TAIWANESE JOURNAL OF MATHEMATICS
Vol. 7, No. 3, pp. 423-432, September 2003
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ESTIMATES OF AN INTEGRAL OPERATOR ON FUNCTION SPACES

Der-Chen Chang† and Stevo Stević

Abstract. In this paper, we shall study the family of operators of the form

$$T_{\vec{g}}(f)(z) = \int_0^{z_1} \cdots \int_0^{z_n} f(\zeta_1, \cdots, \zeta_n) \prod_{j=1}^{n} g_j'(\zeta_j) d\zeta_j$$

on Hardy $H^p(D_n)$, the generalized weighted Bergman $\mathcal{A}_\mu^{p,q}(D_n)$, $p \in (0, \infty)$, and α-Bloch $\mathcal{B}^\alpha(D_n)$ spaces on the polydisk $D_n = \{(z_1, \ldots, z_n) \in \mathbf{C}^n : |z_j| < 1,\ j = 1, \ldots, n\}$.

1. Introduction and preliminaries

Let D be the unit disk in the complex plane $\mathbf{C}$ and $H(D)$ be the set of all analytic functions $f : D \to \mathbf{C}$. The Bloch space $\mathcal{B}$ is the space of all analytic functions f on D such that

$$b(f) = \sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty.$$

Let $\mathcal{B}_{\mathcal{S}}$ denote a subspace of the Bloch space that consists of all analytic functions f on D such that

$$\|f\|_{\mathcal{B}_{\mathcal{S}}} = \sup_{z \in D} |1 - z|\ |f'(z)| < \infty.$$

In the article [1], Aleman and Siskakis studied operators of the form

$$T_g(f)(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta,$$

Received November 14, 2002.
Communicated by S. B. Hsu.
1991 *Mathematics Subject Classification*: 47B38,46E20.
Key words and phrases: Analytic function, Hardy space, Bergman space, Bloch space, integral operator, polydisk.

on weighted Bergman spaces

$$\mathcal{A}^p_\omega = \left\{ f \in H(D) \mid \int_D |f(z)|^p \omega(z) dm(z) < \infty \right\},$$

with $\omega(r)$ other than the standard radial weight $(1-r)^\alpha$. Recently there has been great deal of interest in studying the weighted Bergman spaces with weights other than the standard (see, for example, [1, 4, 11, 12, 18, 20, 21, 23] and the references therein).

The following theorem was proved by Aleman and Siskakis in [1]:

Theorem 1.1. *Let ω be a positive radial weight function on the unit disc and there is a constant C such that*

$$\omega(r) \geq \frac{C}{1-r} \int_r^1 \omega(s) ds, \qquad 0 < r < 1.$$

If $g \in \mathcal{B}$, then T_g is bounded on $\mathcal{A}^p_\omega$ and $||T_g||_{op} \cdot\ C(p)\|g\|_{\mathcal{B}}$ for $p \geq 1$. Here $\|T_g\|_{op}$ is the operator norm of the operator T_g.

Motivated by this theorem, we define and study a family of integral operators $T_{\vec{g}}$, on the polydisk D_n. The operators are defined by

$$T_{\vec{g}}(f)(z) = \int_0^{z_1} \cdots \int_0^{z_n} f(\zeta_1, ..., \zeta_n) \prod_{j=1}^n g'_j(\zeta_j) d\zeta_j,$$

whenever $f(z) = \sum_{|\alpha|=0}^\infty a_\alpha z^\alpha$ is an analytic function on D_n (α is multi-index from $(\mathbf{Z}_+)^n$). Here g_j, $j = 1, ..., n$, are analytic functions on the unit disk. It is easy to see that

$$(1) \qquad T_{\vec{g}}(f)(z) = \prod_{j=1}^n z_j \int_0^1 \cdots \int_0^1 f(\tau_1 z_1, ..., \tau_n z_n) \prod_{j=1}^n g'_j(\tau_j z_j) d\tau_j.$$

If $g_j(\zeta_j) = \ln(1/(1-\zeta_j))$, $j = 1, ..., n$, then $T_{\vec{g}}(f)$ is a natural generalization of the Cesàro operator $\mathcal{C}$ on the unit disk:

$$\vec{\mathcal{C}}(f)(z) = \prod_{j=1}^n z_j \int_0^1 \cdots \int_0^1 f(\tau_1 z_1, \ldots, \tau_n z_n) \prod_{j=1}^n (1 - \tau_j z_j)^{-1} d\tau_1 \cdots d\tau_n.$$

The Cesàro operator on the unit disk has been studied by many mathematicians (see, for example [1, 2, 5, 7, 8, 9, 13, 14, 15, 16, 17, 19, 24, 25] and the references therein). In this paper we continue our investigations of some integral operators defined on analytic functions on the polydisk which were started in the articles [3], [4] and [22].

If $g_j(\zeta_j) = \zeta_j$, $j = 1, ..., n$, then $T_{\vec{g}}(f)$ is the integration operator.

In what follows, we write $z \cdot w$ as an abbreviation for $(z_1 w_1, ..., z_n w_n)$ for $z, w \in \mathbf{C}^n$; $e^{i\theta}$ is an abbreviation for $(e^{i\theta_1}, ..., e^{i\theta_n})$; $d\tau = d\tau_1 \cdots d\tau_n$; $d\theta = d\theta_1 \cdots d\theta_n$ and r, s, τ are vectors in $\mathbf{C}^n$. We write $0 \cdot\ \ r < 1$, where $r = (r_1, ..., r_n)$ it means $0 \cdot\ \ r_j < 1$ for $j = 1, ..., n$.

Our first result is:

Theorem 1.2. *If $g_j \in \mathcal{B}_{\mathcal{S}}$, $j = 1, ..., n$, then there is a constant C depending only on p and n, such that*

$$\int_{[0,2\pi]^n} |T_{\vec{g}}(f)(r \cdot e^{i\theta})|^p d\theta \cdot \ \ C \prod_{j=1}^{n} r_j^p ||g_j||_{\mathcal{B}_{\mathcal{S}}}^p \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta,$$

for $0 < p < \infty$ and for all $f \in H(D_n)$.

The proof of Theorem 1.2 for the case $0 < p \cdot\ \ 1$ relies on Theorem 1.5 in [22]. In the proof of this theorem we use Miao's arguments [13], which are modifications of the corresponding arguments used in the case of the unit disk. Miao's ideas were originated from Hardy and Littlewood [10]. We shall give detailed discussion later.

In order to prove Theorem 1.2, we need three auxiliary results which are incorporated in the following lemmas.

For real y and $\sigma > -1$, set

$$H^\sigma(y) = \frac{1}{1 + |y|} \begin{cases} 1 + |y|^\sigma, & \text{if } \sigma < 0 \\ \log(2 + 1/|y|), & \text{if } \sigma = 0 \\ 1, & \text{if } \sigma > 0. \end{cases}$$

Lemma 1.3. *[2] For $\sigma > -1$, there is a constant $C = C(\sigma)$ such that*

$$\int_0^1 \frac{x^{\sigma+1} dx}{[x^2 + \varphi^2][x^2 + \theta^2]^{(\sigma+1)/2}} \cdot \ \ C \frac{H^\sigma(\varphi/\theta)}{|\theta|}$$

for all real φ and $\theta \neq 0$.

For any measurable function $g(e^{i\theta})$, define $E_s g(e^{i\theta}) = E_{s_1,...,s_n} g(e^{i\theta})$ by

$$E_s g(e^{i\theta}) = \begin{cases} g(e^{i(s+1)\theta}), & \text{if } |s_j \theta_j| \cdot\ \ \pi \text{ for all } j \in \{1, ..., n\}, \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma is a generalization of Lemma 2.2 in [2].

Lemma 1.4 *Let* $\sigma_j > -1$, $j = 1, .., n$, $1 < p < \infty$ *and*

$$A_{\vec{\sigma},p} = 2^{n/p} \int_{\mathbf{R}^n} \prod_{j=1}^{n} \frac{H^{\sigma_j}(s_j)}{|s_j+1|^{1/p}} ds.$$

Then $A_{\vec{\sigma},p} < \infty$ *and*

$$\int_{[-\pi,\pi]^n} \left(\int_{\mathbf{R}^n} \prod_{j=1}^{n} H^{\sigma_j}(s_j) E_s g(e^{i\theta}) ds \right)^p d\theta \cdot \ A^p_{\vec{\sigma},p} \int_{[-\pi,\pi]^n} g^p(e^{i\theta}) d\theta,$$

for all measurable $g \geq 0$.

Proof. The first assertion of Lemma 1.4 can be easily proved. Let $H^{\sigma}(s) = \prod_{j=1}^{n} H^{\sigma_j}(s_j)$. By Minkowski's inequality we obtain

$$\left(\int_{[-\pi,\pi]^n} \left(\int_{\mathbf{R}^n} H^{\sigma}(s) E_s g(e^{i\theta}) ds \right)^p d\theta \right)^{1/p} \tag{2}$$
$$\cdot \ \int_{\mathbf{R}^n} H^{\sigma}(s) \left(\int_{[-\pi,\pi]^n} [E_s g(e^{i\theta})]^p d\theta \right)^{1/p} ds.$$

On the other hand, since for real b, $\min\{|b+1|, |(b+1)/b|\} \cdot \ 2$, for $s_j \neq -1$, $j = 1, ..., n$, we obtain

$$\begin{aligned} & \int_{[-\pi,\pi]^n} [E_s g(e^{i\theta})]^p d\theta \\ = & \int_{ {}^{n}_{j=1}\{\theta_j \,:\, |s_j\theta_j| \ \ \pi\} \cap \{\theta_j \,:\, |\theta_j| \ \ \pi\}} g^p(e^{i(s+1)\theta}) d\theta \\ = & \prod_{j=1}^{n} \frac{1}{|s_j+1|} \int_{ {}^{n}_{j=1}\{\varphi_j \,:\, |s_j\varphi_j| \ \ |s_j+1|\pi\} \cap \{\varphi_j \,:\, |\varphi_j| \ \ |s_j+1|\pi\}} g^p(e^{i\varphi}) d\varphi \end{aligned} \tag{3}$$

$$\begin{aligned} \cdot \ & \prod_{j=1}^{n} \frac{1}{|s_j+1|} \int_{ {}^{n}_{j=1}\{|\varphi_j| \ \ 2\pi\}} g^p(e^{i\varphi}) d\varphi \\ = \ & 2^n \prod_{j=1}^{n} \frac{1}{|s_j+1|} \int_{[-\pi,\pi]^n} g^p(e^{i\varphi}) d\varphi. \end{aligned} \tag{4}$$

From (2) and (4) the result

$$\int_{[-\pi,\pi]^n} \left(\int_{\mathbf{R}^n} \prod_{j=1}^{n} H^{\sigma_j}(s_j) E_s g(e^{i\theta}) ds \right)^p d\theta \cdot \ A^p_{\vec{\sigma},p} \int_{[-\pi,\pi]^n} g^p(e^{i\theta}) d\theta,$$

follows immediately.

2. Proof of Theorem 1.2

Now we are in a position to prove Theorem 1.2.

Proof. **Case 1.** $0 < p \cdot 1$. Let $f \in H(D_n)$ and denote

$$I = \left(\prod_{j=1}^{n} r_j^p\right)^{-1} M_p^p(T_{\vec{g}}(f), r) = \left(\prod_{j=1}^{n} r_j^p\right)^{-1} \int_{[0,2\pi]^n} |T_{\vec{g}}(f)(r \cdot e^{i\theta})|^p d\theta.$$

Since $g_j \in \mathcal{B}_S, j = 1, ..., n$, and by Theorem 1.5 in [22] for case $\vec{\gamma} = \vec{0}$, one has

$$\begin{aligned} I &\cdot \int_{[0,2\pi]^n} \left(\int_{[0,1)^n} |f(\tau \cdot r \cdot e^{i\theta})| \prod_{j=1}^{n} |g'(\tau_j r_j e^{i\theta_j})| d\tau \right)^p d\theta \\ &\cdot \prod_{j=1}^{n} ||g_j||_{\mathcal{B}_S}^p \int_{[0,2\pi]^n} \left(\int_{[0,1)^n} \frac{|f(\tau \cdot r \cdot e^{i\theta})|}{\prod_{j=1}^{n} |1 - \tau_j r_j e^{i\theta_j}|} d\tau \right)^p d\theta \\ &\cdot C \prod_{j=1}^{n} ||g_j||_{\mathcal{B}_S}^p \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta, \end{aligned}$$

for which the result follows.

Case 2. $1 < p < \infty$. Let $f \in H(D_n)$ and $0 < r < 1$, set $f_r(e^{i\varphi}) = f(r \cdot e^{i\varphi})$. Then for $0 < \tau < 1$, $f(\tau \cdot r \cdot e^{i\theta})$ is given by the following integral

$$f(\tau \cdot r \cdot e^{i\theta}) = \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} f_r(e^{i\varphi}) \prod_{j=1}^{n} P(\tau_j, \varphi_j - \theta_j) d\varphi \tag{5}$$

where $P(\rho, \phi)$ is the Poisson kernel *i.e.,*

$$P(\rho, \phi) = \frac{1 - \rho^2}{1 - 2\rho \cos \phi + \rho^2}.$$

Combining (1) and (5) and using Fubini's theorem, we obtain

$$T_{\vec{g}}(f)(r \cdot e^{i\theta}) = \prod_{j=1}^{n} \frac{z_j}{2\pi} \int_{[-\pi,\pi]^n} K_r^{\vec{g}}(\theta, \varphi) f_r(e^{i(\theta+\varphi)}) d\varphi,$$

where

$$K_r^{\vec{g}}(\theta, \varphi) = \prod_{j=1}^{n} \int_0^1 \frac{(1 - \tau_j^2) g_j'(r_j \tau_j e^{i\theta_j})}{(1 - 2\tau_j \cos \varphi_j + \tau_j^2)} d\tau_j.$$

Since $g_j \in \mathcal{B}_\mathcal{S},\ j = 1, ..., n$, we obtain

$$|K_r^{\vec{g}}(\theta, \varphi)| \cdot \prod_{j=1}^{n} ||g_j||_{\mathcal{B}_\mathcal{S}} \int_0^1 \frac{(1-\tau_j^2)}{(1-2\tau_j \cos\varphi_j + \tau_j^2)|1 - r_j\tau_j e^{i\theta_j}|} d\tau_j.$$

Using an estimate in [26, p. 96], we have that there is a constant $C = C(\vec{g})$ such that

$$|K_r^{\vec{g}}(\theta, \varphi)| \cdot \ C\prod_{j=1}^{n} \int_0^1 \frac{x dx}{[x^2+\varphi_j^2][x^2+\theta_j^2]^{1/2}}$$

for $|\theta_j| \cdot \ \pi,\ |\phi_j| \cdot \ \pi,\ j = 1, ..., n$. Thus, by Lemma 1.3, we obtain

$$|K_r^{\vec{g}}(\theta, \varphi)| \cdot \ C\prod_{j=1}^{n} \frac{H^0(\varphi_j/\theta_j)}{|\theta_j|}$$

for $0 < |\theta_j| \cdot \ \pi,\ |\phi_j| \cdot \ \pi,\ 0 < r < 1$. Hence

$$\begin{aligned} |T_{\vec{g}}(f)(r \cdot e^{i\theta})| \ &\cdot \ C\int_{[-\pi,\pi]^n} \prod_{j=1}^{n} \frac{H^0(\varphi_j/\theta_j)}{|\theta_j|} |f_r(e^{i(\theta+\varphi)})| d\varphi \\ &= C\int_{\mathbf{R}^n} \prod_{j=1}^{n} H^0(s_j) E_s|f_r|(e^{i\theta}) ds. \end{aligned} \tag{5}$$

From this, using Lemma 1.4 and 2π periodicity of the subintegral function in θ_j, $j = 1, ..., n$, the result follows.

Remark 1. Throughout the above proof C denotes a constant which may change from line to line.

The Hardy space $H^p(D_n)$ $(0 < p < \infty)$ is defined on D_n as follows:

$$H^p(D_n) = \left\{ f \in H(D_n) \ \ : \ \ ||f||_{H^p(D_n)} = \sup_{0\ \ r<1} \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta < \infty \right\}.$$

From Theorem 1.2 we obtain the following corollaries.

Corollary 2.1. *If $g_j \in \mathcal{B}_\mathcal{S},\ j = 1, ..., n$, then the operator $T_{\vec{g}}$ is bounded on $H^p(D_n)$ for $0 < p < \infty$. Moreover,*

$$\|T_{\vec{g}}(f)\|_{H^p(D_n)} \cdot \ C\prod_{j=1}^{n} \|g_j\|_{\mathcal{B}_\mathcal{S}} \|f\|_{H^p(D_n)}.$$

In particular, the Cesàro operator is bounded on the spaces $H^p(D_n)$ for $0 < p < \infty$.

Given $0 < p, q < \infty$, and positive Borel measures μ_j, $j = 1, ..., n$ on $r_j \in (0,1)$, the weighted space $\mathcal{A}^{p,q}_\mu(D_n)$ consists of those functions f analytic on D_n for which

$$\|f\|_{\mathcal{A}^{p,q}_\mu(D_n)} = \left[\int_{[0,1)^n} \left(\int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta\right)^{\frac{q}{p}} \prod_{j=1}^n d\mu_j(r_j)\right]^{1/q} < \infty.$$

Of particular interest are the absolutely continuous measures of the form $d\mu_j(r_j) = (1 - r_j)^a r_j^b dr_j$. When $a = b = 0$ and $p = q$, the space $A^{p,q}_\mu(D_n)$ is the standard Bergman space $A^p(D_n)$.

Corollary 2.2 *If $g_j \in \mathcal{B}_\mathcal{S}$, $j = 1, ..., n$, then the operator $T_{\vec{g}}$ is bounded on $\mathcal{A}^{p,q}_\mu(D_n)$ for $0 < p, q < \infty$. Moreover, there is a constant C depending only on p and n, such that*

$$||T_{\vec{g}}(f)||_{\mathcal{A}^{p,q}_\mu(D_n)} \cdot \;\; C \prod_{j=1}^n ||g_j||_{\mathcal{B}_\mathcal{S}} \; ||f||_{\mathcal{A}^{p,q}_\mu(D_n)}.$$

In particular, the Cesàro operator is bounded on the spaces $\mathcal{A}^{p,q}_\mu(D_n)$ for $0 < p, q < \infty$.

3. Some invariant spaces of the operator $T_{\vec{g}}$

The α-Bloch space $\mathcal{B}^\alpha(D_n)$ is the space of all analytic functions f on D_n such that

$$b_\alpha(f) = \max_{j=1,...,n} \sup_{z \in D_n} (1 - |z_j|^2)^\alpha \left|\frac{\partial f}{\partial z_j}(z)\right| < \infty.$$

We denote $\mathcal{S}_{\vec{\alpha}}$ the space of all analytic functions f on D_n such that

$$N(f)_{\mathcal{S}_{\vec{\alpha}}} = \sup_{z \in D_n} |f(z)| \prod_{j=1}^n (1 - |z_j|)^{\alpha_j} < \infty,$$

where $\vec{\alpha} = (\alpha_1, ..., \alpha_n)$, $\alpha_j > 0$, $j = 1, ..., n$.

It is well-known that when $n = 1$ and $\alpha > 1$, the following are equivalent:

$$b_\alpha(f) < \infty \Leftrightarrow N(f)_{\mathcal{S}_{\alpha-1}} < \infty.$$

Lemma 3.1 [4]. *Let $\alpha > 1$. Then $\mathcal{B}^{\alpha}(D_n) \subset \mathcal{S}_{\vec{\alpha}-1}(D_n)$, where $\vec{\alpha} - 1 = (\alpha - 1, ..., \alpha - 1)$.*

Remark 2. The function $f(z_1, ..., z_n) = \prod_{k=1}^{n} \frac{c_k}{(1-z_k)^{\alpha-1}}$, shows that the inclusion in this Lemma is proper.

The main result in this section is the following theorem:

Theorem 3.2 *If $g_j \in \mathcal{B}, j = 1, ..., n$, then the space $\mathcal{S}_{\vec{\alpha}}$, $\alpha > 0$ is invariant for the operator $T_{\vec{g}}$ on the polydisk D_n. Moreover there is a constant C independent of f such that*

$$N(T_{\vec{g}}(f))_{\mathcal{S}_{\vec{\alpha}}} \cdot \ C\, N(f)_{\mathcal{S}_{\vec{\alpha}}}.$$

Proof. Let $f \in \mathcal{S}_{\vec{\alpha}}$. Then

$$\begin{aligned}
|T_{\vec{g}} f(z)| \ &\cdot \prod_{j=1}^{n} |z_j| \int_0^1 \cdots \int_0^1 |f(\tau \cdot z) \prod_{j=1}^{n} g_j'(\tau_j z_j)| d\tau \\
&\cdot \prod_{j=1}^{n} |z_j| \int_0^1 \cdots \int_0^1 \frac{|f(\tau \cdot z)| \prod_{j=1}^{n} (1 - \tau_j |z_j|)^{\alpha_j}}{\prod_{j=1}^{n} (1 - \tau_j |z_j|)^{\alpha_j + 1}} \\
&\quad \times \prod_{j=1}^{n} |g_j'(\tau_j z_j)| (1 - \tau_j |z_j|) d\tau \\
&\cdot \ N(f)_{\mathcal{S}_{\vec{\alpha}}} \prod_{j=1}^{n} |z_j| \ ||g_j||_{\mathcal{B}} \int_0^1 \cdots \int_0^1 \frac{1}{\prod_{j=1}^{n} (1 - \tau_j |z_j|)^{\alpha_j + 1}} d\tau \\
&= N(f)_{\mathcal{S}_{\vec{\alpha}}} \prod_{j=1}^{n} |z_j| \ ||g_j||_{\mathcal{B}} \prod_{j=1}^{n} \int_0^1 \frac{1}{(1 - \tau_j |z_j|)^{\alpha_j + 1}} d\tau_j \\
&\cdot \ N(f)_{\mathcal{S}_{\vec{\alpha}}} \prod_{j=1}^{n} \frac{||g_j||_{\mathcal{B}}}{\alpha_j} \prod_{j=1}^{n} \frac{1}{(1 - |z_j|)^{\alpha_j}}
\end{aligned} \tag{5}$$

from which the result follows with $C = N(f)_{\mathcal{S}_{\vec{\alpha}}} \prod_{j=1}^{n} \frac{||g_j||_{\mathcal{B}}}{\alpha_j}$.

From Lemma 3.1 and Theorem 3.2, one obtains the following corollary:

Corollary 3.3 *Let $\alpha > 1$. Then $T_{\vec{g}}$ is bounded operator from $\mathcal{B}^{\alpha}$ to $\mathcal{S}_{\vec{\alpha}-1}$. In particular, the Cesàro operator is bounded from the space $\mathcal{B}^{\alpha}$ into the space $\mathcal{S}_{\vec{\alpha}-1}$.*

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Der-Chen Chang
Department of Mathematics, Georgetown University,
Washington D.C., 20057, USA
e-mail: chang@math.georgetown.edu

Stevo Stević
Matematicki Fakultet, Studentski Trg 16, 11000 Beograd, Serbia, Yugoslavia
e-mail: sstevic@ptt.yu; sstevo@matf.bg.ac.yu