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A NEW CLASS OF DOUBLY NONLINEAR EVOLUTION EQUATIONS

Masayasu Aso, Takesi Fukao and Nobuyuki Kenmochi

Abstract. In this paper we consider the solvability for a new class of doubly nonlinear evolution equations. The motivation of this work comes from a transmission problem of two degenerate parabolic equations with convection term, in which the transmission boundary is time-dependent. We give an abstract existence result, and show that the weak variational formulation for the transmission problem can be solved by applying this abstract result. In our existence proof, the abstract theory of pseudo-monotone operators is useful.

1. Introduction

Let $0 < T < +\infty$. We consider an evolution equation of the form:

$$u'(t) + K(t, \theta(t)) + G(t, u(t)) = f(t) \quad \text{in } V^* \text{ for a.e. } t \in [0, T], \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

where $u' := du/dt$. Here, for each $t \in [0, T]$, $K(t, \cdot)$ is a weakly continuous operator from a reflexive Banach space V into its dual space V^*, $G(t, \cdot)$ is a weakly continuous operator from a Hilbert space H into V^*, where V is imbedded densely and compactly in H, f is a given source function and u_0 is an initial datum. Equation (1.1) is considered with the following relation between θ and u:

$$\theta(t) = \partial\psi^t(u(t)) \quad \text{in } V \text{ for a.e. } t \in [0, T], \tag{1.3}$$

where $\{\psi^t\}$ is a family of proper, lower semicontinuous and convex functions on space V^* and $\partial\psi^t$ is the subdifferential of ψ^t from V^* into V; especially in our setting the subdifferential $\partial\psi^t$ is assumed to be singlevalued.

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Some types of doubly nonlinear evolution equations have been considered so far, for example,

$$A(u'(t)) + B(u(t)) \ni f(t), \tag{1.4}$$

which was considered by Arai [1], Senba [21], Colli and Visintin [8] and Colli [7], where A and B are maximal monotone, possibly nonlinear and multivalued operators from V into V^*; in Colli [7], A is bounded and B is unbounded so that the domain $D(B)$ is contained in a Banach space W imbedded compactly in V. Another type of doubly nonlinear evolution equations is of the form:

$$\frac{d}{dt}A(u(t)) + B(u(t)) \ni f(t), \tag{1.5}$$

which was treated, for instance, by Kenmochi [16], Kenmochi and Pawłow [17],[18], and recently Maitre and Witomski [19]. In our problem, the unknown u can be eliminated by (1.3). In fact, using the conjugate convex function ψ^{t*} of ψ^t, we obtain from (1.3) that

$$u(t) \in \partial\psi^{t*}(\theta(t)) \quad \text{in } V^*, \tag{1.6}$$

where $\partial\psi^{t*}$ is the subdifferential of ψ^{t*} from V into V^*. Hence (1.1) is formally written in the form

$$\frac{d}{dt}\partial\psi^{t*}(\theta(t)) + K(t, \theta(t)) + G\Big(t, \partial\psi^{t*}(\theta(t))\Big) \ni f(t) \quad \text{in } V^*. \tag{1.7}$$

This is a new type of doubly nonlinear evolution equations in respect that the time derivative of $\partial\psi^{t*}(\theta(t))$ is included in the equation as well as a highly nonlinear perturbation $G(t, \partial\psi^{t*}(\theta(t)))$. We have not noticed any results on this class of evolution equations. In this paper we give an existence result for problem $\{(1.1)\text{-}(1.3)\}$. In our construction of a solution, we approximate (1.1) by a time discretization scheme. After getting some uniform estimates we discuss its convergences to obtain a solution of our problem.

The above type of evolution equations arises from transmission problems of two degenerate parabolic equations. This problem has been studied by Fukao, Kenmochi and Pawłow [11]. In their paper it was treated as a system of transmission-Stefan type. Two degenerate parabolic equations are combined by the transmission condition. In this case the problem is formulated as an initial value problem for (1.1) in an abstract Banach space, which will be discussed in detail in the last section.

2. Main Result

Throughout this paper, we denote by V a real reflexive Banach space with norm $|\cdot|_V$, by V^* the dual space of V and by $\langle\cdot,\cdot\rangle_{V^*,V}$ the duality pairing between V^*

and V. Moreover, let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $|\cdot|_H$ such that the following dense and compact imbeddings are satisfied:

$$V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V^*.$$

Without loss of generality we may assume that V and V^* are strictly convex spaces.

Now consider the initial value problem (DN):= {(2.1)-(2.3)}:

$$u'(t) + K(t,\theta(t)) + G(t,u(t)) = f(t) \quad \text{in } V^* \text{ for a.e. } t \in [0,T], \tag{2.1}$$

$$\theta(t) = \partial\psi^t(u(t)) \quad \text{in } V \text{ for a.e. } t \in [0,T], \tag{2.2}$$

$$u(0) = u_0, \tag{2.3}$$

under the following assumptions (A1)-(A3):

(A1) For each $t \in [0,T]$, ψ^t is a proper, lower semicontinuous and convex function on V^* with $D(\psi^t) \subset H$, and ψ^t is coercive, that is, there exists a positive constant C_0, independent of t, such that

$$\psi^t(u) \geq C_0|u|_H^2 \quad \text{for all } u \in D(\psi^t). \tag{2.4}$$

Moreover ψ^t satisfies the following time-dependent condition: For each t, $s \in [0,T]$ and $u \in D(\psi^s)$, there exists $\tilde{u} \in D(\psi^t)$ such that

$$|\tilde{u} - u|_{V^*} \cdot \ C_1|t-s|\left(\psi^s(u)^{\frac{1}{2}} + 1\right), \tag{2.5}$$

$$\psi^t(\tilde{u}) - \psi^s(u) \cdot \ C_1|t-s|(\psi^s(u) + 1), \tag{2.6}$$

where C_1 is a positive constant independent of t, s and u. In addition, ψ^t is strongly monotone in the following sense:

$$\langle u_1 - u_2, \theta_1 - \theta_2\rangle_{V^*,V} \geq C_2|\theta_1 - \theta_2|_H^2 \quad \text{for all } \theta_i = \partial\psi^t(u_i) \text{ and } i = 1,2. \tag{2.7}$$

(A2) The operator $K(t,\cdot) : D(K(t,\cdot)) = V \to V^*$ satisfies that (coerciveness)

$$\langle K(t,\theta),\theta\rangle_{V^*,V} \geq C_3|\theta|_V^2 - C_4 \quad \text{for all } \theta \in V \text{ and } t \in [0,T], \tag{2.8}$$

(boundedness)

$$|K(t,\theta)|_{V^*} \cdot \ C_5|\theta|_V + C_6 \quad \text{for all } \theta \in V \text{ and } t \in [0,T], \tag{2.9}$$

where C_3, C_4, C_5 and C_6 are positive constants. Moreover $K(\cdot,\cdot)$ satisfies the following property: If $\theta_n \to \theta$ in $L^2(0,T;H)$ and weakly in $L^2(0,T;V)$ as $n \to +\infty$, then

$$K(\cdot,\theta_n(\cdot,\cdot)) \to K(\cdot,\theta(\cdot,\cdot)) \quad \text{weakly in } L^2(0,T;V^*) \text{ as } n \to +\infty. \tag{2.10}$$

In addition, for each $\theta \in V$, $K(\cdot,\theta)$ is continuous with respect to t in the following sense: There exists a continuous function $\omega : [0,+\infty) \to [0,+\infty)$ with $\omega(0) = 0$ such that

$$|K(t,\theta) - K(s,\theta)|_{V^*} \cdot \ \omega(|t-s|)(|\theta|_V + 1) \quad \text{for all } t,s \in [0,T]. \tag{2.11}$$

(A3) The operator $G(t,\cdot) : D(G(t,\cdot)) = H \to V^*$ satisfies that (boundedness)

$$|G(t,u)|_{V^*} \cdot \ C_7|u|_H + C_8 \quad \text{for all } u \in H \text{ and } t \in [0,T], \tag{2.12}$$

where C_7 and C_8 are positive constants. Moreover $G(\cdot,\cdot)$ satisfies the following property: If $t_n \to t$ and $u_n \to u$ weakly in H as $n \to +\infty$, then

$$G(t_n,u_n) \to G(t,u) \quad \text{weakly in } V^* \text{ as } n \to +\infty. \tag{2.13}$$

Here we give the definition of a solution of (DN).

Definition 2.1 A pair $\{u,\theta\}$ of functions $u,\theta : [0,T] \to H$ is called a solution of (DN), if $u \in W^{1,2}(0,T;V^*) \cap L^\infty(0,T;H)$, $\theta \in L^2(0,T;V)$ and $\{u,\theta\}$ satisfies (2.1), (2.2) and (2.3).

Our main theorem is formulated now:

Theorem 2.1 *Assume that* (A1)-(A3) *hold. Given* $f \in L^2(0,T;V^*)$ *and* $u_0 \in D(\psi^0)$, (DN) *has at least one solution* $\{u,\theta\}$.

We shall prove our existence theorem in sections 3 and 4. Our main idea for the construction of a solution is to employ the time discretization method for (DN).

3. Approimation of (DN)

In order to construct a solution of (DN), we use the time discretization method to approximate problem (DN). For an arbitrary $N \in \mathbf{N}$, we put $h_N := T/N$, $t_k^N := kh_N$ for $k = 0,1,\dots,N$ and $u_0^N := u_0$. Then, we consider the following time discretization scheme for (DN): Find a pair $\{u_k^N,\theta_k^N\}$ of functions satisfying that

$$\frac{u_k^N - u_{k-1}^N}{h_N} + K(t_k^N,\theta_k^N) = -G(t_{k-1}^N,u_{k-1}^N) + f_{k-1}^N \quad \text{in } V^*, \tag{3.1}$$

with

$$\theta_k^N = \partial\psi^{t_k^N}(u_k^N) \quad \text{in } V \text{ for } k = 1, 2, \dots, N, \tag{3.2}$$

where f_k^N is the discrete approximation of f given by

$$f_k^N := \frac{1}{h_N}\int_{t_k^N}^{t_{k+1}^N} f(s)ds \quad \text{for } k = 0, 1, \dots, N-1. \tag{3.3}$$

Now we discuss the solvability of (DN)$_N$:= {(3.1), (3.2)}. For simplicity, put $A(t,\cdot) := (\partial\psi^t)^{-1}$. Note that (DN)$_N$ can be written in the form:

$$A(t_k^N, \theta_k^N) + h_N K(t_k^N, \theta_k^N) \ni u_{k-1}^N - h_N G(t_{k-1}^N, u_{k-1}^N) + h_N f_{k-1}^N \quad \text{in } V^*. \tag{3.4}$$

In order to construct the time discretization scheme, for each fixed t_k^N, it suffices to check the existence of a solution $\theta \in V$ of

$$A(t_k^N, \theta) + h_N K(t_k^N, \theta) \ni g^* \quad \text{in } V^*, \tag{3.5}$$

where g^* is given in V^*. Now we recall the following general theory of nonlinear operators.

Proposition 3.1 *Let Z be a real reflexive Banach space and Z^* be the dual space of Z. Assume that a multivalued operator $\mathcal{N} : Z \to Z^*$ satisfies the following conditions* (N1) and (N2):

(N1) *For each $z \in Z$, $\mathcal{N}z$ is a non-empty, bounded, convex and closed set in V^*.*

(N2) *$\mathcal{N}$ is weakly sequentially upper semicontinuous and it is coercive, that is,*

$$\lim_{|z|_Z \to +\infty} \inf_{z^* \in \mathcal{N}z} \frac{\langle z^*, z\rangle_{Z^*,Z}}{|z|_Z} = +\infty.$$

Then $\mathcal{N}$ is surjective, that is, $R(\mathcal{N}) = Z^*$.

This proposition is well-known; for instance, it is an immediate consequence of the abstract results on the surjectiveness of nonlinear multivalued operators of Type M (see Brézis [3], Browder and Hess [6], Kenmochi [15] and so on).

Lemma 3.1 (i) *Let $t \in [0,T]$ and $N \in \mathbf{N}$. Then $D(A(t,\cdot)) = V$ and $A(t,\theta) + h_N K(t,\theta)$ is a non-empty, bounded, convex and closed set in V^* for each $\theta \in V$.*
(ii) *$A(t,\cdot) + h_N K(t,\cdot)$ is weakly sequentially upper semicontinuous and coercive as a mapping from V into V^*.*

Proof. (i) Since $A^{-1}(t,\cdot) = \partial\psi^t$, by (2.4) we have $R(A^{-1}(t,\cdot)) = D(A(t,\cdot)) = V$ and the boundedness of $A(t,\cdot)$. From the maximal monotonicity of $A(t,\cdot)$ it follows immediately that $A(t,\theta) + h_N K(t,\theta)$ is non-empty, bounded, convex and closed set in V^* for all $\theta \in V$.
(ii) The coerciveness of $A(t,\cdot) + h_N K(t,\cdot)$ is seen from (2.8) and the monotonicity of $A(t,\cdot)$. Lastly we check that $A(t,\cdot) + h_N K(t,\cdot)$ is weakly sequentially upper semicontinuous in V^*. Now note that $K(t,\cdot)$ is a singlevalued and weakly continuous operator from V into V^*, and $A(t,\cdot)$ is a subdifferential operator with $R(A(t,\cdot)) = D(A^{-1}(t,\cdot)) \subset D(\psi^t) \subset H$. Therefore, by the compact imbedding $V \hookrightarrow H$, we see that the graph $G(A(t,\cdot) + h_N K(t,\cdot))$ is weakly closed in $V \times V^*$, which implies the conclusion. ■

This lemma and Proposition 3.1 imply that for each fixed $N \in \mathbf{N}$, $k = 0, 1, \ldots, N$, and any given g^* in V^*, there exists $\theta \in V$ which satisfies (3.5). In our setting, the subdifferential operator $\partial\psi^t$ is singlevalued. Thus, Lemma 3.1 and Proposition 3.1 show that our approximation scheme (DN)$_N$ has solutions $u_k^N \in H$ and $\theta_k^N \in V$ for each $N \in \mathbf{N}$ and $k = 0, 1, \ldots, N$.

Moreover we obtain the following estimates for $\{u_k^N, \theta_k^N\}$.

Lemma 3.2 *There exists a positive constant M_1 independent of N such that*

(3.6) $$\psi^{t_k^N}(u_k^N) \cdot \ M_1 \quad \textit{for all } k = 0, 1, \ldots, N,$$

(3.7) $$h_N \sum_{k=1}^{N} |\theta_k^N|_V^2 \cdot \ M_1.$$

Proof. Multiplying (DN)$_N$ by θ_k^N we obtain

(3.8) $$\begin{aligned} &\frac{1}{h_N}(u_k^N, \theta_k^N)_H - \frac{1}{h_N}(u_{k-1}^N, \theta_k^N)_H + \langle K(t_k^N, \theta_k^N), \theta_k^N \rangle_{V^*,V} \\ &= -\langle G(t_{k-1}^N, u_{k-1}^N), \theta_k^N \rangle_{V^*,V} + \langle f_{k-1}^N, \theta_k^N \rangle_{V^*,V}. \end{aligned}$$

By conditions (2.5) and (2.6) in (A1) for $t := t_k^N$, $s := t_{k-1}^N$ and $u := u_{k-1}^N \in D(\psi^{t_{k-1}^N})$ we can find an element $\tilde{u}_k^N \in D(\psi^{t_k^N})$ such that

(3.9) $$|\tilde{u}_k^N - u_{k-1}^N|_{V^*} \cdot \ C_1 h_N \Big(\psi^{t_{k-1}^N}(u_{k-1}^N)^{\frac{1}{2}} + 1 \Big),$$

(3.10) $$\psi^{t_k^N}(\tilde{u}_k^N) - \psi^{t_{k-1}^N}(u_{k-1}^N) \cdot \ C_1 h_N (\psi^{t_{k-1}^N}(u_{k-1}^N) + 1).$$

By the definition of subdifferential $\partial\psi^{t_k^N}$ and (3.9) we get

$$
\begin{aligned}
&\frac{1}{h_N}(u_k^N,\theta_k^N)_H - \frac{1}{h_N}(u_{k-1}^N,\theta_k^N)_H \\
&\quad = \frac{1}{h_N}(u_k^N - \tilde{u}_k^N,\theta_k^N)_H + \frac{1}{h_N}(\tilde{u}_k^N - u_{k-1}^N,\theta_k^N)_H \\
&\quad \geq \frac{1}{h_N}\Big(\psi^{t_k^N}(u_k^N) - \psi^{t_k^N}(\tilde{u}_k^N)\Big) - C_1\left(\psi^{t_{k-1}^N}(u_{k-1}^N)^{\frac{1}{2}} + 1\right)|\theta_k^N|_V.
\end{aligned}
$$

Moreover, by Schwarz's inequality and (3.10),

$$
\begin{aligned}
&\frac{1}{h_N}\Big(\psi^{t_k^N}(u_k^N) - \psi^{t_k^N}(\tilde{u}_k^N)\Big) - C_1\left(\psi^{t_{k-1}^N}(u_{k-1}^N)^{\frac{1}{2}} + 1\right)|\theta_k^N|_V \\
&\quad \geq \frac{1}{h_N}\Big(\psi^{t_k^N}(u_k^N) - \psi^{t_{k-1}^N}(u_{k-1}^N)\Big) + \frac{1}{h_N}\Big(\psi^{t_{k-1}^N}(u_{k-1}^N) - \psi^{t_k^N}(\tilde{u}_k^N)\Big) \\
&\qquad - C_1^2 C_{\varepsilon_1}\psi^{t_{k-1}^N}(u_{k-1}^N) - \varepsilon_1|\theta_k^N|_V^2 - C_1^2 C_{\varepsilon_1} - \varepsilon_1|\theta_k^N|_V^2 \\
&\quad \geq \frac{1}{h_N}\Big(\psi^{t_k^N}(u_k^N) - \psi^{t_{k-1}^N}(u_{k-1}^N)\Big) - C_1(\psi^{t_{k-1}^N}(u_{k-1}^N) + 1) \\
&\qquad - 2\varepsilon_1|\theta_k^N|_V^2 - C_1^2 C_{\varepsilon_1}\psi^{t_{k-1}^N}(u_{k-1}^N) - C_1^2 C_{\varepsilon_1} \\
&\quad = \frac{1}{h_N}\Big(\psi^{t_k^N}(u_k^N) - \psi^{t_{k-1}^N}(u_{k-1}^N)\Big) \\
&\qquad - 2\varepsilon_1|\theta_k^N|_V^2 - (C_1 + C_1^2 C_{\varepsilon_1})\psi^{t_{k-1}^N}(u_{k-1}^N) - (C_1 + C_1^2 C_{\varepsilon_1});
\end{aligned}
$$

hereafter ε_i is an arbitrary positive constant and $C_{\varepsilon_i} := 1/(4\varepsilon_i)$ for each $i \in \mathbf{N}$. Next, by the coerciveness (2.8) of K we have

$$
\langle K(t_k^N,\theta_k^N),\theta_k^N\rangle_{V^*,V} \geq C_3|\theta_k^N|_V^2 - C_4.
$$

By the boundedness (2.12) of G we have

$$
\begin{aligned}
&\langle -G(t_{k-1}^N,u_{k-1}^N),\theta_k^N\rangle_{V^*,V} + \langle f_{k-1}^N,\theta_k^N\rangle_{V^*,V} \\
&\quad \cdot\ C_7^2 C_{\varepsilon_2}|u_{k-1}^N|_H^2 + C_8^2 C_{\varepsilon_2} + 3\varepsilon_2|\theta_k^N|_V^2 + C_{\varepsilon_2}|f_{k-1}^N|_{V^*}^2.
\end{aligned}
$$

We obtain from (3.8) with the help of (2.4) that

$$
\begin{aligned}
&\frac{1}{h_N}\Big(\psi^{t_k^N}(u_k^N) - \psi^{t_{k-1}^N}(u_{k-1}^N)\Big) + (C_3 - 2\varepsilon_1 - 3\varepsilon_2)|\theta_k^N|_V^2 \\
&\quad \cdot\ \left(C_1 + C_1^2 C_{\varepsilon_1} + \frac{C_7^2 C_{\varepsilon_2}}{C_0}\right)\psi^{t_{k-1}^N}(u_{k-1}^N) + C_{\varepsilon_2}|f_{k-1}^N|_{V^*}^2 \\
&\qquad + (C_1 + C_1^2 C_{\varepsilon_1} + C_4 + C_8^2 C_{\varepsilon_2}),
\end{aligned}
\tag{3.11}
$$

where ε_1 and ε_2 are chosen so as to satisfy $C_3 - 2\varepsilon_1 - 3\varepsilon_2 > 0$. This inequality can be written in the form

$$\psi^{t_k^N}(u_k^N) + M_2 \cdot \left(1 + \frac{TM_3}{N}\right)(\psi^{t_{k-1}^N}(u_{k-1}^N) + M_2) \quad \text{for all } k = 1, \dots, N, \tag{3.12}$$

where M_2 and M_3 are positive constants independent of N; M_2 depends on $|f|_{L^2(0,T;V^*)}$ and $M_3 := C_1 + C_1^2 C_{\varepsilon_1} + (C_7^2 C_{\varepsilon_2}/C_0)$. Here, applying the discrete Gronwall's inequality to (3.12), we obtain

$$\psi^{t_k^N}(u_k^N) \cdot \ \exp(TM_3)(\psi^0(u_0) + M_2) \quad \text{for all } k = 1, 2, \dots, N. \tag{3.13}$$

Next, we multiply (3.11) and (3.13) by h_N and sum them up for $k = 1, 2, \dots, N$ to obtain

$$\psi^T(u_N^N) + (C_3 - 2\varepsilon_1 - 3\varepsilon_2) h_N \sum_{k=1}^N |\theta_k^N|_V^2 \cdot \ M_3 h_N \sum_{k=1}^N \psi^{t_{k-1}^N}(u_{k-1}^N) + M_2 T + \psi^0(u_0).$$

Hence (3.7) is obtained from this inequality together with (3.13). ■

Now, let $\{u_i^N, \theta_i^N\}_{i=0}^N$ be the discretization scheme of (3.1), (3.2) and (3.3) constructed above and define the piecewise linear H-valued function u^N generated by $\{u_i^N\}$ on $[0,T]$, that is, $u^N(0) := u_0$ and

$$u^N(t) := \frac{(t - t_{k-1}^N) u_k^N + (t_k^N - t) u_{k-1}^N}{h_N} \quad \text{if } t \in (t_{k-1}^N, t_k^N] \text{ for } k = 1, 2, \dots, N.$$

It is clear that $u^N \in C([0,T]; H)$ for each $N \in \mathbf{N}$. Moreover, we have the following lemma:

Lemma 3.3 *$\{u^N\}_{N=1}^\infty$ is equicontinuous as a family of functions from $[0,T]$ into V^* and uniformly bounded on $[0,T]$ as H-valued functions.*

Proof. Using Lemma 3.2, we have $|u_k^N|_H \cdot \ (M_1/C_0)^{1/2}$ for $k = 1, 2, \dots, N$, so that $\{u^N\}$ is uniformly bounded on $[0,T]$ as H-valued functions. Next we take $s, t \in [0,T]$ with $s \cdot \ t$ such that $s \in [t_{k-1}^N, t_k^N]$ and $t \in [t_{j-1}^N, t_j^N]$, where $1 \cdot \ k \cdot \ j \cdot \ N$. We see from the boundedness (2.9) of K and (2.12) of G together with (2.4), (3.6) and (3.7) that there exists a positive constant M_4, which depends on C_6 and C_8, such that

$$
\begin{aligned}
&|u^N(t) - u^N(s)|_{V^*} \\
&\cdot \quad 2\sum_{\ell=k}^{j} |u_\ell^N - u_{\ell-1}^N|_{V^*} \\
&\cdot \quad 2\Big(h_N \sum_{\ell=k}^{j} |K(t_\ell^N, \theta_\ell^N)|_{V^*} + h_N \sum_{\ell=k-1}^{j} |G(t_\ell^N, u_\ell^N)|_{V^*} + h_N \sum_{\ell=k-1}^{j} |f_\ell^N|_{V^*} \Big) \\
&\cdot \quad 2\Big(h_N(j-k+1)\Big)^{\frac{1}{2}} \Big\{ C_5 \Big(h_N \sum_{k=1}^{N} |\theta_k^N|_V^2 \Big)^{\frac{1}{2}} + C_7 \Big(h_N \sum_{k=1}^{N} \frac{\psi^{t_k^N}(u_k^N)}{C_0} \Big)^{\frac{1}{2}} \\
&\quad\quad + M_4 + |f|_{L^2(0,T;V^*)} \Big\} \\
&\cdot \quad 2(|t-s| + 2h_N)^{\frac{1}{2}} \Big\{ C_5 M_1^{\frac{1}{2}} + C_7 \Big(\frac{TM_1}{C_0} \Big)^{\frac{1}{2}} + M_4 + |f|_{L^2(0,T;V^*)} \Big\},
\end{aligned}
$$

which shows the equicontinuity of $\{u^N\}$ on $[0, T]$. ■

By virtue of Lemmas 3.2, 3.3 and the Ascoli-Arzela's theorem it is easily seen that there exists a subsequence $\{N_n\} \subset \{N\}$ and there exists a function $u \in C([0, T]; V^*)$ such that

$$(3.14) \qquad u^{N_n} \to u \quad \text{in } C([0, T]; V^*) \text{ as } n \to +\infty.$$

Moreover, define the V^*-valued step function $\bar{u}^N$ and the V-valued step function $\bar{\theta}^N$ on $[0, T]$ by putting $\bar{u}^N(0) := u_0$, $\bar{\theta}^N(0) := \partial\psi^0(u_0)$ and

$$(3.15) \qquad \bar{u}^N(t) := u_k^N, \quad \bar{\theta}^N(t) := \theta_k^N \quad \text{if } t \in (t_{k-1}^N, t_k^N] \quad \text{for } k = 1, 2, \ldots, N.$$

Then (3.14) implies

$$(3.16) \qquad \sup_{t\in[0,T]} |\bar{u}^{N_n}(t) - u(t)|_{V^*} \to 0 \quad \text{as } n \to +\infty.$$

Next, we prove the following convergence and relation between u and θ:

Lemma 3.4 *There exists a subsequence $\{N_m\} \subset \{N_n\}$ with $N_m \to \infty$ as $m \to +\infty$ and there exists a function $\theta \in L^2(0, T; V)$ such that*

$$(3.17) \qquad \bar{\theta}^{N_m} \to \theta \quad \text{weakly in } L^2(0, T; V) \text{ as } m \to +\infty.$$

Moreover the pair $\{u, \theta\}$ satisfies that $\theta(t) = \partial\psi^t(u(t))$ in V for a.e. $t \in [0, T]$.

Proof. By (3.7) in Lemma 3.2, we see that $\{\bar{\theta}^{N_n}\}_{n=1}^{\infty} = \{\partial\psi^t(\bar{u}^{N_n})\}_{n=1}^{\infty}$ is bounded in $L^2(0,T;V)$, which implies (3.17) for some subsequence $\{N_m\}$. Next, we show that $\theta(t) = \psi^t(u(t))$ for a.e. $t \in [0,T]$ which is equivalent to

$$\int_0^T \langle \eta(t) - u(t), \theta(t)\rangle_{V^*,V} dt \cdot \int_0^T \psi^t(\eta(t))dt - \int_0^T \psi^t(u(t))dt \tag{3.18}$$

$$\text{for all } \eta \in L^2(0,T;V^*).$$

Let η be any function in $L^2(0,T;V^*)$ such that $\int_0^T \psi^t(\eta(t))dt < +\infty$; hence $\eta \in L^2(0,T;H)$ (cf. (2.4)). Then there are step functions $\bar{\eta}^{N_m} \in L^2(0,T;H)$ for all $m \in \mathbf{N}$ such that

$$\bar{\eta}^{N_m} \to \eta \quad \text{in } L^2(0,T;H) \text{ as } m \to +\infty, \tag{3.19}$$

$$\limsup_{m\to+\infty} \sum_{k=1}^{N_m} \int_{t_{k-1}^{N_m}}^{t_k^{N_m}} \psi^{t_k^{N_m}}(\bar{\eta}^{N_m}(t))dt \cdot \int_0^T \psi^t(\eta(t))dt; \tag{3.20}$$

see Kenmochi [16]. Now, from the definition of subdifferential it follows that

$$\sum_{k=1}^{N_m} \int_{t_{k-1}^{N_m}}^{t_k^{N_m}} \langle \bar{\eta}^{N_m}(t) - \bar{u}^{N_m}(t), \bar{\theta}^{N_m}(t)\rangle_{V^*,V} dt \tag{3.21}$$
$$\cdot \sum_{k=1}^{N_m} \int_{t_{k-1}^{N_m}}^{t_k^{N_m}} \psi^{t_k^{N_m}}(\bar{\eta}^{N_m}(t))dt - \sum_{k=1}^{N_m} \int_{t_{k-1}^{N_m}}^{t_k^{N_m}} \psi^{t_k^{N_m}}(\bar{u}^{N_m}(t))dt.$$

Let t be any time in $(0,T]$. By virtue of (2.5) and (2.6) in (A1) for $t \in (t_{k-1}^{N_m}, t_k^{N_m}]$, $s := t_k^{N_m}$ and $\bar{u}^{N_m}(t) \in D(\psi^{t_k^{N_m}})$ there exists a function $\tilde{z}_m \in D(\psi^t)$ such that

$$|\tilde{z}_m - \bar{u}^{N_m}(t_k^{N_m})|_{V^*} \cdot C_1|t - t_k^{N_m}|\left(\psi^{t_k^{N_m}}(\bar{u}^{N_m}(t_k^{N_m}))^{\frac{1}{2}} + 1\right), \tag{3.22}$$

$$\psi^t(\tilde{z}_m) - \psi^{t_k^{N_m}}(\bar{u}^{N_m}(t_k^{N_m})) \cdot C_1|t - t_k^{N_m}|(\psi^{t_k^{N_m}}(\bar{u}^{N_m}(t_k^{N_m})) + 1). \tag{3.23}$$

We now observe from (3.16) and (3.22) that

$$\tilde{z}_m \to u(t) \quad \text{in } V^* \text{ as } m \to +\infty. \tag{3.24}$$

On the other hand, using (3.23), (3.24) and the fact that ψ^t is lower semicontinuous on V^*, we see

$$\begin{aligned}
&\liminf_{m\to+\infty} \psi^{t_k^{N_m}}(\bar{u}^{N_m}(t_k^{N_m})) \\
&\geq \liminf_{m\to+\infty} \left(\frac{1}{1 + C_1|t - t_k^{N_m}|}\psi^t(\tilde{z}_m) - \frac{C_1|t - t_k^{N_m}|}{1 + C_1|t - t_k^{N_m}|} \right) \\
&\geq \psi^t(u(t)).
\end{aligned}$$

Thus, taking the limit as $m \to +\infty$ in (3.21) and using (3.16), (3.17), (3.19) and (3.20) with the help of Fatou's lemma, we get (3.18). ■

4. Proof of Theorem

In this section we prove the main theorem after preparing two lemmas. Let $\{\bar{u}^{N_m}, \bar{\theta}^{N_m}\}_{m=1}^{\infty}$ and $\{u, \theta\}$ be such as obtained in the last section.

Lemma 4.1 *Let $\{\theta_j\}$ be any sequence in $C([0,T];V)$ such that $\theta_j \to \theta$ in $L^2(0,T;V)$ as $j \to +\infty$. Put*

$$u_j(t) := \left(\partial\psi^t + F^{-1}\right)^{-1}(\theta_j(t) + F^{-1}u(t)) \quad \text{in } V^* \text{ for all } t \in [0,T]. \tag{4.1}$$

where F is the duality mapping from V to V^*. Then $u_j \in C([0,T];V^*)$ and

$$u_j \to u \quad \text{weakly in } L^2(0,T;V^*) \text{ as } j \to +\infty. \tag{4.2}$$

Proof. First note that $\partial\psi^t + F^{-1}$ is a singlevalued and surjective operator from V^* into V as well as from $L^2(0,T;V^*)$ into $L^2(0,T;V)$. Clearly $u_j \in L^2(0,T;V^*)$. We show $u_j \in C([0,T];V^*)$ as follows: For each fixed $j \in \mathbf{N}$ and $t \in [0,T]$, let $\{t_n\} \subset [0,T]$ be any sequence such that $t_n \to t$ as $n \to +\infty$. Our claim is to show that

$$u_j(t_n) \to u_j(t) \quad \text{in } V^* \text{ as } n \to +\infty. \tag{4.3}$$

Take a function $w \in C([0,T];V^*)$ with $w(t) \in D(\psi^t)$ such that

$$t \to \psi^t(w(t)) \quad \text{is bounded in } [0,T],$$

for example, by a result in Kenmochi [16] we can find w as a solution of the Cauchy problem

$$\frac{\partial w}{\partial t}(t) + \partial_H\psi^t(w(t)) = 0 \quad \text{in } H,$$
$$w(0) = w_0 \quad \text{in } H,$$

where $\partial_H\psi^t$ is the subdifferential of ψ^t in H and w_0 is an element of $D(\psi^0)$. Put $g_j(t) := \theta_j(t) + F^{-1}u(t)$ in V for all $t \in [0,T]$. Then, from the definition of subdifferential and assumption (2.4) in (A1) it follows that

$$\begin{aligned}
&\langle u_j(t_n) - w(t_n), g_j(t_n)\rangle_{V^*,V} \\
&\quad = \langle u_j(t_n) - w(t_n), \partial\psi^{t_n}(u_j(t_n)) + F^{-1}u_j(t_n)\rangle_{V^*,V} \\
&\quad \geq \psi^{t_n}(u_j(t_n)) - \psi^{t_n}(w(t_n)) + |u_j(t_n)|_{V^*}^2 - |w(t_n)|_{V^*}|u_j(t_n)|_{V^*} \\
&\quad \geq C_0|u_j(t_n)|_H^2 - \psi^{t_n}(w(t_n)) + |u_j(t_n)|_{V^*}^2 - \frac{1}{2}|w(t_n)|_{V^*}^2 - \frac{1}{2}|u_j(t_n)|_{V^*}^2 \\
&\quad \geq C_0|u_j(t_n)|_H^2 - \psi^{t_n}(w(t_n)) - \frac{1}{2}|w(t_n)|_{V^*}^2.
\end{aligned}$$

On the other hand, there exists a positive constant M_5 such that

$$\begin{aligned}
&\langle u_j(t_n) - w(t_n), g_j(t_n)\rangle_{V^*,V} \\
&\cdot \quad M_5|u_j(t_n)|_H|g_j(t_n)|_V + |w(t_n)|_{V^*}|g_j(t_n)|_V \\
&\cdot \quad M_5^2\varepsilon_3|u_j(t_n)|_H^2 + C_{\varepsilon_3}|g_j(t_n)|_V^2 + \frac{1}{2}|w(t_n)|_{V^*}^2 + \frac{1}{2}|g_j(t_n)|_V^2.
\end{aligned}$$

Since $|g_j(\tau)|_V \cdot |\theta_j(\tau)|_V + |u(\tau)|_{V^*} =: M_6$ for all $\tau \in [0,T]$, we have for any $\varepsilon_3 > 0$ that

$$\begin{aligned}
(C_0 - M_5^2\varepsilon_3)|u_j(t_n)|_H^2 \quad &\cdot \quad \psi^{t_n}(w(t_n)) + |w(t_n)|_{V^*}^2 + \left(C_{\varepsilon_3} + \frac{1}{2}\right)|g_j(t_n)|_V^2 \\
&\cdot \quad \max_{\tau\in[0,T]}\left\{\psi^\tau(w(\tau)) + |w(\tau)|_{V^*}^2\right\} + \left(C_{\varepsilon_3} + \frac{1}{2}\right)M_6^2.
\end{aligned}$$

Therefore $\{u_j(t_n)\}_{n=1}^\infty$ is bounded in H. Hence, the compact imbedding $H \hookrightarrow V^*$ implies that there exist a subsequence $\{t_{n_i}\} \subset \{t_n\}$ with $t_{n_i} \to t$ as $i \to +\infty$ and an element $z_j(t) \in V^*$ such that

$$u_j(t_{n_i}) \to z_j(t) \quad \text{in } V^* \text{ as } i \to +\infty. \tag{4.4}$$

Now, let $i \to +\infty$ in the relation

$$\partial\psi^{t_{n_i}}(u_j(t_{n_i})) + F^{-1}u_j(t_{n_i}) = g_j(t_{n_i}) \quad \text{in } V \text{ for all } i \in \mathbf{N}. \tag{4.5}$$

Then, (4.4) and (4.5) imply that

$$\partial\psi^t(z_j(t)) + F^{-1}z_j(t) = g_j(t) \quad \text{in } V. \tag{4.6}$$

On the other hand, we have by (4.1) that

$$\begin{aligned}
g_j(t) &:= \theta_j(t) + F^{-1}u(t) \\
&= \left\{\partial\psi^t(u_j(t)) + F^{-1}u_j(t) - F^{-1}u(t)\right\} + F^{-1}u(t) \\
&= \partial\psi^t(u_j(t)) + F^{-1}u_j(t).
\end{aligned}$$

On account of the strict monotonicity of $\partial\psi^t + F^{-1}$, (4.6) implies that $z_j(t) = u_j(t)$. Therefore we see that u_j is continuous in V^* at t. Thus $u_j \in C([0,T];V^*)$. Finally letting $j \to +\infty$ in (4.1), we obtain (4.2). ■

Lemma 4.2 *There exists a subsequence $\{N_m\} \subset \{N_n\}$ with $N_m \to +\infty$ as $m \to +\infty$ such that*

$$\bar\theta^{N_m} \to \theta \quad \text{in } L^2(0,T;H) \text{ as } m \to +\infty. \tag{4.7}$$

Proof. Using Lemma 4.1, we consider step functions $\bar{u}_j^{N_m}$ and $\bar{\theta}_j^{N_m}$ given by

$$\begin{aligned}\bar{u}_j^{N_m}(t) &:= u_j(t_k^{N_m}) \quad \text{if } t \in (t_{k-1}^{N_m}, t_k^{N_m}] \quad \text{for } k = 1, 2, \ldots, N_m, \\ \bar{\theta}_j^{N_m}(t) &:= \theta_j(t_k^{N_m}) \\ &= \partial\psi^{t_k^{N_m}}(u_j(t_k^{N_m})) + F^{-1}u_j(t_k^{N_m}) - F^{-1}u(t_k^{N_m}) \\ &\qquad \text{if } t \in (t_{k-1}^{N_m}, t_k^{N_m}] \quad \text{for } k = 1, 2, \ldots, N_m.\end{aligned}$$

Then, for each $j \in \mathbf{N}$ we see from the above definitions that

$$\bar{u}_j^{N_m}(t) \to u_j(t) \quad \text{in } V^* \text{ uniformly } t \in [0, T], \tag{4.8}$$

$$\bar{\theta}_j^{N_m}(t) \to \theta_j(t) \quad \text{in } V \text{ uniformly } t \in [0, T], \tag{4.9}$$

as $m \to +\infty$. Now we note that

$$\begin{aligned} I &:= \int_0^T \langle \bar{u}^{N_m}(t) - u(t), \bar{\theta}^{N_m}(t) - \theta(t) \rangle_{V^*,V} dt \\ &= \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), \bar{\theta}^{N_m}(t) - \theta(t) \rangle_{V^*,V} dt \\ &\quad + \int_0^T \langle \bar{u}_j^{N_m}(t) - u_j(t), \bar{\theta}^{N_m}(t) - \theta(t) \rangle_{V^*,V} dt \\ &\quad + \int_0^T \langle u_j(t) - u(t), \bar{\theta}^{N_m}(t) - \theta(t) \rangle_{V^*,V} dt \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

The first term I_1 of I is estimated as follows: With the help of the strong monotonicity (2.7) in (A1) we have

$$\begin{aligned} I_1 &= \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), \bar{\theta}^{N_m}(t) - \theta(t) \rangle_{V^*,V} dt \\ &= h_{N_m} \sum_{k=1}^{N_m} \langle \bar{u}^{N_m}(t_k^{N_m}) - \bar{u}_j^{N_m}(t_k^{N_m}), \bar{\theta}^{N_m}(t_k^{N_m}) - \bar{\theta}_j^{N_m}(t_k^{N_m}) \rangle_{V^*,V} \\ &\quad + \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), \bar{\theta}_j^{N_m}(t) - \theta_j(t) \rangle_{V^*,V} dt \\ &\quad + \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), \theta_j(t) - \theta(t) \rangle_{V^*,V} dt \end{aligned}$$

$$
\begin{aligned}
&\geq C_2 \int_0^T |\bar{\theta}^{N_m}(t) - \bar{\theta}_j^{N_m}(t)|_H^2 dt \\
&\quad + C_2 \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), F^{-1}u(t) - F^{-1}\bar{u}_j^{N_m}(t) \rangle_{V^*,V} dt \\
&\quad + \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), \bar{\theta}_j^{N_m}(t) - \theta_j(t) \rangle_{V^*,V} dt \\
&\quad + \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), \theta_j(t) - \theta(t) \rangle_{V^*,V} dt \\
&\geq \frac{C_2}{3} \int_0^T |\bar{\theta}^{N_m}(t) - \theta(t)|_H^2 dt \\
&\quad - C_2 \int_0^T |\bar{\theta}_j^{N_m}(t) - \theta_j(t)|_H^2 dt - C_2 \int_0^T |\theta_j(t) - \theta(t)|_H^2 dt \\
&\quad + C_2 \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), F^{-1}u(t) - F^{-1}\bar{u}_j^{N_m}(t) \rangle_{V^*,V} dt \\
&\quad + \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), \bar{\theta}_j^{N_m}(t) - \theta_j(t) \rangle_{V^*,V} dt \\
&\quad + \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), \theta_j(t) - \theta(t) \rangle_{V^*,V} dt.
\end{aligned}
$$

Adding the second and third terms I_2 and I_3 to the above inequality, we obtain

$$
\begin{aligned}
&\frac{C_2}{3} \int_0^T |\bar{\theta}^{N_m}(t) - \theta(t)|_H^2 dt \\
&\cdot \quad I - I_2 - I_3 + C_2 \int_0^T |\theta(t) - \theta_j(t)|_H^2 dt + C_2 \int_0^T |\theta_j(t) - \bar{\theta}_j^{N_m}(t)|_H^2 dt \\
&\quad - C_2 \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), F^{-1}u(t) - F^{-1}\bar{u}_j^{N_m}(t) \rangle_{V^*,V} dt \\
&\quad - \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), \bar{\theta}_j^{N_m}(t) - \theta_j(t) \rangle_{V^*,V} dt \\
&\quad - \int_0^T \langle \bar{u}^{N_m}(t) - \bar{u}_j^{N_m}(t), \theta_j(t) - \theta(t) \rangle_{V^*,V} dt.
\end{aligned}
$$

Now, let $m \to +\infty$. Then, by convergences (3.16), (3.17), (4.8), (4.9) and the monotonicity of F^{-1},

$$
\begin{aligned}
&\limsup_{m\to+\infty}\int_0^T |\bar{\theta}^{N_m}(t)-\theta(t)|_H^2 dt \\
&\cdot\quad 3\int_0^T |\theta(t)-\theta_j(t)|_H^2 dt \\
&\qquad -3\int_0^T \langle u(t)-u_j(t), F^{-1}u(t)-F^{-1}u_j(t)\rangle_{V^*,V} dt \\
&\qquad -\frac{3}{C_2}\int_0^T \langle u(t)-u_j(t), \theta_j(t)-\theta(t)\rangle_{V^*,V} dt \\
&\cdot\quad 3\int_0^T |\theta(t)-\theta_j(t)|_H^2 dt \\
&\qquad -\frac{3}{C_2}\int_0^T \langle u(t)-u_j(t), \theta_j(t)-\theta(t)\rangle_{V^*,V} dt \quad \text{for all } j\in\mathbf{N}.
\end{aligned}
$$

Moreover, let $j\to+\infty$ in the above inequality. Then, by virtue of Lemma 4.1, we obtain that

$$\limsup_{m\to+\infty}\int_0^T |\bar{\theta}^{N_m}(t)-\theta(t)|_H^2 dt \cdot\ 0.$$

Thus $\bar{\theta}^{N_m}\to\theta$ in $L^2(0,T;H)$ as $m\to+\infty$. ■

Proof of Theorem 2.1 As was shown above, there exist a subsequence $\{N_m\}\subset\{N\}$, $\theta\in L^2(0,T;V)$ and $u\in C([0,T];V^*)\cap L^\infty(0,T;H)$ such that

$$
\begin{aligned}
\bar{\theta}^{N_m}&\to\theta \quad \text{in } L^2(0,T;H),\\
\bar{\theta}^{N_m}&\to\theta \quad \text{weakly in } L^2(0,T;V),\\
u^{N_m}&\to u \quad \text{in } C([0,T];V^*) \quad \text{as } m\to+\infty,
\end{aligned}
$$

and

$$\theta(t)=\partial\psi^t(u(t)) \quad \text{in } V \text{ for a.e. } t\in[0,T].$$

Now, define V^*-valued step functions $\bar{K}^{N_m}$, $\bar{G}^{N_m}$ and $\bar{f}^{N_m}$ on $[0,T]$ by

$$
\begin{aligned}
\bar{K}^{N_m}(t)&=K(t_k^{N_m},\theta_k^{N_m}) \quad \text{if } t\in(t_{k-1}^{N_m},t_k^{N_m}] \quad \text{for } k=1,2,\dots,N_m,\\
\bar{G}^{N_m}(t)&=G(t_k^{N_m},u_k^{N_m}) \quad \text{if } t\in(t_{k-1}^{N_m},t_k^{N_m}] \quad \text{for } k=1,2,\dots,N_m,\\
\bar{f}^{N_m}(t)&=f_k^{N_m} \quad \text{if } t\in(t_{k-1}^{N_m},t_k^{N_m}] \quad \text{for } k=1,2,\dots,N_m.
\end{aligned}
$$

Then our approximate problem $(\mathrm{DN})_{N_m}$ can be written in the form:

$$\frac{d}{dt}u^{N_m}(t) + \bar{K}^{N_m}(t) + \bar{G}^{N_m}(t) = \bar{f}^{N_m}(t) \quad \text{in } V^* \text{ for a.e. } t \in [0,T]. \tag{4.10}$$

Our claim is to show that (4.10) converges to (2.1) as $m \to +\infty$. To this end, we show that

$$\bar{K}^{N_m} \to K(\cdot,\theta(\cdot)) \quad \text{weakly in } L^2(0,T;V^*) \text{ as } m \to +\infty. \tag{4.11}$$

By (2.10) in the assumptions (A2) of K we see that

$$K(\cdot,\bar{\theta}^{N_m}(\cdot)) \to K(\cdot,\theta(\cdot)) \quad \text{weakly in } L^2(0,T;V^*) \text{ as } m \to +\infty.$$

Moreover, by (2.11) in the assumptions (A2) of K,

$$|\bar{K}^{N_m}(t) - K(t,\bar{\theta}^{N_m}(t))|_{V^*} \cdot \ \omega(|t_k^{N_m} - t|)(|\bar{\theta}^{N_m}(t)|_V + 1)$$
$$\text{for all } t \in (t_{k-1}^{N_m}, t_k^{N_m}] \text{ and } k = 1,2,\ldots,N_m.$$

Hence (4.11) holds. Similarly, by (2.13) in the assumption (A3) of G, we have

$$\bar{G}^{N_m} \to G(\cdot,u(\cdot)) \quad \text{weakly in } L^2(0,T;V^*) \text{ as } m \to +\infty,$$

and clearly

$$\bar{f}^{N_m} \to f \quad \text{in } L^2(0,T;V^*) \text{ as } m \to +\infty.$$

These convergences imply that $\{(d/dt)u^{N_m}\}$ is bounded in $L^2(0,T;V^*)$, so that

$$\frac{d}{dt}u^{N_m} \to u' \quad \text{weakly in } L^2(0,T;V^*) \text{ as } m \to +\infty,$$

Finally, passing to the limit in m of (4.10), we have

$$u'(t) + K(t,\theta(t)) + G(t,u(t)) = f(t) \quad \text{in } V^* \text{ for a.e. } t \in [0,T].$$

Thus $\{u,\theta\}$ is a solution of (DN). ■

5. Application

In this section we give an application of our abstract result.

Let $0 < T < +\infty$, and $\quad \subset \mathbf{R}^3$ be a bounded domain with smooth boundary $\Gamma := \partial\quad$. Assume that for each $t \in [0,T]$, $\quad$ is divided into two subdomains $\ _1(t)$ and $\ _2(t)$ by a time-dependent interface $\Gamma_{12}(t)$, that is,

$$= \ _1(t) \cup \Gamma_{12}(t) \cup \ _2(t) \quad \text{for all } t \in [0,T].$$

We denote by $\Gamma_i(t)$ the set $\partial \ _i(t) \setminus \Gamma_{12}(t)$ for $i = 1,2$ (see Fig. 1).

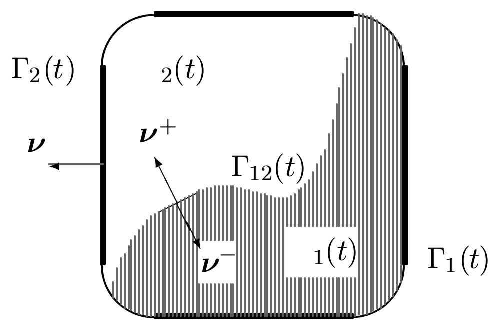

Fig. 1

Under this setting of the domains, we consider the following system of two degenerate parabolic equations:

$$(5.1)\quad \left(\frac{\partial u_1}{\partial t} + \mathbf{v}_1 \cdot \nabla u_1\right) - \mathrm{div}\Big(k_1(\theta_1)\nabla\theta_1\Big) = h \quad \text{in } Q_1 := \bigcup_{t\in(0,T)} \{t\} \times {}_1(t),$$

$$(5.2)\quad \theta_1 = \beta_1(u_1) \quad \text{in } Q_1,$$

$$(5.3)\quad \left(\frac{\partial u_2}{\partial t} + \mathbf{v}_2 \cdot \nabla u_2\right) - \mathrm{div}\Big(k_2(\theta_2)\nabla\theta_2\Big) = h \quad \text{in } Q_2 := \bigcup_{t\in(0,T)} \{t\} \times {}_2(t),$$

$$(5.4)\quad \theta_2 = \beta_2(u_2) \quad \text{in } Q_2,$$

where div means the divergence with respect to space variable x, h is a given function in $Q := (0,T) \times$, and for $i = 1, 2$, $\mathbf{v}_i$ are given vector fields on $\overline{Q}_i$. Moreover, for each $i = 1, 2$, β_i and k_i are functions such that

- $\beta_i : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and non-decreasing, and satisfies that

$$(5.5)\quad |\beta_i(r)| \geq C_{\beta_i}|r| - C'_{\beta_i} \quad \text{for all } r \in \mathbf{R},$$

where C_{β_i} and C'_{β_i} are positive constants;
- $k_i : \mathbf{R} \to \mathbf{R}$ is positive and non-decreasing, and satisfies that

$$(5.6)\quad 0 < C_{k_i} \cdot\ k_i(r) \cdot\ C'_{k_i} \quad \text{for all } r \in \mathbf{R},$$

where C_{k_i} and C'_{k_i} are positive constants.

We combine these two equations (5.1) and (5.3) on the moving boundary $\Gamma_{12}(t)$ by the transmission condition

$$(5.7)\quad \theta_1 = \theta_2, \quad k_1(\theta_1)\frac{\partial \theta_1}{\partial \boldsymbol{\nu}^+} + k_2(\theta_2)\frac{\partial \theta_2}{\partial \boldsymbol{\nu}^-} = 0 \quad \text{on } \Sigma_{12} := \bigcup_{t\in(0,T)} \{t\} \times \Gamma_{12}(t),$$

where $\boldsymbol{\nu}^+ := \boldsymbol{\nu}^+(t,x)$ is the unit normal vector on $\Gamma_{12}(t)$ pointing to $\ _2(t)$ and $\boldsymbol{\nu}^- := -\boldsymbol{\nu}^+$. We consider the system $\{$(5.1)-(5.4),(5.7)$\}$, subject to the initial and boundary conditions for $i = 1, 2$:

$$(5.8)\qquad u_i(0) = u_{i0} \quad \text{in} \quad _i(0),$$

$$(5.9)\qquad k_i(\theta_i)\frac{\partial \theta_i}{\partial \boldsymbol{\nu}} + n_0\theta_i = p \quad \text{on } \Sigma_i := \bigcup_{t\in(0,T)} \{t\} \times \Gamma_i(t),$$

where u_0 and p are given functions, n_0 is a positive constant and $\boldsymbol{\nu} := \boldsymbol{\nu}(x)$ is the unit vector outward normal to Γ. The system $\{$(5.1)-(5.4),(5.7)-(5.9)$\}$ is referred as (TDP).

It is well-known that equations (5.1) and (5.2) (resp. (5.3) and (5.4)) describe a Stefan problem in non-cylindrical domain Q_1 (resp. Q_2), in which there is a convective vector field $\mathbf{v}_1$ (resp. $\mathbf{v}_2$). We call such a problem the transmission-Stefan problem. This problem is discussed under the following assumptions (A4) and (A5) (cf. [10,11]):

(A4) The domain $\ _1(t)$ and $\ _2(t)$ depend smoothly on time t in the following sense: There is a transformation $y = \mathbf{y}(t,x) := (y_1(t,x), y_2(t,x), y_3(t,x))$ which is a function of C^2-class from $\overline{Q}$ into $\mathbf{R}^3$ satisfying that $y = \mathbf{y}(t,x)$ is a C^2-diffeomorphism from $\overline{\ }$ onto itself for each $t \in [0,T]$ and

$$\mathbf{y}(t, \overline{\ _i(t)}) = \overline{\ _i(0)} \quad \text{for all } t \in [0,T] \quad \text{for } i = 1,2, \quad \mathbf{y}(0,\cdot) = \mathbf{I} \text{ (identity) in } \overline{\ }.$$

(A5) The convective vector fields $\mathbf{v}_1$ and $\mathbf{v}_2$, which may occur by the motion of domains, are prescribed so that $\mathbf{v}_i \in C^1(\overline{Q_i})^3$ for $i = 1, 2$ and the following properties are satisfied:

$$(5.10)\qquad \text{div}\mathbf{v}_i(t,\cdot) = 0 \quad \text{in} \quad _i(t) \quad \text{for all } t \in (0,T) \quad \text{for } i = 1,2,$$

$$(5.11)\qquad \mathbf{v}_1 \cdot \boldsymbol{\nu}^+ = -\mathbf{v}_2 \cdot \boldsymbol{\nu}^- = v_{\Sigma_{12}} \quad \text{on } \Sigma_{12},$$

$$(5.12)\qquad \mathbf{v}_1 \cdot \boldsymbol{\nu} = \mathbf{v}_2 \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Sigma := (0,T) \times \Gamma,$$

where $v_{\Sigma_{12}} := v_{\Sigma_{12}}(t,x)$ is the normal speed of $\Gamma_{12}(t)$ at $x \in \Gamma_{12}(t)$.

Using our abstract theory, we show the existence of a weak solution of (TDP). Let us take the Sobolev space $H^1(\ \ )$ and the Hilbert space $L^2(\ \ )$ as V and H, respectively. Then we have

$$V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V^*,$$

where $(H^1(\ \ ))^* =: V^*$ is the dual space of V. Moreover ψ^t, K, G and f are defined as follows:

• ψ^t is a proper, lower semicontinuous and convex function on V^* defined by

$$\psi^t(u) := \begin{cases} \displaystyle\sum_{i=1,2} \int_{\ \ i(t)} \hat{\beta}_i(u)dx & \text{if } u \in H, \\ +\infty & \text{if } u \in V^* \setminus H, \end{cases}$$

where $\hat{\beta}_i(r) := \int_0^r \beta_i(s)ds$;

• K is a singlevalued operator from $[0,T] \times V$ into V^* defined by

$$\langle K(t,\theta), z\rangle_{V^*,V} := \int k(t,\theta)(\nabla\theta \cdot \nabla z)dx + n_0 \int_\Gamma \theta z d\Gamma \quad \text{for all } z \in V,$$

where $k(t,\theta) := k(t,x,\theta(x))$ and $k : [0,T] \times \ \ \times \mathbf{R} \to \mathbf{R}$ is defined by

$$k(t,x,r) := \begin{cases} k_1(r) \ \text{ for all } \ r \in \mathbf{R} \ \text{ if } x \in \ \ _1(t), \\ k_2(r) \ \text{ for all } \ r \in \mathbf{R} \ \text{ if } x \in \overline{\ \ _2(t)}; \end{cases}$$

• G is a singlevalued operator from $[0,T] \times H$ into V^* defined by

$$\langle G(t,u), z\rangle_{V^*,V} := -\sum_{i=1,2} \int_{\ \ i(t)} u(\mathbf{v}_i(t) \cdot \nabla z)dx \quad \text{for all } z \in V;$$

• f is a function in $L^2(0,T;V^*)$ given by

$$\langle f(t), z\rangle_{V^*,V} := \int h(t)zdx + \int_\Gamma p(t)zd\Gamma \quad \text{for all } z \in V \text{ and a.e. } t \in [0,T],$$

where $h \in L^2(Q)$ and $p \in L^2(\Sigma)$ are prescribed.

It is easily seen that (A1), (A2) and (A3) are satisfied (cf. T. Fukao, N. Kenmochi and I. Pawłow [10]). Now our problem (TDP) is reformulated in the following weak variational form:

$$(5.13) \quad -\int_Q u\frac{\partial\eta}{\partial t}dxdt - \int_Q u(\mathbf{v}\cdot\nabla\eta)dxdt + \int_Q k(\theta)(\nabla\theta\cdot\nabla\eta)dxdt + n_0\int_\Sigma \theta\eta d\Gamma dt$$

$$= \int_Q h\eta dxdt + \int_\Sigma p\eta d\Gamma dt + \int u_0\eta(0)dx \quad \text{for all } \eta \in W,$$

where

$$k(\theta) := k(t, x, \theta(t, x)),$$
$$W := \{\eta \in H^1(Q);\ \eta(T, \cdot) = 0 \text{ on } \quad \},$$

and

$$u := u_i, \quad \theta := \theta_i, \quad \mathbf{v} := \mathbf{v}_i \quad \text{on } Q_i \quad \text{and } u_0 := u_{i0} \quad \text{on} \quad {}_i(0) \quad \text{for } i = 1, 2.$$

In fact, the variational identity (5.13) is derived from (5.1), (5.2), (5.3) and (5.4) in the following way. Let η be any test function in W. First, multiply (5.1) and (5.3) by η and integrate their resultants over Q_1 and Q_2, respectively. Then, with the help of conditions (5.8), (5.10), (5.11), (5.12) and the Green-Stokes' formula we have

$$\begin{aligned}
&\sum_{i=1,2}\int_{Q_i}\left(\frac{\partial u_i}{\partial t} + \mathbf{v}_i \cdot \nabla u_i\right)\eta dxdt \\
= \quad & -\int_{Q_1} u_1\left(\frac{\partial \eta}{\partial t} + \mathbf{v}_1 \cdot \nabla\eta\right)dxdt \\
& + \int_{\Sigma_{12}} u_1\eta(\mathbf{v}_1 \cdot \boldsymbol{\nu}^+)d\Gamma_{12}(t)dt - \int_{{}_1(0)} u_{10}\eta(0)dx \\
& - \int_{Q_2} u_2\left(\frac{\partial \eta}{\partial t} + \mathbf{v}_2 \cdot \nabla\eta\right)dxdt \\
& + \int_{\Sigma_{12}} u_2\eta(\mathbf{v}_2 \cdot \boldsymbol{\nu}^-)d\Gamma_{12}(t)dt - \int_{{}_2(0)} u_{20}\eta(0)dx \\
= \quad & -\int_Q u\left(\frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \nabla\eta\right)dxdt - \int u_0\eta(0)dx.
\end{aligned}$$

Next, by (5.7) and (5.9),

$$\begin{aligned}
&-\sum_{i=1,2}\int_{Q_i} \mathrm{div}\Big(k_i(\theta_i)\nabla\theta_i\Big)\eta dxdt \\
= \quad & \int_{Q_1} k_1(\theta_1)(\nabla\theta_1 \cdot \nabla\eta)dxdt \\
& - \int_{\Sigma_1} k_1(\theta_1)\frac{\partial \theta_1}{\partial \boldsymbol{\nu}}\eta d\Gamma_1(t)dt + \int_{\Sigma_{12}} k_1(\theta_1)\frac{\partial \theta_1}{\partial \boldsymbol{\nu}^+}\eta d\Gamma_{12}(t)dt \\
& + \int_{Q_2} k_2(\theta_2)(\nabla\theta_2 \cdot \nabla\eta)dxdt \\
& - \int_{\Sigma_2} k_2(\theta_2)\frac{\partial \theta_2}{\partial \boldsymbol{\nu}}\eta d\Gamma_2(t)dt + \int_{\Sigma_{12}} k_2(\theta_2)\frac{\partial \theta_2}{\partial \boldsymbol{\nu}^-}\eta d\Gamma_{12}(t)dt
\end{aligned}$$

$$= \int_Q k(\theta)(\nabla\theta\cdot\nabla\eta)dxdt + \int_\Sigma (n_0\theta - p)\eta d\Gamma(t)dt.$$

Here, add the above two equations. Then we get the variational identity (5.13). Moreover by (5.2), (5.4) and the definition of ψ^t we see that

$$\theta(t) = \partial\psi^t(u(t)) \quad \text{in } V \text{ for a.e. } t \in [0,T],$$

and (5.13) can be transformed to a doubly nonlinear problem of the form (DN) in the space V^*. Therefore, applying Theorem 2.1, we conclude that there exist $u \in W^{1,2}(0,T;V^*) \cap L^\infty(0,T;H)$ and $\theta \in L^2(0,T;V)$ which give a weak solution of (TDP) in the variational sense (5.13).

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Masayasu Aso
Graduate School of Science and Technology, Chiba University
1-33 Yayoi-cho, Inage-ku, Chiba, 263-8522 Japan
E-mail: masayasu.aso@graduate.chiba-u.jp

Takesi Fukao
Department of Mathematics, Faculty of Education,
Chiba University,
1-33 Yayoi-cho, Inage-ku, Chiba, 263-8522, Japan
E-mail: fukao@faculty.chiba-u.jp

Nobuyuki Kenmochi
Department of Mathematics, Faculty of Education,
Chiba University,
1-33 Yayoi-cho, Inage-ku, Chiba, 263-8522, Japan
E-mail: kenmochi@faculty.chiba-u.jp