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SOME ENTIRE SOLUTIONS OF THE ALLEN–CAHN EQUATION

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Abstract. This paper is dealing with entire solutions of a bistable reaction-diffusion equation with Nagumo type nonlinearity, so called the Allen–Cahn equation. Here the entire solutions are meant by the solutions defined for all $(x, t) \in \mathbb{R} \times \mathbb{R}$. In this article we first show the existence of an entire solution which behaves as two traveling front solutions coming from both sides of x-axis and annihilating in a finite time, using the explicit expression of the traveling front and the comparison theorem. We also show the existence of an entire solution emanating from the unstable standing pulse solution and converges to the pair of diverging traveling fronts as the time tends to infinity. Then in terms of the comparison principle we prove a rather general result on the existence of an unstable set of an unstable equilibrium to apply to the present case.

1. Introduction

We are concerned with the following scalar reaction-diffusion equation:

$$u_t = u_{xx} + u(u-a)(1-u), \quad x \in \mathbb{R} \tag{1.1}$$

with the constant a satisfying

$$0 < a < \frac{1}{2}.$$

This equation is called the Allen-Cahn equation in a phase transition problem while called the Nagumo equation in a propagation phenomenon of nerve excitation.

The equation (1.1) has the characteristic feature such that the dynamics generated by the diffusion–free equation of (1.1) admits two stable equilibria $u = 0, 1$ and an

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unstable one $u = a$ separating the basins of the two equilibria. Thus it is a called a bistable reaction-diffusion equation or a reaction-diffusion equation with bistable nonlinearity.

One of the most basic but important nontrivial behavior of solutions to such a bistable reaction-diffusion equation on $\mathbb{R}$ is the propagation of a traveling front wave, which is characterized by a solution having the form $\Phi(x-ct)$ or $\Phi(-x-ct)$, where c is a positive constant and $\Phi(z)$ is monotone decreasing with $\Phi(-\infty) = 1$ and $\Phi(\infty) = 0$. We note that the existence and the stability of the traveling front wave for reaction-diffusion equations with more general bistable nonlinearity were extensively studied in [1], [2], [4] and [6]. (We also refer to the study for the dynamics of the transition layer with small diffusion coefficient, for instances, see [3] and [12] and references therein).

Other interesting phenomena concerning the traveling front wave are annihilation of two front waves and generation of diverging fronts. We explain these phenomena more precisely. Let two facing fronts be created. Then the left front and the right front travel from left to right and from right to left respectively. The two fronts eventually collide and annihilate in a finite time. On the other hand there is a solution developing into a pair of two diverging fronts for appropriate initial data. In this case as $t \to \infty$, the two fronts converges traveling front waves $\Phi(x-ct-x_0)$ in $x \geq 0$ and $\Phi(-x-ct-x_1)$ in x · 0 respectively, where x_0 and x_1 are some constants. These dynamics were also mathematically studied in detail. For instance, as for the former case see [3] and [12] and the latter case can be found in [6].

We, however, arrive at the query that there exists an entire solution characterizing the annihilation of two fronts or the diverging fronts, where the entire solution is meant by a solution which defined for all $(x,t) \in \mathbb{R} \times \mathbb{R}$. We remark that an equilibrium solution and a traveling wave solution are simple examples of entire solutions. Thus we have to look for a different type of entire solutions. Yagisita [13] gave a positive answer to the former case, that is, he proved that there is an entire solution $u(x,t)$ such that

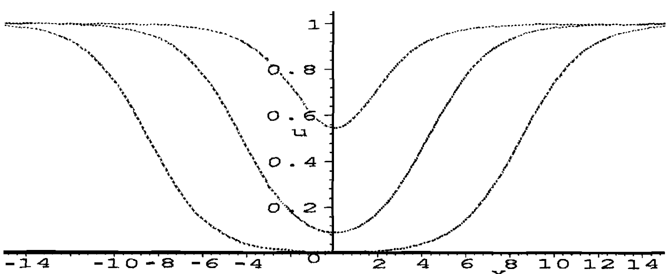

Fig. 1. Annihilation of fronts.

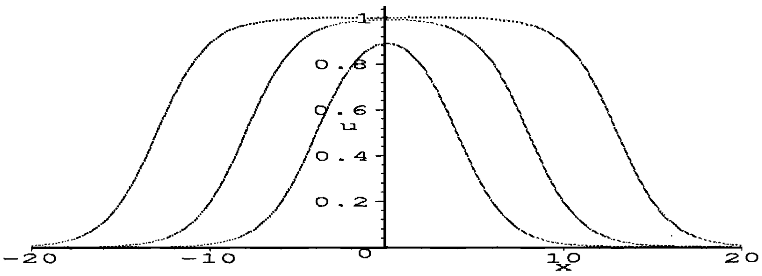

Fig. 2. Diverging fronts.

$$\lim_{t\to-\infty}\left\{\sup_{x\ 0}\Big|u(x,t)-\Phi(x-ct-x_0)\Big|+\sup_{x\geq 0}\Big|u(x,t)-\Phi(-x-ct-x_1)\Big|\right\}=0,$$

and

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|u(x,t)-1|=0,$$

for given x_0, x_1 (see [7] for Fisher-KPP equation). He also discussed the stability and the uniqueness. His study revealed a new aspect of the dynamical property of the bistable reaction-diffusion equation. His main tools used in the proof are the spectrum theorem for the linearized problem of the traveling front wave, the invariant manifold theorem and the comparison theorem. In fact he proved the above result in skillful. Unfortunately his argument is rather technical and lengthy.

The aim of this article is to propose a simple proof for the existence of the entire solution found in [13] and to show the other entire solution characterizing the diverging fronts in the specific equation (1.1). In our argument we use only the comparison principle for a parabolic equation together with the continuity of the semiflow defined by the time translation of solutions. For instance in the former case we propose appropriate supersolution and subsolution defined globally in time such that they both have the same asymptotic profile and the same convergence rate as $t\to-\infty$. Once we establish them, we can show the existence of the desired entire solution by the comparison theorem and the continuity of the semiflow.

The main result is the following:

Theorem 1.1 *Consider the reaction-diffusion equation* (1.1).

(i) *Given constants x_1 and x_2 there exists an entire solution $u(x,t)$ satisfying $0<u(x,t)<1$ and*

$$\lim_{t\to-\infty}\left\{\sup_{x\ 0}\Big|u(x,t)-\Phi(x-ct-x_1)\Big|+\sup_{x\geq 0}\Big|u(x,t)-\Phi(-x-ct+x_2)\Big|\right\}=0,$$

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|u(x,t)-1|=0.$$

(ii) *Let $v(x)$ be a positive standing pulse solution. Given x_1 and x_2 there exists an entire solutions $u_1(x,t)$ satisfying $v(x-\bar{x}) < u_1(x,t) < 1, \bar{x} = (x_1 + x_2)/2$ and*

$$\lim_{t\to-\infty} \sup_{x\in\mathbb{R}} |u_1(x,t) - v(x-\bar{x})| = 0,$$
$$\lim_{t\to\infty} \{\sup_{x\geq 0} |u_1(x,t) - \Phi(x-ct-x_1)| + \sup_{x\quad 0} |u_1(x,t) - \Phi(-x-ct+x_2)|\} = 0.$$

(iii) *Given $\tilde{x}$ there exists an entire solution $u_2(x,t)$ satisfying $0 < u_2(x,t) < v(x-\tilde{x})$ and*

$$\lim_{t\to-\infty} \sup_{x\in\mathbb{R}} |u_2(x,t) - v(x-\tilde{x})| = 0,$$
$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} u_2(x,t) = 0\,.$$

Note that (1.1) has the unstable standing pulse solution which is written as

$$v(x) := \frac{3a}{1 + a + \sqrt{(1-2a)(2-a)/2}\cosh(\sqrt{a}x)}\,. \tag{1.2}$$

We state an idea in the proof of Theorem 1.1 below. First notice that the traveling front solutions to (1.1) are explicitly written as

$$\Phi(\pm x - ct) := \frac{\exp\{\pm(x-ct)/\sqrt{2}\}}{1 + \exp\{\pm(x-ct)/\sqrt{2}\}}\,,$$

where

$$c := \sqrt{2}\left(\frac{1}{2} - a\right) \ (> 0).$$

We also see from [12] that

$$\underline{u}(x,t) = \frac{\exp\{(x-x_0)/\sqrt{2} + \omega t\} + \exp\{-(x+x_0)/\sqrt{2} + \omega t\}}{1 + \exp\{(x-x_0)/\sqrt{2} + \omega t\} + \exp\{-(x+x_0)/\sqrt{2} + \omega t\}}$$

is a subsolution for $(x,t) \in \mathbb{R}$, where we put

$$\omega := \frac{1}{2} - a \ \left(= \frac{c}{\sqrt{2}}\right).$$

This subsolution has the asymptotic profile such that as $t \to -\infty$, it converges to $\Phi(x-ct)$ in $x > 0$ and $\Phi(-x-ct)$ in $x < 0$ respectively. Moreover it is a good approximate solution for a wide range of t. This consideration suggested us to construct a supersolution with the form:

$$(1.3)\quad \overline{u}(x,t) := \frac{\exp\{(x-x_0)/\sqrt{2}+p(t)\}+\exp\{-(x+x_0)/\sqrt{2}+p(t)\}}{1+\exp\{(x-x_0)/\sqrt{2}+p(t)\}+\exp\{-(x+x_0)/\sqrt{2}+p(t)\}},$$

with

$$\lim_{t\to-\infty}|p(t)-\omega t| = 0\ .$$

Using the comparison argument and the continuity of the semiflow defined by time shift of solutions to (1.1), we easily prove that there is a unique entire solution between the subsolution and the supersolution. Since they has the same asymptotic profile as $t\to-\infty$, we can assert that the entire solution also does.

Next we go to the second and the third results of Theorem 1.1. When we prove the existence of a solution converging to the unstable equilibrium $v(x)$ as $t\to-\infty$, we apply the next general result.

Theorem 1.2 *Let $f(x,u)$ be a continuous function defined in $\mathbb{R}^N\times I_0$ and assume that $f(\cdot,u)$ is C^1-Lipschitz in u, where I_0 is an open interval of $\mathbb{R}$. Consider a reaction-diffusion equation*

$$(1.4)\qquad u_t = \triangle u + f(x,u),\quad x\in\mathbb{R}^N.$$

If the equation

$$\triangle u + f(x,u) = 0,\quad x\in\mathbb{R}^N$$

admits a nonconstant C^2-solution $v(x)$ such that there exist a positive $\mu > 0$ and a positive C^2-function φ which solve the linearized eigenvalue problem

$$\triangle\varphi + f_u(x,v(x))\varphi = \mu\varphi,\qquad \lim_{|x|\to\infty}\varphi(x) = 0,$$

then the equation (1.4) *has solutions $u^+(x,t), u^-(x,t)$ satisfying*

$$v(x)+\epsilon\varphi(x)\exp(p_1(t)) \cdot\ u^+(x,t) \cdot\ v(x)+\epsilon\varphi(x)\exp(p_2(t)),$$

$$v(x)-\epsilon\varphi(x)\exp(p_2(t)) \cdot\ u^-(x,t) \cdot\ v(x)-\epsilon\varphi(x)\exp(p_1(t)),$$

for $(x,t)\in\mathbb{R}^N\times(-\infty,0]$ where ϵ is a sufficiently small positive number and $p_j(t)$ $(j=1,2)$ are monotone increasing functions satisfying

$$p_1(t) < p_2(t)\quad (-\infty < t \cdot\ 0),\qquad \lim_{t\to-\infty}|p_j(t)-\mu t| = 0.$$

The assertion of (iii) of Theorem 1.1 immediately follows from this theorem while the proof of (ii) can be completed by the argument in [6] for large t.

We remark that the first result of Theorem 1.1 is due to the nice explicit form of supersolution (2.8). Thus we have a difficulty if we extend our argument to a general bistable reaction-diffusion equation. This generalization of our arguement will be a future problem. On the other hand the second and the third results of Theorem 1.1 can be easily generalized to the reaction-diffusion equation with more general bistable nonlinearity. Indeed our argument does not depend on the explicit form of the standing pulse solution (1.2) but on the existence of the unstable positive eigenfunction with decay at $|x| = \infty$. Moreover the convergence to the diverging fronts as $t \to \infty$ was established by [6] for the general bistable nonlinearity.

We also remark on Theorem 1.2. Matano proved in [10] the existence of an unstable set for an unstable equilibrium provided that the semiflow generated by a parabolic equation is 'strongly' order-preserving and compact. In the present case this condition is not clealy met. Thus we needed our theorem to apply to the present problem. (He also discussed a stable set in [11] when the domain is unbounded).

The readers might suspect that they could prove Theorem 1.2 by using the unstable manifold theorem. A standard way to prove the existence of an unstable set for the equilibrium solution is certainly the application of the unstable manifold theorem in an infinite-dimensional dynamical system. However, to carry out it under the condition in Theorem 1.2, we would be puzzled by the choice of an appropriate phase space for the semiflow. In addition it would need a lengthy preparation for setting up to apply the theorem. Thus it would be helpful if there is a simple alternative argument. The readers will find our simple proof of Theorem 1.2 by the comparison principle though the assertion is only related to the most unstable direction for the positive eigenfunction.

We finally remark on the related work [7] where entire solutions of the KPP-Fisher equation are studied (also see [8]). Their equation is monostable, that is,

$$u_t = u_{xx} + f(u), \quad f(0) = f(1) = 0,\ f(u) > 0\ (0 < u < 1),\ f'(0) > 0, f'(1) < 0.$$

Then the diffusion–free equation has a unique asymptotically stable equilibrium and an unstable one. It is known that under the condition $f(u) \cdot \; f'(0)u$ $(0 \cdot \; u \cdot \; 1)$ there are a family of traveling front solutions connecting $u = 0$ and $u = 1$; namely there are traveling front solutions $U^c(x - ct), U^c(-x - ct)$ with the speed $c, c \geq 2\sqrt{f'(0)}$. One can also see that there is a uniform entire solution $V(t)$ satisfying

$$\lim_{t \to -\infty} V(t) = 0, \quad \lim_{t \to \infty} V(t) = 1.$$

They prove the existence of an entire solution by using a combination of

$$U^{c_1}(x - c_1 t + \xi_1),\ U^{c_2}(-x - c_2 t + \xi_2),\ V(t + \xi_3) \quad \left(c_1, c_2 > 2\sqrt{f'(0)}\right)$$

as $t \to -\infty$. They also show the continuity of the family of the entire solution with respect to ξ_1, ξ_2, ξ_3, c_1 and c_2 in some topology. Their proof for the existence of the entire solution is similar to our proof since they also use the comparison principle with appropriate subsolution and upper estimate. However their argument for the upper estimate is not applicable to the bistable case. Thus we need a new idea of it. Moreover we prove the different type of the entire solution with diverging fronts.

This paper is organized as follows. In the next section we construct the supersolution (1.3) precisely and estimate the asymptotic behavior as $t \to -\infty$. In the third section we prove the existence of the entire solution of (i) in Theorem 1.1. Then the proof of Theorem 1.2 is given in the forth section. In the final section the remaining results of Theorem 1.1 is shown with the aid of Theorem 1.2.

2. Globally Defined Suppersolution and Subsolution

Recall that the equation (1.1) possesses the traveling front solutions connecting between the two equilibrium states $u = 0$ and $u = 1$:

$$\Psi(\pm x, t) := \frac{\exp\{(\pm x - x_0)/\sqrt{2} + \omega t\}}{1 + \exp\{(\pm x - x_0)/\sqrt{2} + \omega t\}} \tag{2.1}$$

where

$$\omega = \frac{1}{2} - a.$$

and the speed of the traveling fronts is given by $c = \sqrt{2}\omega$. Here we fix x_0 as satisfying $\Psi(\pm x_0, 0) = 1/2$. Note that each traveling front solution is unique up to the translation of time or space shift.

As mentioned in the previous section, two facing fronts collide and eventually annihilate in a finite time (for instance see [3], [6], [12] and references therein) and a time-globally defined smooth subsolution which exhibits such a collision and annihilation was found in [12]. We also rewrite the subsolution below

$$\underline{u}(x,t) = \frac{\exp\{(x - x_0)/\sqrt{2} + \omega t\} + \exp\{-(x + x_0)/\sqrt{2} + \omega t\}}{1 + \exp\{(x - x_0)/\sqrt{2} + \omega t\} + \exp\{-(x + x_0)/\sqrt{2} + \omega t\}},$$

which is simply written as

$$\underline{u}(x,t) = 1 - \frac{1}{1 + \psi_1(x,t)}, \tag{2.2}$$

where

$$\psi_1(x,t) := \eta_1 \cosh(x/\sqrt{2}) \exp(\omega t), \qquad \eta_1 := 2\exp(-x_0/\sqrt{2}). \tag{2.3}$$

For convenience sake, we put

$$\mathcal{F}[u] := u_t - u_{xx} - u(u-a)(1-u).$$

Then we have the following lemma.

Lemma 2.1 *The function*

$$u(x,t) = 1 - \frac{1}{1+\psi(x,t)}$$

satisfies

$$\mathcal{F}[u] = \frac{1}{(1+\psi)^2}\left\{\psi_t - \psi_{xx} + a\psi + \frac{2{\psi_x}^2 - \psi^2}{1+\psi}\right\}.$$

Moreover, if $\psi(x,t) = \eta \cosh(x/\sqrt{2})\exp(p(t))$, *then*

$$\mathcal{F}[u] = \frac{\psi}{(1+\psi)^2}\left\{p_t - \omega - \frac{\eta^2\exp(2p)}{\psi(1+\psi)}\right\}. \tag{2.4}$$

Since the computation to verify the lemma is straight forward, we skip it.

Applying (2.2) and (2.3) to Lemma 2.1 immediately yields

$$\mathcal{F}[\underline{u}] = -\frac{\eta_1^2\exp(2\omega t)}{(1+\psi_1(x,t))^3} < 0.$$

Thus $\underline{u}(x,t)$ is a subsolution for any $t \in (-\infty,\infty)$ and it gives a good approximate solution for large $|t|$. In [12] they also construct a supersolution having two fronts, and estimate the annihilation time for the fronts. However, the supersolution in [12] is only defined in a finite time interval.

Now we shall construct a supersolution $\overline{u}(x,t)$ defined for any $t \in (-\infty,\infty)$ which satisfies

$$\lim_{t\to-\infty}\sup_{x\in\mathbb{R}} |\overline{u}(x,t) - \underline{u}(x,t)| = 0,$$

for an appropriate η_1 or x_0. The identity (2.4) suggests us to find the supersolution with the form (1.3). In fact we get to an ordinary differential equation for $p(t)$ in the next lemma.

Lemma 2.2 *Given* $\eta_2 > 0$, *consider the ordinary differential equation*

$$\begin{cases} \dot{p}(t) = \omega + \eta_2\exp(p(t)), \\ p(0) = 0. \end{cases} \tag{2.5}$$

Then the solution $p(t)$ *of* (2.5) *over* $(-\infty, \infty)$ *satisfies*

$$\exp(p(t) - \omega t) = \frac{\omega}{\omega + \eta_2}\Big(1 + O(\exp(\omega t))\Big), \ (t \to -\infty). \tag{2.6}$$

Proof. First notice that the solution $p(t)$ is monotone increasing. We put

$$y(t) = \eta_2 \exp(p(t)).$$

Then $y(t)$ satisfies

$$\begin{cases} \dot{y}(t) = (\omega + y)\, y, \\ y(0) = \eta_2. \end{cases}$$

A simple integration yields

$$y(t) = \frac{c_0 \omega \exp(\omega t)}{1 - c_0 \exp(\omega t)}, \qquad c_0 := \frac{\eta_2}{\omega + \eta_2}, \tag{2.7}$$

which leads to (2.6). ∎

It is easily seen that $y(t)$ is monotone increasing and that the solution exists for $t < t_*$, where

$$t_* := -\frac{\log c_0}{\omega} = \frac{1}{\omega} \log \frac{\omega + \eta_2}{\eta_2} > 0$$

and that $\overline{u}(x,t) \to 1$ as $t \to t_* - 0$. With the aid of $p(t)$ of (2.5) we define

$$\overline{u}(x,t) := \begin{cases} 1 - \dfrac{1}{1 + \psi_2(x,t)}, & (t < t_*), \\ 1 & (t \geq t_*), \end{cases} \tag{2.8}$$

where

$$\psi_2(x,t) := \eta_2 \cosh(x/\sqrt{2}) \exp(p(t)).$$

It follows from Lemma 2.1 that

$$\mathcal{F}[\overline{u}] \ \geq \frac{\psi_2(x,t)}{(1 + \psi_2(x,t))^2} \left\{ \dot{p} - \omega - \frac{\eta_2 \exp(p(t))}{1 + \eta_2 \exp(p(t))} \right\} \geq 0$$

for $t < t_*$.

If $u(x,t)$ is a solution of (1.1) satisfying $u(x,t_0) \cdot \ \overline{u}(x,t_0)$, then

$$u(x,t) \cdot \ \overline{u}(x,t)$$

for $t \geq t_0$. Indeed by the assumption, $u(x,t) \le 1$ for $t \geq t_0$, the inequality holds even if t is greater than t_*.

In the end of this section we show the following lemma:

Lemma 2.3 *Suppose that*

$$\eta_1 = \frac{\omega}{\omega + \eta_2}\eta_2.$$

Then $\overline{u}(x,t) > \underline{u}(x,t)$ *for any* $(x,t) \in \mathbb{R} \times \mathbb{R}$ *and there exists a positive constant* K_1 *such that*

$$\sup_{x \in \mathbb{R}} |\overline{u}(x,t) - \underline{u}(x,t)| \le K_1 \exp(\omega t) \qquad (t \le 0).$$

Proof. By the definition of $\underline{u}$ and $\overline{u}$ we can compute

$$\overline{u}(x,t) - \underline{u}(x,t) = \frac{\cosh(x/\sqrt{2})\exp(\omega t)\{\eta_2 \exp(p(t) - \omega t) - \eta_1\}}{(1 + \psi_1(x,t))(1 + \psi_2(x,t))}.$$

We see from (2.7) that

$$y(t)\exp(-\omega t) = \eta_2 \exp(p(t) - \omega t)$$

is monotone increasing and that by (2.6)

$$\eta_2 \exp(p(t) - \omega t) = \eta_1 + O(\exp(\omega t))$$

as $t \to -\infty$. These facts immediately lead us to the assertion of the lemma. ■

Remark 2.4 We first note that in Lemma 2.3

$$\eta_1 < \eta_2. \tag{2.9}$$

Next we give a remark on the supersolution. Instead of (2.5), we can use a solution of

$$\dot{p} = \omega + \frac{\eta_2 \exp(p(t))}{1 + \eta_2 \exp(p(t))}.$$

Then $\overline{u}$ with this $p(t)$ can be defined globally and it certainly a supersolution. We thereby obtain the same result for the existence of the entire solution proved in the next section by using the new supersolution. Although this supersolution would be helpful in studying of the behavior of the entire solution for large $t > 0$, it suffice to consider the present one for the later arguments. In addition we will use a similar equation to (2.5) in the proof of Theorem 1.2.

3. An Entire Solution with Annihilating Fronts

In this section we show the existence of an entire solution which behaves like two traveling front solutions coming from both sides of the real axis and annihilating in a finite time.

Lemma 3.1 *Let* $\underline{u}(x,t)$ *and* $\overline{u}(x,t)$ *be the subsolution and the supersolution defined by* (2.2) *and* (2.8) *respectively. Assume the condition in Lemma 2.3. Then the equation* (1.1) *has a unique entire solution* $u^*(x,t)$ *satisfying*

$$\underline{u}(x,t) < u^*(x,t) < \overline{u}(x,t), \quad (x,t) \in \mathbb{R}^2. \tag{3.1}$$

It suffices to prove this lemma for the assertion of Theorem 1.1 (i). Indeed since the traveling wave solution of (2.1) can be written as

$$\Psi(x,t) = 1 - \frac{1}{1 + (\eta_1/2)\exp(x/\sqrt{2})\exp(\omega t)},$$

we can easily verify $\overline{u}(x,t) > \Psi(x,t)$ with (2.9) and estimate

$$\begin{aligned}
&\sup_{x\geq 0}(\overline{u}(x,t) - \Psi(x,t)) \\
&= \frac{\exp(x/\sqrt{2})\exp(\omega t)}{2\{1 + (\eta_1/2)\exp(x/\sqrt{2})\exp(\omega t)\}(1+\psi_2)}\left\{\eta_2\exp(p(t)-\omega t) - \eta_1\right\} \\
&\quad + \frac{\exp(-x/\sqrt{2})\eta_2\exp(p(t)-\omega t)}{2\{1 + (\eta_1/2)\exp(x/\sqrt{2})\exp(\omega t)\}(1+\psi_2)}\exp(\omega t) \\
&\cdot \; K\exp(\omega t)
\end{aligned}$$

for $t \cdot 0$ as in the proof of Lemma 2.3. Hence from replacing x by $x-(x_1+x_2)/2$ and choosing $x_0 = (x_2-x_1)/2$ in Lemma 3.1, Theorem 1.1 (i) immediately follows.

We prove the lemma in the rest of this section. Denote by $u(x,t;t_0,v_0)$ a solution to (1.1) with the initial condition

$$u(x,t_0;t_0,v_0) = v_0(x), \qquad v_0 \in C^0(\mathbb{R}), \quad 0 \cdot v_0(x) \cdot 1.$$

Define the mapping $S(t) : C^0(\mathbb{R}) \to C^0(\mathbb{R})$ for $t \geq 0$ by

$$[S(t)v_0](x) := u(x,t;0,v_0).$$

We often use the notation $u(\cdot) \cdot v(\cdot)$ for $u(x) \cdot v(x),\ x \in \mathbb{R}$. We note

$$u(\cdot,t;t_0,v_0) = S(t-t_0)v_0.$$

Thus it follows from the maximum principle and the inequalities

$$\underline{u}(\cdot,t_0) \cdot v_0(\cdot) \cdot \overline{u}(\cdot,t_0)$$

that

$$\underline{u}(\cdot, t) \cdot \;\; S(t-t_0)v_0 \cdot \;\; \overline{u}(\cdot, t).$$

First we observe some continuity of $S(t)$.

Lemma 3.2 *Given a positive function $w(x)$ satisfying $\lim_{|x|\to\infty} w(x) = 0$, consider a set*

$$W := \{u \in C^0(\mathbb{R}) : |u(x) - 1| \cdot \;\; w(x) \;\; (x \in \mathbb{R})\},$$

and assume that $S(t)W \subset W$ $(t \geq 0)$. Let $\|u\|_{C^0} := \sup_{x\in\mathbb{R}} |u(x)|$. Then for any $u_0 \in W$ the map $t \mapsto S(t)u_0$ is continuous from $[0, \infty)$ into $C^0(\mathbb{R})$ equipped with norm $\|\cdot\|_{C^0}$. Moreover if $\{u_n\} \subset C^0(\mathbb{R})$ converges to v with respect to $\|\cdot\|_{C^0}$, then given $T > 0$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|S(t)u_n - S(t)v\|_{C^0} = 0.$$

Proof. Since $u_0 \in W$ is equicontinuous, we easily obtain the first assertion. We prove the second one. Set $u_i := S(t)u_{0i}$ $(i = 1, 2)$. Using the heat kernel

$$K(x, t) = \frac{1}{2\sqrt{\pi t}} \exp(-|x|^2/4\pi t)$$

we can write

$$u_i(x,t) = \int_{-\infty}^{\infty} K(x-y, t)u_{0i}(y)dy + \int_0^t ds \int_{-\infty}^{\infty} K(x-y, t-s)g(u_i(y, s))dy,$$

for $t \geq 0$ $(i = 1, 2)$, where we put $g(u) = u(u-a)(1-u)$. Then we obtain

$$|u_1(x, t) - u_2(x,t)| \cdot \;\; \sup_{y\in\mathbb{R}} |u_{01}(y) - u_{02}(y)| + L \int_0^t \sup_{y\in\mathbb{R}} |u_1(y, s) - u_2(y, s)|ds,$$

where

$$L := \sup_{0 \;\; u \;\; 1} |g_u(u)|.$$

Applying the Gronwall inequality yields

$$\sup_{y\in\mathbb{R}} |u_1(y, t) - u_2(y, t)| \cdot \;\; \sup_{y\in\mathbb{R}} |u_{01}(y) - u_{02}(y)| \exp(Lt), \;\; t \geq 0.$$

This implies the desired continuity. ■

We also need the following lemma for the proof of the uniqueness.

Lemma 3.3 *For any positive integer n, there exists a positive number ϵ_n such that*

$$\overline{u}(x,-n) \cdot \quad \underline{u}(x,-n+\epsilon_n)$$

for all $x \in \mathbb{R}$ and

$$\lim_{n\to\infty} \epsilon_n = 0$$

Proof. We first compute

$$\begin{aligned} \underline{u}(x,t') - \overline{u}(x,t) &= \frac{\eta_1 \exp(\omega t') \cosh(x/\sqrt{2})}{1+\eta_1 \exp(\omega t') \cosh(x/\sqrt{2})} - \frac{\eta_2 \exp p(t) \cosh(x/\sqrt{2})}{1+\eta_2 \exp p(t) \cosh(x/\sqrt{2})} \\ \text{(3.2)} \qquad &= \frac{(\eta_1 \exp(\omega t') - \eta_2 \exp p(t)) \cosh(x/\sqrt{2})}{(1+\eta_1 \exp(\omega t') \cosh x/\sqrt{2})(1+\eta_2 \exp p(t) \cosh x/\sqrt{2})}. \end{aligned}$$

Let

$$\epsilon_n := \frac{p(-n)+\omega n}{\omega} + \frac{1}{\omega}\log\frac{\eta_2}{\eta_1}.$$

Then by Lemmas 2.2 and 2.3, we have $\epsilon_n > 0$ and

$$\lim_{n\to\infty} \epsilon_n = 0.$$

Substituting $t' = -n+\epsilon_n$ and $t = -n$ into (3.2) yields $\underline{u}(x,-n+\epsilon_n) \geq \overline{u}(x,-n)$. ■

Now we are in the position to prove Lemma 3.1. Set

$$v_n(\cdot) = S(n)\underline{u}(\cdot,-n), \quad w_n(\cdot) = S(n)\overline{u}(\cdot,-n), \quad n = 0,1,2\ldots.$$

We show that the sequences $\{v_n\}$ and $\{w_n\}$ are monotone increasing and decreasing respectively. Since $\underline{u}(x,t)$ is a subsolution, we see

$$\underline{u}(\cdot,-n) < S(1)\underline{u}(\cdot,-(n+1)).$$

Thus applying the maximum principle yields

$$v_n = S(n)\underline{u}(\cdot,-n) < S(n)S(1)\underline{u}(\cdot,-(n+1)) = v_{n+1},$$

which implies the monotonicity of $\{v_n\}$. We can also see that $\{w_n\}$ is monotone decreasing in n.

Since

$$0 \le v_n(x) \le w_0(x) = \overline{u}(x,0),$$

there is a function $v^* \in C^0(\mathbb{R})$ such that v_n uniformly converges to v^*. Thus $u^*(\cdot,t) := S(t)v^*$ is a solution for $t \geq 0$.

We show that $u^*(x,t)$ is defined for any $t \le 0$. Given $T > 0$, take an integer n_1 so that $T < n_1$. We fix T and n_1. Then, for $n \geq n_1$, we have

$$S(-T)v_n = S(-T)S(n)\underline{u}(\cdot,-n) = S(n-T)\underline{u}(\cdot,-n).$$

Since

$$S(n+1-T)\underline{u}(\cdot,-(n+1)) = S(n-T)S(1)\underline{u}(\cdot,-(n+1)) > S(n-T)\underline{u}(\cdot,-n),$$

the sequence $\{S(n-T)\underline{u}(\cdot,-n)\}_{n\geq n_1}$ is monotone increasing. Thus the argument similar to the above implies that there is a continuous function v^T to which $S(n-T)\underline{u}(\cdot,-n)$ converges uniformly. In terms of the fact

$$v^* = \lim_{n\to\infty} S(T)S(-T)v_n = S(T)v^T,$$

we obtain $v^T = S(-T)v^*$. Since $T > 0$ is arbitrary number, we can conclude that $u^*(\cdot,t) := S(t)v^*$ is a defined globally for $t \in \mathbb{R}$.

Next we show the uniqueness of such entire solutions between the supersolution and the subsolution. Let $u_1^*(x,t), u_2^*(x,t)$ be two entire solutions such that

$$\underline{u}(x,t) \le u_j^*(x,t) \le \overline{u}(x,t) \qquad (j = 1,2),$$

for all $(x,t) \in \mathbb{R}^2$. By Lemma 3.3,

$$\underline{u}(x,-n) \le u_1^*(x,-n) \le \overline{u}(x,-n) \le \underline{u}(x,-n+\epsilon_n) \le u_2^*(x,-n+\epsilon_n).$$

The maximum principle implies that

$$u_1^*(x,t) \le u_2^*(x,t+\epsilon_n)$$

for $t \geq -n$. We similarly obtain

$$u_2^*(x,t) \le u_1^*(x,t+\epsilon_n)$$

for $t \geq -n$. Specifically

$$u_2^*(x,-\epsilon_n) \le u_1^*(x,0) \le u_2^*(x,\epsilon_n)$$

holds for large n. It follows from Lemma 3.3 that $u_1^*(x,0) \equiv u_2^*(x,0)$ for all $x \in \mathbb{R}$. We thereby assert that the entire solution satisfying (3.1) is unique. This concluded the proof of Theorem 1.1. ■

4. Proof of Theorem 1.2

First notice that for the function $f(x, u)$ in Theorem 1.2 there are positive constants M, δ such that

$$|f_u(x, v(x)) - f_u(x, v(x) + w)| \cdot \; M|w|, \qquad |w| < \delta. \tag{4.1}$$

We only prove the existence of the solution $u^+(x, t)$ in the theorem since the other case can be easily obtained by reversing the signs.

We consider a couple of ordinary differential equations

$$\begin{cases} \dot{p_1}(t) = \mu - \epsilon M \exp(p_1(t)), & p_1(0) = -p_1^0, \\ \dot{p_2}(t) = \mu + \epsilon M \exp(p_2(t)), & p_2(0) = -p_2^0. \end{cases} \tag{4.2}$$

With the conditions

$$p_1^0 > \log\left(\frac{\epsilon M}{\mu}\right), \qquad p_2^0 > 0, \tag{4.3}$$

there are solutions

$$\begin{aligned} p_1(t) &= \mu t - \log\left\{\exp(p_1^0) - \frac{\epsilon M(1 - \exp(\mu t))}{\mu}\right\}, \qquad t \cdot \; 0 \\ p_2(t) &= \mu t - \log\left\{\exp(p_2^0) + \frac{\epsilon M(1 - \exp(\mu t))}{\mu}\right\}, \qquad t \cdot \; 0, \end{aligned}$$

which can be easily obtained by integration of the equations. We also see that both of $p_1(t)$ and $p_2(t)$ are monotone increasing. With the aid of these solutions we have the following lemma.

Lemma 4.1 *Assume the hypotheses in Theorem 1.2 and let M and δ are constants in* (4.1). *Set*

$$u_i(x, t) := v(x) + \epsilon\varphi(x)\exp(p_i(t)) \quad (i = 1, 2),$$

where $p_1(t)$ and $p_2(t)$ $(t \cdot \; 0)$ are the solutions to (4.2) *with* (4.3). *Let the eigenfunction $\varphi(x) > 0$ be normalized so that*

$$\sup_{x \in \mathbb{R}^N} \varphi(x) = 1.$$

Then $u_1(x, t)$ (resp. *$u_2(x, t)$*) *is a subsolution* (resp. *supersolution*) *of* (1.4) *for $0 < \epsilon < \delta$ and $t \in (-\infty, 0]$. Moreover, if p_i^0, $(i = 1, 2)$ are chosen as satisfying*

$$\exp(p_1^0) - \exp(p_2^0) = \frac{2\epsilon M}{\mu}, \tag{4.4}$$

then

$$0 \cdot \; u_2(x,t) - u_1(x,t) \cdot \; K_2 \exp(\mu t), \quad (x,t) \in \mathbb{R}^N \times (-\infty, 0] \tag{4.5}$$

where K_2 *is a positive constant.*

Proof. We can compute

$$\begin{aligned}
&(u_1)_t - \triangle u_1 - f(x,u_1) \\
&\quad = \epsilon\varphi \exp(p_1(t))\dot{p}_1 - \epsilon\triangle\varphi \exp(p_1(t)) + f(x,v) - f(x, v + \epsilon\varphi \exp(p_1(t))) \\
&\quad \cdot \; \epsilon\varphi \exp(p_1(t)) \{\dot{p}_1 - \mu + \epsilon M \exp(p_1(t))\} = 0 \quad (t \cdot \; 0).
\end{aligned}$$

This implies that u_1 is a subsolution of (1.4) for $t \cdot \; 0$. Similarly we can check that u_2 is a supersolution. Moreover under the condition (4.4) we obtain

$$\lim_{t \to -\infty} |p_1(t) - \mu t| = \lim_{t \to -\infty} |p_2(t) - \mu t|.$$

Thus the first inequality of (4.5) follows from $\dot{p}_1 < \dot{p}_2$.

The second inequality of (4.5) follows from the similar computation found in the proof of Lemmas 2.2 and 2.3. ■

Proof of Theorem 1.2 With the new variable $U = u - v(x)$ we write the equation as

$$U_t = \Delta U + \tilde{F}(x,U), \qquad \tilde{F}(x,U) := F(x, v(x) + U) - F(x, v(x)).$$

Set

$$W := \{U \in C^0(\mathbb{R}^N) : |U(x)| \cdot \; \varphi(x) \ (x \in \mathbb{R}^N)\},$$

and let $U(x,t;U_0)$ be a solution with $U(x,0;U_0) = U_0$. Then for the map $t \mapsto S(t)U_0 := U(\cdot, t; U_0)$, $U_0 \in W$ the assertion of Lemma 3.2 holds. Thus by virtue of the above lemma we can apply the same argument as in the proof of Theorem 1.1 (i) to obtain the desired solution. This concludes the proof of Theorem 1.2. ■

5. An Entire Solution with Diverging Fronts

In this section we give the proof of Theorem 1.1 (ii) and (iii), using the result of Theorem 1.2.

By the translation invariance of the equation we may assume $\tilde{x} = 0$. Thus we consider the linearized eigenvalue problem

$$\varphi_{xx} + f_u(v(x))\varphi = \mu\varphi, \tag{5.1}$$

in $L^2(\mathbb{R})$, where we put $f(u) = u(u-a)(1-u)$. Since $(\varphi, \mu) = (v_x(x), 0)$ meets (5.1) and $v_x(x)$ changes the sign, we can assert that there exist a positive eigenvalue

μ and the corresponding positive eigenfunction $\varphi(x)$. Moreover since $v(x)$ decay exponentially as $|x| \to \infty$, we can see that as $|x| \to \infty$, $\varphi(x)$ asymptotically satisfies

$$\varphi_{xx} + f_u(0)\varphi = \varphi_{xx} - a\varphi = \mu\varphi.$$

Thus

$$\varphi(x) = O(\exp(-\sqrt{a+\mu}|x|)), \quad |x| \to \infty \tag{5.2}$$

(see [9, Section 5.4]). Hence we can apply Theorem 1.2 to obtain solutions $u^{\pm}(x,t)$ defined in $(-\infty, 0]$. Since these solutions are extended forward in time, it turns out that both solutions are entire solutions. It therefore suffices to prove that they satisfies the asymptotic behaviors as $t \to \infty$.

By virtue of (5.2) $u^{-}(x,t)$ is positive and monotone decreasing in time by the comparison theorem (indeed Lemma 4.1 shows $u = v(x) - \epsilon\varphi(x)\exp(p_1(t))$ is a supersolution). The solution $u^{-}(x,t)$ converges to $u = 0$ since there is no equilibrium solution between $u = v(x)$ and $u = 0$ and the convergence is uniform. This implies the result (iii) of the theorem.

Next we consider $u^{+}(x,t)$. Recall the result by [6] (or [4]). Lemma 6.1 in [6] tells that there are positive numbers ξ, β and q_0 with $q_0 < a$ such that

$$u^{+}(x,t) < \Psi(x-\xi,t) + \Psi(-x-\xi,t) - 1 + q_0\exp(-\beta t), \quad t \geq 0.$$

Thus for any fixed $t \geq 0$,

$$\limsup_{x\to\pm\infty} u^{+}(x,t) < a$$

holds. On the other hand we see that $u^{+}(x,t)$ is monotone increasing (see Lemma 4.1 again). We can assert that u^{+} converges to $u = 1$ uniformly in any compact interval. Thus for sufficiently large time the condition of Theorem 3.2 in [6] are enjoyed so that we obtain the desired convergence result as $t \to \infty$. This concludes the rest of the proof of Theorem 1.1. ∎

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References

1. D. G. Aronson and H. F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in *Partial Differential Equations and Related Topics*, ed. J. A. Goldstein Lecture Notes in Mathematics, **446**, 5-49, Springer 1975.

2. D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.* **30(1)** (1978), 33-76.

3. X. Chen, Generation and propagation of interfaces for reaction-diffusion equations, *J. Differential Equations* **96** (1992), 116-141.

4. X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in non-local evolution equations, *Adv. in Differential Equations*, **2** (1997), 125-160.

5. J.-P. Eckmann and J. Rougemont, Coarsening by Ginzburg-Landau dynamics, *Commun. Math. Phys.* **199** (1998), 441-470.

6. P. C. Fife and J. B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Ration. Mech. Anal.* **65(4)** (1977), 335-361.

7. F. Hamel and N. Nadirashvili, Entire solutions of the KPP equation, *Comm. Pure Appl. Math.* **52** (1999), 1255-1276.

8. F. Hamel and N. Nadirashvili, Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$. *Arch. Ration. Mech. Anal.* **157(2)** (2001), 91-163.

9. D. Henry, *Geometric Theory of Semilinear Parabolic Equations,* Springer-Verlag, New York, 1981.

10. H. Matano, Existence of nontrivial unstable sets for equilibriums of strongly order-preserving systems, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **30(3)** (1984), 645-673.

11. H. Matano, L^∞ stability of an exponentially decreasing solution of the problem $\Delta u + f(x,u) = 0$ in R^n. *Japan J. Appl. Math.* **2(1)** (1985), 85-110.

12. Y. Morita and Y. Mimoto, Collision and collapse of layers in a 1D scalar reaction-diffusion equation, *Physica D* **140** (2000), 151-170.

13. H. Yagisita, Backward global solutions characterizing annihilation dynamics of travelling fronts, *Publ. Res. Inst. Math. Sci.* **39** (2003), 117-164.

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