TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 5, pp. 1293-1302, August 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

VERTICAL TANGENT VECTORS TO THE GRAPH OF A MULTIFUNCTION

N. D. Yen and Jen-Chih Yao*

Abstract. The sets of vertical vectors in the contingent cone, the intermediate tangent cone and the Clarke tangent cone to the graph of a multifunction between normed spaces at a given point are estimated or computed by exact formulae under some suitable assumptions. The obtained results sharpen and complement the results of Dien and Yen [Acta Math., Vietnam. Vol. 10(1) (1985), 144-147] where the set of vertical vectors in the Clarke tangent cone was considered.

1. INTRODUCTION

Consider a multifunction $F : X \rightrightarrows Y$ between normed spaces with the graph $\mathrm{gph}\, F := \{(x, y) \in X \times Y \, : \, y \in F(x)\}$. Given a point $z_0 = (x_0, y_0) \in \mathrm{gph}\, X$ we denote the Clarke tangent cone, the intermediate tangent cone, and the contingent cone to gph F respectively by $C_{\mathrm{gph}\, F}(z_0)$, $T^b_{\mathrm{gph}\, F}(z_0)$, and $T_{\mathrm{gph}\, F}(z_0)$. Following the primal-space approach to differentiation in [1], one uses each of the tangent cones to define a *graphical derivative* of multifunctions. In fact, the differential theory of set-valued analysis in [3] is built upon these graphical derivatives. Although the dual-space approach to generalized differentiation has had a very successful development (see [4, 9, 11]), Aubin's method of defining derivatives of a multifunction via tangent cones to its graph at a given point continues to play an important role and to attract an immense attention (see for instance [6-8, 11]).

Received January 2, 2008.
Communicated by Mau-Hsiang Shih.
2000 *Mathematics Subject Classification*: 49J53, 54C60.
Key words and phrases: Multifunction, Graph, Vertical tangent vector, Upper estimation, Exact formula.
This work was supported by a research grant from the National Sun Yat-Sen University, Kaohsiung, Taiwan.
*Correspounding author.

By definition [3], a vector $w = (u, v) \in Z := X \times Y$ belongs to $T_{\mathrm{gph}\,F}(z_0)$ if and only if

$$\liminf_{t\to 0^+} \frac{\mathrm{dist}(z_0 + tw, \mathrm{gph}\,F)}{t} = 0, \tag{1.1}$$

where $\mathrm{dist}(z, \Omega) = \inf\{\|z - z'\| \,:\, z' \in \Omega\}$ denotes the distance from $z \in Z$ to a subset $\Omega \subset Z$ and $\|z\| := \|x\| + \|y\|$ for any $z = (x, y) \in Z$. The inclusions $w \in T^b_{\mathrm{gph}\,F}(z_0)$ and $w \in C_{\mathrm{gph}\,F}(z_0)$, respectively, mean

$$\lim_{t\to 0^+} \frac{\mathrm{dist}(z_0 + tw, \mathrm{gph}\,F)}{t} = 0 \tag{1.2}$$

and

$$\lim_{t\to 0^+,\, z\xrightarrow{\mathrm{gph}\,F} z_0} \frac{\mathrm{dist}(z + tw, \mathrm{gph}\,F)}{t} = 0, \tag{1.3}$$

where $z \xrightarrow{\mathrm{gph}\,F} z_0$ denotes the limit in $\mathrm{gph}\,F \cup \{z_0\}$. The conditions (1.1), (1.2) and (1.3) can be rewritten, respectively, as the following

$$\forall\varepsilon > 0, \forall\delta > 0, \exists t \in (0, \delta) \text{ such that } \big[z_0 + t\big(w + B_Z(0, \varepsilon)\big)\big] \cap \mathrm{gph}\,F \neq \emptyset,$$

$$\forall\varepsilon > 0, \exists\delta > 0 \text{ such that } \big[z_0 + t\big(w + B_Z(0, \varepsilon)\big)\big] \cap \mathrm{gph}\,F \neq \emptyset \quad \forall t \in (0, \delta),$$

and

$$\forall\varepsilon > 0, \exists\delta > 0 \text{ such that}$$
$$\big[z + t\big(w + B_Z(0, \varepsilon)\big)\big] \cap \mathrm{gph}\,F \neq \emptyset \quad \forall t \in (0, \delta),\ \forall z \in B_Z(z_0, \delta) \cap \mathrm{gph}\,F.$$

Here $B_Z(z_0, \delta)$ denotes the closed ball centered at z_0 with radius $\delta > 0$. The closed unit ball in Z will be denoted by B_Z.

It is well-known [3] that $T^b_{\mathrm{gph}\,F}(z_0)$ and $T_{\mathrm{gph}\,F}(z_0)$ are closed cones (may be nonconvex), $C_{\mathrm{gph}\,F}(z_0)$ is a closed convex cone, and

$$C_{\mathrm{gph}\,F}(z_0) \subset T^b_{\mathrm{gph}\,F}(z_0) \subset T_{\mathrm{gph}\,F}(z_0) \subset \overline{\mathrm{cone}}\,\big(\mathrm{gph}\,F - z_0\big), \tag{1.4}$$

where $\overline{\mathrm{cone}}\,\Omega$ denotes the closure of the cone generated by a subset $\Omega \subset Z$. The inclusions become equalities when $\mathrm{gph}\,F$ is a convex set.

By abuse of terminology, we say that an element $w = (u, v) \in X \times Y$ of a cone $K \subset Z = X \times Y$ a *vertical vector* if $u = 0$. The set of vertical vectors of K is abbreviated to $\mathrm{Vert}(K)$, that is

$$\mathrm{Vert}(K) = K \cap \big(\{0\} \times Y\big) = \{w = (u, v) \in K \,:\, u = 0\}.$$

Our aim in this paper is to find some upper estimates for the sets of vertical vectors in the contingent cone, the intermediate tangent cone and the Clarke tangent cone to the graph of a multifunction between normed spaces at a given point. Besides, it will shown that, under some suitable assumptions, the sets of vertical vectors can be computed by exact formulae. Our results sharpen and complement the results of Dien and Yen [5] where the set of vertical vectors in the Clarke tangent cone was investigated. As shown by Sach and Craven [12], the results of [5] can be used for studying invexity of multifunctions and for obtaining duality theorems for mathematical programming problems under inclusion constraints. We observe also that, due to the importance of the graphical derivatives of multifunctions, exact information about the "vertical part" of the tangent cones $C_{\mathrm{gph}\,F}(z_0)$, $T^b_{\mathrm{gph}\,F}(z_0)$ and $T_{\mathrm{gph}\,F}(z_0)$ might be useful in other aspects of set-valued analysis and its applications.

The rest of this paper has two sections. Sect. 2 establishes upper estimates for the above-mentioned sets of vertical vectors. Sect. 3 derives some exact formulae for computing the sets. During the course, we will consider some interesting examples.

2. Upper Estimation for the Set of Vertical Vectors

Definition 2.1.

(1) F is *locally Lipschitz* at x_0 if there exist a constant $\ell > 0$ and a neighborhood U of x_0 such that

$$F(x) \subset F(u) + \ell\|x-u\|B_Y \quad \forall x, u \in U.$$

(2) (See [10]) F is *upper Lipschitz* at x_0 if there exist a constant $\ell > 0$ and a neighborhood U of x_0 such that

$$F(x) \subset F(x_0) + \ell\|x-x_0\|B_Y \quad \forall x \in U.$$

(3) (See [2, 9]) F is *Lipschitz-like* at $(x_0, y_0) \in \mathrm{gph}\,F$ if there exist a constant $\ell > 0$, a neighborhood U of x_0 and a neighborhood V of y_0 such that

$$F(x) \cap V \subset F(u) + \ell\|x-u\|B_Y \quad \forall x, u \in U.$$

(4) F is *upper Lipschitz-like* at $(x_0, y_0) \in \mathrm{gph}\,F$ if there exist a constant $\ell > 0$, a neighborhood U of x_0 and a neighborhood V of y_0 such that

$$F(x) \cap V \subset F(x_0) + \ell\|x-x_0\|B_Y \quad \forall x \in U. \tag{2.1}$$

(5) F is *upper Hölder-like* of an order $\alpha > 0$ at $(x_0, y_0) \in \operatorname{gph} F$ if there exist a constant $\ell > 0$, a neighborhood U of x_0 and a neighborhood V of y_0 such that

$$F(x) \cap V \subset F(x_0) + \ell \|x - x_0\|^{\alpha} B_Y \quad \forall x \in U. \tag{2.2}$$

Clearly, if F is locally Lipschitz at x_0 then it is upper Lipschitz at x_0, and the latter implies that F is upper Lipschitz-like at any (x_0, y_0) with $y_0 \in F(x_0)$. The reverse implications are not true in general. Note also that the Lipschitz-like property implies the upper Lipschitz-like property, but the reverse implication does not hold.

Example 2.2. The multifunction $F : I\!R \rightrightarrows I\!R$ given by $F(x) = \{1\}$ for $x > 0$, $F(x) = \{0\}$ for $x < 0$, and $F(0) = \{\frac{1}{2}, 1\}$, is upper Lipschitz-like at the points $(x_0, y_0) := (0, 1)$ and $(x_0, y_0) := (0, \frac{1}{2})$, but not upper Lipschitz at $x_0 = 0$. The multifunction $G : I\!R \rightrightarrows I\!R$ defined by $G(x) = \{1\}$ for $x > 0$, $G(x) = \{-1\}$ for $x < 0$, and $G(0) = [-1, 1]$, is upper Lipschitz but not locally Lipschitz around $x_0 = 0$. It is easy to see that G is not Lipschitz-like at the points $(x_0, y_0) := (0, 1)$ and $(x_0, y_0) := (0, \frac{1}{2})$.

We now obtain a sharp upper estimate for the set of vertical vectors in the contingent cone $T_{\operatorname{gph} F}(x_0, y_0)$.

Proposition 2.3. *If F is upper Lipschitz-like at $(x_0, y_0) \in \operatorname{gph} F$, then*

$$\Big\{(0, v) \, : \, v \in Y, \ (0, v) \in T_{\operatorname{gph} F}(x_0, y_0)\Big\} \subset \{0\} \times \overline{\operatorname{cone}}\big(F(x_0) - y_0\big). \tag{2.3}$$

Proof. Let $v \in Y$. To obtain (2.3), we will show that if $v \notin \overline{\operatorname{cone}}\big(F(x_0) - y_0\big)$ then $(0, v) \notin T_{\operatorname{gph} F}(x_0, y_0)$. Suppose that $v \notin \overline{\operatorname{cone}}\big(F(x_0) - y_0\big)$. Choose $\eta > 0$ such that $[v + \bar{B}_Y(0, \eta)] \cap [\operatorname{cone}(F(x_0) - y_0)] = \emptyset$. Then, for any $t > 0$, it holds

$$\big(t[v + B_Y(0, \eta)]\big) \cap [F(x_0) - y_0] = \emptyset.$$

Hence

$$\big[tv + \frac{t}{2} B_Y(0, \eta)\big] \cap \big[F(x_0) - y_0 + \frac{t}{2} B_Y(0, \eta)\big] = \emptyset \quad \forall t > 0. \tag{2.4}$$

By our assumption, there exist $\ell > 0, \rho > 0$, and a neighborhood V of y_0 satisfying (2.1) with $U := B_X(x_0, \rho)$. Put $W_1 = U_1 \times V_1$ where

$$U_1 = B_X(0, \rho) \cap B_X\left(0, \frac{\eta}{2\ell}\right), \quad V_1 = B_Y\left(0, \frac{\eta}{2}\right).$$

To show that $(0, v) \notin T_{\text{gph}\, F}(x_0, y_0)$, it suffices to verify that for any $\lambda > 0$ there exists $t \in (0, \lambda)$ such that

$$\big[(x_0, y_0) + t((0, v) + W_1)\big] \cap \text{gph}\, F = \emptyset. \tag{2.5}$$

Let $\lambda > 0$ be given arbitrarily. We choose $t \in (0, \lambda)$ as small as $y_0 + t\Big(v + B_Y(0, \frac{\eta}{2})\Big) \subset V$. From (2.4) it follows that

$$t(v + y) \notin F(x_0) - y_0 + \frac{t}{2} B_Y(0, \eta)$$

for any $y \in V_1$. Therefore

$$y_0 + t(v + y) \notin F(x_0) + \frac{t}{2} B_Y(0, \eta) \quad \forall y \in V_1. \tag{2.6}$$

By (2.1), for any $x \in U_1$,

$$\begin{aligned} F(x_0 + tx) \cap V &\subset F(x_0) + \|tx\| B_Y(0, \ell) \\ &\subset F(x_0) + t\frac{\eta}{2\ell} B_Y(0, \ell) \\ &\subset F(x_0) + \frac{t}{2} B_Y(0, \eta). \end{aligned} \tag{2.7}$$

Combining (2.6) with (2.7) and remembering that $y_0 + t\Big(v + B_Y(0, \frac{\eta}{2})\Big) \subset V$, we have

$$y_0 + t(v + y) \notin F(x_0 + tx) \quad \forall x \in U_1,\ \forall y \in V_1.$$

Hence

$$(x_0, y_0) + t((0, v) + (x, y)) \notin \text{gph}\, F \quad \forall (x, y) \in W_1;$$

thus (2.5) is satisfied. ∎

The notation $\text{Vert}\Big(T_{\text{gph}\, F}(x_0, y_0)\Big)$ will be used for denoting the set on the left-hand side of (2.3). Similarly, the sets of vertical tangent vectors in $T^b_{\text{gph}\, F}(x_0, y_0)$ and $C_{\text{gph}\, F}(x_0, y_0)$ will be abbreviated, respectively, to $\text{Vert}\Big(T^b_{\text{gph}\, F}(x_0, y_0)\Big)$ and $\text{Vert}\Big(C_{\text{gph}\, F}(x_0, y_0)\Big)$. The inclusions in (1.4) imply

$$\text{Vert}\Big(C_{\text{gph}\, F}(x_0, y_0)\Big) \subset \text{Vert}\Big(T^b_{\text{gph}\, F}(x_0, y_0)\Big) \subset \text{Vert}\Big(T_{\text{gph}\, F}(x_0, y_0)\Big).$$

Hence, the next statement is immediate from Proposition 2.3.

Corollary 2.4. *If F is upper Lipschitz-like at $(x_0, y_0) \in \operatorname{gph} F$, then*

$$\text{(2.8)} \qquad \operatorname{Vert}\Big(T^b_{\operatorname{gph} F}(x_0, y_0)\Big) \subset \{0\} \times \overline{\operatorname{cone}}\big(F(x_0) - y_0\big)$$

and

$$\text{(2.9)} \qquad \operatorname{Vert}\Big(C_{\operatorname{gph} F}(x_0, y_0)\Big) \subset \{0\} \times \overline{\operatorname{cone}}\big(F(x_0) - y_0\big).$$

Remark 2.5. The upper estimate (2.9) sharpens a result of Dien and Yen [5, Proposition 2] where it was assumed that F is locally Lipschitz at x_0 and $F(x)$ is convex for all x from a neighborhood of x_0.

The assumption that F is upper Lipschitz-like at $(x_0, y_0) \in \operatorname{gph} F$ seems to be weakest possible for the validity of the above proposition and corollary. One cannot replace it by the requirement that F is upper Hölder-like of an order $\alpha \in (0, 1)$ at $(x_0, y_0) \in \operatorname{gph} F$.

Example 2.6. Consider a (single-valued, continuous) map $F : I\!\!R \rightrightarrows I\!\!R$ with $F(x) = \{\sqrt{|x|}\}$ for all $x \in I\!\!R$. It is clear that F is upper Hölder-like of the order $\alpha = \frac{1}{2}$ at $(x_0, y_0) := (0, 0) \in \operatorname{gph} F$. Since

$$C_{\operatorname{gph} F}(x_0, y_0) = T^b_{\operatorname{gph} F}(x_0, y_0) = T_{\operatorname{gph} F}(x_0, y_0) = \{0\} \times [0, +\infty)$$

and $\overline{\operatorname{cone}}\big(F(x_0) - y_0\big) = \{0\}$, none of the estimates (2.3), (2.8), (2.9) holds true.

3. Exact Formulae

The following statement describes some sufficient conditions for having exact formulae for computing the sets of vertical vectors in the intermediate tangent cone $T^b_{\operatorname{gph} F}(x_0, y_0)$ and the contingent cone $T_{\operatorname{gph} F}(x_0, y_0)$.

Theorem 3.1. *If F is upper Lipschitz-like at $(x_0, y_0) \in \operatorname{gph} F$ and the set $F(x_0)$ is convex, then*

$$\text{(3.1)} \quad \operatorname{Vert}\Big(T^b_{\operatorname{gph} F}(x_0, y_0)\Big) = \operatorname{Vert}\Big(T_{\operatorname{gph} F}(x_0, y_0)\Big) = \{0\} \times \overline{\operatorname{cone}}\big(F(x_0) - y_0\big).$$

Proof. By the definition of the intermediate tangent cone, for any $v \in T^b_{F(x_0)}(y_0)$ we have $\lim\limits_{t \to 0^+} \dfrac{\operatorname{dist}\big(y_0 + tv, F(x_0)\big)}{t} = 0$. The last property yields the equality

$$\lim_{t \to 0^+} \frac{\operatorname{dist}\big((x_0, y_0 + tv), \mathrm{gphF}\big)}{t} = 0$$

which shows that $(0, v) \in T^b_{\text{gph}\,F}(x_0, y_0)$. Thus

$$\{0\} \times T^b_{F(x_0)}(y_0) \subset \text{Vert}\Big(T^b_{\text{gph}\,F}(x_0, y_0)\Big). \tag{3.2}$$

The convexity of $F(x_0)$ implies that $T^b_{F(x_0)}(y_0) = \overline{\text{cone}}\big(F(x_0) - y_0\big)$. Hence, by (2.3),

$$\text{Vert}\Big(T^b_{\text{gph}\,F}(x_0, y_0)\Big) \subset \text{Vert}\Big(T_{\text{gph}\,F}(x_0, y_0)\Big) \subset \{0\} \times T^b_{F(x_0)}(y_0). \tag{3.3}$$

It is clear that the desired expression (3.1) follows from (3.3) and (3.2). ■

We say that the restriction of F on $\text{dom}\,F := \{x \in X \,:\, F(x) \neq \emptyset\}$ is *lower semicontinuous* at x_0 if for any $y \in F(x_0)$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(y + B_Y(0, \varepsilon)) \cap F(x) \neq \emptyset \quad \forall x \in B_X(x_0, \delta) \cap \text{dom}\,F.$$

If x_0 belongs to the interior of $\text{dom}\,F$ and the restriction of F on $\text{dom}\,F$ is lower semicontinuous at the point, then we simply say that F is lower semicontinuous at x_0.

Sufficient conditions for having an exact formula for computing the set of vertical vectors in the Clarke tangent cone $C_{\text{gph}\,F}(x_0, y_0)$ are given in the next statement. Although the first assertion is a known result [5, Proposition 1], a proof is provided to make the presentation self-contained.

Theorem 3.2. *Let $z_0 = (x_0, y_0) \in \text{gph}\,F$ and $F(x)$ be convex for all x from a neighborhood of x_0. The following properties hold:*

(i) (see [5]) *If the restriction of F on* $\text{dom}\,F$ *is lower semicontinuous at x_0, then*

$$\{0\} \times \overline{\text{cone}}\big(F(x_0) - y_0\big) \subset \text{Vert}\Big(C_{\text{gph}\,F}(x_0, y_0)\Big). \tag{3.4}$$

(ii) *If the restriction of F on* $\text{dom}\,F$ *is lower semicontinuous at x_0 and F is upper Lipschitz-like at (x_0, y_0), then*

$$\text{Vert}\Big(C_{\text{gph}\,F}(x_0, y_0)\Big) = \{0\} \times \overline{\text{cone}}\big(F(x_0) - y_0\big). \tag{3.5}$$

(iii) *If F is Lipschitz-like at any point $(x_0, y) \in \text{gph}\,F$, where $y \in F(x_0)$, then the equality* (3.5) *is valid.*

Proof. (i) We will follow the proof scheme of [5]. Since

$$\mathrm{Vert}\Big(C_{\mathrm{gph}\,F}(x_0, y_0)\Big)$$

is a closed convex cone, the inclusion (3.4) will be established if we can show that

$$(0, v) \in C_{\mathrm{gph}\,F}(x_0, y_0) \quad \forall v \in \big(F(x_0) - y_0\big) \setminus \{0\}. \tag{3.6}$$

Take any $v \in \big(F(x_0) - y_0\big) \setminus \{0\}$. Let $v = y_1 - y_0$ for some $y_1 \in F(x_0)$, $y_1 \neq y_0$. Fix an $\varepsilon > 0$. Since the restriction of F on $\mathrm{dom}\,F$ is lower semicontinuous at x_0, there exists $\delta \in \big(0, \frac{\varepsilon}{3}\big)$ such that

$$\Big(y_1 + B_Y\Big(0, \frac{\varepsilon}{3}\Big)\Big) \cap F(x') \neq \emptyset \quad \forall x' \in B_X(x_0, \delta) \cap \mathrm{dom}\,F. \tag{3.7}$$

Since $F(x)$ is convex for all x from a neighborhood of x_0, there is no loss of generality in assuming that $F(x')$ is convex for every $x' \in B_X(x_0, \delta)$. We set

$$\lambda = \frac{\varepsilon}{3\|v\|} \tag{3.8}$$

and let (z', t) be a pair satisfying the conditions

$$z' = (x', y') \in B_Z(z_0, \delta) \cap \mathrm{gph}\,F, \quad t \in (0, \lambda). \tag{3.9}$$

According to (3.7), there exists $y'' \in F(x')$ such that

$$\|y_1 - y''\| \leq \frac{\varepsilon}{3}. \tag{3.10}$$

Setting

$$\bar{x} = x', \quad \bar{y} = \frac{1}{1+t}y' + \frac{t}{1+t}y'',$$

we have $(\bar{x}, \bar{y}) \in \mathrm{gph}\,F$ by the convexity of $F(x')$. Using (3.7)-(3.10) we have

$$\begin{aligned}
&\|(x', y') + t(0, v) - (\bar{x}, \bar{y})\| \\
&= (1+t)^{-1}\|(1+t)y' + (1+t)tv - (1+t)\bar{y}\| \\
&= (1+t)^{-1}\|(1+t)y' + tv + t^2v - y' - ty''\| \\
&= (1+t)^{-1}\|t(y' - y_0) + t^2v + t\big((v + y_0) - y''\big)\| \\
&\leq t\Big(\|y' - y_0\| + t\|v\| + \|y_1 - y''\|\Big) \\
&\leq t\Big(\delta + t\|v\| + \frac{\varepsilon}{3}\Big) \\
&< t\varepsilon.
\end{aligned}$$

Thus

$$(x', y') + t(0, v) \in \operatorname{gph} F + tB_Z(0, \varepsilon)$$

or, equivalently,

$$\Big[(x', y') + t\big((0, v) + B_Z(0, \varepsilon)\big)\Big] \cap \operatorname{gph} F \neq \emptyset. \tag{3.11}$$

We have proved that for every $\varepsilon > 0$ there exist $\lambda > 0$ and $\delta > 0$ such that (3.11) holds for any pair (z', t) satisfying (3.9). This establishes (3.6).

(ii) It suffices to combine (i) with Corollary 2.4.

(iii) If F is Lipschitz-like at any point $(x_0, y) \in \operatorname{gph} F$, where $y \in F(x_0)$, then F is lower semicontinuous at x_0. Moreover, F is upper Lipschitz-like at (x_0, y_0). Hence, the claim follows from (ii). ■

The assumption saying that the restriction of F on $\operatorname{dom} F$ is lower semicontinuous at x_0 is crucial for the validity of (3.4) and (3.5).

Example 3.3. Consider the multifunction $F : I\!R \rightrightarrows I\!R$ defined by setting $F(x) = \{1\}$ for all $x \neq 0$ and $F(0) = (-\infty, 1]$. Let $(x_0, y_0) = (0, 1)$. Since $C_{\operatorname{gph} F}(x_0, y_0) = \{(0, 0)\}$, the formulae (3.4) and (3.5) fail to hold. The unique damage to the assumptions of the first two assertions in Theorem 3.2 is that F is not lower semicontinuous at x_0. A similar effect is achieved with the multifunction G in Example 2.2 and the points $(x_0, y_0) = (0, 1)$ and $(x_0, y_0) = (0, -1)$.

References

1. J.-P. Aubin, Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions, in: *Mathematical Analysis and Applications*, Part A, pp. 159-229, Adv. in Math. Suppl. Stud., 7a, Academic Press, 1981.

2. J.-P. Aubin, Lipschitz behavior of solutions to convex minimization problems, *Math. Oper. Res.*, **9** (1984), 87-111.

3. J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

4. J. M. Borwein and Q. J. Zhu, *Techniques of Variational Analysis*, Springer, New York, 2005.

5. P. H. Dien and N. D. Yen, A remark on Clarke's tangent cone, *Acta Math. Vietnam.*, **10(1)** (1985), 144-147.

6. A. L. Dontchev, M. Quicampoix and N. Zlateva, Aubin criterion for metric regularity, *J. Convex Anal.*, **13** (2006), 281-297.

7. G. M. Lee and N. Q. Huy, On proto-differentiability of generalized perturbation maps, *J. Math. Anal. Appl.*, **324** (2006), 1297-1309.

8. G. M. Lee and N. Q. Huy, On sensitivity analysis in vector optimization, *Taiwanese J. Math.*, **11** (2007), 945-958.

9. B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation, Vol. I: Basic Theory, Vol. II: Applications*, Springer, Berlin, 2006.

10. S. M. Robinson, Some continuity properties of polyhedral multifunctions, *Math. Program. Study*, **14** (1981), 206-214.

11. R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer, Berlin, 1998.

12. P. H. Sach and B. D. Craven, Invex multifunctions and duality, *Numer. Funct. Anal. Optim.*, **12** (1991), 575-591.

N. D. Yen
Institute of Mathematics,
Vietnamese Academy of Science and Technology,
18 Hoang Quoc Viet,
Hanoi 10307, Vietnam
E-mail: ndyen@math.ac.vn

Jen-Chih Yao
Department of Applied Mathematics,
National Sun Yat-sen University,
Kaohsiung 804, Taiwan
E-mail: yaojc@math.nsysu.edu.tw