i=1}^{j} |L'_i|$, where $|P'|$ denotes the length of P'. We claim that $|P'| \geq \lceil \frac{3k}{4} \rceil - 1$. Consider elements x_1 and x_k to be either vertices or edges of G by the following three cases:

Case 1: $x_1, x_k \in V(G)$. In this case, j is odd and each subpath L_i for i even (resp. odd) is edge-unified (resp. vertex-unified). Thus we have

$$|P'| = \sum_{i=1}^{\frac{j+1}{2}} (|V(L_{2i-1})| - 1) + \sum_{i=1}^{\frac{j-1}{2}} |V(L_{2i})| = \sum_{i=1}^{j} |V(L_i)| - \frac{j+1}{2} = k - \frac{j+1}{2}.$$

Case 2: $x_1, x_k \in E(G)$. In this case, j is odd and each subpath L_i for i even (resp. odd) is vertex-unified (resp. edge-unified). Thus we have

$$|P'| = \sum_{i=1}^{\frac{j-1}{2}} (|V(L_{2i})| - 1) + \sum_{i=1}^{\frac{j+1}{2}} |V(L_{2i-1})| = k - \frac{j-1}{2}.$$

Case 3: $x_1 \in V(G)$ and $x_k \in E(G)$ or $x_1 \in E(G)$ and $x_k \in V(G)$. In this case, j is even. Without loss of generality, we assume that $x_1 \in V(G)$ and $x_k \in E(G)$. Thus we have

$$|P'| = \sum_{i=1}^{\frac{j}{2}} (|V(L_{2i-1})| - 1) + \sum_{i=1}^{\frac{j}{2}} |V(L_{2i})| = k - \frac{j}{2}.$$

Since $j \leq \lfloor \frac{k}{2} \rfloor + 1$ by Lemma 6, the length of P' in the above three cases is at least

$$k - \frac{j+1}{2} \geq k - \frac{\lfloor \frac{k}{2} \rfloor}{2} - 1 \geq \frac{3k}{4} - 1.$$

Thus, G contains a path of length $\lceil \frac{3k}{4} \rceil - 1$. ■

A necessary condition for $T(G)$ to be P_k-free can readily be made as follows. Since $T(G)$ contains both G and $L(G)$ as induced subgraphs, if $T(G)$ is P_k-free then both G and $L(G)$ are P_k-free. By Proposition 1, this implies that G contains no path of length k and no induced path of length $k-1$. The following theorem improves this bound.

Theorem 6. *Let G be a graph and let $k \geq 2$. If $T(G)$ is P_k-free, then*

(1) *G contains no path of length $k-1$, and*

(2) *G contains no induced path of length $\lceil \frac{3k-2}{4} \rceil$.*

Proof. (1) Assume that G contains a path $v_1v_2\cdots v_k$ of length $k-1$. For $i=1,\ldots,k-1$, let $e_i = v_iv_{i+i}$. Then $e_1\cdots e_{k-1}v_k$ forms an induced path of length $k-1$ in $T(G)$. Thus, $T(G)$ is not P_k-free.

(2) Assume that G has an induced path $P = v_0v_1\cdots v_p$ with $p \geq \lceil\frac{3k-2}{4}\rceil$. We will show that $T(G)$ is not P_k-free. Let $e_i = v_{i-1}v_i$ for $i = 1,\ldots,p$ and let $S = \{v_0, e_1, v_1, e_2, \ldots, v_{p-1}, e_p, v_p\}$ be the set of elements of P. Denote G_S as the subgraph of $T(G)$ induced by the corresponding vertices of the elements of S. We claim that G_S contains an induced path of length at least $k-1$.

To simplify the description, we use $f(i)$ for $i = 1, 2, \ldots, 2p+1$ to denote the vertices of G_S, where

$$f(i) = \begin{cases} v_{\frac{i-1}{2}} & \text{if } i \text{ is odd,} \\ e_{\frac{i}{2}} & \text{if } i \text{ is even.} \end{cases}$$

Since P is an induced path of G, distinct vertices $f(i)$ and $f(j)$ in G_S are adjacent for $|i-j| \leq 2$, and are nonadjacent for $|i-j| \geq 3$. Let $X = x_0, x_1, \ldots, x_h$ be an increasing sequence from the set $\{1, 2, \ldots, 2p+1\}$. Obviously, if $x_{i+1} - x_i \leq 2$ for all $i = 0,\ldots,h-1$, then $f(x_0)\cdots f(x_h)$ forms a path of length h in G_S. Moreover, if additional conditions $x_{i+2} - x_i \geq 3$ hold for all $i = 0,\ldots,h-2$, then $f(x_0)\cdots f(x_h)$ is an induced path of length h in G_S.

Let $q = \lceil\frac{2p+1}{3}\rceil$ and $r = (2p+1) \bmod 3$. We now consider a v_0-v_p induced path P' in G_S that is constructed from an increasing sequence X such that all the terms of X satisfy the conditions:

$$x_{i+1} - x_i \leq 2 \;\text{ and }\; x_{i+2} - x_i \geq 3.$$

Case 1: $r = 0$. In this case, we have $2p + 1 = 3q$ and $p \equiv 1 \pmod 3$. We select $X = 1, 3, 4, 6, 7, 9, \ldots, 3q-2, 3q$. Then $|P'| = 2q - 1 = \frac{4p-1}{3}$.

Case 2: $r = 1$. In this case, we have $2p + 1 = 3q - 2$ and $p \equiv 0 \pmod 3$. Select $X = 1, 3, 4, 6, 7, 9, \ldots, 3q-5, 3q-3, 3q-2$. Then $|P'| = 2q - 2 = \frac{4p}{3}$.

Case 3: $r = 2$. In this case, we have $2p+1 = 3q-1$ and $p \equiv 2 \pmod 3$. Select $X = 1, 2, 4, 5, 7, 8, \ldots, 3q-5, 3q-4, 3q-2, 3q-1$. Then $|P'| = 2q - 1 = \frac{4p+1}{3}$.

In the above three cases, the length of P' can be expressed in term $\lceil\frac{4p-1}{3}\rceil$ by considering the congruence of p. Since $p \geq \lceil\frac{3k-2}{4}\rceil \geq \frac{3k-2}{4}$, we have

$$\lceil\frac{4p-1}{3}\rceil \geq \frac{4p-1}{3} \geq k-1.$$

From the above argument, we obtain that the induced subgraph G_S of $T(G)$ contains an induced path of length at least $k-1$. Thus, $T(G)$ is not P_k-free. ■

Corollary 3. *The following statements are equivalent for a graph* G:

(1) $T(G)$ *is* P_4*-free.*

(2) $L(T(G))$ *is hinge-free.*

(3) $T(G)$ *is both hinge-free and* P_4*-free.*

(4) $T^2(G)$ *is hinge-free.*

Moreover, the only connected graph G *for which* $T(G)$ *is* P_4*-free are* K_2 *and* K_3.

Proof. The equivalences (1)⇔(2) and (3)⇔(4) follow directly from Theorems 1 and 2, respectively. (3)⇒(1) is trivial. We prove (1)⇒(3) as follows.

By Theorem 6, if $T(G)$ is P_4-free then G has no path of length 3. The nontrivial connected graphs containing a path of length at most 2 are K_2, K_3, P_3, and $K_{1,n}$ for $n \geq 3$. Clearly, $T(P_3)$ is not P_4-free. Since every $T(K_{1,n})$ for $n > 3$ contains $T(P_3)$ as an induced subgraph, it is not P_4-free. Also, it is easy to check that $T(K_2)$ and $T(K_3)$ are both P_4-free and hinge-free. ■

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