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ON THE RECURSIVE SEQUENCE $x_{n+1} = \frac{x_{n-(5k+9)}}{1+x_{n-4}x_{n-9}...x_{n-(5k+4)}}$

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Abstract. In this paper a solution of the following difference equation was investigated

$$x_{n+1} = \frac{x_{n-(5k+9)}}{1+x_{n-4}x_{n-9}...x_{n-(5k+4)}},\ n = 0, 1, 2, ...$$

where $x_{-(5k+9)}, x_{-(5k+8)}, ..., x_{-1}, x_0 \in (0, \infty)$.

1. Introduction

Recently there has been a lot of interest in studying the periodic nature of nonlinear difference equations. For some recent results concerning among other problems, the periodic nature of scalar nonlinear difference equations see, for examples [1, 6-8].

In [2] the following problem was posed. Is there a solution of the following difference equation

$$x_{n+1} = \frac{\beta x_{n-1}}{\beta + x_n} \text{ for } n = 0, 1, 2, ...$$

where $x_{-1}, x_0, \beta \in (0, \infty)$ such that $x_n \to 0$ as $n \to \infty$.

In [9] Stevic assumed that $\beta = 1$ and solved the following problem

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n} \text{ for } n = 0, 1, 2, ...$$

where $x_{-1}, x_0 \in (0, \infty)$. Also, this result was generalized to the equation of the following form:

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)} \text{ for } n = 0, 1, 2, ...$$

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where $x_{-1}, x_0 \in (0, \infty)$.

In [3-5] we solved the following three problems for the positive initial values

$$x_{n+1}=\frac{x_{n-3}}{1+x_{n-1}},\ x_{n+1}=\frac{x_{n-5}}{1+x_{n-2}},\ x_{n+1}=\frac{x_{n-5}}{1+x_{n-1}x_{n-3}} \text{ for } n=0,1,2,\ldots .$$

In this paper we investigate the following nonlinear difference equation

$$x_{n+1}=\frac{x_{n-(5k+9)}}{1+x_{n-4}x_{n-9}\ldots x_{n-(5k+4)}},\ n=0,1,2,\ldots . \tag{1}$$

where $x_{-(5k+9)}, x_{-(5k+8)}, \ldots, x_{-1}, x_0 \in (0, \infty)$.

2. Main Result

Theorem 1. *Consider the difference equation* (1). *Then the following statements are true:*

(a) *The sequences* $(x_{(5k+10)n-(5k+9)}), (x_{(5k+10)n-(5k+8)}), \ldots, (x_{(5k+10)n})$ *are decreasing and there exist* $a_1, a_2, \ldots, a_{5k+10} \geqslant 0$ *such that*

$$\lim_{n\to\infty} x_{(5k+10)n-(5k+9)} = a_1,\ \lim_{n\to\infty} x_{(5k+10)n-(5k+8)} = a_2, \ldots, \lim_{n\to\infty} x_{(5k+10)n} = a_{5k+10}.$$

(b) $(a_1, a_2, \ldots, a_{5k+10}, a_1, a_2, \ldots, a_{5k+10}, \ldots)$ *is a solution with period 5k+10 of equation* (1).

(c) $\lim_{n\to\infty} x_{(5k+10)n-(5k+9)} \lim_{n\to\infty} x_{(5k+10)n-(5k+4)} \cdots \lim_{n\to\infty} x_{(5k+10)n-4} = 0$,
$\lim_{n\to\infty} x_{(5k+10)n-(5k+8)} \lim_{n\to\infty} x_{(5k+10)n-(5k+3)} \cdots \lim_{n\to\infty} x_{(5k+10)n-3} = 0$,
$\lim_{n\to\infty} x_{(5k+10)n-(5k+7)} \lim_{n\to\infty} x_{(5k+10)n-(5k+2)} \cdots \lim_{n\to\infty} x_{(5k+10)n-2} = 0$,
$\lim_{n\to\infty} x_{(5k+10)n-(5k+6)} \lim_{n\to\infty} x_{(5k+10)n-(5k+1)} \cdots \lim_{n\to\infty} x_{(5k+10)n-1} = 0$ and
$\lim_{n\to\infty} x_{(5k+10)n-(5k+5)} \lim_{n\to\infty} x_{(5k+10)n-(5k)} \cdots \lim_{n\to\infty} x_{(5k+10)n} = 0$
or $a_1.a_6 \ldots a_{5k+6} = 0$, $a_2.a_7 \ldots a_{5k+7} = 0$, $a_3.a_8 \ldots a_{5k+8} = 0$,
$a_4.a_9 \ldots a_{5k+9} = 0$ and $a_5.a_{10} \ldots a_{5k+10} = 0$.

(d) *If there exists* $n_0 \in N$ *such that*

$$x_{n-4}x_{n-9}\ldots x_{n-(5k+4)} \geq x_{n+1}x_{n-4}x_{n-9}\ldots x_{n-(5k-1)}$$

for all $n \geqslant n_0$, *then*

$$\lim_{n\to\infty} x_n = 0.$$

(e) *The following formulas*

$$x_{(5k+10)n+1} = x_{-(5k+9)}(1 - \frac{x_{-4}x_{-9}\ldots x_{-(5k+4)}}{1+x_{-4}x_{-9}\ldots x_{-(5k+4)}} \sum_{j=0}^{n}\prod_{i=1}^{(k+2)j}\frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}})$$

$$x_{(5k+10)n+2} = x_{-(5k+8)}(1 - \frac{x_{-3}x_{-8}\ldots x_{-(5k+3)}}{1+x_{-3}x_{-8}\ldots x_{-(5k+3)}} \sum_{j=0}^{n}\prod_{i=1}^{(k+2)j}\frac{1}{1+x_{5i-3}x_{5i-8}\ldots x_{5i-(5k+3)}})$$

$$x_{(5k+10)n+3} = x_{-(5k+7)}(1 - \frac{x_{-2}x_{-7}\ldots x_{-(5k+2)}}{1+x_{-2}x_{-7}\ldots x_{-(5k+2)}} \sum_{j=0}^{n}\prod_{i=1}^{(k+2)j}\frac{1}{1+x_{5i-2}x_{5i-7}\ldots x_{5i-(5k+2)}})$$

$$x_{(5k+10)n+4} = x_{-(5k+6)}(1 - \frac{x_{-1}x_{-6}\ldots x_{-(5k+1)}}{1+x_{-1}x_{-6}\ldots x_{-(5k+1)}} \sum_{j=0}^{n}\prod_{i=1}^{(k+2)j}\frac{1}{1+x_{5i-1}x_{5i-6}\ldots x_{5i-(5k+1)}})$$

$$x_{(5k+10)n+5} = x_{-(5k+5)}(1 - \frac{x_{0}x_{-5}\ldots x_{-(5k)}}{1+x_{0}x_{-5}\ldots x_{-(5k)}} \sum_{j=0}^{n}\prod_{i=1}^{(k+2)j}\frac{1}{1+x_{5i}x_{5i-5}\ldots x_{5i-(5k)}})$$

$$\vdots$$

$$x_{(5k+10)n+(5k+6)} = x_{-4}(1 - \frac{x_{-9}x_{-14}\ldots x_{-(5k+9)}}{1+x_{-4}x_{-9}\ldots x_{-(5k+4)}} \sum_{j=0}^{n}\prod_{i=1}^{(k+2)j+(k+1)}\frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}})$$

$$x_{(5k+10)n+(5k+7)} = x_{-3}(1 - \frac{x_{-8}x_{-13}\ldots x_{-(5k+8)}}{1+x_{-3}x_{-8}\ldots x_{-(5k+3)}} \sum_{j=0}^{n}\prod_{i=1}^{(k+2)j+(k+1)}\frac{1}{1+x_{5i-3}x_{5i-8}\ldots x_{5i-(5k+3)}})$$

$$x_{(5k+10)n+(5k+8)} = x_{-2}(1-\frac{x_{-7}x_{-12}...x_{-(5k+7)}}{1+x_{-2}x_{-7}...x_{-(5k+2)}}\sum_{j=0}^{n}\prod_{i=1}^{(k+2)j+(k+1)}\frac{1}{1+x_{5i-2}x_{5i-7}...x_{5i-(5k+2)}})$$

$$x_{(5k+10)n+(5k+9)} = x_{-1}(1-\frac{x_{-6}x_{-11}...x_{-(5k+6)}}{1+x_{-1}x_{-6}...x_{-(5k+1)}}\sum_{j=0}^{n}\prod_{i=1}^{(k+2)j+(k+1)}\frac{1}{1+x_{5i-1}x_{5i-6}...x_{5i-(5k+1)}})$$

$$x_{(5k+10)n+(5k+10)} = x_{0}(1-\frac{x_{-5}x_{-10}...x_{-(5k+5)}}{1+x_{0}x_{-5}...x_{-(5k)}}\sum_{j=0}^{n}\prod_{i=1}^{(k+2)j+(k+1)}\frac{1}{1+x_{5i}x_{5i-5}...x_{5i-(5k)}})$$

hold.

(f) *If* $x_{(5k+10)n+i} \to a_i \neq 0$, $x_{(5k+10)n+i+5} \to a_{i+5} \neq 0$,..., $x_{(5k+10)n+(5k+i)} \to a_{5k+i} \neq 0$ *then* $x_{(5k+10)n+(5k+i+5)} \to 0$ *as* $n \to \infty$ (*for* $i = 1, 2, 3, 4, 5$).

Proof.

(a) Firstly, from the equation (1), we obtain

$$x_{n+1}(1+x_{n-4}x_{n-9}...x_{n-(5k+4)}) = x_{n-(5k+9)}.$$

If $x_{n-4}, x_{n-9}, ..., x_{n-(5k+4)} \in (0,+\infty)$, then $(1+x_{n-4}x_{n-9}...x_{n-(5k+4)}) \in (1,+\infty)$. Since $x_{n+1} < x_{n-(5k+9)}$, $n \in N$, we obtain that there exist

$$\lim_{n\to\infty} x_{(5k+10)n-(5k+9)} = a_1,\ \lim_{n\to\infty} x_{(5k+10)n-(5k+8)} = a_2, ...,\ \lim_{n\to\infty} x_{(5k+10)n} = a_{5k+10}.$$

(b) $(a_1, a_2, ..., a_{5k+10}, a_1, a_2, ..., a_{5k+10}, ...)$ a solution with period 5k+10 of equation (1).

(c) In view of the equation (1), we obtain

$$x_{(5k+10)n+1} = \frac{x_{(5k+10)n-(5k+9)}}{1+x_{(5k+10)n-4}...x_{(5k+10)n-(5k+4)}}.$$

Take the limits on both sides of the above equality

$$\lim_{n\to\infty} x_{(5k+10)n+1} = \lim_{n\to\infty}\frac{x_{(5k+10)n-(5k+9)}}{1+x_{(5k+10)n-4}...x_{(5k+10)n-(5k+4)}}$$

$\lim_{n\to\infty} x_{(5k+10)n-(5k+9)} \lim_{n\to\infty} x_{(5k+10)n-(5k+4)} \cdots \lim_{n\to\infty} x_{(5k+10)n-4} = 0$ or $a_1.a_6 \ldots a_{5k+6} = 0$.

Similarly;

$$x_{(5k+10)n+2} = \frac{x_{(5k+10)n-(5k+8)}}{1 + x_{(5k+10)n-3} \cdots x_{(5k+10)n-(5k+3)}}.$$

Take the limits on both sides of the above equality

$$\lim_{n\to\infty} x_{(5k+10)n+2} = \lim_{n\to\infty} \frac{x_{(5k+10)n-(5k+8)}}{1 + x_{(5k+10)n-3} \cdots x_{(5k+10)n-(5k+3)}}$$

$\lim_{n\to\infty} x_{(5k+10)n-(5k+8)} \lim_{n\to\infty} x_{(5k+10)n-(5k+3)} \cdots \lim_{n\to\infty} x_{(5k+10)n-3} = 0$ or $a_2.a_7 \ldots a_{5k+7} = 0$.

Similarly;

$$x_{(5k+10)n+3} = \frac{x_{(5k+10)n-(5k+7)}}{1 + x_{(5k+10)n-2} \cdots x_{(5k+10)n-(5k+2)}}.$$

Take the limits on both sides of the above equality

$$\lim_{n\to\infty} x_{(5k+10)n+3} = \lim_{n\to\infty} \frac{x_{(5k+10)n-(5k+7)}}{1 + x_{(5k+10)n-2} \cdots x_{(5k+10)n-(5k+2)}}$$

$\lim_{n\to\infty} x_{(5k+10)n-(5k+7)} \lim_{n\to\infty} x_{(5k+10)n-(5k+2)} \cdots \lim_{n\to\infty} x_{(5k+10)n-2} = 0$ or $a_3.a_8 \ldots a_{5k+8} = 0$.

Similarly;

$$x_{(5k+10)n+4} = \frac{x_{(5k+10)n-(5k+6)}}{1 + x_{(5k+10)n-1} \cdots x_{(5k+10)n-(5k+1)}}.$$

Take the limits on both sides of the above equality

$$\lim_{n\to\infty} x_{(5k+10)n+4} = \lim_{n\to\infty} \frac{x_{(5k+10)n-(5k+6)}}{1 + x_{(5k+10)n-1} \cdots x_{(5k+10)n-(5k+1)}}$$

$\lim_{n\to\infty} x_{(5k+10)n-(5k+6)} \lim_{n\to\infty} x_{(5k+10)n-(5k+1)} \cdots \lim_{n\to\infty} x_{(5k+10)n-1} = 0$ or $a_4.a_9 \ldots a_{5k+9} = 0$.

Similarly;

$$x_{(5k+10)n+5} = \frac{x_{(5k+10)n-(5k+5)}}{1 + x_{(5k+10)n} \cdots x_{(5k+10)n-(5k)}}.$$

Take the limits on both sides of the above equality

$$\lim_{n\to\infty} x_{(5k+10)n+5} = \lim_{n\to\infty} \frac{x_{(5k+10)n-(5k+5)}}{1 + x_{(5k+10)n} \cdots x_{(5k+10)n-(5k)}}$$

$\lim_{n\to\infty} x_{(5k+10)n-(5k+5)} \lim_{n\to\infty} x_{(5k+10)n-(5k)} \ldots \lim_{n\to\infty} x_{(5k+10)n} = 0$ or $a_5.a_{10} \ldots a_{5k+10} = 0$.

(d) If there exists $n_0 \in N$ such that

$$x_{n-4}x_{n-9}...x_{n-(5k+4)} \geq x_{n+1}x_{n-4}x_{n-9}...x_{n-(5k-1)}$$

for all $n \geqslant n_0$, then $a_i.a_{i+5}...a_{5k+i+5} = 0$ Since $a_i \leq a_{i+5} \leq ... \leq a_{5k+i+5} \leq a_i$ (for $i = 1, 2, 3, 4, 5$).

(e) Subracting $x_{n-(5k+9)}$ from both the left and right-hand sides of the equation (1) we obtain

$$x_{n+1} - x_{n-(5k+9)} = \frac{1}{1 + x_{n-4}x_{n-9}...x_{n-(5k+4)}}(x_{n-4} - x_{n-(5k+14)})$$

and the following formula

(2)

$$\text{for } n \geqslant 5 \begin{cases} x_{5n-24} - x_{5n-(5k+34)} = (x_1 - x_{-(5k+9)}) \prod_{i=1}^{n-5} \frac{1}{1 + x_{5i-4}x_{5i-9}...x_{5i-(5k+4)}} \\ x_{5n-23} - x_{5n-(5k+33)} = (x_2 - x_{-(5k+8)}) \prod_{i=1}^{n-5} \frac{1}{1 + x_{5i-3}x_{5i-8}...x_{5i-(5k+3)}} \\ x_{5n-22} - x_{5n-(5k+32)} = (x_3 - x_{-(5k+7)}) \prod_{i=1}^{n-5} \frac{1}{1 + x_{5i-2}x_{5i-7}...x_{5i-(5k+2)}} \\ x_{5n-21} - x_{5n-(5k+31)} = (x_4 - x_{-(5k+6)}) \prod_{i=1}^{n-5} \frac{1}{1 + x_{5i-1}x_{5i-6}...x_{5i-(5k+1)}} \\ x_{5n-20} - x_{5n-(5k+30)} = (x_5 - x_{-(5k+5)}) \prod_{i=1}^{n-5} \frac{1}{1 + x_{5i}x_{5i-5}...x_{5i-(5k)}} \end{cases}$$

holds.

Replacing n by $(k+2)j$ in (2) and summing up them from $j = 0$ to $j = n$. We obtain for $n = 0, 1, 2, \cdots$

(3)

$$x_{(5k+10)n+1} - x_{-(5k+9)} = (x_1 - x_{-(5k+9)}) \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i-4}x_{5i-9}...x_{5i-(5k+4)}}.$$

$$x_{(5k+10)n+2} - x_{-(5k+8)} = (x_2 - x_{-(5k+8)}) \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i-3}x_{5i-8}...x_{5i-(5k+3)}}.$$

$$x_{(5k+10)n+3} - x_{-(5k+7)} = (x_3 - x_{-(5k+7)}) \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i-2}x_{5i-7}...x_{5i-(5k+2)}}.$$

$$x_{(5k+10)n+4} - x_{-(5k+6)} = (x_4 - x_{-(5k+6)}) \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i-1}x_{5i-6}...x_{5i-(5k+1)}}.$$

$$x_{(5k+10)n+5} - x_{-(5k+5)} = (x_5 - x_{-(5k+5)}) \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i}x_{5i-5}...x_{5i-(5k)}}.$$

$$\vdots$$

Also, replacing n by $(k+2)j+(k+1)$ in (2) and summing up them from $j=0$ to $j=n$. We obtain

$$x_{(5k+10)n+(5k+6)} - x_{-4} = (x_1 - x_{-(5k+9)}) \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1+x_{5i-4}x_{5i-9}...x_{5i-(5k+4)}}.$$

$$x_{(5k+10)n+(5k+7)} - x_{-3} = (x_2 - x_{-(5k+8)}) \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1+x_{5i-3}x_{5i-8}...x_{5i-(5k+3)}}.$$

$$x_{(5k+10)n+(5k+8)} - x_{-2} = (x_3 - x_{-(5k+7)}) \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1+x_{5i-2}x_{5i-7}...x_{5i-(5k+2)}}.$$

$$x_{(5k+10)n+(5k+9)} - x_{-1} = (x_4 - x_{-(5k+6)}) \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1+x_{5i-1}x_{5i-6}...x_{5i-(5k+1)}}.$$

$$x_{(5k+10)n+(5k+10)} - x_{0} = (x_5 - x_{-(5k+5)}) \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1+x_{5i}x_{5i-5}...x_{5i-(5k)}}.$$

Now, we obtain from the above formulas;

(4)

$$x_{(5k+10)n+1} = x_{-(5k+9)}\Big(1 - \frac{x_{-4}x_{-9}...x_{-(5k+4)}}{1+x_{-4}x_{-9}...x_{-(5k+4)}} \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i-4}x_{5i-9}...x_{5i-(5k+4)}}\Big)$$

$$x_{(5k+10)n+2} = x_{-(5k+8)}\Big(1 - \frac{x_{-3}x_{-8}...x_{-(5k+3)}}{1+x_{-3}x_{-8}...x_{-(5k+3)}} \sum_{j=0}^{n} \prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i-3}x_{5i-8}...x_{5i-(5k+3)}}\Big)$$

$$x_{(5k+10)n+3} = x_{-(5k+7)}(1 - \frac{x_{-2}x_{-7}...x_{-(5k+2)}}{1 + x_{-2}x_{-7}...x_{-(5k+2)}}$$
$$\sum_{j=0}^{n} \prod_{i=1}^{(k+2)j} \frac{1}{1 + x_{5i-2}x_{5i-7}...x_{5i-(5k+2)}})$$
$$x_{(5k+10)n+4} = x_{-(5k+6)}(1 - \frac{x_{-1}x_{-6}...x_{-(5k+1)}}{1 + x_{-1}x_{-6}...x_{-(5k+1)}}$$
$$\sum_{j=0}^{n} \prod_{i=1}^{(k+2)j} \frac{1}{1 + x_{5i-1}x_{5i-6}...x_{5i-(5k+1)}})$$
$$x_{(5k+10)n+5} = x_{-(5k+5)}(1 - \frac{x_{0}x_{-5}...x_{-(5k)}}{1 + x_{0}x_{-5}...x_{-(5k)}}$$
$$\sum_{j=0}^{n} \prod_{i=1}^{(k+2)j} \frac{1}{1 + x_{5i}x_{5i-5}...x_{5i-(5k)}})$$
$$\vdots$$
$$x_{(5k+10)n+(5k+6)} = x_{-4}(1 - \frac{x_{-9}x_{-14}...x_{-(5k+9)}}{1 + x_{-4}x_{-9}...x_{-(5k+4)}}$$
$$\sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1 + x_{5i-4}x_{5i-9}...x_{5i-(5k+4)}})$$
$$x_{(5k+10)n+(5k+7)} = x_{-3}(1 - \frac{x_{-8}x_{-13}...x_{-(5k+8)}}{1 + x_{-3}x_{-8}...x_{-(5k+3)}}$$
$$\sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1 + x_{5i-3}x_{5i-8}...x_{5i-(5k+3)}})$$
$$x_{(5k+10)n+(5k+8)} = x_{-2}(1 - \frac{x_{-7}x_{-12}...x_{-(5k+7)}}{1 + x_{-2}x_{-7}...x_{-(5k+2)}}$$
$$\sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1 + x_{5i-2}x_{5i-7}...x_{5i-(5k+2)}})$$
$$x_{(5k+10)n+(5k+9)} = x_{-1}(1 - \frac{x_{-6}x_{-11}...x_{-(5k+6)}}{1 + x_{-1}x_{-6}...x_{-(5k+1)}}$$
$$\sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1 + x_{5i-1}x_{5i-6}...x_{5i-(5k+1)}})$$
$$x_{(5k+10)n+(5k+10)} = x_{0}(1 - \frac{x_{-5}x_{-10}...x_{-(5k+5)}}{1 + x_{0}x_{-5}...x_{-(5k)}}$$
$$\sum_{j=0}^{n} \prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1 + x_{5i}x_{5i-5}...x_{5i-(5k)}})$$

hold.

(f) Suppose that $a_1 = a_6 = \ldots = a_{5k+1} = a_{5k+6} = 0$. By (e) we have

$$\lim_{n\to\infty} x_{(5k+10)n+1} = \lim_{n\to\infty} x_{-(5k+9)}\Big(1 - \frac{x_{-4}x_{-9}\ldots x_{-(5k+4)}}{1 + x_{-4}x_{-9}\ldots x_{-(5k+4)}} \sum_{j=0}^{n}\prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}}\Big)$$

(5)

$$a_1 = x_{-(5k+9)}\Big(1 - \frac{x_{-4}x_{-9}\ldots x_{-(5k+4)}}{1+x_{-4}x_{-9}\ldots x_{-(5k+4)}} \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}}\Big)$$

$$a_1 = 0 \Rightarrow \frac{1+x_{-4}x_{-9}\ldots x_{-(5k+4)}}{x_{-4}x_{-9}\ldots x_{-(5k+4)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}}$$

$$\lim_{n\to\infty} x_{(5k+10)n+6} = \lim_{n\to\infty} x_{-(5k+4)}\Big(1 - \frac{x_{-4}x_{-9}\ldots x_{-(5k-1)}x_{-(5k+9)}}{1 + x_{-4}x_{-9}\ldots x_{-(5k+4)}} \sum_{j=0}^{n}\prod_{i=1}^{(k+2)j+1} \frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}}\Big)$$

(6)

$$a_6 = x_{-(5k+4)}\Big(1 - \frac{x_{-4}x_{-9}\ldots x_{-(5k-1)}x_{-(5k+9)}}{1+x_{-4}x_{-9}\ldots x_{-(5k+4)}} \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+1} \frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}}\Big)$$

$$a_6 = 0 \Rightarrow \frac{1+x_{-4}x_{-9}\ldots x_{-(5k+4)}}{x_{-4}x_{-9}\ldots x_{-(5k-1)}x_{-(5k+9)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+1} \frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}}\ .$$

$$\vdots$$

Similarly,

$$\lim_{n\to\infty} x_{(5k+10)n+(5k+1)} = \lim_{n\to\infty} x_{-9}\Big(1 - \frac{x_{-4}x_{-14}\ldots x_{-(5k+9)}}{1+x_{-4}x_{-9}\ldots x_{-(5k+4)}} \sum_{j=0}^{n}\prod_{i=1}^{(k+2)j+k} \frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}}\Big)$$

$$a_{5k+1} = \lim_{n\to\infty} x_{-9}\Big(1 - \frac{x_{-4}x_{-14}\ldots x_{-(5k+9)}}{1+x_{-4}x_{-9}\ldots x_{-(5k+4)}} \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+k} \frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}}\Big)$$

(7) $$a_{5k+1} = 0 \Rightarrow \frac{1+x_{-4}x_{-9}\ldots x_{-(5k+4)}}{x_{-4}x_{-14}\ldots x_{-(5k+9)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+k} \frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}}\ .$$

$$\lim_{n\to\infty} x_{(5k+10)n+(5k+6)} = \lim_{n\to\infty} x_{-4}\Big(1 - \frac{x_{-9}x_{-14}\ldots x_{-(5k+9)}}{1+x_{-4}x_{-9}\ldots x_{-(5k+4)}} \sum_{j=0}^{n}\prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}}\Big)$$

$$a_{5k+6} = x_{-4}\Big(1 - \frac{x_{-9}x_{-14}\ldots x_{-(5k+9)}}{1+x_{-4}x_{-9}\ldots x_{-(5k+4)}} \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1+x_{5i-4}x_{5i-9}\ldots x_{5i-(5k+4)}}\Big)$$

$$a_{5k+6}=0 \Rightarrow \frac{1+x_{-4}x_{-9}...x_{-(5k+4)}}{x_{-9}x_{-14}...x_{-(5k+9)}} \tag{8}$$
$$= \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1+x_{5i-4}x_{5i-9}...x_{5i-(5k+4)}}.$$

From the equation (5) and (6),

$$\frac{1+x_{-4}x_{-9}...x_{-(5k+4)}}{x_{-4}x_{-9}...x_{-(5k+4)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i-4}x_{5i-9}...x_{5i-(5k+4)}} > \tag{9}$$
$$\frac{1+x_{-4}x_{-9}...x_{-(5k+4)}}{x_{-4}x_{-9}...x_{-(5k-1)}x_{-(5k+9)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+1} \frac{1}{1+x_{5i-4}x_{5i-9}...x_{5i-(5k+4)}}$$

thus, $x_{-(5k+9)} > x_{-(5k+4)}$.

$\vdots$

From the equation (7) and (8),

$$\frac{1+x_{-4}x_{-9}...x_{-(5k+4)}}{x_{-4}x_{-14}...x_{-(5k+9)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+k} \frac{1}{1+x_{5i-4}x_{5i-9}...x_{5i-(5k+4)}} > \tag{10}$$
$$\frac{1+x_{-4}x_{-9}...x_{-(5k+4)}}{x_{-9}x_{-14}...x_{-(5k+9)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1+x_{5i-4}x_{5i-9}...x_{5i-(5k+4)}}$$

thus, $x_{-9} > x_{-4}$. Which implies $x_{-(5k+9)} > x_{-(5k+4)} > ... > x_{-9} > x_{-4}$. This contradicts our assumption.

We omit the proofs of the equations (11)-(18). Since proofs of the equations (11), (13), (15), (17) can easy be obtained in a way to be proof of equation (9) and proofs of the equations (12), (14), (16), (18) can easy be obtained in a way to be proof of equation (10).

Suppose that $a_2 = a_7 = ... = a_{5k+2} = a_{5k+7} = 0$.

$$\frac{1+x_{-3}x_{-8}...x_{-(5k+3)}}{x_{-3}x_{-8}...x_{-(5k+3)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j} \frac{1}{1+x_{5i-3}x_{5i-8}...x_{5i-(5k+3)}} > \tag{11}$$
$$\frac{1+x_{-3}x_{-8}...x_{-(5k+3)}}{x_{-3}x_{-8}...x_{-(5k-2)}x_{-(5k+8)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+1} \frac{1}{1+x_{5i-3}x_{5i-8}...x_{5i-(5k+3)}}$$

thus, $x_{-(5k+8)} > x_{-(5k+3)}$.

$\vdots$

$$\frac{1+x_{-3}x_{-8}...x_{-(5k+3)}}{x_{-3}x_{-13}...x_{-(5k+8)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+k} \frac{1}{1+x_{5i-3}x_{5i-8}...x_{5i-(5k+3)}} > \tag{12}$$
$$\frac{1+x_{-3}x_{-8}...x_{-(5k+3)}}{x_{-8}x_{-13}...x_{-(5k+8)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+(k+1)} \frac{1}{1+x_{5i-3}x_{5i-8}...x_{5i-(5k+3)}}$$

thus, $x_{-8} > x_{-3}$. Which implies $x_{-(5k+8)} > x_{-(5k+3)} > \ldots > x_{-8} > x_{-3}$. This contradicts our assumption.

Suppose that $a_3 = a_8 = \ldots = a_{5k+3} = a_{5k+8} = 0$.

$$\begin{aligned}\frac{1+x_{-2}x_{-7}\ldots x_{-(5k+2)}}{x_{-2}x_{-7}\ldots x_{-(5k+2)}} &= \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j}\frac{1}{1+x_{5i-2}x_{5i-7}\ldots x_{5i-(5k+2)}} > \\ \frac{1+x_{-2}x_{-7}\ldots x_{-(5k+2)}}{x_{-2}x_{-7}\ldots x_{-(5k-3)}x_{-(5k+7)}} &= \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+1}\frac{1}{1+x_{5i-2}x_{5i-7}\ldots x_{5i-(5k+2)}}\end{aligned} \tag{13}$$

thus, $x_{-(5k+7)} > x_{-(5k+2)}$.

$\vdots$

$$\begin{aligned}\frac{1+x_{-2}x_{-7}\ldots x_{-(5k+2)}}{x_{-2}x_{-12}\ldots x_{-(5k+7)}} &= \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+k}\frac{1}{1+x_{5i-2}x_{5i-7}\ldots x_{5i-(5k+2)}} > \\ \frac{1+x_{-2}x_{-7}\ldots x_{-(5k+2)}}{x_{-7}x_{-12}\ldots x_{-(5k+7)}} &= \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+(k+1)}\frac{1}{1+x_{5i-2}x_{5i-7}\ldots x_{5i-(5k+2)}}\end{aligned} \tag{14}$$

thus, $x_{-7} > x_{-2}$. Which implies $x_{-(5k+7)} > x_{-(5k+2)} > \ldots > x_{-7} > x_{-2}$. This contradicts our assumption.

Suppose that $a_4 = a_9 = \ldots = a_{5k+4} = a_{5k+9} = 0$.

$$\begin{aligned}\frac{1+x_{-1}x_{-6}\ldots x_{-(5k+1)}}{x_{-1}x_{-6}\ldots x_{-(5k+1)}} &= \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j}\frac{1}{1+x_{5i-1}x_{5i-6}\ldots x_{5i-(5k+1)}} > \\ \frac{1+x_{-1}x_{-6}\ldots x_{-(5k+1)}}{x_{-1}x_{-6}\ldots x_{-(5k-4)}x_{-(5k+6)}} &= \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+1}\frac{1}{1+x_{5i-1}x_{5i-6}\ldots x_{5i-(5k+1)}}\end{aligned} \tag{15}$$

thus, $x_{-(5k+6)} > x_{-(5k+1)}$.

$\vdots$

$$\begin{aligned}\frac{1+x_{-1}x_{-6}\ldots x_{-(5k+1)}}{x_{-1}x_{-11}\ldots x_{-(5k+6)}} &= \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+k}\frac{1}{1+x_{5i-1}x_{5i-6}\ldots x_{5i-(5k+1)}} > \\ \frac{1+x_{-1}x_{-6}\ldots x_{-(5k+1)}}{x_{-6}x_{-11}\ldots x_{-(5k+6)}} &= \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+(k+1)}\frac{1}{1+x_{5i-1}x_{5i-6}\ldots x_{5i-(5k+1)}}\end{aligned} \tag{16}$$

thus, $x_{-6} > x_{-1}$. Which implies $x_{-(5k+6)} > x_{-(5k+1)} > \ldots > x_{-6} > x_{-1}$. This contradicts our assumption.

Suppose that $a_5 = a_{10} = ... = a_{5k+5} = a_{5k+10} = 0$.

$$\frac{1+x_0x_{-5}...x_{-(5k)}}{x_0x_{-5}...x_{-(5k)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j}\frac{1}{1+x_{5i}x_{5i-5}...x_{5i-(5k)}} > \tag{17}$$
$$\frac{1+x_0x_{-5}...x_{-(5k)}}{x_0x_{-5}...x_{-(5k-5)}x_{-(5k+5)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+1}\frac{1}{1+x_{5i}x_{5i-5}...x_{5i-(5k)}}$$

thus, $x_{-(5k+5)} > x_{-(5k)}$.

$$\vdots$$

$$\frac{1+x_0x_{-5}...x_{-(5k)}}{x_0x_{-10}...x_{-(5k+5)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+k}\frac{1}{1+x_{5i}x_{5i-5}...x_{5i-(5k)}} > \tag{18}$$
$$\frac{1+x_0x_{-5}...x_{-(5k)}}{x_{-5}x_{-10}...x_{-(5k+5)}} = \sum_{j=0}^{\infty}\prod_{i=1}^{(k+2)j+(k+1)}\frac{1}{1+x_{5i}x_{5i-5}...x_{5i-(5k)}}$$

thus, $x_{-5} > x_0$. Which implies $x_{-(5k+5)} > x_{-(5k)} > ... > x_{-5} > x_0$. This contradicts our assumption. Then the proof is complete. ■

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