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THE ORTHOGONALITY IN THE LOCALLY CONVEX SPACES

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Abstract. The purpose of this paper is to introduce and to discuss the concept of orthogonality in the locally convex spaces, and obtaining some results on orthogonality in locally convex spaces similar to orthogonality of normed spaces. We shall obtain some characterizations of the best approximation and the best coapproximation in locally convex spaces.

1. Introduction

Suppose that X is a vector space over $\phi \in \{\mathbf{C}, \mathbf{R}\}$. A seminorm is a function $p: X \to [0, \infty)$ having the following properties:

(a) $p(x + y) \leq p(x) + p(y)\ \forall x, y \in X$;

(b) $p(\alpha x) = |\alpha| p(x),\ \forall \alpha \in \phi$ and $\forall x \in X$. It follows from (b) that $p(0) = 0$. A norm is a seminorm p such that

(c) $x = 0\ if\ p(x) = 0$.

A topological vector space is a vector space X together with respect to this topology

(a) the map $X \times X \longrightarrow X$ defined by $(x, y) \mapsto x + y$ is continuous;

(b) the map $\phi \times X \longrightarrow X$ defined by $(\alpha, x) \mapsto \alpha x$ is continuous.
A topological vector space X is a locally convex space whose topology is defined by seminorms P such that

$$\bigcap_{p \in P} \{x \in X : p(x) = 0\} = \{0\}.$$

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Many authors have introduced the concept of orthogonality in different ways (see [1-2], [5], [14]). In [1] G. C. Birkhoff modified the concept of orthogonality, in fact, if X is a normed linear space and $x, y \in X$, x is said to be orthogonal to y and is denoted by $x \perp y$ if and only if $\|x\| \leq \|x + \alpha y\|$ for all scalar α. In [5], we define the concept of orthogonality in vector spaces with a seminorm p, we said that x is orthogonal to y if and only if

$$p(x) \neq 0,\ p(x) = \inf{}_{\alpha} p(x + \alpha y).$$

In this note, we shall define orthogonality in the locally convex spaces.

Definition 1.1. Let (X, P) be a locally convex space, and $x, y \in X$. We say that x is orthogonal to y if and only if for all $p \in P$

$$p(x) \leq p(x + \alpha y)\ (\alpha \in \phi),$$

in this case we write $x \perp y$. If M_1 and M_2 are subsets of X, we say that M_1 is orthogonal to M_2 if and only if $g_1 \perp g_2$ for all $g_1 \in M_1,\ g_2 \in M_2$. If M_1 is orthogonal to M_2, we write $M_1 \perp M_2$. Suppose p is a seminorm on X. If $x \in X$, put

$$M_x^p = \{\Lambda :\ X \overset{linear}{\rightarrow} \phi :\ |\Lambda(z)| \leq p(z), \forall z \in X,\ \Lambda(x) = p(x)\}.$$

At first we state the following lemmas which is needed in the proof of the main results.

Lemma 1.2. ([13]) *Let M be a subspace of a vector space X and let f be a linear functional on M such that;*

$$|f(x)| \leq p(x)\ (x \in M).$$

Then f extends to a linear functional Λ on X that satisfies

$$|\Lambda(x)| \leq p(x)\ (x \in X).$$

Lemma 1.3. ([5]) *Let X be a locally convex space, G be a subspace of X, $x \in X \backslash \bar{G}$. Then the following statements are equivalent: (a) $x \perp G$*
(b) For all $p \in P$ there exists a linear functional Λ_p on X such that $\Lambda_p \in M_x^p$ and $\Lambda_p|_G = 0$.

2. Main Results

In this section we state and prove our main results.

Corollary 2.1. *Let (X, P) be a locally convex space and $x, y \in X$. Then the following two conditions are equivalent:*

(a) *$x \perp y$.*

(b) *For all $p \in P$ there exists a linear functional Λ_p on X such that, $\Lambda_p \in M_x^p$ and $\Lambda_p(y) = 0$.*

Lemma 2.2. *Let (X, P) be a locally convex space and $x, y \in X$. Then the following statements are true:*

(a) *$x \perp y$ then $< x > \cap < y >= \{0\}$.*

(b) *For all $x \in X$, $0 \perp x$ and $x \perp 0$.*

Proof.

(a) Suppose $z \in < x > \cap < y >$. Then $z = \alpha x = \beta y$ for some scalars $\alpha, \beta \in \phi$. If $\alpha = 0$, then $z = 0$. In the otherwise, suppose $\alpha \neq 0$ and $p \in P$ From Corollary 2.2, there exists a linear functional Λ_p on X such that, $\Lambda_p \in M_x^p$ and $\Lambda_p(y) = 0$. Therefore $\Lambda_p(z) = 0$ and $p(x) = 0$. It follows that $z = x = 0$, since X is a locally convex space.

(b) It is trivial. ■

Corollary 2.3. *Let (X, P) be a locally convex space, G be a nonempty subset of X and $x \in X \backslash G$. Then the following two conditions are equivalent:*

(a) *$G \perp x$.*

(b) *For all $p \in P$ and all $g \in G$ there exists a linear functional Λ_p^g on X with $\Lambda_p^g \in M_g^p$ and $\Lambda_p^g(x) = 0$.*

Example 2.4. Let X be a linear space of dimension greater than 1. Suppose $\|.,.\|$ is a real-valued function on $X \times X$ satisfying the following conditions:

(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.

(b) $\|x, y\| = \|y, x\|$ for all $x, y \in X$.

(c) $\|\lambda x, y\| = |\lambda|\|x, y\|$ for all $\lambda \in \mathbf{R}$ and all $x, y \in X$.

(d) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.

Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space. Some of the basic properties of 2-norms, that they are non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and all $\alpha \in \mathbf{R}$.

Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X$, $p_b(x) = \|x, b\|$, $x \in X$, is a seminorm and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex topology on X.

Suppose $(X, \|.,.\|)$ is a 2-normed space and $x, y \in X$. Then from Definition 1.1,

$$x \perp y \Leftrightarrow \forall b \in X \ \forall \alpha \in \phi \ \|x, b\| \leq \|x + \alpha y\|.$$

Example 2.5. Let X be a normed space and $x \in X$. Define $p_x : X^* \longrightarrow [0, \infty)$, by $p_x(x^*) = |x^*(x)|$. Then p_x is a seminorm and $P = \{p_x : \ x \in X\}$ makes X^* into a locally convex space., the topology defined by these seminorms is called weak-star topology on X^*, it often denoted by $\sigma(x^*, x)$. For $y_1^*, y_2^* \in X^*$ we say

$$y_1^* \perp y_2^* \Leftrightarrow \forall x \in X \ \forall \alpha \in \mathbf{C} \ |y_1^*(x)| \leq |(y_1^* + \alpha y_2^*)(x)|.$$

Example 2.6. If X is a topology vector space and M is a closed linear space. Then $\frac{X}{M}$ with the quotient topology is a topology vector space. If p is a seminorm on X define $\overline{p}$ on $\frac{X}{M}$ by $\overline{p}(x + M) = \inf\{p(x + y) : \ y \in M\}$, $\overline{p}$ is a seminorm on $\frac{X}{M}$ and if X is a locally convex space by seminorm $P = \{p\}$ then $\frac{X}{M}$ is a locally convex space by seminorm $\overline{P} = \{\overline{p}\}$. The orthogonality in $\frac{X}{M}$ is defined by

$$z_1, z_2 \in X, \ z_1 + M \perp z_2 + M \Leftrightarrow \forall y \in M \ \forall \alpha \in \mathbf{C}, \ \overline{p}(z_1) \leq \overline{p}(z_1 + \alpha z_2 + y).$$

In the normed linear spaces, the concepts of best approximation and best coapproximation have been defined. (see [3-12, 14]) we shall define these concepts for the locally convex spaces.

Let G be a subspace of the locally convex space X. we will define

$$\hat{G} = \{x \in X : \ x \perp G\},$$

and

$$\breve{G} = \{x \in X : \ G \perp x\},$$

also a point $g_0 \in G$ is said to be a best approximation (resp. best coapproximation) for $x \in X$ if and only if $x - g_0 \in \hat{G}$ $(resp.\ x - g_0 \in \breve{G})$.

The set of all best approximations (resp. best coapproximations) of $x \in X$ in G is shown by $P_G(x)$ $(resp.\ R_G(x))$. In other words

$$P_G(x) = \{g_0 \in G \ : \ x - g_0 \in \hat{G}\}$$

and

$$R_G(x) = \{g_0 \in G \ : \ x - g_0 \in \breve{G}\}.$$

If $P_G(x)$ $(resp.\ R_G(x))$ is non-empty for every $x \in X$, then G is called a Proximinal (resp. coproximinal) set. The set G is Chebyshev (resp. cochebyshev) if $P_G(x)$ $(resp.\ R_G(x))$ is a singleton set for every $x \in X$.

Corollary 2.7. *Let (X, P) be a locally convex space, G be a subspace of X and $x \in X \backslash \bar{G}$. Then the following statements are equivalent: (a) $g_0 \in P_G(x)$ (b)*

For all $p \in P$ there exists a linear functional Λ_p on X such that $\Lambda_p \in {M^p}_{x-g_0}$ and $\Lambda_p|_G = 0$.

Corollary 2.8. *Let (X, P) be a locally convex space, G be a subspace of X and $x \in X \backslash \bar{G}$. Then the following statements are equivalent:*

(a) $g_0 \in R_G(x)$

(b) *For all $p \in P$ and all $g \in G$ there exists a linear functional Λ_p^g on X such that $\Lambda_p^g \in {M^p}_g$ and $\Lambda_p^g(x - g_0) = 0$.*

Lemma 2.9. *Let (X, P) be a locally convex space, G be a subspace of X. Then the following statements are true:*

(a) $G \cap \hat{G} = \{0\}$

(b) $G \cap \breve{G} = \{0\}$.

(c) *$\alpha x \in \hat{G}$, if $x \in \hat{G}$ and $\alpha \in \phi$.*

Proof. The parts (*c*) and (*d*) are consequences of definition of orthogonality and Lemma 2.2. Suppose $x \in G \cap \hat{G}$ (resp. $x \in G \cap \breve{G}$), then $x \perp G$ (resp. $G \perp x$) and $x \in G$. Therefore $x \perp x$, form Lemma 2.2, $x = 0$. ■

Theorem 2.10. *Let X be a locally convex space. Then if G is a proximinal subspace of X and $\hat{G}$ is a convex set, then G is Chebyshev.*

Proof. Suppose $x \in X$ and $g_1, g_2 \in P_G(x)$, then $x - g_1, x - g_2 \in \hat{G}p$. Since $\hat{G}$ is convex, it follows that $\frac{1}{2}(g_1 - g_2) \in \hat{G}$. Since $\frac{1}{2}(g_1 - g_2) \in G$. The Lemma 2.8 shows that $g_1 = g_2$. ■

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