TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 5, pp. 1067-1086, August 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

ON THE L_p-VERSION OF THE PETTY'S CONJECTURED PROJECTION INEQUALITY AND APPLICATIONS

Weidong Wang and Gangsong Leng

Abstract. Petty's conjectured projection inequality is a famous open problem in convex bodies theory. In this paper, it is shown that a L_p-version of the Petty's conjectured projection inequality. As its applications, we give a reverse of the Blaschke-Santaló inequality and consider the monotony of volumes for convex body and its L_p- Petty projection body, respectively. Otherwise, we also give the reverses of the L_p-Petty projection inequality.

1. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies(compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$, for the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we respectively write $\mathcal{K}_o^n$ and $\mathcal{K}_s^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, denote by $V(K)$ the $n-$dimensional volume of body K, for the standard unit ball B in $\mathbb{R}^n$, denote $\omega_n = V(B)$.

The classical projection bodies were introduced at the turn of the previous century by Minkowski. He showed that corresponding to each $K \in \mathcal{K}^n$, the projection body of K, ΠK, is aunique origin-symmetric convex body, which can be defined: the length of the image of the orthogonal projection of ΠK onto 1-dimensional subspace l of $\mathbb{R}^n$, to the (n-1)-dimensional volume of the image of the orthogonal projection of K onto the codimension 1 subspace $l^{\perp}$. Interest in projection bodies was rekindled by Bolker ([5]), Petty ([25]) and Schneider ([29]). During the past

Received May 18, 2006, accepted September 14, 2006.
Communicated by Bor-Luh Lin.
2000 *Mathematics Subject Classification*: 52A40 52A20.
Key words and phrases: L_p-version, Petty projection inequality, Petty's conjecture, L_p-projection body, reverse.
Research is supported in part by the Natural Science Foundation of China (Grant No. 10671117) and Academic Mainstay Foundation of Hubei Province of China (Grant No. D200729002).

three decades, projection bodies have received considerable attention (for example see articles [1, 2, 4, 6, 9, 11, 14, 23, 26, 32, 34] or books [10, 13, 31]).

The fundamental inequality for projection bodies in the field of affine isoperimetric inequalities is the Petty projection inequality (see [26]): *If $K \in \mathcal{K}^n$, then*

$$V(K)^{n-1}V(\Pi^*K) \leq (\frac{\omega_n}{\omega_{n-1}})^n, \tag{1.1}$$

with equality if and only if K is an ellipsoid. Where Π^*K denote the polar of the projection body ΠK, rather than $(\Pi K)^*$. The reverse of Petty projection inequality was established by Zhang (see [34]), it is called the Zhang projection inequality.

One of the outstanding unsolved problems in the field of affine isoperimetric inequalities is Petty's conjecture for the volume of projection bodies. In 1971, Petty in [26] conjectured that: *If $K \in \mathcal{K}^n$, is it true that*

$$V(\Pi K)V(K)^{1-n} \geq \omega_{n-1}^n \omega_n^{2-n}, \tag{1.2}$$

with equality if and only if K is an ellipsoid? Petty's conjecture is called the Petty's conjectured projection inequality in [10]. Petty's conjectured projection inequality has been studied by Lutwak [16], Schneider [30] and Brannen [8].

Recently, Petty projection inequality has been studied extensively (for example see [17-20, 24, 33]). In particular, Lutwak, Yang and Zhang in [19] have extended the Petty projection inequality to the L_p-projection body.

In [19] the authors showed the notion of L_p-projection body as follows: For $K \in \mathcal{K}_o^n$ and real number $p \geq 1$, the L_p-projection body, $\Pi_p K$, of K is origin-symmetric convex body whose support function is given by

$$h_{\Pi_p K}(u) = [\frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(K, v)]^{\frac{1}{p}}, \tag{1.3}$$

for all $u \in S^{n-1}$, where $u \cdot v$ denotes the standard inner product of u and v, and

$$c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1}. \tag{1.4}$$

$S_p(K, \cdot)$ is a positive Borel measure on S^{n-1}, called the L_p-surface area measure of K, $S_1(K)$ is just the classical surface area measure, $S(K, \cdot)$, of K. It turns out that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to the surface area measure $S(K, \cdot)$ of K, and has Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h(K, \cdot)^{1-p}. \tag{1.5}$$

Regarding the study of the L_p-projection body also see [15, 28].

Remark 1. The unusual normalization of definition (1.3) is chosen so that for the unit ball, B, we have $\Pi_p B = B$. In particular, for $p = 1$ and $K \in \mathcal{K}^n$, then the convex body $\Pi_1 K$ is the classical projection body ΠK of K under the normalization of definition (1.3), and $\Pi B = B$ (rather than the classical $\omega_{n-1}B$, see [19]).

Further, Lutwak, Yang and Zhang in [19] established the L_p-Petty projection inequality: *If $K \in \mathcal{K}_o^n$, and $p \geq 1$, then*

$$V(K)^{(n-p)/p} V(\Pi_p^* K) \leq \omega_n^{n/p}, \tag{1.6}$$

with equality if and only if K is an ellipsoid centered at the origin. Where $\Pi_p^* K$ denote the polar of the projection body $\Pi_p K$, rather than $(\Pi_p K)^*$. The Petty projection inequality for L_p-mixed projection body was obtained by Wang and Leng (see [33]).

Remark 2. For $p = 1$ and $K \in \mathcal{K}^n$, we adopt the normalization of definition (1.3), then Petty projection inequality (1.1) can be rewritten as follows (see [19]): *If $K \in \mathcal{K}^n$, then*

$$V(K)^{n-1} V(\Pi^* K) \leq \omega_n^n,$$

with equality if and only if K is an ellipsoid. This is just the form of inequality (1.6) for $p = 1$.

Remark 3. Under the normalization of definition (1.3), Petty's conjecture (1.2) can be rewritten that: *If $K \in \mathcal{K}^n$, is it true that*

$$V(\Pi K) V(K)^{1-n} \geq \omega_n^{2-n}, \tag{1.7}$$

with equality if and only if K is an ellipsoid?

Now a nature question is: if $K \in \mathcal{K}_o^n$, what is the largest lower bound of the affine invariant

$$f_p(K) = V(\Pi_p K) V(K)^{(p-n)/p} \tag{1.8}$$

for $p \geq 1$? When $p = 1$ and $K \in \mathcal{K}^n$, Petty conjecture that $f_p(K)$ is minimized by ellipsoids.

In this paper, we shall study this question. Our main woke is to find a lower bound of $f_p(K)$, i.e., we give a L_p-version of the Petty's conjectured projection inequality. it can be stated:

Theorem 1. *If $K \in \mathcal{K}_o^n$, then for $1 \leq p \leq 2$,*

$$f_p(K) = V(\Pi_p K) V(K)^{(p-n)/p} \geq (n c_{n-2,p})^{-n/p} \omega_n^{(2p-n)/p}; \tag{1.9}$$

for $2 \le p \le \infty$,

$$f_p(K) = V(\Pi_p K)V(K)^{(p-n)/p} \ge n^{-n/2}(c_{n-2,p})^{-n/p}\omega_n^{(2p-n)/p}. \tag{1.10}$$

with equality in inequality (1.9) *if and only if* $p = 2$ *and* K *is an ellipsoid centered at the origin, with equality in inequality* (1.10) *for* $p \neq 2$ *if and only if* $n = 1$ *and* K *is an origin-symmetric segment or for* $p = 2$ *if and only if* K *is an ellipsoid centered at the origin.*

Let $p = 1$ in inequality (1.9), we immediately obtain a right form of the Petty's conjectured projection inequality as follows:

Corollary 1. *If* $K \in \mathcal{K}_o^n$, *then*

$$V(\Pi K)V(K)^{1-n} \ge (2\omega_{n-1})^{-n}\omega_n^2. \tag{1.11}$$

Compare with the Petty's conjectured projection inequality (1.7) and inequality (1.11), Petty's conjecture (1.7) is stronger than inequality (1.11).

In the section 2 of this paper, for each $K \in \mathcal{K}_s^n$, we give the definition of $\Pi_\infty K$. From this, let $p \longrightarrow \infty$ in inequality (1.10), we obtain that

Corollary 2. *If* $K \in \mathcal{K}_s^n$, *then*

$$V(K)V(K^*) \ge n^{-n/2}\omega_n^2, \tag{1.12}$$

with equality if and only if $n = 1$ *where* K *is an origin-symmetric segment and* K^* *denotes the polar of* K.

Inequality (1.12) is of interest. It not only is just a reverse of the well-known Blaschke-Santaló inequality (see [10, 31]) but also may be regarded as an analogy form of the well-known Bourgain-Milman inequality (see [7]).

If $K \in \mathcal{K}_s^n$, we also give a upper bound of $f_p(K)$ in (1.8) as follows:

Theorem 2. *If* $K \in \mathcal{K}_s^n$, $1 \le p \le \infty$, *then*

$$f_p(K) = V(\Pi_p K)V(K)^{(p-n)/p} \le (c_{n-2,p})^{-n/p}\omega_n^{(2p-n)/p}, \tag{1.13}$$

with equality for $1 \le p < \infty$ *if and only if* $n = 1$ *and* K *is an origin-symmetric segment, for* $p = \infty$ *if and only if* K *is an ellipsoid centered at the origin.*

Applying Theorem 1 and Theorem 2, we further study the monotony of volumes for convex body K and its L_p- projection body $\Pi_p K$ in the section 4 (Theorem 3 and Theorem 4).

Analogy to the proof method of Theorem 1, together with the results of Lutwak, Yang and Zhang (see [21, 22]), we also obtain two reverses of the L_p-Petty projection inequality (Theorem 5 and Theorem 6) in the last section.

2. Preliminaries

2.1. Support Function, Radial Function and Polar Body

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \longrightarrow (0, \infty)$, is defined by (see [10, 27])

$$h(K, x) = max\{x \cdot y : y \in K\}, \qquad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \longrightarrow [0, +\infty)$, is defined by (see [10, 27])

$$\rho(K, x) = max\{\lambda \geq 0 : \lambda x \in K\}, \qquad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

From the definition of radial function, we know that for $\lambda > 0$, $\rho_K(u) \leq \lambda \rho_L(u)$ for any $u \in S^{n-1}$ if and only if $K \subseteq \lambda L$.

If $K \in \mathcal{K}_o^n$, the polar body of K, K^*, is defined by (see [10, 31])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}. \tag{2.1}$$

Obviously, we have $(K^*)^* = K$.

Regard to $K \in \mathcal{K}_o^n$ and its polar body, the well-known Blaschke-Santaló inequality can be stated that (see [10]): *If K is an origin-symmetric convex body, then*

$$V(K)V(K^*) \leq \omega_n^2, \tag{2.2}$$

with equality if and only if K is an ellipsoid.

From the definition (2.1), we also know that: If $K \in \mathcal{K}_o^n$, then the support and radial functions of K^*, the polar body of K, are defined respectively by

$$h_{K^*} = \frac{1}{\rho_K} \quad and \quad \rho_{K^*} = \frac{1}{h_K}. \tag{2.3}$$

If $K, L \in \mathcal{K}_o^n$, $\lambda > 0$, then

$$K \subseteq \lambda L \quad \Longleftrightarrow \quad K^* \supseteq \frac{1}{\lambda} L^*. \tag{2.4}$$

In particular,

$$K = \lambda L \quad \Longleftrightarrow \quad K^* = \frac{1}{\lambda} L^*. \tag{2.5}$$

2.2. L_p John Ellipsoid and the Body $\Gamma_{-p}K$

The notion of L_p John ellipsoid is shown by Lutwak, Yang and Zhang in [22]. Suppose $K \in \mathcal{K}_o^n$ and $0 < p \leq \infty$, for each origin-symmetric ellipsoid E, the unique ellipsoid E_pK that solves the constrained maximization problem

$$V(E_pK) = maxV(E) \ subject \ to \ \bar{V}_p(K, E) \leq 1,$$

is called the L_p John ellipsoid of K. Where $\bar{V}_p(K, L) = (V_p(K, L)/V(K))^{1/p}$, $V_p(K, L)$ is the L_p-mixed volume of $K, L \in \mathcal{K}_o^n$.

When $p = 2$, the E_2K is just the new ellipsoid $\Gamma_{-2}K$ (see [22]) which be posed by Lutwak, Yang and Zhang in [21]; When $p = \infty$, the $E_\infty K$ is just the well-known classical John ellipsoid (see [22]).

For the L_p John ellipsoid, E_pK, of K, the authors in [22] proved that: *If $K \in \mathcal{K}_o^n$, $1 \leq p \leq \infty$, then*

$$V(K) \geq V(E_pK), \tag{2.6}$$

with equality for $p = 1$ if and only if K is an ellipsoid, and equality for $p > 1$ if and only if K is an ellipsoid centered at the origin.

Obviously, let $p = 2$ in inequality (2.6), we have that: *If $K \in \mathcal{K}_o^n$, then*

$$V(K) \geq V(\Gamma_{-2}K), \tag{2.7}$$

with equality if and only if K is an ellipsoid centered at the origin. The inequality (2.7) first was established by Lutwak, Yang and Zhang in [21].

Lutwak, Yang and Zhang in [22] also showed the notion of body $\Gamma_{-p}K$ as follows: If $K \in \mathcal{K}_o^n$ and $p > 0$, then body $\Gamma_{-p}K$ is an origin-symmetric body whose radial function is given by

$$\rho_{\Gamma_{-p}K}^{-p}(u) = \frac{1}{V(K)} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(K, v), \tag{2.8}$$

for all $u \in S^{n-1}$. Where $S_p(K, \cdot)$ is the L_p-surface area measure of K.

Note for $p \geq 1$ the body $\Gamma_{-p}K$ is an origin-symmetric convex body (see [22]). In particular, for $p = 2$, we have

$$\rho_{\Gamma_{-2}K}^{-2}(u) = \frac{1}{V(K)} \int_{S^{n-1}} | u \cdot v |^2 \, dS_2(K, v),$$

for all $u \in S^{n-1}$. This is just the definition of the new ellipsoid $\Gamma_{-2}K$ (see [21]).

Using (1.5), then definition (2.8) may be rewritten that

$$n^{-\frac{1}{p}} \rho_{\Gamma_{-p}K}^{-1}(u) = [\frac{1}{nV(K)} \int_{S^{n-1}} (\frac{| u \cdot v |}{h_K(v)})^p h_K(v) dS(K, v)]^{\frac{1}{p}},$$

for all $u \in S^{n-1}$. Thus for $p = \infty$, define $\Gamma_{-\infty}K$ by (see [22])

$$\Gamma_{-\infty}K = \lim_{p \longrightarrow \infty} \Gamma_{-p}K \tag{2.9}$$

and

$$\rho^{-1}_{\Gamma_{-\infty}K}(u) = max\{\frac{\mid u \cdot v \mid}{h_K(v)} : v \in suppS(K, \cdot)\},$$

for all $u \in S^{n-1}$. From this, if K is an origin-symmetric convex body, then

$$\Gamma_{-\infty}K = K. \tag{2.10}$$

2.3. The Definition of $\Pi_\infty K$

Analogous to the definition of $\Gamma_{-\infty}K$, we now give the definition of $\Pi_\infty K$.

From the definition (1.3) of $\Pi_p K$, together with (1.5) and (2.3), the definition (1.3) may be rewritten as follows: For $K \in \mathcal{K}^n_o$ and $p \geq 1$, then

$$\begin{aligned}(c_{n-2,p}\omega_n V(K))^{-\frac{1}{p}} \rho^{-1}_{\Pi^*_p K}(u) &= (c_{n-2,p}\omega_n V(K))^{-\frac{1}{p}} h_{\Pi_p K} \\ &= \left[\tfrac{1}{nV(K)} \int_{S^{n-1}} \left(\tfrac{|u \cdot v|}{h_K(v)}\right)^p h_K(v) dS(K, v)\right]^{\frac{1}{p}},\end{aligned}$$

for all $u \in S^{n-1}$. Thus for $p = \infty$, we define $\Pi^*_\infty K$ by

$$\Pi^*_\infty K = \lim_{p \longrightarrow \infty} \Pi^*_p K \tag{2.11}$$

and

$$\rho^{-1}_{\Pi^*_\infty K}(u) = max\{\frac{\mid u \cdot v \mid}{h_K(v)} : v \in suppS(K, \cdot)\}$$

for all $u \in S^{n-1}$. But using equality (1.4), we know

$$\lim_{p \longrightarrow \infty} (c_{n-2,p})^{1/p} = 1. \tag{2.12}$$

Hence, if $K \in \mathcal{K}^n_s$, then $\Pi^*_\infty K = K$, namely

$$\Pi_\infty K = K^*. \tag{2.13}$$

2.4. Jensen's Inequality

Suppose $p \neq 0$, μ is a finite Borel measure in a set X, and f is a nonnegative μ-integrable function on X. The pth mean $M_p f$ of f is

$$M_p f = \left[\frac{1}{\mu(X)} \int_X f(x)^p d\mu(x)\right]^{\frac{1}{p}},$$

$$\lim_{p\longrightarrow\infty} M_p f = max\{f(x) : x \in X\}$$

and

$$\lim_{p\longrightarrow 0} M_p f = exp[\frac{1}{\mu(X)}\int_X logf(x)d\mu(x)].$$

Jensen's inequality may be stated that (see [12]):

If $p \le q$ and $M_q f$ exists, then

$$M_p f \le M_q f, \tag{2.14}$$

with equality for $p \neq q$ if and only if f is a constant or if and only if $p = q$.

3. The Proofs of Theorem 1 and Theorem 2

In the section, we shall prove the Theorem 1 and Theorem 2. In order to prove the two Theorems, the following Lemmas are essential.

Lemma 1. *If $K \in \mathcal{K}_o^n$, $0 < p \le q \le \infty$, then*

$$n^{\frac{1}{q}}\Gamma_{-q}K \subseteq n^{\frac{1}{p}}\Gamma_{-p}K, \tag{3.1}$$

with equality for $p \neq q$ if and only if $n = 1$ and K is an origin-symmetric segment or if and only if $p = q$.

Proof. From definition (2.8) and formula (1.5), together with Jensen's inequality (2.14), it follows that for $0 < p \le q < \infty$,

$$\begin{aligned}
\rho^{-1}_{\Gamma_{-p}K}(u) &= \left(\frac{1}{V(K)}\int_{S^{n-1}} \mid u\cdot v\mid^p dS_p(K,v)\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{V(K)}\int_{S^{n-1}} \mid u\cdot v\mid^p h_K^{1-p}(v)dS(K,v)\right)^{\frac{1}{p}} \\
&= n^{\frac{1}{p}}\left(\frac{1}{nV(K)}\int_{S^{n-1}} (\frac{\mid u\cdot v\mid}{h_K(v)})^p h_K(v)dS(K,v)\right)^{\frac{1}{p}} \\
&\le n^{\frac{1}{p}}\left(\frac{1}{nV(K)}\int_{S^{n-1}} (\frac{\mid u\cdot v\mid}{h_K(v)})^q h_K(v)dS(K,v)\right)^{\frac{1}{q}} \\
&= n^{\frac{1}{p}-\frac{1}{q}}\left(\frac{1}{V(K)}\int_{S^{n-1}} \mid u\cdot v\mid^q dS_q(K,v)\right)^{\frac{1}{q}} \\
&= n^{\frac{1}{p}-\frac{1}{q}}\rho^{-1}_{\Gamma_{-q}K}(u),
\end{aligned}$$

for all $u \in S^{n-1}$. From this, we get

$$n^{\frac{1}{q}}\rho_{\Gamma_{-q}K}(u) \le n^{\frac{1}{p}}\rho_{\Gamma_{-p}K}(u),$$

for all $u \in S^{n-1}$. Further, we have

$$n^{\frac{1}{q}}\Gamma_{-q}K \subseteq n^{\frac{1}{p}}\Gamma_{-p}K,$$

this is just (3.1). For case $q = \infty$ follows from the real case together with definition (2.9).

According to the condition of equality holds in Jensen's inequality, we know the equality holds in (3.1) for $p \neq q$ if and only if $| u \cdot v | / h_K(v)$, for any give $u \in S^{n-1}$ and all $v \in S^{n-1}$, is a constant or if and only if $p = q$, i.e., for $p \neq q$ if and only if $n = 1$ and K is an origin-symmetric segment or if and only if $p = q$.■

From (3.1), we immediately obtain that:

Lemma 2. *If $K \in \mathcal{K}_o^n$, then for $1 \leq p \leq 2$,*

$$n^{\frac{1}{p}}\Gamma_{-p}K \supseteq n^{\frac{1}{2}}\Gamma_{-2}K; \tag{3.2}$$

for $2 \leq p \leq \infty$,

$$n^{\frac{1}{p}}\Gamma_{-p}K \subseteq n^{\frac{1}{2}}\Gamma_{-2}K. \tag{3.3}$$

In each inequality, with equality for $p \neq 2$ if and only if $n = 1$ and K is an origin-symmetric segment or if and only if $p = 2$.

Lemma 3. [22] *If $K \in \mathcal{K}_o^n$, then for $1 \leq p \leq 2$,*

$$\Gamma_{-p}K \subseteq E_pK; \tag{3.4}$$

for $2 \leq p \leq \infty$,

$$\Gamma_{-p}K \supseteq E_pK. \tag{3.5}$$

In each inequality, with equality if and only if $p = 2$.

Lemma 4. *If $K \in \mathcal{K}_o^n$, $1 \leq p \leq \infty$, then*

$$\Gamma^*_{-p}K = (\frac{nc_{n-2,p}\omega_n}{V(K)})^{\frac{1}{p}}\Pi_pK. \tag{3.6}$$

Proof. According to definition (1.3), equality (2.3) and definition (2.8), we respectively have that for $1 \leq p < \infty$,

$$\rho^{-p}_{\Pi^*_pK}(u) = h^p_{\Pi_pK}(u) = \frac{1}{n\omega_n c_{n-2,p}}\int_{S^{n-1}} | u \cdot v |^p \, dS_p(K, v)$$

and

$$\rho_{\Gamma_{-p}K}^{-p}(u)=\frac{1}{V(K)}\int_{S^{n-1}}\mid u\cdot v\mid^{p}dS_p(K,v),$$

for all $u\in S^{n-1}$. Thus

$$\rho_{\Gamma_{-p}K}^{-p}(u)=\frac{nc_{n-2,p}\omega_n}{V(K)}\rho_{\Pi_p^*K}^{-p}(u),$$

for all $u\in S^{n-1}$, hence

$$\Gamma_{-p}K=(\frac{V(K)}{nc_{n-2,p}\omega_n})^{\frac{1}{p}}\Pi_p^*K. \tag{3.7}$$

Combining with (2.5) and (3.7), we immediately obtain (3.6).

For $p=\infty$, using definition (2.9) and (2.11), and together with equality (2.10), (2.12) and (2.13), we easily know equality (3.6) and (3.7) are true. ■

Proof of Theorem 1. For $1\le p\le 2$, using (2.4) and (3.4), we get

$$\Gamma_{-p}^*K\supseteq E_p^*K. \tag{3.8}$$

Because of the L_p John ellipsoid E_pK is an origin-symmetric ellipsoid, together with the condition of equality holds in the Blaschke-Santaló inequality (2.2), we have

$$V(E_pK)V(E_p^*K)=\omega_n^2. \tag{3.9}$$

From (3.8) and (3.9), combining with inequality (2.6), we obtain that

$$V(\Gamma_{-p}^*K)\ge V(E_p^*K)=\frac{\omega_n^2}{V(E_pK)}\ge\frac{\omega_n^2}{V(K)}. \tag{3.10}$$

But equality (3.6) immediately gives that for $1\le p\le\infty$,

$$V(\Gamma_{-p}^*K)=(\frac{nc_{n-2,p}\omega_n}{V(K)})^{\frac{n}{p}}V(\Pi_pK). \tag{3.11}$$

Thus (3.10) can be rewritten by

$$(\frac{nc_{n-2,p}\omega_n}{V(K)})^{\frac{n}{p}}V(\Pi_pK)\ge\frac{\omega_n^2}{V(K)},$$

namely,

$$V(\Pi_pK)V(K)^{(p-n)/p}\ge(nc_{n-2,p})^{-n/p}\omega_n^{(2p-n)/p},$$

this is just inequality (1.9). According to the conditions of equality hold in (2.6) and (3.4), we see with equality in inequality (1.9) if and only if $p = 2$ and K is an ellipsoid centered at the origin.

For $2 \leq p \leq \infty$, according to (3.3) in Lemma 2, and using (2.4), we have

$$n^{-\frac{1}{p}}\Gamma^*_{-p}K \supseteq n^{-\frac{1}{2}}\Gamma^*_{-2}K. \tag{3.12}$$

Analogy with the L_p John ellipsoid E_pK, the volume of the new ellipsoid $\Gamma_{-2}K$ satisfy (3.9). From this, together with (3.12) and inequality (2.7), then

$$n^{-\frac{n}{p}}V(\Gamma^*_{-p}K) \geq n^{-\frac{n}{2}}V(\Gamma^*_{-2}K) = \frac{n^{-\frac{n}{2}}\omega_n^2}{V(\Gamma_{-2}K)} \geq \frac{n^{-\frac{n}{2}}\omega_n^2}{V(K)}.$$

Using (3.11), above inequality may be rewritten as follows:

$$n^{-n/p}(\frac{nc_{n-2,p}\omega_n}{V(K)})^{n/p}V(\Pi_pK) \geq \frac{n^{-n/2}\omega_n^2}{V(K)},$$

thus

$$V(\Pi_pK)V(K)^{(p-n)/p} \geq n^{-n/2}(c_{n-2,p})^{-n/p}\omega_n^{(2p-n)/p}.$$

Hence, inequality (1.10) is obtained. Together with the cases of equality hold in (3.3) and (2.7), we know with equality in inequality (1.10) for $p \neq 2$ if and only if $n = 1$ and K is an origin-symmetric segment or for $p = 2$ if and only if K is an ellipsoid centered at the origin.

To sum up, the proof of Theorem 1 is completed. ■

Proof of Corollary 1. Taking $p = 1$ in inequality (1.9), then

$$V(\Pi K)V(K)^{1-n} \geq (nc_{n-2,1})^{-n}\omega_n^{(2-n)},$$

using (1.5), we know

$$nc_{n-2,1} = (n+1)c_{n,1} = \frac{2\omega_{n-1}}{\omega_n}.$$

From this, inequality (1.11) of Corollary 1 is given. ■

Proof of Corollary 2. From inequality (1.10), together with definition (2.11), equality (2.12) and (2.13), we immediately obtain inequality (1.12) of Corollary 2 when $K \in \mathcal{K}^n_s$. The case of equality holds in inequality (1.10) immediately gives that with equality in inequality (1.12) if and only if $n = 1$ and K is an origin-symmetric segment. ■

The proof of Theorem 2 require the following a Lemma.

If $K \in \mathcal{K}_s^n$, let $q = \infty$ in Lemma 1, using (3.1) and together with definition (2.9) and equality (2.10), we immediately get that

Lemma 5. *If $K \in \mathcal{K}_s^n$, $1 \leq p \leq \infty$, then*

$$K \subseteq n^{\frac{1}{p}} \Gamma_{-p} K, \tag{3.13}$$

with equality for $1 \leq p < \infty$ if and only if $n = 1$ and K is an origin-symmetric segment or if and only if $p = \infty$.

Proof of Theorem 2. From (3.13), together with (2.4), then

$$n^{-\frac{1}{p}} \Gamma^*_{-p} K \subseteq K^*,$$

thus

$$n^{-\frac{n}{p}} V(\Gamma^*_{-p} K) \leq V(K^*),$$

using the Blaschke-Santaló inequality (2.2), we have

$$n^{-\frac{n}{p}} V(\Gamma^*_{-p} K) V(K) \leq V(K) V(K^*) \leq \omega_n^2.$$

Above inequality together with (3.11) immediately give that

$$n^{-\frac{n}{p}} \left(\frac{n c_{n-2,p} \omega_n}{V(K)}\right)^{\frac{n}{p}} V(\Pi_p K) V(K) \leq \omega_n^2,$$

namely,

$$V(\Pi_p K) V(K)^{(p-n)/p} \leq (c_{n-2,p})^{-n/p} \omega_n^{(2p-n)/p},$$

this is just inequality (1.13). According to the conditions of equality hold in inequality (3.13) and (2.2), we easily see with equality in inequality (1.13) for $1 \leq p < \infty$ if and only if $n = 1$ and K is an origin-symmetric segment, for $p = \infty$ if and only if K is an ellipsoid centered at the origin. Theorem 2 is proven. ∎

4. The Applications of Theorem 1 and Theorem 2

As the applications of Theorem 1 and Theorem 2, we give the monotony of volumes of K and $\Pi_p K$ in this section .

Theorem 3. *If $K \in \mathcal{K}_o^n$, $L \in \mathcal{K}_s^n$, and $V(\Pi_p K) \leq V(\Pi_p L)$, then for $1 \leq p \leq 2$ and $n > p$,*

$$V(K) \leq n^{\frac{n}{n-p}} V(L); \tag{4.1}$$

for $2 \le p \le \infty$ *and* $n > p$,

$$V(K) \le n^{\frac{np}{2(n-p)}} V(L); \tag{4.2}$$

for $2 \le p \le \infty$ *and* $2 \le n < p$,

$$V(K) \ge n^{\frac{np}{2(n-p)}} V(L). \tag{4.3}$$

Theorem 4. *Suppose* $K \in \mathcal{K}_o^n$, $L \in \mathcal{K}_s^n$, *if* $V(K) \ge V(L)$ *and* $n > p$, *then for* $1 \le p \le 2$,

$$V(\Pi_p K) \ge n^{-n/p} V(\Pi_p L), \tag{4.4}$$

for $2 \le p \le \infty$,

$$V(\Pi_p K) \ge n^{-n/2} V(\Pi_p L); \tag{4.5}$$

if $V(K) \le V(L)$, *then for* $2 \le p \le \infty$ *and* $2 \le n < p$,

$$V(\Pi_p K) \ge n^{-n/2} V(\Pi_p L). \tag{4.6}$$

Proof of Theorem 3. For $1 \le p \le 2$, since $K \in \mathcal{K}_o^n$, using inequality (1.9) of Theorem 1, then

$$V(K)^{(n-p)/p} \le (nc_{n-2,p})^{n/p} \omega_n^{(n-2p)/p} V(\Pi_p K), \tag{4.7}$$

but $V(\Pi_p K) \le V(\Pi_p L)$, thus

$$V(K)^{(n-p)/p} \le (nc_{n-2,p})^{n/p} \omega_n^{(n-2p)/p} V(\Pi_p L). \tag{4.8}$$

Because of $L \in \mathcal{K}_s^n$, together with inequality (1.13) of Theorem 2, we have

$$V(\Pi_p L) \le (c_{n-2,p})^{-n/p} \omega_n^{(2p-n)/p} V(L)^{(n-p)/p}. \tag{4.9}$$

From inequalities (4.8) and (4.9), we obtain

$$V(K)^{(n-p)/p} \le n^{n/p} V(L)^{(n-p)/p}.$$

Hence when $n > p$, we get inequality (4.1).

For $2 \le p \le \infty$, since $K \in \mathcal{K}_o^n$, then using inequality (1.10) of Theorem 1, we give

$$V(K)^{(n-p)/p} \le n^{n/2} (c_{n-2,p})^{n/p} \omega_n^{(n-2p)/p} V(\Pi_p K). \tag{4.10}$$

According to the condition $V(\Pi_p K) \le V(\Pi_p L)$ and (4.10), we get

$$V(K)^{(n-p)/p} \le n^{n/2}(c_{n-2,p})^{n/p}\omega_n^{(n-2p)/p}V(\Pi_p L), \tag{4.11}$$

since $L \in \mathcal{K}_s^n$, then using inequalities (4.9) and (4.11), we have

$$V(K)^{(n-p)/p} \le n^{n/2}V(L)^{(n-p)/p}.$$

From this, when $n > p$, we obtain inequality (4.2); when $2 \le n < p$, we get inequality (4.3).

To sum up, the proof of Theorem 3 is completed. ∎

Proof of Theorem 4. For $1 \le p \le 2$, since $L \in \mathcal{K}_s^n$, $V(K) \ge V(L)$ and $n > p$, then using (4.9), we have that

$$\begin{aligned} V(\Pi_p L) &\le (c_{n-2,p})^{-n/p}\omega_n^{(2p-n)/p}V(L)^{(n-p)/p} \\ &\le (c_{n-2,p})^{-n/p}\omega_n^{(2p-n)/p}V(K)^{(n-p)/p}, \end{aligned} \tag{4.12}$$

but $K \in \mathcal{K}_o^n$, thus inequality (4.7) and inequality (4.12) immediately give that

$$V(\Pi_p L) \le n^{n/p}V(\Pi_p K),$$

hence inequality (4.4) is obtained.

For $2 \le p \le \infty$ and $n > p$, since $L \in \mathcal{K}_s^n$, $V(K) \ge V(L)$, then using (4.9), we have inequality (4.12), but $K \in \mathcal{K}_o^n$, then together with (4.10) and inequality (4.12), we obtain

$$V(\Pi_p L) \le n^{n/2}V(\Pi_p K),$$

namely inequality (4.5) is true.

For $2 \le p \le \infty$ and $2 \le n < p$, since $V(K) \le V(L)$, thus

$$V(K)^{(n-p)/p} \ge V(L)^{(n-p)/p}.$$

But $K \in \mathcal{K}_o^n$, then using (4.10), we have

$$\begin{aligned} V(\Pi_p K) &\ge n^{-n/2}(c_{n-2,p})^{-n/p}\omega_n^{(2p-n)/p}V(K)^{(n-p)/p} \\ &\ge n^{-n/2}(c_{n-2,p})^{-n/p}\omega_n^{(2p-n)/p}V(L)^{(n-p)/p}. \end{aligned}$$

Because of $L \in \mathcal{K}_s^n$, then using (4.9) and above inequalities, we obtain

$$V(\Pi_p K) \ge n^{-n/2}V(\Pi_p L),$$

this is inequality (4.6).

From this, we complete the proof of Theorem 4. ∎

5. The Reverses of L_p-Petty Projection Inequality

Now, analogy to the proof method for Theorem 1, we give two reverses of the L_p-Petty projection inequality:

Theorem 5. *If $K \in \mathcal{K}_s^n$, then for $1 \le p \le 2$,*

$$V(\Pi_p^* K)V(K)^{(n-p)/p} \ge 2^{-n} n^{n/2} (c_{n-2}, p)^{n/p} \omega_n^{(n+p)/p}; \tag{5.1}$$

for $2 \le p \le \infty$,

$$V(\Pi_p^* K)V(K)^{(n-p)/p} \ge 2^{-n} n^{n/p} (c_{n-2}, p)^{n/p} \omega_n^{(n+p)/p}. \tag{5.2}$$

With equality in inequality (5.1) for $p \neq 2$ if and only if $n = 1$ and K is an origin-symmetric segment, for $p = 2$ if and only if K is a parallelotope, with equality in inequality (5.2) if and only if $p = 2$ and K is a parallelotope.

Theorem 6. *If $K \in \mathcal{K}_o^n$, is positioned so that its John point is at the origin, then for $1 \le p \le 2$,*

$$V(\Pi_p^* K)V(K)^{(n-p)/p} \ge \frac{n!(c_{n-2}, p)^{n/p}}{(n+1)^{(n+1)/2}} \omega_n^{(n+p)/p}; \tag{5.3}$$

for $2 \le p \le \infty$,

$$V(\Pi_p^* K)V(K)^{(n-p)/p} \ge \frac{n! n^{n/p} (c_{n-2}, p)^{n/p}}{n^{n/2}(n+1)^{(n+1)/2}} \omega_n^{(n+p)/p}. \tag{5.4}$$

With equality in inequality (5.3) for $p \neq 2$ if and only if $n = 1$ and K is an origin-symmetric segment, for $p = 2$ if and only if K is a simplex, with equality in inequality (5.4) if and only if $p = 2$ and K is a simplex.

Here, we establish the following lemmas:

Lemma 6. *If $K \in \mathcal{K}_o^n$, then for $1 \le p \le 2$,*

$$V(\Pi_p^* K)V(K)^{(n-p)/p} \ge n^{n/2} (c_{n-2,p})^{n/p} \omega_n^{n/p} \frac{V(\Gamma_{-2} K)}{V(K)}; \tag{5.5}$$

for $2 \le p \le \infty$,

$$V(\Pi_p^* K)V(K)^{(n-p)/p} \ge (n c_{n-2,p})^{n/p} \omega_n^{n/p} \frac{V(E_p K)}{V(K)}. \tag{5.6}$$

With equality in inequality (5.5) *for* $p \neq 2$ *if and only if* $n = 1$ *and* K *is an origin-symmetric segment or if and only if* $p = 2$*, with equality in inequality* (5.6) *if and only if* $p = 2$.

Proof. For $1 \leq p \leq 2$, using (3.2) of Lemma 2, then

$$n^{n/p} V(\Gamma_{-p}K) \geq n^{n/2} V(\Gamma_{-2}K),$$

from (3.7), we get

$$V(\Gamma_{-p}K) = \left(\frac{V(K)}{nc_{n-2,p}\omega_n}\right)^{\frac{n}{p}} V(\Pi_p^* K), \tag{5.7}$$

thus

$$n^{\frac{n}{p}} \left(\frac{V(K)}{nc_{n-2,p}\omega_n}\right)^{\frac{n}{p}} V(\Pi_p^* K) \geq n^{\frac{n}{2}} V(\Gamma_{-2}K),$$

this inequality immediately gives inequality (5.5). According to the case of equality holds in (3.2), we know with equality in (5.5) for $p \neq 2$ if and only if $n = 1$ and K is an origin-symmetric segment or if and only if $p = 2$.

For $2 \leq p \leq \infty$, from (3.5), we have

$$V(\Gamma_{-p}K) \geq V(E_p K),$$

this inequality together with (5.7), we obtain

$$\left(\frac{V(K)}{nc_{n-2,p}\omega_n}\right)^{\frac{n}{p}} V(\Pi_p^* K) \geq V(E_p K).$$

From this, inequality (5.6) is obtained. The condition of equality holds in inequality (5.6) is the same as (3.5). ■

Regard to the L_p John ellipsoid $E_p K$, Lutwak, Yang and Zhang (see [22]) proved the following result.

Lemma 7. *If* $K \in \mathcal{K}_s^n$*, then for* $0 < p \leq \infty$,

$$V(E_p K) \geq 2^{-n} \omega_n V(K), \tag{5.8}$$

with equality if and only if K *is a parallelotope.* In particular, for $E_2 K = \Gamma_{-2}K$, inequality (5.8) has been proven in [21].

Proof of Theorem 5. For $1 \leq p \leq 2$, taking $p = 2$ in inequality (5.8), then

$$V(\Gamma_{-2}K) \geq 2^{-n} \omega_n V(K), \tag{5.9}$$

with equality if and only if K is a parallelotope. This inequality together with inequality (5.5) give inequality (5.1). According to the conditions of equality hold in (5.5) and (5.9), we easily see with equality in (5.1) for $p \neq 2$ if and only if $n = 1$ and K is an origin-symmetric segment (1-dimensional parallelotope), for $p = 2$ if and only if K is a parallelotope.

For $2 \leq p \leq \infty$, again using inequality (5.8), and combining with (5.6), inequality (5.2) is immediately given. Obviously, with equality in (5.2) if and only if $p = 2$ and K is a parallelotope by the cases of equality hold in (5.6) and (5.8). ∎

The proof of Theorem 6 also require several Lemmas.

Lemma 8. [22] *If $K \in \mathcal{K}_o^n$, $0 < p \leq q \leq \infty$, then*

$$V(E_q K) \leq V(E_p K). \tag{5.10}$$

Lemma 9. [3] *If K is positioned so that its John points at the origin, then there exists an ellipsoid $E \subseteq K$, centered at the origin, such that*

$$V(E) \geq \frac{n!\omega_n}{n^{n/2}(n+1)^{(n+1)/2}} V(K). \tag{5.11}$$

with equality if and only if K is a simplex.

Using Lemma 8 and Lemma 9, we may obtain the following result:

Lemma 10. *If $K \in \mathcal{K}_o^n$, is positioned so that its John points at the origin, $0 < p \leq \infty$, then*

$$V(E_p K) \geq \frac{n!\omega_n}{n^{n/2}(n+1)^{(n+1)/2}} V(K), \tag{5.12}$$

with equality if and only if K is a simplex.

When $p = 2$, the proof of Lemma 10 by Lutwak, Yang and Zhang (see [21]).

Proof. Because of $E_\infty K$ is just the John's ellipsoid, write $E_J K$, of convex body K (see [22]), then inequality (5.10) in Lemma 8 gives that

$$V(E_p K) \geq V(E_J K).$$

Since each convex body K, its John's ellipsoid $E_J K$ is a unique ellipsoid of maximal volume contained in K. Thus together with inequality (5.11), we immediately obtain inequality (5.12). According to the condition of equality holds in (5.11), we know with equality in (5.12) if and only if K is a simplex. ∎

Proof of Theorem 6. For $1 \leq p \leq 2$, taking $p = 2$ in inequality (5.12) and using inequality (5.5), we get inequality (5.3). Because of an origin-symmetric segment may be regarded as a 1-dimensional simplex, thus together with the conditions of equality hold in (5.5) and (5.12), we see with equality in (5.3) for $p \neq 2$ if and only if $n = 1$ and K is an origin-symmetric segment, for $p = 2$ if and only if K is a simplex.

For $2 \leq p \leq \infty$, inequalities (5.6) and (5.12) immediately give inequality (5.4), and with equality in (5.4) if and only if $p = 2$ and K is a simplex.

The proof of Theorem 6 is over. ∎

REFERENCES

1. R. Alexauder, Zonoid theory and Hilbert's fouth problem, *Geom. Dedicata*, **28** (1988), 199-211.
2. K. Ball, Shadows of convex bodies, *Tans. Amer. Math. Soc.*, **327** (1991), 891-901.
3. K. Ball, Volume ratios and reverse isoperimeyric inequality, *J. London Math. Soc.*, **44** (1991), 351-359.
4. U. Betke and P. Mcmullen, Estimating the sizes of convex bodies from projection, *J. London. Math. Soc.*, **27** (1983), 525-538.
5. E. D. Bolker, A class of convex bodies, *Trans. Amer. Math. Soc.*, **145** (1969), 323-345.
6. J. Bourgain and J. Lindenstrauss, Projection bodies, Geometric Aspects of Functional Analysis, *Springer Lecture Notes in Math.*, **1317** (1988), 250-270.
7. J. Bourgain and V. D. Milman, New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$, *Invent. Math.*, **88** (1987), 319-340.
8. N. S. Brannen, Volumes of projection bodies, *Mathematika*, **43** (1996), 255-264.
9. G. D. Chakerian and E. Lutwak, Bodies with similar projections, *Tans. Amer. Math. Soc.*, **349** (1997), 1811-1820.
10. R. J. Gardner, *Geometric Tomography*, Cambridge Univ. Press, Cambridge, 1995.
11. P. R. Goodey and W. Weil, *Zonoids and Generalizations,* Handbook of Convex Geometry, North-Holland, Amsterdam, 1993, pp. 1297-1326.
12. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1959.
13. K. Leichtweiβ, *Affine Geometry of Convex Bodies*, J. A. Barth, Heidelberg, 1998.
14. M. Ludwig, Projection bodies and valuations, *Adv. Math.*, **172** (2002), 158-168.
15. M. Ludwig, Minkowski valuations, *Trans. Amer. Math. Soc.*, **357** (2005), 4191-4213.

16. E. Lutwak, On a conjectured projection inequality of Petty, *Contemp. Math.*, **113** (1990), 171-181.

17. E. Lutwak, *Selected affine isoperimetric inequalities*, In Handbook of Convex Geometry. ed., North-Holand. Amsterdam, 1993, pp. 151-176.

18. E. Lutwak, Inequalities for mixed projection bodies, *Amer. Math. Soc.*, **339** (1993), 901-916.

19. E. Lutwak, D. Yang and G. Y. Zhang, L_p affine isoperimetric inequalities, *J. Differential Geom.*, **56** (2000), 111-132.

20. E. Lutwak, D. Yang and G. Y. Zhang, A new affine invariant for polytopes and Schneider's projection problem, *Tran. Amer. Math.*, **353** (2001), 1767-1779.

21. E. Lutwak, D. Yang and G. Y. Zhang, A new ellipsoid associated with convex dodies, *Duke. Math. J.*, **104** (2000), 375-390.

22. E. Lutwak, D. Yang and G. Y. Zhang, L_p John ellipsoids, *Proc. London Math. Soc.*, **90** (2005), 497-520.

23. H. Martini, Zur Bestimmung Konvexer Polytope durch the Inhalte ihrer Projection, *Beiträge Zur Algebra und Geometrie*, **18** (1984), 75-85.

24. S. Micheal, Petty's projection inequality and Santaló's affine isoperimetric inequality, *Geom. Dedicata*, **57** (1995), 285-295.

25. C. M. Petty, *Projection bodies*, Proc. Coll. Convexity, Copenhagen, 1965, Kϕbenhavns Univ. Math. Inst., 1967, pp. 234-241.

26. C. M. Petty, *Isoperimetric problems*, Proc. Conf. Convexty and Combinatorial Geometry (Univ. Oklahoma, 1971), University of Oklahoma, 1972, pp. 26-41.

27. Delin Ren, *Introduction of Integral Geometry*, Shanghai Science and Technology Press, Shanghai, 1988.

28. D. Ryabogin and A. Zvavitch, The Fourier transform and Firey projections of convex bodies, *Indiana Univ. Math. Journal*, **53** (2004), 667-682.

29. R. Schneider, Zu einem problem von shephard über die projectionen konvexer körper, *Math. Z.*, **101** (1967), 71-82.

30. R. Schneider, Geometric inequalities for Poisson processes of convex bodies and cylinders, *Results in Math.*, **11** (1987), 165-185.

31. R. Schneider, *Convex Bodies: The Brunn-Minkowski theory*, Cambridge Univ. Press, Cambridge, 1993.

32. R. Schneider and W. Weil, *Zonoids and Related Topics*, Convexity and its applications, Birkhäuser, Basel, 1983, pp. 296-317.

33. W. D. Wang and G. S. Leng, The Petty projection inequality for L_p-mixed projection bodies, *Acta Mathematica Sinica* (*English series*), **23** (2007), 1485-1494.

34. G. Y. Zhang, Restricted chord projection and affine inequalities, *Geom. Dedicata*, **39** (1991), 213-222.

Weidong Wang
Department of Mathematics,
China Three Gorges University,
Hubei Yichang, 443002,
P. R. China
and
Department of Mathematics,
Hubei Institute for nationalities,
Hubei Enshi 445000,
P. R. China
E-mail: wdwxh722@163.com

Gangsong Leng
Department of Mathematics,
Shanghai University,
Shanghai 200436,
P. R. China